Absorption of Fixed Scalars and the D-brane Approach to Black Holes

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Abstract

We calculate the emission and absorption rates of fixed scalars by the near-extremal five-dimensional black holes that have recently been modeled using intersecting D-branes. We find agreement between the semi-classical and D-brane computations. At low energies the fixed scalar absorption cross-section is smaller than for ordinary scalars and depends on other properties of the black hole than just the horizon area. In the D-brane description, fixed scalar absorption is suppressed because these scalars must split into at least four, rather than two, open strings running along the D-brane. Consequently, this comparison provides a more sensitive test of the effective string picture of the D-brane bound state than does the cross-section for ordinary scalars. In particular, it allows us to read off the value of the effective string tension. That value is precisely what is needed to reproduce the near-extremal 5-brane entropy.

October 1996

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1. Introduction

Many conventional wisdoms of general relativity are being reconsidered in the context of string theory simply because the string effective actions for gravity coupled to matter are more general than those considered in the past. One of the important differences is the presence of non-minimal scalar–gauge field couplings, leading to a breakdown of the 'no hair' theorem (see the discussion in [1]). Another new effect is the existence of certain scalars which, in the presence of an extremal charged black hole with regular horizon [2,3,4,5], acquire an effective potential [6] which fixes their value at the horizon [7,8]. These are the fixed scalars. The absorption of fixed scalars into $D = 4$ extremal black holes was recently considered in [9] and found to be suppressed compared to ordinary scalars: whereas the absorption cross-section of the latter approaches the horizon area $A_{\rm h}$ as $\omega \to 0$ [10], the fixed scalar cross-section was found to vanish as ω^2.

The main result of this paper is the demonstration that the fixed scalar emission and absorption rates, as calculated using the methods of semi-classical gravity, are exactly reproduced by the effective string model of black holes based on intersecting D-branes. The D-brane description of the five-dimensional black holes involves n_1 1-branes and n_5 5-branes with some left-moving momentum along the intersection [11,12]. The low-energy dynamics of the resulting bound state is believed to be well described by an effective string wound $n_1 n_5$ times around the compactification volume [13,14,15,16,17]. This model has been successful in matching not only the extremal [11,12] and near-extremal [18,19,13] entropies, but the rate of Hawking radiation of ordinary scalars as well [15,16,17].

As part of our study, we have computed the semi-classical absorption cross-section of fixed scalars from both extremal and near-extremal $D = 5$ black holes. In general, we find cross-sections with a non-trivial energy dependence. In particular, for the extremal $D = 5$ black holes with two charges equal,

$$\sigma_{\rm abs} = \frac{\pi^2}{2} R^2 r_K^3 \omega^2 \frac{\frac{\omega}{2T_L}}{1 - e^{-\frac{\omega}{2T_L}}} \left(1 + \frac{\omega^2}{16\pi^2 T_L^2} \right)$$

where r_K, $R \gg r_K$ and T_L are parameters related to the charges. At low energies the cross-section vanishes as ω^2, just as in the $D = 4$ case studied in [9]. For non-extremal black holes, however, the cross-section no longer vanishes as $\omega \to 0$. For near-extremal $D = 5$ black holes, we find (for $\omega \sim T_H \ll T_L$)

$$\sigma_{\rm abs}(\omega) = \frac{1}{4} A_{\rm h} r_K^2 (\omega^2 + 4\pi^2 T_H^2) \ ,$$

where T_H is the Hawking temperature. A similar formula holds for the $D = 4$ case. Thus, even at low energies, the fixed scalar cross-section is sensitive to several features of the black hole geometry. By comparison, the limiting value of the ordinary scalar cross-section

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is given by the horizon area alone. All of the complexities of the fixed scalar emission and absorption will be reproduced by, and find a simple explanation in, the effective string picture.

The absorption cross-section for ordinary scalars finds its explanation in the D-brane description in terms of the process $scalar \to L+R$ together with its time-reversal $L+R \to scalar$, where L and R represent left-moving and right-moving modes on the effective string [12,14,15,16,17]. The absorption cross-section for fixed scalars is so interesting because, as we will show, it depends on the existence of eight kinematically permitted processes:

$$
\begin{aligned}
1) & \quad scalar \to L + L + R + R \\
2) & \quad scalar + L \to L + R + R \\
3) & \quad scalar + R \to L + L + R \\
4) & \quad scalar + L + R \to L + R
\end{aligned}
\tag{1}
$$

and their time-reversals. One of the main results of this paper is that competition among 1–4 and their time-reversals gives the following result for the fixed scalar absorption cross-section,

$$
\sigma_{\text{abs}}(\omega) = \frac{\pi r_1^2 r_5^2}{256 T_{\text{eff}}^2} \frac{\omega \left(e^{\frac{\omega}{T_H}} - 1 \right)}{\left(e^{\frac{\omega}{2 T_L}} - 1 \right) \left(e^{\frac{\omega}{2 T_R}} - 1 \right)} (\omega^2 + 16\pi^2 T_L^2)(\omega^2 + 16\pi^2 T_R^2) \ ,
\tag{2}
$$

where T_L and T_R are the left and right-moving temperatures, T_{eff} is the effective string tension [13,20], and r_1^2 and r_5^2 are essentially the 1-brane and 5-brane charges. The only restriction on the validity of (2) is that $T_L, T_R, \omega \ll 1/r_1 \sim 1/r_5$ so that we stay in the dilute gas regime and keep the wavelength of the fixed scalar much larger than the longest length scale of the black hole. Remarkably, the very simple effective string result (2) is in complete agreement with the rather complicated calculations in semi-classical gravity! The semi-classical calculations involve no unknown parameters, so comparison with (2) allows us to infer T_{eff}. The result is in agreement with the fractional string tension necessary to explain the entropy of near-extremal 5-branes [20].

To set up the semi-classical calculations, we will develop in section 2 an effective action technique for deriving the equations of motion for fixed scalars. This technique shows how the fixed scalar equation couples with Einstein's equations when $r_1 \neq r_5$; therefore, we restrict ourselves to the regime $r_1 = r_5 = R$ where the fixed scalar equation is straightforward. We briefly digress to four dimensions, demonstrating how the same techniques lead to similar equations for fixed scalars. Clearly, comparisons analogous to the ones made in this paper are possible for the four-dimensional case, where the effective string appears at the triple intersection of M-theory 5-branes [21]. In section 3 we use the

Dirac-Born-Infeld (DBI) action to see how various scalars in $D = 5$ couple to the effective string. The main result of section 3 is that the leading coupling of the fixed scalar is to *four* fluctuation modes of the string. This highlights its difference from the moduli which couple to two fluctuation modes. In section 4 we return to five dimensions and exhibit approximate solutions to the fixed scalar equation, deriving the semi-classical emission and absorption rates. In section 5 we calculate the corresponding rates with D-brane methods, finding complete agreement with semi-classical gravity. We conclude in section 6. In the Appendix we discuss the absorption rate as implied by the effective string action of section 3 of some other 'off-diagonal' scalars present in the system.

2. Field Theory Effective Action Considerations

1. $D = 5$ case

First we shall concentrate on the case of a $D = 5$ black hole representing the bound state of n_1 RR strings and n_5 RR 5-branes compactified on a 5-torus [12]. This black hole may be viewed as a static solution corresponding to the following truncation of type IIB superstring effective action compactified on 5-torus (cf. [22,23])

$$S_5 = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left[R - \frac{4}{3}(\partial_\mu \phi_5)^2 - \frac{1}{4} G^{pl} G^{qn} (\partial_\mu G_{pq} \partial^\mu G_{ln} + e^{2\phi_5} \sqrt{G} \partial_\mu B_{pq} \partial^\mu B_{ln}) \right. \tag{3}$$

$$\left. - \frac{1}{4} e^{-\frac{4}{3}\phi_5} G_{pq} F_{\mu\nu}^{(K)p} F_{\mu\nu}^{(K)q} - \frac{1}{4} e^{\frac{2}{3}\phi_5} \sqrt{G} G^{pq} H_{\mu\nu p} H_{\mu\nu q} - \frac{1}{12} e^{\frac{4}{3}\phi_5} \sqrt{G} H_{\mu\nu\lambda}^2 \right] ,$$

where $\mu, \nu, ... = 0, 1, .., 4; p, q, ... = 5, ..., 9$. ϕ_5 is the 5-d dilaton and G_{pq} is the metric of 5-torus,

$$\phi_5 \equiv \phi_{10} - \frac{1}{4} \ln G , \qquad G = \det G_{pq} ,$$

and B_{pq} are the internal components of the RR 2-form field. $F_{\mu\nu}^{(K)p}$ is the Kaluza-Klein vector field strength, while $H_{\mu\nu p}$ and $H_{\mu\nu\lambda}$ are given explicitly by [22]

$$H_{\mu\nu p} = F_{\mu\nu p} - B_{pq} F_{\mu\nu}^{(K)q} , \quad F_p = dA_p , \quad F^{(K)p} = dA^{(K)p} , \tag{4}$$

$$H_{\mu\nu\lambda} = \partial_\mu B'_{\nu\lambda} - \frac{1}{2} A_\mu^{(K)p} F_{\nu\lambda p} - \frac{1}{2} A_{\mu p} F_{\nu\lambda}^{(K)p} + \text{cycles} ,$$

where $A_{\mu p} = B_{\mu p} + B_{pq} A_\mu^{(K)q}$ and $B'_{\mu\nu} = B_{\mu\nu} + A_{[\mu}^{(K)p} A_{\nu]p} - A_\mu^{(K)p} B_{pq} A_\nu^{(K)q}$ are related to the components of the $D = 10$ RR 2-form field B_{MN}.

The 'shifts' in the field strengths in (4) will vanish for the black hole background considered below (for which the internal components of the 2-form B_{pq} will be zero and the two vector fields $A^{(K)p}$ and A_p will be electric), and, as it turns out, are also not relevant for the discussion of perturbations.

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For comparison, a similar truncated $D = 5$ action with B_{pq}, $F_{\mu\nu p}$ and $H_{\mu\nu\lambda}$ from the NS-NS sector has the following antisymmetric tensor terms (the full action in general contains both RR and NS-NS antisymmetric tensor parts) [22]

$$-\frac{1}{4}G^{pl}G^{qn}\partial_\mu B_{pq}\partial^\mu B_{ln} - \frac{1}{4}e^{-\frac{4}{3}\phi_5}G^{pq}H_{\mu\nu p}H_{\mu\nu q} - \frac{1}{12}e^{-\frac{8}{3}\phi_5}H_{\mu\nu\lambda}^2 \ .$$

We shall assume that there are non-trivial electric charges in only one of the five internal directions and that the metric corresponding to the internal 5-torus (over which the 5-brane will be wrapped) is

$$(ds_{10}^2)_{T^5} = e^{2\nu_5}dx_5^2 + e^{2\nu}(dx_6^2 + dx_7^2 + dx_8^2 + dx_9^2) \ , \tag{5}$$

where x_5 is the string direction and ν is the 'scale' of the four 5-brane directions transverse to the string. It is useful to introduce a different basis for the scalars, defining the 'six-dimensional' dilaton, ϕ, and the 'scale' λ of the x_5 (string) direction as measured in the $D = 6$ Einstein-frame metric:

$$\phi = \phi_{10} - 2\nu = \phi_5 + \frac{1}{2}\nu_5 \ , \quad \lambda = \nu_5 - \frac{1}{2}\phi = \frac{3}{4}\nu_5 - \frac{1}{2}\phi_5 \ . \tag{6}$$

The action (3) can be expressed either in terms of ϕ_5, ν_5, ν or ϕ, λ, ν (in both cases the kinetic term is diagonal). In the latter case (we set $B_{pq} = 0$)

$$S_5 = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\bigg[R - (\partial_\mu\phi)^2 - \frac{4}{3}(\partial_\mu\lambda)^2 - 4(\partial_\mu\nu)^2 \tag{7}$$

$$- \frac{1}{4}e^{\frac{8}{3}\lambda}F_{\mu\nu}^{(K)2} - \frac{1}{4}e^{-\frac{4}{3}\lambda+4\nu}F_{\mu\nu}^2 - \frac{1}{12}e^{\frac{4}{3}\lambda+4\nu}H_{\mu\nu\lambda}^2\bigg] \ .$$

Here $F_{\mu\nu}^{(K)} \equiv F_{\mu\nu}^{(K)5}$ is the KK vector field strength corresponding to the string direction, while $F_{\mu\nu} \equiv F_{\mu\nu 5}$ and $H_{\mu\nu\lambda}$ correspond the 'electric' (D1-brane) and 'magnetic' (D5-brane) components of the field strength of the RR 2-form field. Evidently ϕ is an ordinary 'decoupled' scalar while λ and ν are different: they interact with the gauge charges. We shall see that they are examples of the so-called 'fixed scalars'.

