

## Some new results in $O(a)$ improved lattice QCD \*

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It is shown how on-shell  $O(a)$  improvement can be implemented non-perturbatively in lattice QCD with Wilson quarks. Improvement conditions are obtained by requiring the PCAC relation to hold exactly in certain matrix elements. These are derived from the QCD Schrödinger functional which enables us to simulate directly at vanishing quark masses. In the quenched approximation and for bare couplings in the range  $0 \leq g_0 \leq 1$ , we determine the improved action, the improved axial current, the additive renormalization of the quark mass and the isospin current normalization constants  $Z_A$  and  $Z_V$ .

### 1. INTRODUCTION

The leading cutoff effects in lattice QCD with Wilson quarks [1] are proportional to the lattice spacing  $a$  and can be rather large for typical values of the Monte Carlo (MC) simulation parameters [2]. Decreasing the lattice spacing at constant physical length scales means larger lattices and therefore rapidly increasing costs of the MC simulations.

An alternative method to reduce cutoff effects in lattice field theories is due to Symanzik [3,4]. He has shown that the approach of Green's functions to their continuum limit can be accelerated by using an improved action and improved (composite) fields. A considerable simplification is achieved if the improved continuum approach is only required for on-shell quantities such as particle masses and matrix elements of improved fields between physical states [5-7].

Symanzik improvement may be viewed as an extension of the renormalization programme to the level of irrelevant operators. While the

structure of the possible counterterms is dictated by the symmetries, their coefficients have to be fixed by appropriate improvement conditions. Although they can be estimated in perturbation theory, a non-perturbative determination of the improvement coefficients through MC simulations is clearly preferable.

To  $O(a)$ , we have recently carried out the non-perturbative improvement programme in quenched lattice QCD, thus leaving residual cutoff effects of  $O(a^2)$  only. The detailed results are given elsewhere [7-10]. In this report we emphasize the general concepts and give a short account of the main results.

We first review on-shell improvement in the framework of Symanzik's local effective theory. The set-up in finite space-time volume with Schrödinger functional boundary conditions is introduced in sect. 3. We are then in the position to state the improvement conditions and discuss their evaluation in MC simulations (sect. 4). Finally we present our results for the critical quark mass (subsect. 4.2) and the current normalization constants in the  $O(a)$  improved theory (sect. 5).

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## 2. ON-SHELL IMPROVEMENT

### 2.1. Lattice QCD with Wilson quarks

In this section we consider QCD on an infinitely extended lattice with two degenerate light Wilson quarks of bare mass  $m_0$  [1]. The action is the sum of the usual Wilson plaquette action and the quark action

$$S_F[U, \bar{\psi}, \psi] = a^4 \sum_x \bar{\psi}(x)(D + m_0)\psi(x), \quad (1)$$

where  $a$  denotes the lattice spacing. The Wilson-Dirac operator

$$D = \frac{1}{2} \sum_{\mu=0}^3 [(\nabla_\mu^* + \nabla_\mu)\gamma_\mu - a\nabla_\mu^* \nabla_\mu], \quad (2)$$

contains the lattice covariant forward and backward derivatives,  $\nabla_\mu$  and  $\nabla_\mu^*$ . The last term in eq. (2) eliminates the unwanted doubler states but also breaks chiral symmetry. As a consequence, both additive and multiplicative renormalization of the quark mass are necessary, i.e. any renormalized quark mass  $m_R$  is of the form

$$m_R = Z_m m_q, \quad m_q = m_0 - m_c, \quad (3)$$

where  $m_c$  is the so-called critical quark mass.

Chiral symmetry violation is more directly seen by studying the conservation of the isovector axial current  $A_\mu^a$ . The current and the associated axial density on the lattice are defined through

$$A_\mu^a(x) = \bar{\psi}(x)\gamma_\mu\gamma_5\frac{\tau^a}{2}\psi(x), \quad (4)$$

$$P^a(x) = \bar{\psi}(x)\gamma_5\frac{\tau^a}{2}\psi(x), \quad (5)$$

where  $\tau^a$  are Pauli matrices acting on the flavour index of the quark field. The PCAC relation

$$\tilde{\partial}_\mu A_\mu^a(x) = 2mP^a(x) + O(a), \quad (6)$$

$$\tilde{\partial}_\mu = \frac{1}{2}(\partial_\mu^* + \partial_\mu), \quad (7)$$

then includes an error term of order  $a$ , which can be rather large on the accessible lattices [2].

The isospin symmetry remains unbroken on the lattice and there exists an associated conserved vector current. However, it is often advantageous to use the current which is strictly local,

$$V_\mu^a(x) = \bar{\psi}(x)\gamma_\mu\frac{\tau^a}{2}\psi(x). \quad (8)$$

The conservation of this current is then also violated by cutoff effects, and a finite renormalization is required to ensure that the associated charge takes half-integral values.

### 2.2. Symanzik's local effective theory

Near the continuum limit the lattice theory can be described in terms of a local effective theory [4],

$$S_{\text{eff}} = S_0 + aS_1 + a^2S_2 + \dots, \quad (9)$$

where  $S_0$  is the action of the continuum theory, defined e.g. on a lattice with spacing  $\varepsilon \ll a$ . The terms  $S_k$ ,  $k = 1, 2, \dots$ , are space-time integrals of lagrangians  $\mathcal{L}_k(x)$ . These are given as general linear combinations of local gauge invariant composite fields which respect the exact symmetries of the lattice theory and have canonical dimension  $4 + k$ . We use the convention that explicit (non-negative) powers of the quark mass  $m$  are included in the dimension counting. A possible basis of fields for the lagrangian  $\mathcal{L}_1(x)$  then reads

$$\begin{aligned} \mathcal{O}_1 &= \bar{\psi} \sigma_{\mu\nu} F_{\mu\nu} \psi, \\ \mathcal{O}_2 &= \bar{\psi} D_\mu D_\mu \psi + \bar{\psi} \overleftarrow{D}_\mu \overleftarrow{D}_\mu \psi, \\ \mathcal{O}_3 &= m \text{tr} \{ F_{\mu\nu} F_{\mu\nu} \}, \\ \mathcal{O}_4 &= m \{ \bar{\psi} \gamma_\mu D_\mu \psi - \bar{\psi} \overleftarrow{D}_\mu \gamma_\mu \psi \}, \\ \mathcal{O}_5 &= m^2 \bar{\psi} \psi, \end{aligned} \quad (10)$$

where  $F_{\mu\nu}$  is the field tensor and  $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ .

When considering correlation functions of local gauge invariant fields the action is not the only source of cutoff effects. If  $\phi(x)$  denotes such a lattice field (e.g. the axial density or the isospin currents of subsect. 2.1), one expects the connected  $n$ -point function

$$G_n(x_1, \dots, x_n) = (Z_\phi)^n \langle \phi(x_1) \dots \phi(x_n) \rangle_{\text{con}} \quad (11)$$

to have a well-defined continuum limit, provided the renormalization constant  $Z_\phi$  is correctly tuned and the space-time arguments  $x_1, \dots, x_n$  are kept at a physical distance from each other.

In the effective theory the