To study spherically symmetric configurations corresponding to this action it is sufficient to choose the five-dimensional metric in the '2+3' form

$$ds_5^2 = g_{mn}dx^mdx^n + ds_3^2 = -e^{2a}dt^2 + e^{2b}dr^2 + e^{2c}d\Omega_3^2 \ , \tag{8}$$

where a, b, c are functions of r and t. Solving first the equations for $H_{\mu\nu\lambda}$, $F_{\mu\nu}$ and $F_{\mu\nu}^{(K)}$ and assuming that the first two have, respectively, the electric and the magnetic components (with the charges P and Q corresponding to the D5-brane and the D1-brane), while the third has only the electric component with the Kaluza-Klein charge Q_K, we may

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eliminate them from the action (7). The result is an effective two-dimensional theory with coordinates $x^m = (t, r)$ and the action given (up to the constant prefactor) by[1]

$$S_2 = \int d^2x \sqrt{-g} e^{3c} \left[R + 6(\partial_m c)^2 - (\partial_m \phi)^2 - \frac{4}{3}(\partial_m \lambda)^2 - 4(\partial_m \nu)^2 + V(c, \nu, \lambda) \right]$$

$$= \int dt dr \left[-e^{3c+b-a}(6\dot{c}\dot{b} + 6\dot{c}^2 - \dot{\phi}^2 - \frac{4}{3}\dot{\lambda}^2 - 4\dot{\nu}^2) \right. \tag{9}$$

$$+ e^{3c+a-b}(6c'a' + 6c'^2 - \phi'^2 - \frac{4}{3}\lambda'^2 - 4\nu'^2)$$

$$\left. + 6e^{a+b+c} - 2e^{a+b-3c} f(\nu, \lambda) \right] .$$

The first term in the potential originates from the curvature of the 3-sphere while the second is produced by the non-trivial charges,

$$f(\nu, \lambda) = Q_K^2 e^{-\frac{8}{3}\lambda} + e^{\frac{4}{3}\lambda}(P^2 e^{4\nu} + Q^2 e^{-4\nu}) . \tag{10}$$

This is a special case of the more general expression following from (3): if the electric charges corresponding to the vector fields in (3) are Q_{Kp} and Q^p we get

$$f(\phi_5, G_{pq}) = e^{\frac{4}{3}\phi_5} Q_{Kp} Q_{Kq} G^{pq} + e^{-\frac{2}{3}\phi_5} \left(P^2 G^{1/2} + Q^p Q^q G_{pq} G^{-1/2} \right) . \tag{11}$$

The potential f in (10) has the global minimum at $e^{4\nu} = QP^{-1}$, $e^{4\lambda} = Q_K^2 Q^{-1} P^{-1}$. These values of ν and λ are thus 'fixed points' to which these fields are attracted on the horizon, which is why such fields can be called 'fixed scalars.' By contrast, the decoupled scalar ϕ can be chosen to be equal to an arbitrary constant.

As an aside, we note that this structure of the potential (10) explains why one needs at least *three* different charges to get an extremal $D = 5$ black hole with a regular horizon (i.e. with scalar fields that do not blow up): it is necessary to have at least three exponential terms to 'confine' the two fixed scalars. If the number of non-vanishing charges is smaller than three, then one or both scalars will blow up at the horizon.

Equivalent actions and potentials are found for theories that are obtained from the one above by U-duality. For example, in the case of the NS-NS truncation of type II action, which has a $D = 5$ black hole solution representing a bound state of NS-NS strings

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[1] The full set of equations and constraints is derived by first keeping the 2-d metric g_{mn} general and using its diagonal gauge-fixed form only after the variation. In addition to choosing g_{mn} diagonal as in (8), one can use the gauge freedom to impose one more relation between a and b.

and solitonic 5-branes, we can put the action in the form (9), where λ is still the scale of the string direction as measured by the 6-d metric, while the roles of 2ν (the scale of the 4-torus) and $-\phi$ are interchanged.[2]

In order to find the static black hole solution to (9), we define $\rho = 2c + a$, $d\tau = -2e^{-3c-a+b}dr$. Now (9) reduces to a 'particle' action (we choose $\phi = $ const)

$$S_1 = \int d\tau \left[\frac{3}{2}(\partial_\tau \rho)^2 - \frac{3}{2}(\partial_\tau a)^2 - \frac{4}{3}(\partial_\tau \lambda)^2 - 4(\partial_\tau \nu)^2 + \frac{3}{2}e^{2\rho} - \frac{1}{2}e^{2a}f(\nu,\lambda) \right] , \qquad (12)$$

which should be supplemented by the 'zero-energy' constraint,

$$\frac{3}{2}(\partial_\tau \rho)^2 - \frac{3}{2}(\partial_\tau a)^2 - \frac{4}{3}(\partial_\tau \lambda)^2 - 4(\partial_\tau \nu)^2 - \frac{3}{2}e^{2\rho} + \frac{1}{2}e^{2a}f(\nu,\lambda) = 0 .$$

The special structure of f in (10) makes it possible to find a simple analytic solution of this 'Toda-type' system. Introducing new variables $\alpha = a - \frac{4}{3}\lambda$, $\beta = a + \frac{2}{3}\lambda + 2\nu$, $\gamma = a + \frac{2}{3}\lambda - 2\nu$ and using the special form (10) of f, we can convert (12) to four non-interacting Liouville-like systems (related only through the constraint)

$$S_1 = \int d\tau \left[\frac{3}{2}(\partial_\tau \rho)^2 - \frac{1}{2}(\partial_\tau \alpha)^2 - \frac{1}{2}(\partial_\tau \beta)^2 - \frac{1}{2}(\partial_\tau \gamma)^2 \right. \qquad (13)$$

$$\left. + \frac{3}{2}e^{2\rho} - \frac{1}{2}Q_K^2 e^{2\alpha} - \frac{1}{2}P^2 e^{2\beta} - \frac{1}{2}Q^2 e^{2\gamma} \right] .$$

The general solution depends on the three gauge charges P, Q, Q_K and one parameter which we will call μ which governs the degree of non-extremality. In a convenient gauge, the solution reads [24,12,5,19]

$$e^{2a} = h\mathcal{H}^{-2/3} , \qquad e^{2b} = h^{-1}\mathcal{H}^{1/3} , \qquad e^{2c} = r^2\mathcal{H}^{1/3} , \qquad \mathcal{H} \equiv H_{\hat{P}}H_{\hat{Q}}H_{\hat{Q}_K} , \qquad (14)$$

$$e^{2\lambda} = H_{\hat{Q}_K}(H_{\hat{Q}}H_{\hat{P}})^{-\frac{1}{2}} , \qquad e^{4\nu} = H_{\hat{Q}}H_{\hat{P}}^{-1} , \qquad e^{2\phi} = e^{2\phi_{10,\infty}} , \qquad (15)$$

$$h = 1 - \frac{2\mu}{r^2} , \qquad H_{\hat{q}} = 1 + \frac{\hat{q}}{r^2} , \qquad \hat{q} \equiv \sqrt{q^2 + \mu^2} - \mu , \qquad q = (P, Q, Q_K) .$$

We have chosen the asymptotic values λ_∞ and ν_∞ to be zero. To compare with previous equations, we also note that $e^{2\rho} = r^2(r^2 - 2\mu)$.

In the extremal limit, $\mu = 0$, one finds

$$e^{-\alpha} = H_{Q_K}, \qquad e^{-\beta} = H_P, \qquad e^{-\gamma} = H_Q ,$$

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[2] There exists an equivalent representation of this NS-NS action where the fixed scalars are the 5-d dilaton and the scale of the string direction, while the scale of the 4-torus is decoupled.

where $H_q = c_q + q\tau$ and $\tau = 1/r^2$. The constants c_{Q_K}, c_P, c_Q must satisfy $c_{Q_K} c_P c_Q = 1$ in order for the 5-d metric to approach the Minkowski metric at infinity. The two remaining arbitrary constants correspond to the asymptotic values of λ and ν. As is clear from (9),(10), shifting λ and ν by constants is equivalent to a rescaling of Q_K, Q, P. The assumption that $\nu_\infty = 0$ and $\lambda_\infty = \nu_{5\infty} + \nu_\infty - \frac{1}{2}\phi_{10,\infty} = 0$ implies (setting $\alpha' = 1$):

$$V_4 = e^{4\nu_\infty} = 1 \ , \qquad \mathcal{R}^2 = e^{2\nu_{5\infty}} = g = e^{\phi_{10,\infty}} \ , \qquad \kappa_5^2 = \frac{2\pi^2 g^2}{\mathcal{R}V_4} \ , \tag{16}$$

where V_4 is the volume of T^4 in the (6789) directions, while $\mathcal{R}$ is the radius of the circle in direction 5. Then the 'charges' Q_K, Q, P are related to the quantized charges n_1, n_5, n_K as follows:

$$n_1 = \frac{V_4 Q}{g} = \frac{Q}{g} \ , \qquad n_5 = \frac{P}{g} \ , \qquad n_K = \frac{\mathcal{R}^2 V_4 Q_K}{g^2} = \frac{Q_K}{g} \ . \tag{17}$$

The somewhat unusual form of the last relation is due to our choice $\lambda_\infty = 0$ instead of more standard $\nu_{5\infty} = 0$.

In using the black hole solution (14), (15), we will often find it convenient to work in terms of characteristic radii rather than the charges, so we define

$$r_1^2 = \hat{Q} \ , \qquad r_5^2 = \hat{P} \ , \qquad r_K^2 = \hat{Q}_K \ , \qquad r_0^2 = 2\mu \ . \tag{18}$$

From the classical GR point of view, these parameters can take on any values. Recent experience has shown, however, that when the radii satisfy [17]

$$r_0, r_K \ll r_1, r_5 \tag{19}$$

the black hole can be successfully matched to a bound state of D1-branes and D5-branes (with no antibranes present) carrying a dilute gas of massless excitations propagating along the bound D1-branes. Evidence for this gas can be seen directly in the energy, entropy and temperature of the black hole solution. Introducing a new parameter σ through

$$r_K^2 = r_0^2 \sinh^2 \sigma$$

one finds the following expressions [19,17] for the ADM mass, Hawking temperature and the entropy in the parameter region (19):

$$M = \frac{2\pi^2}{\kappa_5^2}\left(r_1^2 + r_5^2 + \frac{1}{2}r_0^2 \cosh 2\sigma\right) \ , \qquad T_H^{-1} = \frac{2\pi r_1 r_5}{r_0} \cosh \sigma \ , \tag{20}$$

$$\mathrm{S} = \frac{2\pi A_\mathrm{h}}{\kappa_5^2} = \frac{4\pi^3}{\kappa_5^2} r_1 r_5 r_0 \cosh \sigma \ .$$

The entropy and energy are those of a gas of massless one-dimensional particles with the left-movers and right-movers each having its own temperature [17]:

$$T_L = \frac{r_0 e^\sigma}{2\pi r_1 r_5} \ , \qquad T_R = \frac{r_0 e^{-\sigma}}{2\pi r_1 r_5} \ . \tag{21}$$

The Hawking temperature is related to these two temperatures by

$$\frac{2}{T_H} = \frac{1}{T_L} + \frac{1}{T_R} \ , \tag{22}$$

a fact which also has a natural thermodynamic interpretation. These results will be heavily used in later comparisons of classical GR results with D-brane calculations of corresponding quantities.

Let us now turn to the discussion of the propagation of perturbations on this black hole background. The goal will be to calculate the classical absorption cross-section of various scalar fields and eventually to compare them with comparable D-brane quantities. The behavior of 'free' scalars, like ϕ, is quite different from that of 'fixed' scalars, like λ and ν. The spherically symmetric fluctuations of ϕ obey the standard massless Klein-Gordon equation in this background. Namely, if $\delta\phi = e^{i\omega t}\tilde{\phi}(r)$, then

$$\left[r^{-3} \frac{d}{dr} (hr^3 \frac{d}{dr}) + \omega^2 h^{-1} H_{\hat{P}} H_{\hat{Q}} H_{\hat{Q}_K} \right] \tilde{\phi} = 0 \ , \tag{23}$$

This scattering problem, and its D-brane analog, have been analyzed at length recently and we will have no more to say about it. The spherically symmetric fluctuations of the metric functions a, b, c and the scalars λ, ν in general obey a complicated set of coupled differential equations.[3] However, when the charges P and Q are set equal, a dramatic simplification occurs: the background value of ν in (15) (i.e. the 'scale' of the transverse 4-torus) becomes constant and its small fluctuations $\delta\nu$ decouple from those of the other fields.[4] The gaussian effective action for $\delta\nu$ extracted from (9) is

$$\delta S_2 = \int d^2x \sqrt{-g} e^{3c} \left[-4(\partial_m \delta\nu)^2 - 32P^2 e^{-6c+\frac{4}{3}\lambda}(\delta\nu)^2 + ... \right] \tag{24}$$

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[3] The spherically symmetric fluctuations of the gauge fields need not be considered explicitly: since the dependence on $H_{\mu\nu\lambda}$ and $F_{\mu\nu}$ is gaussian, they are automatically included when going from (7) to (9).

[4] Similar simplification occurs when any two of the three charges are equal. For example, if $P = Q_K$ we may introduce $\lambda' = -\frac{1}{2}(\lambda - 3\nu)$, $\nu' = -\frac{1}{2}(\nu + \lambda)$ (in terms of which the kinetic part in the action (9) preserves its diagonal form) to discover that ν' has decoupled fluctuations. The resulting equation for $\delta\nu'$ has the same form as the equation for $\delta\nu$ in the case of $P = Q$.

and spherically symmetric fluctuations $\delta\nu = e^{i\omega t}\tilde{\nu}$ obey

$$\left[r^{-3}\frac{d}{dr}(hr^3\frac{d}{dr}) + \omega^2 h^{-1}H_{\hat{P}}^2 H_{\hat{Q}_K} - 8P^2 r^{-6}H_{\hat{P}}^{-2} \right]\tilde{\nu} = 0 \ . \tag{25}$$

This is the standard Klein-Gordon equation (23) augmented by a space-dependent mass term originating from the expansion of the effective potential $f(\nu,\lambda)$ in (10). This mass term falls off as r^{-6} at large r, and, in the extremal case, blows up like $8/r^2$ near the horizon at $r = 0$. This is the $l(l+2)/r^2$ angular momentum barrier for an $l = 2$ partial wave in $D = 5$. This 'transmutation' of angular momentum plays an important role in the behavior of the fixed scalar cross-section. For later analysis, it will be convenient to rewrite this equation using the coordinate $\tau = 1/r^2$:

$$\left[[(1 - 2\mu\tau)\frac{d}{d\tau}]^2 + \frac{1}{4}\omega^2\tau^{-3}(1 + \hat{P}\tau)^2(1 + \hat{Q}_K\tau) - 2\frac{P^2(1 - 2\mu\tau)}{(1 + \hat{P}\tau)^2} \right]\tilde{\nu} = 0 \ . \tag{26}$$

Remarkably, the extremal fixed scalar equation is *identical* to the equation for the fluctuations of the components of the antisymmetric tensor, B_{ij}, in the uncompactified spatial directions. Taking $\mu = 0$, $i, j, k = 1, 2, 3, 4$, and making the appropriate reduction of (7),(14), we find

$$\delta S_5 \sim \int dt dr r^3 e^{-a + \frac{4}{3}\lambda + 4\nu}\left[-(\partial_t \delta B_{ij})^2 + e^{3a}\partial_k \delta B_{ij}\partial_k \delta B_{ij} + ... \right] \ . \tag{27}$$

Defining

$$\delta B_{ij} = e^{-a - \frac{2}{3}\lambda - 2\nu}C_{ij} = H_P^{l/2}C_{ij}, \qquad l = 2 \ ,$$

we obtain the following equation for $C_{ij}(r,t) = e^{i\omega t}\tilde{C}_{ij}(r)$

$$\left[r^{-3}\frac{d}{dr}(r^3\frac{d}{dr}) + \omega^2 H_P H_Q H_{Q_K} - 8P^2 r^{-6}H_P^{-2} \right]\tilde{C}_{ij} = 0 \ . \tag{28}$$

Note that the mass term comes from

$$H_P^{l/2}r^{-3}\frac{d}{dr}(r^3\frac{d}{dr})H_P^{-l/2} = l(l+2)P^2 r^{-6}H_P^{-2} = 8P^2 r^{-6}H_P^{-2}$$

and turns out to be the same as in the extremal limit of (25). Had we started with a vector field in $D = 5$ we would instead have $l = 1$ and the mass term would be $3P^2 r^{-6}H_P^{-2}$. We conjecture that the antisymmetric tensor B_{ij} is related to the fixed scalar by the residual supersymmetry of the extremal black hole background. As we discuss in the next section, a similar identity holds in $D = 4$ between the fixed scalar and the vector, A_i, equations. [5]

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[5] This raises the question of why the supersymmetry explanation applies to the scalar-tensor pair, but not to the scalar-vector one in $D = 5$. The answer presumably lies in the 'electric' nature of the two vector fields in (7). In general, the equations for spherically-symmetric perturbations in a non-trivial background need not be invariant under S-duality relating $B_{\mu\nu}$ and A_μ in $D = 5$.

Note that, when all the three charges are equal, $P = Q = Q_K$, the background value of the other scalar, λ, is constant as well. Then the small fluctuations of this field decouple from gravitational perturbations and satisfy the same equation as ν, (25). If only two of the charges are equal, then there is only one fixed scalar which has a constant background value and decouples from gravitational perturbations. We would also like to know the fixed scalar scattering equations (and solutions) for the general $Q_K \neq Q \neq P$ black hole. This problem is surprisingly complicated due to mixing with gravitational perturbations, and we have yet to solve it.

To summarize, we have identified a set of scalars around the familiar type II string $D = 5$ black hole solution which merit the name of 'fixed scalars' in that their horizon values are fixed by the background charges. Their fluctuations in the black hole background satisfy the Klein-Gordon equation, augmented by a position-dependent mass term. In section 4 we will solve the new equations to find the absorption cross-section by the black hole for these special scalars.

2. $D = 4$ case

Previous experience [25,26,21,27,16,28] suggests that one may be able to extend the $D = 5$ successes in reproducing entropies and radiation rates with D-brane methods to $D = 4$ black holes carrying 4 charges. Although we will not pursue the p-brane approach to $D = 4$ black hole dynamics in this paper, this is a natural place to discuss $D = 4$ fixed scalars and to record their scattering equations for later use.

A convenient representation of the $D = 4$ black hole with four different charges [3,4] is the $D = 11$ supergravity configuration $5\perp5\perp5$ of three 5-branes intersecting over a common string [21,29]. The three magnetic charges are related to the numbers of 5-branes in three different hyperplanes, while the electric charge has Kaluza-Klein origin. The reduction to $D = 4$ of the relevant part of the $D = 11$ supergravity (or $D = 10$ type IIA) action has the form

$$S_4 = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g} \left[R - 2(\partial_\mu \nu)^2 - \frac{3}{2}(\partial_\mu \zeta)^2 - \frac{4}{3}(\partial_\mu \xi)^2 - (\partial_\mu \eta)^2 \right. \tag{29}$$

$$\left. - \frac{1}{4} e^{3\zeta}(F_{\mu\nu}^{(K)})^2 - \frac{1}{4} e^{\zeta} \left(e^{-\frac{8}{3}\xi}(F_{\mu\nu}^{(1)})^2 + e^{\frac{4}{3}\xi}[e^{2\eta}(F_{\mu\nu}^{(2)})^2 + e^{-2\eta}(F_{\mu\nu}^{(3)})^2] \right) \right] .$$

The scalar fields are expressed in terms of components of the internal 7-torus part of the $D = 11$ metric. By the logic of the previous section, the 'scale' ν of the 6-torus transverse to the intersection string is a decoupled scalar, while the fields ζ, ξ, η (related to the scale of the string direction and the ratios of sizes of 2-tori shared by pairs of 5-branes) are fixed scalars. If the internal part of the $D = 11$ metric is

$$ds_7^2 = e^{2\nu_4}dx_4^2 + e^{2\nu_1}(dx_5^2 + dx_6^2) + e^{2\nu_2}(dx_7^2 + dx_8^2) + e^{2\nu_3}(dx_9^2 + dx_{10}^2) ,$$

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where x_4 is the direction of the common string, then

$$\nu = \nu_1 + \nu_2 + \nu_3, \quad \xi = \nu_1 - \frac{1}{2}\nu_2 - \frac{1}{2}\nu_3, \quad \eta = \nu_3 - \nu_2, \quad \zeta = \nu_4 + \frac{2}{3}(\nu_1 + \nu_2 + \nu_3).$$

Using an ansatz for the 4-d metric similar to (8), solving for the vector fields, and substituting the result back into the action, we get the following effective two-dimensional action (cf. (9))

$$S_2 = \int d^2 x \sqrt{-g} e^{2c} \left[R + 2(\partial_m c)^2 - 2(\partial_\mu \nu)^2 - \frac{3}{2}(\partial_m \zeta)^2 - \frac{4}{3}(\partial_m \xi)^2 - (\partial_m \eta)^2 \right. \tag{30}$$

$$\left. + 2e^{-2c} - \frac{1}{2} e^{-4c} f(\zeta, \xi, \eta) \right] ,$$

where

$$f(\zeta, \eta, \xi) = Q_K^2 e^{-3\zeta} + e^\zeta \left[P_1^2 e^{-\frac{8}{3}\xi} + e^{\frac{4}{3}\xi}(P_2^2 e^{2\eta} + P_3^2 e^{-2\eta}) \right] . \tag{31}$$

As in the $D = 5$ case, one finds that the special structure of f makes it possible to diagonalise the interaction term by a field redefinition and thus find the static solution in a simple factorised form [3,4,29] (cf. (14))

$$ds_4^2 = -e^{2a} dt^2 + e^{2b} dr^2 + e^{2c} d\Omega_2^3 , \tag{32}$$

$$e^{2a} = h\mathcal{H}^{-1/2} , \quad e^{2b} = h^{-1}\mathcal{H}^{1/2} , \quad e^{2c} = r^2 \mathcal{H}^{1/2} , \quad \mathcal{H} \equiv H_{\hat{Q}_K} H_{\hat{P}_1} H_{\hat{P}_2} H_{\hat{P}_3} ,$$

$$e^{2\eta} = H_{\hat{P}_3} H_{\hat{P}_2}^{-1} , \quad e^{2\xi} = H_{\hat{P}_1}(H_{\hat{P}_2} H_{\hat{P}_3})^{-1/2} , \quad e^{2\zeta} = H_{\hat{Q}_K}(H_{\hat{P}_1} H_{\hat{P}_2} H_{\hat{P}_3})^{-1/3}, \tag{33}$$

$$h = 1 - \frac{2\mu}{r} , \quad H_{\hat{q}} = 1 + \frac{\hat{q}}{r} , \quad \hat{q} \equiv \sqrt{q^2 + \mu^2} - \mu , \quad q = (Q_K, P_1, P_2, P_3) .$$

As in the $D = 5$ case, for the generic values of charges the spherically symmetric perturbations of this solution obey a complicated system of equations (for discussions of perturbations of single-charged dilatonic black holes see, e.g., [30]). However, when the three magnetic charges are equal, η and ξ have constant background values, and so their small spherically-symmetric fluctuations decouple from the metric perturbations,

$$\delta S_2 = \int d^2 x \sqrt{-g} e^{2c} \left[-(\partial_m \delta\eta)^2 - 2P^2 e^{-4c+\zeta+\frac{4}{3}\xi}(\delta\eta)^2 + ... \right] , \tag{34}$$

leading to the following radial Klein-Gordon equation with an extra mass term ($\delta\eta(r,t) = e^{i\omega t}\tilde{\eta}(r)$; cf. (25))

$$\left[r^{-2}\frac{d}{dr}(hr^2\frac{d}{dr}) + \omega^2 h^{-1} H_{\hat{P}}^3 H_{\hat{Q}_K} - 2P^2 r^{-4} H_{\hat{P}}^{-2} \right] \tilde{\eta} = 0 . \tag{35}$$

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The same universal equation is found for $\delta\xi$. In terms of $\tau = 1/r$ this becomes

$$\left[[(1 - 2\mu\tau)\frac{d}{d\tau}]^2 + \omega^2\tau^{-2}(1 + \hat{P}\tau)^3(1 + \hat{Q}_K\tau) - 2\frac{P^2(1 - 2\mu\tau)}{(1 + \hat{P}\tau)^2} \right] \tilde{\eta} = 0 \ . \qquad (36)$$

Represented in this form this is very similar to (26) found in the $D = 5$ case: the differential operator and mass terms are exactly the same, while the frequency terms are related by $\omega \to 2\omega$, $\tau^{-3}(1 + \hat{P}\tau)^2 \to \tau^{-2}(1 + \hat{P}\tau)^3$.

In the extremal case and with all four charges chosen to be equal, $Q_K = P$, (35) reduces to the equation studied in [9]. The characteristic coefficient 2 in the mass term gives the effective potential of the form $l(l + 1)/r^2$ near the horizon, with $l = 1$. Away from the horizon, the fixed scalar equation differs from that of the $l = 1$ partial wave of the ordinary scalar. Remarkably, however, in the extremal limit the fixed scalar equation (35) is *identical* to that for the vector perturbations in the extremal black hole background. This is true not only when all charges are equal (so that all scalars have constant background values) but also in the above case of $P_i = P \neq Q_K$. Consider perturbations $\delta A_i(r, t)$ ($i = 1, 2, 3$; $\delta A_0 = 0$, $\nabla_i A^i = \partial_i A_i = 0$) of any of the three 'magnetic' vector fields in (29),

$$\delta S_4 \sim \int dtdr r^2 e^\zeta \left[-e^{-2a}(\partial_t \delta A_i)^2 + e^{2a}\partial_i \delta A_j \partial_i \delta A_j + ... \right] \ . \qquad (37)$$

Redefining the field to absorb the prefactor and using (33), $\delta A_i = e^{-a - \frac{1}{2}\zeta}C_i = H_P^l C_i$, $l = 1$, we obtain the Klein-Gordon-type equation for $C_i(r, t)$ with an extra mass term

$$H_P^l \Delta_3 H_P^{-l} = l(l + 1)P^2 r^{-4} H_P^{-2} = 2P^2 r^{-4} H_P^{-2} \ , \qquad (38)$$

which is exactly the same as in (35) in the $\mu = 0$ limit.

This immediately implies that the absorption cross-section for the fixed scalars should, in the extremal case, have the same soft behavior $\sim \omega^2$ (see [9] and below) as the vector cross-section [31]. The two cross-sections differ, however, in non-extremal case. Indeed, using methods similar to those in section 4, we find (for $\omega \sim T_H$)

$$\sigma_{\rm abs} = A_{\rm h}\hat{P}\hat{Q}_K(\omega^2 + 4\pi^2 T_H^2) \ , \qquad (39)$$

which no longer vanishes at $\omega = 0$. The facts presented above are consistent with a possible explanation of the relation between the coupled scalar and vector perturbations as being due to the residual supersymmetry (the unbroken 1/8 of maximal supersymmetry [3]) present in the black hole background in the extremal limit.

Finally, let us note that there exist other representations of the 4-charge $D = 4$ black hole. For example in the case of the $2\perp 2\perp 5\perp 5$ representation [21], or, equivalently, the U-dual $D = 4$ configuration in the NS-NS sector with two (electric and magnetic) charges

coming from the $D = 10$ antisymmetric tensor and two (electric and magnetic) charges being of Kaluza-Klein origin, we may parametrize the metric as

$$ds_{10}^2 = e^{2\nu_4} dx_4^2 + e^{2\nu_5} dx_5^2 + e^{2\nu}(dx_6^2 + dx_7^2 + dx_8^2 + dx_9^2) \ .$$

This leads to the effective Lagrangian related to the above one (30) by a linear field redefinition and re-interpretation of the charges. The potential is (cf. (31))

$$f(\phi, \nu_4, \nu_5) = e^{2\phi}(Q_K^2 e^{2\nu_4} + Q^2 e^{-2\nu_4}) + e^{-2\phi}(P_K^2 e^{2\nu_5} + P^2 e^{-2\nu_5}) \ ,$$

where ϕ is the 4-d dilaton. This shows that the scale of the remaining 4-torus, ν, decouples.

3. Effective String Couplings

We now turn to a discussion of the effective action governing the absorption and emission of fixed scalars by the bound state of D1- and D5-branes. We use the same framework as the recent demonstrations of agreement between GR and D-brane treatments of the absorption of generic decoupled scalars [12,14,15,16]. We assume that: (i) the $D = 5$ black hole is equivalent to n_1 D1-branes bound to n_5 D5-branes, with some left-moving momentum; (ii) that the low-energy dynamics of this system is described by the DBI action for a string with an effective tension T_{eff}, and (iii) that the relevant bosonic oscillations of this effective string are only in the four 5-brane directions ($i = 6, 7, 8, 9$) transverse to the 1-brane. These assumptions serve to specify the detailed couplings of external closed string fields, in particular the fixed scalars, to the D-brane degrees of freedom. This is an essential input to any calculation of absorption and emission rates and, as we shall see, brings fairly subtle features of the effective action into play.

Specifically, we assume that the low-energy excitations of our system are described by the standard D-string action

$$I = -T_{\text{eff}} \int d^2\sigma \ e^{-\phi_{10}} \sqrt{-\det \gamma_{ab}} + ... \ , \quad \gamma_{ab} = G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu \ , \quad (40)$$

where ϕ_{10} and $G_{\mu\nu}$ are the $D = 10$ dilaton and string-frame metric. The specific dependence on ϕ_{10} is motivated by the expected $1/g_{str}$ behavior of the D-string tension. The normalization constant of the tension, T_{eff}, is subtle and will be discussed later. Our goal is to read off the couplings between excitations of the effective string and the fluctuations of the metric and dilaton that correspond to the fixed scalars.

It should be noted that the essential structure of the effective string action we are interested in can be, at least qualitatively, understood using semi-classical effective field theory methods. A straightforward generalization of the extremal classical solution [24,12]

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describing a BPS bound state of a string and 5-brane in which the string is localized on the 5-brane[6] has the $D = 10$ string metric ($m = 1, 2, 3, 4; i = 6, 7, 8, 9$)

$$ds_{10}^2 = H_1^{-1/2} H_5^{-1/2} (-dt^2 + dx_5^2) + H_1^{1/2} H_5^{1/2} dx_m dx_m + H_1^{1/2} H_5^{-1/2} dx_i dx_i \ ,$$

where $H_5 = H_5(x_m) = 1 + P/x_m^2$ and $H_1 = H_1(x_m, x_i)$ is a solution of

$$[\partial^m \partial_m + H_5(x_n) \partial^i \partial_i] H_1 = 0$$

such that for $P \to 0$ it approaches the standard string harmonic function, $H_1 \to 1 + Q/(x_i^2 + x_m^2)^3$. If one averages over the x_i-dependence of H_1 one returns to the original 'delocalised' case, $H_1 = 1 + Q/x_i^2$, which corresponds to the extremal limit of the $D = 5$ black hole (14),(15) with $Q_K = 0$ (here we consider the 'unboosted' string). The presence of the 5-brane breaks the $O(1,1) \times O(8)$ symmetry of the standard RR string solution down to $O(1,1) \times O(4) \times O(4)$. Since the localized solution also breaks 4+4 translational invariances, the string soliton has 4+4 collective coordinates: $X^m(x_5, t), X^i(x_5, t)$. The corresponding $O(4) \times O(4)$ invariant effective string action thus should have the following form in the static gauge,

$$I = \int d^2\sigma [T_0 + T_\parallel \partial^a X^i \partial_a X^i + T_\perp \partial^a X^m \partial_a X^m + ...] \ .$$

The constants $T_0, T_\parallel, T_\perp$ can be determined using standard methods (see, e.g., [32]) by substituting the perturbed solution into the $D = 10$ effective field theory action, etc. T_0 is proportional to the ADM mass of the background, $T_0 \sim P + Q$. The same should be true also for $T_\perp$, $T_\perp \sim P + Q$, since X^m describe oscillations of the whole bound state in the common transverse 4-space. At the same time, $T_\parallel$ is the effective tension of the string within the 5-brane, so that $T_\parallel \sim Q$. In the special cases $P = 0$ and $Q = 0$ these expressions are in obvious agreement with the standard results for a free string and a free 5-brane. In the case when $P \gg Q$, i.e. $n_5 \gg n_1$, we learn that $T_\perp \gg T_\parallel$, so that oscillations of the string in the four directions X^m transverse to the 5-brane can be ignored. If we further assume, following [13,33], that the string lying within the 5-brane has the effective length $L_{\text{eff}} \sim QP \sim n_1 n_5$, we finish with the following expression for the effective string tension in the relevant directions parallel to the 5-brane: $T_{\text{eff}} \sim T_\parallel / L_{\text{eff}} \sim 1/P \sim 1/n_5$. This picture is consistent with that suggested in [13,20] and will pass a non-trivial test below.

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[6] Instead of talking about a bound state of several single-charged D-strings and D5-branes with coinciding centers, it is sufficient, at the classical level, to consider just a single string and a single 5-brane having charges $Q \sim n_1$ and $P \sim n_5$.

In accord with the assumption (iii), we thus drop terms involving derivatives of $X^m = X^{1,2,3,4}$ (i.e. motions in the uncompactified directions). We also eliminate two more string coordinates by choosing the static gauge $X^0 = \sigma^0$, $X^5 = \sigma^1$ and write

$$\gamma_{ab} \equiv \eta_{ab} + \hat{\gamma}_{ab} = G_{ab}(x) + 2G_{i(a}(x)\partial_{b)}X^i + G_{ij}(x)\partial_a X^i \partial_b X^j \ . \tag{41}$$

We make the Kaluza-Klein assumption that the background fields ϕ_{10} and $G_{\mu\nu}$ depend only on the external coordinates $x^m \equiv X^m, m = 0, 1, ..., 5$. Since we are interested in linear absorption and emission processes, we make a weak-field expansion in powers of ϕ_{10} and $h_{mn} \equiv G_{mn} - \eta_{mn}$, splitting h_{ij} into traceless and trace parts: $h_{ij} = \bar{h}_{ij} + \frac{1}{4}\delta_{ij}h$. Finally, we distinguish L and R string excitations by introducing $\partial_+ = \partial_0 + \partial_1$, $\partial_- = -\partial_0 + \partial_1$. We can then use the formula

$$\sqrt{-\det(\eta_{ab} + \hat{\gamma}_{ab})} = 1 + \frac{1}{2}\hat{\gamma}_{+-} - \frac{1}{8}\hat{\gamma}_{++}\hat{\gamma}_{--} + \frac{1}{16}\hat{\gamma}_{+-}\hat{\gamma}_{++}\hat{\gamma}_{--} + \dots$$

to expand (40), finding the following action for X^i:

$$I_X = -T_{\text{eff}} \int d^2\sigma \left[1 + \frac{1}{2}\partial_+ X\partial_- X - \frac{1}{8}(\partial_+ X)^2(\partial_- X)^2 + \dots \right. \tag{42}$$

$$\left. + L_0 + L_1 + L_2 + L_3 + L_4 + L_4' + \dots \right] \ ,$$

$$L_0 = \frac{1}{2}(h_{55} - h_{00}) - \phi_{10} \ , \qquad L_1 = \frac{1}{2}h_{5i}(\partial_+ + \partial_-)X^i \ , \tag{43}$$

$$L_2 = -\frac{1}{2}\phi\partial_+ X^i \partial_- X^i + \frac{1}{2}\bar{h}_{ij}\partial_+ X^i \partial_- X^j - \frac{1}{8}(h_{00} + h_{55})[(\partial_+ X)^2 + (\partial_- X)^2] \ , \tag{44}$$

$$L_3 = -\frac{1}{4}h_{5i}[\partial_- X^i(\partial_+ X)^2 + \partial_+ X^i(\partial_- X)^2] \ , \tag{45}$$

$$L_4 = \frac{1}{8}(\phi_{10} - \frac{1}{2}h)(\partial_+ X)^2(\partial_- X)^2 = \frac{1}{8}\phi(\partial_+ X)^2(\partial_- X)^2 - \frac{1}{4}\nu(\partial_+ X)^2(\partial_- X)^2 \ , \tag{46}$$

$$L_4' = \frac{1}{16}(h_{55} + h_{00})\partial_+ X\partial_- X[(\partial_+ X)^2 + (\partial_- X)^2] + \frac{1}{16}(h_{55} - h_{00})(\partial_+ X)^2(\partial_- X)^2$$

$$= \frac{1}{8}\lambda[\partial_+ X\partial_- X((\partial_+ X)^2 + (\partial_- X)^2) + (\partial_+ X)^2(\partial_- X)^2] \tag{47}$$

$$+ \frac{1}{16}\phi[\partial_+ X\partial_- X((\partial_+ X)^2 + (\partial_- X)^2) + (\partial_+ X)^2(\partial_- X)^2] + O(h_{00}) \ .$$

The expansion has been organized in powers of derivatives of X^i and we have kept terms at most linear in the external fields (since we don't use them in what follows, we have

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dropped higher-order terms involving $\bar{h}_{ij}$). We have also reorganized those fields in a way appropriate for the compactification on a five-torus:

$$\nu \equiv \frac{1}{8}h \ , \quad \phi \equiv \phi_{10} - \frac{1}{4}h \ , \quad \lambda \equiv \frac{1}{2}h_{55} + \frac{1}{8}h - \frac{1}{2}\phi_{10} = \frac{1}{2}h_{55} - \frac{1}{2}\phi \ , \quad h \equiv h_{ii} \ , \quad (48)$$

where ν is the scale of the 4-torus part of the 5-brane (if $G_{ij} = e^{2\nu}\delta_{ij}$ then, in the linearized approximation, $h_{ii} = 8\nu$), ϕ is the corresponding six-dimensional dilaton and λ is the scale of the fifth (string) dimension measured in the six-dimensional Einstein metric. These are the same three scalar fields that appear in the GR effective action (7), (9).

Since the kinetic terms in the effective action (7), (9) are diagonal in ϕ, ν and λ, we can immediately read off some important conclusions from (44), (46). The expansion in powers of worldsheet derivatives is a low-energy expansion and, of the fields we have kept, only the dilaton ϕ is coupled at leading order. It is also easy to see that the 'off-diagonal' components of the metric $\bar{h}_{ij}$ have the same coupling as ϕ to lowest order in energy. (These are the fields whose emission and absorption were considered in [14,15]). What is more interesting is that the scalar ν only couples at the next-to-leading order (fourth order in derivatives). Note that its interaction term can be written in terms of worldsheet energy-momentum tensors as $\nu T^X_{++} T^X_{--}$.[7] The scalar λ likewise does not get emitted at the leading order and does couple at the same order as ϕ, but with a different vertex.[8] The 'graviton' components in the time and string directions h_{00} and h_{55} couple to the string in a way similar to λ, which reflects their mixing with λ in the effective action (9). Indeed, the vertex $(h_{00} + h_{55})(T^X_{++} + T^X_{--})$ gives a vanishing contribution to the amplitude of production of a closed string state: it only couples a pair of left-movers or a pair of right-movers so that the production is forbidden kinematically. The important (and non-trivial) point is that the simplest DBI action for the coupling of the external fields to the D-brane gives the fields ν and λ, previously identified as the 'fixed scalars', different (and weaker) couplings to the effective string than the fields like ϕ previously identified as 'decoupled scalars.' The precise couplings will shortly be used to make precise calculations of absorption and emission rates.

The action (42) is at best the bosonic part of a supersymmetric action. In previous discussions of D-brane emission and absorption, it has been possible to ignore the coupling of external fields to the massless fermionic excitations of the D-brane. For the questions that interest us, that will no longer be possible and we make a specific proposal for the couplings of worldsheet fermions. In the successful D-brane description of the entropy of

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[7] There is a similar coupling for ϕ which produces a subleading correction to its emission rate.

[8] The different vertices for ν and λ probably reflect the different behavior of their fluctuations (the non-decoupling of $\delta\lambda$ from metric perturbations) in the case when Q_K is not equal to $Q = P$.

rotating black holes [34], five-dimensional angular momentum is carried by the fermions alone. There are two worldsheet fermion doublets, one right-moving and one left-moving. The $SO(4)$ rotations of the uncompactified (1234) coordinates are decomposed as $SU(2)_L \times SU(2)_R$ and the obvious (and correct) choice is to take the left-moving fermions, S^a, to be a doublet under $SU(2)_L$ and the right-moving fermions, $S^{\dot{a}}$ to be a doublet under $SU(2)_R$. This set of fermions may be bosonized as two boson fields, ϕ^1 and ϕ^2. As mentioned above, the next-to-leading terms in (45),(46) can be written in terms of the X-field stress-energy tensor,

$$T_{++}^X = (\partial_+ \vec{X})^2 \ , \qquad T_{--}^X = (\partial_- \vec{X})^2 \ .$$

The obvious guess for the supersymmetric completion of these interaction terms is simply to add the bosonization fields $\vec{\phi}$ to the worldsheet energy-momentum tensors:

$$T_{++}^X \to T_{++}^{\text{tot}} = (\partial_+ \vec{X})^2 + (\partial_+ \vec{\phi})^2 \ ,$$

and similarly for T_{--}^X. This will have a crucial effect on the normalization of the fixed scalar absorption rate.

We also observe that the scalars h_{5i} ('mixing' the string direction with the four transverse directions) which also couple to the gauge fields in the effective action approach (see (3)), have a yet different coupling to the effective string. The rates produced by this coupling are calculated in the Appendix, with the conclusion that h_{5i} is neither a fixed scalar of the kind studied in [9] nor an ordinary scalar. Thus, the Nambu action predicts the existence of a variety of massless scalar fields which interact differently with the black hole.

4. Semi-Classical Description of Absorption

In this section we will mainly discuss the solution of the radial differential equation one obtains for s-wave perturbations in the fixed scalar ν related to the volume of the internal T^4 in string metric (5). Let us start by restating some results of section 2. From (8), (14) and (15) one can read off the five-dimensional Einstein metric:

$$ds_5^2 = -(H_{\hat{Q}} H_{\hat{P}} H_{\hat{Q}_K})^{-2/3} h dt^2 + (H_{\hat{Q}} H_{\hat{P}} H_{\hat{Q}_K})^{1/3} \left(h^{-1} dr^2 + r^2 d\Omega_3^2 \right) \ . \qquad (49)$$

To avoid the mixing between gravitational perturbations and the fixed scalar, we will restrict ourselves to the case $P = Q$, i.e. $r_1 = r_5 = R$, $P^2 = R^2(R^2 + r_0^2)$ (see (18)). The small fluctuation equation, (25), may be written as

$$\left[(hr^3 \partial_r)^2 + (r^2 + R^2)^2 (r^2 + r_K^2)\omega^2 - \frac{8r^4 R^4}{(r^2 + R^2)^2} h \left(1 + \frac{r_0^2}{R^2} \right) \right] \tilde{\nu} = 0 \ , \qquad (50)$$

where $h = 1 - \frac{r_0^2}{r^2}$. Since we work in the regime $r_0 \ll R$, we will neglect the last factor in the last term.

Several different radial coordinates are useful in different regions. The ones we will use most often are u and y defined by the relations

$$1 - \frac{r_0^2}{r^2} = e^{-r_0^2/u^2} , \qquad y = \frac{R^2 r_K \omega}{2u^2} . \tag{51}$$

Note that $u \to 0$ and $y \to \infty$ at the horizon.

The most efficient tool for obtaining the absorption cross-section is the ratio of fluxes method used in [17]. In all the cases we will treat, the solution to (50) whose near-horizon limit represents purely infalling matter has the limiting forms

$$\tilde{\nu} \approx e^{iy} \tag{52}$$

near the horizon and

$$\tilde{\nu} \approx \alpha \frac{J_1(\omega r)}{\omega r} = \frac{\alpha}{2} \frac{H_1^{(1)}(\omega r) + H_2^{(2)}(\omega r)}{\omega r} \tag{53}$$

far from the black hole, where J_1 is the Bessel function.[9] The term in (53) containing $H_2^{(2)}(\omega r)$ is the incoming wave. Once the constant α is known, one can compute the flux for the incoming wave and compare it to the flux for the infalling wave (52) to find the absorption probability. In the present instance, fluxes are purely radial:

$$J = \frac{1}{2i} (\tilde{\nu}^* d\tilde{\nu} - \text{c.c.}) = J_r dr . \tag{54}$$

Observing that the number of particles passing through a sphere S_r^3 at radius r in a time interval $[0, t]$ is

$$\int_{S_r^3 \times [0,t]} *J = 2\pi^2 h r^3 J_r t , \tag{55}$$

one concludes that the flux per unit solid angle is

$$\mathcal{F} = h r^3 J_r = \frac{1}{2i} (\tilde{\nu}^* h r^3 \partial_r \tilde{\nu} - \text{c.c.}) . \tag{56}$$

The absorption probability is

$$1 - |S_0|^2 = \frac{\mathcal{F}_\text{h}}{\mathcal{F}_\infty^\text{incoming}} = \frac{2\pi}{|\alpha|^2} R^2 r_K \omega^3 . \tag{57}$$

---

[9] In fact there can be phase shifts in the arguments of the exponential in (52) and the Bessel functions in (53), but they are immaterial for computing fluxes.

We will always be interested in cases where this probability is small. By the Optical Theorem, the absorption cross-section is

$$\sigma_{\text{abs}} = \frac{4\pi}{\omega^3} \left(1 - |S_0|^2\right) = \frac{8\pi^2}{|\alpha|^2} R^2 r_K. \tag{58}$$

Readers unfamiliar with the solution matching technology may be helped by the analogy to tunneling through a square potential barrier in one dimension. If particles come from the left side of the barrier, the wave function is to a good approximation a standing wave on the left side of the barrier, a decreasing exponential inside the barrier, and a purely right moving exponential on the right side of the barrier.

To obtain the familiar result $\sigma_{\text{abs}} = A_{\text{h}}$ for low-energy, ordinary scalars falling into an extremal black hole, it suffices to match the limiting value of (52) for small y directly to the limiting value of (53) for small r [10]. Due to non-extremality and to the presence of the potential term in (50), this naive matching scheme is invalid. A more refined approximate solution must be used, and a more physically interesting low-energy cross-section will be obtained.

We will now present approximate solutions to (50) in two regimes most easily characterized in D-brane terms: we shall first consider $T_R = 0$ with ω/T_L arbitrary; then we shall consider T_R much less than T_L but not zero, and allow ω/T_R to vary arbitrarily.

When $T_R = 0$, the black hole is extremal: $r_0 = 0$ and $r = u$. As usual, one proceeds by joining a near horizon solution **I** to a far solution **III** using an exact solution **II** to the $\omega = 0$ equation [15,9]. The dominant terms of (50) and the approximate solutions in the three regions are

$$\textbf{I.} \quad \left[\partial_y^2 + 1 - \frac{2\eta}{y} - \frac{2}{y^2}\right] \tilde{\nu}_{\textbf{I}} = 0 \qquad\qquad \tilde{\nu}_{\textbf{I}} = G_1(y) - iF_1(y)$$

$$\textbf{II.} \quad \left[(r^3\partial_r)^2 - 8\frac{R^4}{H^2}\right] \tilde{\nu}_{\textbf{II}} = 0 \qquad\qquad \tilde{\nu}_{\textbf{II}} = \frac{C}{H(r)} + DH^2(r) \tag{59}$$

$$\textbf{III.} \quad \left[(r^3\partial_r)^2 + r^6\omega^2\right] \tilde{\nu}_{\textbf{III}} = 0 \qquad\qquad \tilde{\nu}_{\textbf{III}} = \alpha\frac{J_1(\omega r)}{\omega r} + \beta\frac{N_1(\omega r)}{\omega r} \;,$$

where C, D, α and β are constants and F_1 and G_1 are Coulomb functions [35] whose charge parameter η is given by

$$\eta = -\tfrac{1}{4}\left(2\omega r_K + \frac{\omega R^2}{r_K}\right) = -\frac{1}{4\pi}\frac{\omega}{T_L}\left(1 + 2\frac{r_K^2}{R^2}\right) \;. \tag{60}$$

In the last equality we have used the definition of T_L in (21). The quantity r_K^2/R^2 is small in the dilute gas approximation, and we will neglect it when comparing the final semi-classical cross-section with the D-brane answer.

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By design, $\tilde{\nu}_{\mathbf{I}} \to e^{iy}$ as $y \to \infty$ up to a phase shift in y. An approximate solution can be patched together from $\tilde{\nu}_{\mathbf{I},\mathbf{II},\mathbf{III}}$ if one sets

$$ C = \frac{\alpha}{2} = \frac{2}{3C_1(\eta)\omega r_K} \ , \qquad D = 0 \ , \qquad \beta = 0 \ , \tag{61} $$

where $C_1(\eta) = \frac{1}{3}e^{-\pi\eta/2}|\Gamma(2+i\eta)|$ [35]. A slightly better matching can be obtained by allowing D and β to be nonzero, but the changes in the final solution do not affect the fluxes $\mathcal{F}_{\mathrm{h}}$ and $\mathcal{F}_{\infty}$ (these changes are however crucial in determining S_0 by the old methods of [36], and give phase information on the scattered wave which the flux method does not). Having only $C \neq 0$ in region $\mathbf{II}$ is analogous to the fact that for right-moving particles incident on a square potential barrier, the wave function under the barrier can be taken as a purely falling exponential with no admixture of the rising exponential.

From (61) and (58) the cross-section is immediate:

$$ \sigma_{\mathrm{abs}} = \frac{\pi^2}{2} r_K R^2 (\omega r_K)^2 |3C_1(\eta)|^2 = \tfrac{1}{4} A_{\mathrm{h}} (\omega r_K)^2 (1+\eta^2) \frac{2\pi\eta}{e^{2\pi\eta}-1} \ , \tag{62} $$

where A_{h} is the area of the horizon (given in (20)). Note that the derivation of (62) does not require the assumption that $r_K \ll R$.

To make the comparison with the D-brane approach, we can write (62) in the following suggestive form

$$ \sigma_{\mathrm{abs}} = \frac{\pi^2}{2} r_K R^2 (\omega r_K)^2 \frac{\frac{\omega}{2T_L}}{1-e^{-\frac{\omega}{2T_L}}} \left(1 + \frac{\omega^2}{16\pi^2 T_L^2}\right) \left[1 + O(r_K^2/R^2)\right] \ . \tag{63} $$

In section 5 we will compute the same quantity using effective D-string method and will find agreement to the indicated order of accuracy. To obtain $O(r_K^2/R^2)$ corrections on the D-brane side one would have to go beyond the dilute gas approximation.

Now let us continue on to the second regime in which an approximate solution to the radial equation (50) is fairly straightforward to obtain: $\omega, T_R \ll T_L$ with ω/T_R arbitrary. A quantity which enters more naturally into the differential equations than ω/T_R is

$$ B = \frac{R^2 r_K \omega}{r_0^2} = \frac{\omega}{\kappa}\tanh\sigma \approx \frac{\omega}{\kappa} \ , \qquad \kappa \equiv 2\pi T_H \ , \tag{64} $$

where κ is the surface gravity at the horizon, and in the last step we used the fact that $\sigma \gg 1$ when $T_R \ll T_L$. In dropping terms from the exact equation (50) to obtain soluble forms in the three matching regions, it is essential to retain B as a quantity of $O(1)$; however, r_0/r_K and $\omega R^2/r_K$ can be regarded as small because $T_R \ll T_L$ and $\omega \ll T_L$. In regions $\mathbf{II}$ and $\mathbf{III}$, the approximate equations turn out to be precisely the same as in (59), but in $\mathbf{I}$ one obtains a more complicated differential equation:

$$ \left[\partial_y^2 + 1 - \frac{8}{B^2}\frac{e^{-2y/B}}{(1-e^{-2y/B})^2}\right]\tilde{\nu}_{\mathbf{I}} = 0 \ . \tag{65} $$

This equation can be cast in the form of a supersymmetric quantum mechanics eigenfunction problem. Define a rescaled variable $z = y/B$ and supercharge operators

$$\mathcal{Q} = -\partial_z + \coth z \;, \qquad \mathcal{Q}^\dagger = \partial_z + \coth z \;. \tag{66}$$

Then (65) can be rewritten in the form

$$\mathcal{Q}\mathcal{Q}^\dagger \tilde{\nu}_{\mathbf{I}} = \left[-\partial_z^2 + 2\mathrm{csch}^2\, z + 1 \right] \tilde{\nu}_{\mathbf{I}} = \left(1 + B^2 \right) \tilde{\nu}_{\mathbf{I}} \;. \tag{67}$$

The eigenfunctions of the related Hamiltonian $\mathcal{Q}^\dagger \mathcal{Q} = -\partial_z^2 + 1$ are just exponentials, and from them one can read off the solutions to (67): the infalling solution is

$$\tilde{\nu}_{\mathbf{I}} = \frac{\mathcal{Q}\, e^{iBz}}{1 - iB} = \frac{\coth z - iB}{1 - iB}\, e^{iBz} = \frac{\coth \frac{y}{B} - iB}{1 - iB}\, e^{iy} \;. \tag{68}$$

The factor in the denominator is chosen so that $\tilde{\nu}_{\mathbf{I}} \approx e^{iy}$ for large y.

Performing the matching as usual, one obtains

$$\sigma_{\mathrm{abs}} = \tfrac{1}{4} A_{\mathrm{h}}(\omega r_K)^2 (1 + B^{-2}) = \tfrac{1}{4} A_{\mathrm{h}}(\omega r_K)^2 \left(1 + \frac{4\pi^2 T_H^2}{\omega^2} \right) \;. \tag{69}$$

In section 5 we will show that the effective string calculation gives the same result when $\omega, T_R \ll T_L$.

5. D-brane Absorption and Emission Cross-Sections

In this section we give a detailed calculation of the emission and absorption of the fixed scalar ν, using the interaction vertices computed in section 3. We recall that ν is related (see (5)) to the volume (measured in the string metric) of the compactification 4-torus orthogonal to the string. To study the leading coupling of ν, it is sufficient to retain the following two terms in the string effective action (cf. (42)):

$$
\begin{aligned}
I = \int d^2\sigma \Bigg\{ &\frac{1}{2} (\partial_+ \vec{X} \cdot \partial_- \vec{X} + \partial_+ \vec{\phi} \cdot \partial_- \vec{\phi}) \\
&+ \frac{1}{4T_{\mathrm{eff}}} \left[(\partial_+ \vec{X})^2 + (\partial_+ \vec{\phi})^2 \right] \left[(\partial_- \vec{X})^2 + (\partial_- \vec{\phi})^2 \right] \nu(x) \Bigg\}
\end{aligned}
\tag{70}
$$

where we have absorbed $\sqrt{T_{\mathrm{eff}}}$ into the fields to make them properly normalized. From (7) we see that the scalar field with the proper bulk kinetic term is $2\nu/\kappa_5$. Consider the invariant amplitude for processes mediated by the quartic interaction in (70). If p_1 and p_2 are the left-moving energies, while q_1 and q_2 are the right-moving ones, the matrix element among properly normalized states is

$$\frac{\sqrt{2}\kappa_5}{T_{\mathrm{eff}}} \sqrt{\frac{q_1 q_2 p_1 p_2}{\omega}} \;. \tag{71}$$

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The basic assumption of the D-brane approach to black hole physics is that the left-movers and right-movers can be treated as thermal ensembles [12,19]. Strictly speaking, they are microcanonical ensembles, but for our purposes the canonical ensemble is good enough and we proceed as if we are dealing with a massless one-dimensional gas of left-movers of temperature T_L and right-movers with temperature T_R. The motivation for this assumption has been explained at length in several recent papers [12,19,17]. To compute the rate for the process $scalar \to L + L + R + R$ we have to square the normalized matrix element (71) and integrate it over the possible energies of the final state particles. Because of the presence of the thermal sea of left-movers and right-movers, we must insert Bose enhancement factors: for example, each left-mover in the final state picks up a factor of $1 + \rho_L(p_i) = -\rho_L(-p_i)$, where

$$\rho_L(p_i) = \frac{1}{e^{\frac{p_i}{T_L}} - 1} \tag{72}$$

is the Bose-Einstein distribution. If there were a left-mover of energy p_i in the initial state, it would pick up an enhancement factor $\rho_L(p_i)$. Similar factors attach to right-movers.

Conservation of energy and of momentum parallel to the effective string introduces the factor

$$(2\pi)^2 \delta(p_1+p_2+q_1+q_2-\omega)\delta(p_1+p_2-q_1-q_2) = \frac{1}{2}(2\pi)^2 \delta(p_1+p_2-\frac{\omega}{2})\delta(q_1+q_2-\frac{\omega}{2}) \tag{73}$$

into the integrals over p_1, p_2, q_1, and q_2. Putting everything together, we find that the rate for $scalar \to L + L + R + R$ is given by

$$\begin{aligned}
\Gamma(1) &= \Gamma(scalar \to L + L + R + R) \\
&= \frac{36}{4} \frac{\kappa_5^2 L_{\text{eff}}}{4\pi^2 T_{\text{eff}}^2 \omega} \int_0^\infty dp_1 dp_2 \, \delta\left(p_1 + p_2 - \frac{\omega}{2}\right) \frac{p_1}{1 - e^{-\frac{p_1}{T_L}}} \frac{p_2}{1 - e^{-\frac{p_2}{T_L}}} \\
&\quad \times \int_0^\infty dq_1 dq_2 \, \delta\left(q_1 + q_2 - \frac{\omega}{2}\right) \frac{q_1}{1 - e^{-\frac{q_1}{T_R}}} \frac{q_2}{1 - e^{-\frac{q_2}{T_R}}} \,,
\end{aligned} \tag{74}$$

where L_{eff} is the length of the effective string. The factor of $36 = 6^2$ arises from the presence of six species of left-movers (four bosons and two bosonized fermions) and six species of right-movers. We divide by $4 = 2^2$ because of particle identity: because the two left-movers in the final state are identical particles, the integral over p_1, p_2 counts every left-moving final state twice (similarly for the right-movers).

To write down the rates for the three other absorptions processes (that is, processes 2, 3, and 4 in eq. (1)), it is convenient to define the integrals

$$\begin{aligned}
I_L(s_1, s_2) &= \int_0^\infty dp_1 dp_2 \, \delta\left(s_1 p_1 + s_2 p_2 + \frac{\omega}{2}\right) s_1 p_1 \rho_L(s_1 p_1) \cdot s_2 p_2 \rho_L(s_2 p_2) \\
I_R(s_1, s_2) &= \int_0^\infty dq_1 dq_2 \, \delta\left(s_1 q_1 + s_2 q_2 + \frac{\omega}{2}\right) s_1 q_1 \rho_L(s_1 q_1) \cdot s_2 q_2 \rho_L(s_2 q_2) \,.
\end{aligned} \tag{75}$$

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The choices $s_i = 1$ and $s_i = -1$ correspond, respectively, to putting a particle in the initial or final state. Then the total absorption rate, including all four competing processes of (1), is

$$
\begin{aligned}
\Gamma_{\mathrm{abs}}(\omega) &= \Gamma(1) + \Gamma(2) + \Gamma(3) + \Gamma(4) \\
&= 36 \frac{\kappa_5^2 L_{\mathrm{eff}}}{4\pi^2 T_{\mathrm{eff}}^2 \omega} \left[\tfrac{1}{4} I_L(-1,-1) I_R(-1,-1) + \tfrac{1}{2} I_L(-1,1) I_R(-1,-1) \right. \\
&\qquad\qquad \left. + \tfrac{1}{2} I_L(-1,-1) I_R(-1,1) + I_L(-1,1) I_R(-1,1) \right] \\
&= \frac{9\kappa_5^2 L_{\mathrm{eff}}}{4\pi^2 T_{\mathrm{eff}}^2 \omega} \int_{-\infty}^{\infty} dp_1 dp_2\, \delta\left(p_1 + p_2 - \frac{\omega}{2} \right) \frac{p_1}{1 - e^{-\frac{p_1}{T_L}}} \frac{p_2}{1 - e^{-\frac{p_2}{T_L}}} \\
&\qquad\qquad \times \int_{-\infty}^{\infty} dq_1 dq_2\, \delta\left(q_1 + q_2 - \frac{\omega}{2} \right) \frac{q_1}{1 - e^{-\frac{q_1}{T_R}}} \frac{q_2}{1 - e^{-\frac{q_2}{T_R}}} \\
&= \frac{\kappa_5^2 L_{\mathrm{eff}}}{(32\pi)^2 T_{\mathrm{eff}}^2} \frac{\omega}{\left(1 - e^{-\frac{\omega}{2T_L}}\right)\left(1 - e^{-\frac{\omega}{2T_R}}\right)} \left(\omega^2 + 16\pi^2 T_L^2\right)\left(\omega^2 + 16\pi^2 T_R^2\right) \ .
\end{aligned}
\tag{76}
$$

The fractional coefficients inside the square brackets on the second line of (76) are symmetry factors for the final states (the initial states are always simple enough so that their symmetry factors are unity). It is remarkable that although the individual processes 1–4 have rates which cannot be expressed in closed form, their sum is expressible in terms of integrals which can be performed analytically [37] because they run over all p_1, p_2, q_1, and q_2.

A similar calculation may be performed for the four emission processes, with the result

$$
\Gamma_{\mathrm{emit}}(\omega) = e^{-\frac{\omega}{T_H}} \Gamma_{\mathrm{abs}}(\omega) = -\Gamma_{\mathrm{abs}}(-\omega) \ ,
\tag{77}
$$

where the Hawking temperature characterizing the distribution of the emitted scalars is related to T_R and T_L by (22). Our convention has been to compute $\Gamma_{\mathrm{abs}}(\omega)$ assuming that the flux of the incoming scalar is unity. We have also suppressed the phase space factor $d^4 k/(2\pi)^4$ for the outgoing scalar in computing $\Gamma_{\mathrm{emit}}(\omega)$, and we have assumed that the outgoing scalar is emitted into the vacuum state, so that $\Gamma_{\mathrm{emit}}(\omega)$ includes no Bose enhancement factors. These conventions were chosen because they lead to simple expressions for $\Gamma_{\mathrm{emit}}(\omega)$ (77) and σ_{abs} below, but they must be borne carefully in mind when considering questions of detailed balance. Suppose we put the black hole in a thermal bath of scalars at temperature T_H. Then $\Gamma_{\mathrm{abs}}(\omega)$ and $\Gamma_{\mathrm{emit}}(\omega)$ pick up Bose enhancement factors for the scalars: those factors are, respectively, $1/(e^{\omega/T_H} - 1)$ and $1/(1 - e^{-\omega/T_H})$. Once these factors are included, the emission and absorption rates become equal by virtue of the first equality in (77). The fact that calculating $\Gamma_{\mathrm{emit}}(\omega)$ in the same way that we calculated $\Gamma_{\mathrm{abs}}(\omega)$ leads to (77) is a nontrivial check on detailed balance. This check is

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analogous to verifying that QED reproduces the Einstein A and B coefficients for the decay of the first excited state of hydrogen.

Because $\Gamma_{\text{abs}}(\omega)$ was computed assuming unit flux, one would naively guess that the absorption cross-section to be compared with a semi-classical calculation is $\sigma_{\text{abs}} = \Gamma_{\text{abs}}(\omega)$. (Now we are back to the conventions where $\Gamma_{\text{abs}}(\omega)$ and $\Gamma_{\text{emit}}(\omega)$ do not include Bose enhancement factors for the scalars). This is not quite right; instead,

$$\sigma_{\text{abs}}(\omega) = \Gamma_{\text{abs}}(\omega) - \Gamma_{\text{emit}}(\omega) = \Gamma_{\text{abs}}(\omega) + \Gamma_{\text{abs}}(-\omega) \ . \tag{78}$$

To see why (78) is right, we have to remember what we are doing in a semi-classical computation. We send in a classical wave in the field whose quanta are the scalars of interest, and we watch to see what fraction of it is sucked up by the black hole and what fraction is re-emitted. The quantum field theory analog is to send in a coherent state of scalars with large average particle number, so that the flux is almost fixed at its classical expectation value $\mathcal{F}$. The dominant processes are then absorption and stimulated emission. The Bose enhancement factors collapse to $\mathcal{F}$ for both absorption and emission, up to errors which are insignificant in the semi-classical limit. The net rate at which particles are absorbed is then $\Gamma_{\text{abs}}(\omega)\mathcal{F} - \Gamma_{\text{emit}}(\omega)\mathcal{F}$. But this rate is $\sigma_{\text{abs}}\mathcal{F}$ by definition, whence (78). Note that the last expression in (78) is manifestly invariant under time-reversal, which takes $\omega \to -\omega$.

In order to obtain definite results for the absorption cross-section, we must supply values for the effective length L_{eff} of the string, as well as its effective tension T_{eff}. It is a by-now-familiar story that multiple D-strings bound to multiple five-branes behave like a single D-string multiply wound about the compactification direction [13]. In the case at hand it is well-understood that the effective string length is [13,17]

$$\kappa_5^2 L_{\text{eff}} = 4\pi^3 r_1^2 r_5^2 \ . \tag{79}$$

With this substitution, the fixed scalar ν absorption cross-section becomes

$$\sigma_{\text{abs}}(\omega) = \frac{\pi r_1^2 r_5^2}{256 T_{\text{eff}}^2} \frac{\omega \left(e^{\frac{\omega}{T_H}} - 1 \right)}{\left(e^{\frac{\omega}{2 T_L}} - 1 \right) \left(e^{\frac{\omega}{2 T_R}} - 1 \right)} (\omega^2 + 16\pi^2 T_L^2)(\omega^2 + 16\pi^2 T_R^2) \ . \tag{80}$$

This is similar to, but not quite the same as, the absorption cross-section for the ordinary 'unfixed' scalar calculated in [17].

The object of our exercise is to offer further evidence that the D-brane configuration *is* the corresponding black hole by showing that (80) is identical to the corresponding quantity calculated by standard classical GR methods. For technical reasons, the GR calculation in a general black hole background is quite difficult and the results we have

been able to obtain (presented in section 4) are only valid in certain simplifying limits. The most important simplification is to take equal brane charges $r_1 = r_5 = R$.

First we consider the extremal limit, $T_R = 0$. Here (80) reduces to

$$\sigma_{\rm abs}(\omega) = \frac{\pi^3 r_1^2 r_5^2 T_L^3}{8 T_{\rm eff}^2} \omega^2 \frac{\frac{\omega}{2 T_L}}{1 - e^{-\frac{\omega}{2 T_L}}} \left(1 + \frac{\omega^2}{16 \pi^2 T_L^2} \right) \ . \tag{81}$$

This is to be compared with the classical fixed scalar absorption cross-section in the extremal background (eqn. (63)):

$$\sigma_{\rm abs}(\omega) = \frac{\pi^2}{2} R^2 r_K^3 \omega^2 \frac{\frac{\omega}{2 T_L}}{1 - e^{-\frac{\omega}{2 T_L}}} \left(1 + \frac{\omega^2}{16 \pi^2 T_L^2} \right) \ . \tag{82}$$

Using that in the extremal limit

$$T_L = \frac{r_K}{\pi r_1 r_5} \ ,$$

and remembering that we were only able to do the classical calculation for $r_1 = r_5 = R$, we see that the D-branes and GR match if we take the effective string tension to be

$$T_{\rm eff} = \frac{1}{2 \pi R^2} = \frac{1}{2 \pi \alpha' g n_5} \ , \tag{83}$$

where we have restored the dependence on α' that we have been suppressing since (16). This value for $T_{\rm eff}$ is precisely equal to the tension of the 'fractionated' D-string moving inside n_5 5-branes [13,20]. This is a highly non-trivial independent check on the applicability of the effective string model to fixed scalars, and also on the idea of D-string 'fractionation!'

Another interesting comparison to be made is for near-extremal black holes. For $\omega, T_R \ll T_L$ but with ratio of ω to T_R otherwise arbitrary, (80) becomes, using (83) for the tension,

$$\sigma_{\rm abs}(\omega) = \frac{\pi^2}{2} R^2 r_K^3 (\omega^2 + 4 \pi^2 T_H^2) \ . \tag{84}$$

This is in exact agreement with the absorption cross-section on non-extremal black holes (69) computed using general relativity.

For the fixed scalar the coupling to $(\partial X)^2$ is absent from the D-brane action, and the cross-section we found is the leading effect. For an ordinary 'decoupled' scalar, such as the 6-d dilaton, both terms are present. So, the cross-section computed above should be part of the correction to the leading effect determined in [17]. This is an interesting topic for future investigation.

6. Conclusions

Let us try to recapitulate in a few words what it has taken many equations for us to state. The main thrust of the paper is to explore the behavior of the type of fixed scalar studied earlier in [6,8,38], and most recently in [9] – but now in the context of five-dimensional black holes that can be modeled by bound states of D1-branes and D5-branes [11-17]. For the most part we have focused on the fixed scalar ν which corresponds to the volume of the internal four-torus as measured by the string metric. Through an interesting interplay between semi-classical computations (where the basic theory is well known but analytically intractable in general) and D-brane computations (where the theory is less well known but very tractable), we have arrived at a general formula (80) for the cross-section for low-energy fixed scalars to be absorbed into the black hole.

The absorption cross-section (80) has a much richer and more interesting functional form than the simple ω^2 dependence found in [9]. Even in the simple limit $\omega, T_R \ll T_L$ in which comparison calculations between GR and D-branes were initially performed [14,15], the fixed scalar cross-section goes not as ω^2 but as $\omega^2 + \kappa^2$, where $\kappa = 2\pi T_H$ is the surface gravity at the horizon. While we have derived the expression (80) in full generality only in the D-brane picture, we have demonstrated that it agrees with semi-classical calculations of the cross-section in the two regimes: one regime reproduces this novel $\omega^2 + \kappa^2$ behavior, while the other deals with absorption into extremal black holes. Because the equations for the gravitational perturbations and fixed scalar perturbations couple unless two of the three charges, e.g., the 1-brane charge and the 5-brane charge, are equal to each other, our semi-classical computations are limited to the equal charge case (similar equal-charge assumption was used in $D = 4$ case in [9]). Modulo this limitation, we have confidence that a full greybody factor computation along the lines of [17] would reproduce the general result (80).

One of the reasons why the extension of the semi-classical calculations to unequal 1-brane and 5-brane charges (but with both still greater than the third charge, $P, Q \gg Q_K$, to remain in the dilute gas region) would be interesting, is that the D-brane computations involve one free parameter, the tension T_{eff} of the effective string, which can be read off from a comparison with a semi-classical calculation. The expectation, based on the work of [13,20] and on the arguments given at the beginning of section 3, is that $T_{\text{eff}} = \frac{1}{2\pi\alpha' g n_5}$. Our work confirms this relation when the 1-brane and 5-brane charges are equal. What a semi-classical calculation with unequal charges should confirm is that T_{eff} is independent of the number of 1-branes.

Although our ultimate goal has been to demonstrate a new agreement between semi-classical GR and a perturbative treatment of the effective string, we have along the way studied interesting facets of both formalisms separately. On the D-brane side, we have been forced to go beyond the leading quadratic terms in the expansion of the DBI action

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and examine terms quartic in the derivatives of the string collective coordinate fields X^i. As we argued in section 3, the generic form of the quadratic terms is practically inevitable given the invariances of the problem. But the decoupling of the fixed scalar from quadratic terms and the precise form of its coupling to quartic terms is a signature of the DBI action. The agreement between the D-brane and GR cross-sections for fixed scalars is thus a more stringent test of the DBI action than the agreements obtained previously [15,16,17] for ordinary scalars.

From the open string theory point of view, the $(\partial X)^2$ term in the D-string action (40),(42) originates upon dimensional reduction from the $F_{\mu\nu}^2$ term in the $D = 10$ Born-Infeld action, while the $(\partial X)^4$ terms correspond to the $F_{\mu\nu}^4$ -terms. It is amusing to note that the fixed scalars, which are coupled to the Maxwell terms of the closed string vector fields in the space-time effective action (7), thus do not couple to the Maxwell term of the open string vector field in the effective D-string action, while exactly the opposite is true for the 'decoupled' scalars. It is thus the $F_{\mu\nu}^4$ -terms in the DBI action (which are important also in some other contexts) that are effectively responsible for the leading contribution to the cross-section of fixed scalars.

At the relevant $(\partial X)^4$ order, the processes involving fermionic excitations of the effective string contribute in the same way as purely bosonic processes. Fortunately, the coupling of bosonic excitations to the fixed scalar field predicted by the DBI action is of a particularly simple form, $T_{++}^X T_{--}^X \nu(x)$, which admits an obvious generalization to include fermions: $T_{++}^{\text{tot}} T_{--}^{\text{tot}} \nu(x)$. Obtaining precise agreement with GR using this coupling and the normalization of T_{eff} as in [13,20] may be viewed as determining a partial supersymmetrization of the DBI action via D-brane spectroscopy.

On the GR side, we have to some extent systematized the study of spherical black hole configurations, including spherically symmetric perturbations around the basic $D = 5$ black hole with three charges, by reducing the problem to an effective two-dimensional one. For time-independent configurations, this gives a straightforward derivation of the basic black hole solution. We were disappointed to find, however, that, despite relative simplicity of the effective two-dimensional theory compared to the full supergravity equations, it still leads to complicated coupled differential equations for time-dependent fluctuations around the static solution. So far, we have been able to extract simple equations from the intractable general case only when some pair of charges are equal. Then the background value of one fixed scalar becomes constant and its fluctuations decouple from the other fields, leading to a non-extremal five-dimensional generalization of the equation studied in [9]. Similar two-dimensional effective theory techniques with similar equal charge limitations were applied to the basic four-dimensional black hole with four charges. In this paper, we have taken the four-dimensional calculations only far enough to see that fixed scalars whose background values become constant when three of the four charges are equal have an absorption cross-section with the characteristic $\omega^2 + \kappa^2$ dependence.

One final comment is that we have focused almost exclusively on absorption rather than Hawking emission. This is not because Hawking emission is any more difficult, but rather because agreement between the semi-classical Hawking calculation and the D-brane result is inevitable once a successful comparison of absorption cross-sections is made. To see this, one must only note that detailed balance between emission and absorption is built into the Hawking calculation and that it can be checked explicitly in the D-brane description. Once detailed balance is established in both descriptions, it obviously suffices to check that the absorption cross-section agrees between the two in order to be sure that emission rates must agree as well.

Appendix A.

The effective string action (42) may be used to classify various scalar fields according to their coupling to the black hole. While this action makes it clear that the 'fixed scalar' ν couples differently from the ordinary scalars, $\bar{h}_{ij}$ and ϕ, we also observe that there are scalars with yet different properties, such as h_{5i}. The purpose of this Appendix is to calculate what their coupling to the effective string, given in (45), implies for the absorption rate. We see that h_{5i} couples either to two left-movers and one right-mover or to two right-movers and one left-mover. The absorption processes due to the first type of coupling are $scalar \rightarrow L + L + R$ and $scalar + L \rightarrow L + R$. The relevant matrix element between properly normalized states is found to be

$$\frac{\kappa_5}{\sqrt{T_{\mathrm{eff}}}}\sqrt{\frac{q_1 p_1 p_2}{\omega}} \quad . \tag{A.1}$$

Adding up the absorption rate for the two processes gives

$$\frac{3\kappa_5^2 L_{\mathrm{eff}}}{8\pi T_{\mathrm{eff}}\omega}\frac{\omega}{1 - e^{-\frac{\omega}{2T_R}}}\int_{-\infty}^{\infty} dp_1 dp_2\, \delta\left(p_1 + p_2 - \frac{\omega}{2}\right)\frac{p_1}{1 - e^{-\frac{p_1}{T_L}}}\frac{p_2}{1 - e^{-\frac{p_2}{T_L}}}$$
$$= \frac{\kappa_5^2 L_{\mathrm{eff}}}{128\pi T_{\mathrm{eff}}}\frac{\omega}{\left(1 - e^{-\frac{\omega}{2T_L}}\right)\left(1 - e^{-\frac{\omega}{2T_R}}\right)}\left(\omega^2 + 16\pi^2 T_L^2\right) \quad . \tag{A.2}$$

The absorption rate due to the processes $scalar \rightarrow R + R + L$ and $scalar + R \rightarrow R + L$ is calculated analogously, so that the total absorption rate for a scalar h_{5i} is

$$\Gamma_{\mathrm{abs}}(\omega) = \frac{\kappa_5^2 L_{\mathrm{eff}}}{64\pi T_{\mathrm{eff}}}\frac{\omega}{\left(1 - e^{-\frac{\omega}{2T_L}}\right)\left(1 - e^{-\frac{\omega}{2T_R}}\right)}\left(\omega^2 + 8\pi^2 T_L^2 + 8\pi^2 T_R^2\right) \quad . \tag{A.3}$$

Now the classical absorption cross-section is found from the relation (78). Using (79) and (83), and setting $r_1 = r_5 = R$, we obtain

$$\sigma_{\mathrm{abs}}(\omega) = \frac{\pi^3 R^6}{8}\frac{\omega\left(e^{\frac{\omega}{T_H}} - 1\right)}{\left(e^{\frac{\omega}{2T_L}} - 1\right)\left(e^{\frac{\omega}{2T_R}} - 1\right)}(\omega^2 + 8\pi^2 T_L^2 + 8\pi^2 T_R^2) \tag{A.4}$$

This cross-section has a number of interesting properties. In the limit $\omega \rightarrow 0$ it approaches

$$\sigma_{\mathrm{abs}}(0) = \pi^2(2r_K^2 + r_0^2)\sqrt{r_K^2 + r_0^2} \quad . \tag{A.5}$$

This is clearly different from the behavior found for the fixed scalar ν: the $\omega = 0$ cross-section of h_{5i} does not vanish at extremality. The expression (A.5) is also different from

the cross-section $\sigma_{\mathrm{abs}}(0) = A_{\mathrm{h}}$ which is found for ordinary scalars. We conclude that h_{5i} is neither the fixed scalar of the type exhibited in [9] nor the ordinary massless scalar.

It is interesting to see what (A.4) reduces to in the extremal limit where r_0 and T_R are sent to zero. Here we find

$$\sigma_{\mathrm{abs}}(\omega) = 2\pi^2 r_K^3 \frac{\frac{\omega}{2T_L}}{1 - e^{-\frac{\omega}{2T_L}}} \left(1 + \frac{\omega^2}{8\pi^2 T_L^2}\right) \ . \tag{A.6}$$

Using the parameter η introduced in (60) this is equal to

$$\sigma_{\mathrm{abs}}(\omega) = 2\pi^2 r_K^3 \left(1 + 2\eta^2\right) \frac{2\pi\eta}{e^{2\pi\eta} - 1} \left[1 + O(r_K^2/R^2)\right] \ . \tag{A.7}$$

It would be very interesting to compare the cross-section (A.3) calculated for h_{5i} using the effective string methods to the corresponding classical GR cross-section. The calculation of the latter is a rather complicated exercise which we postpone for the future.

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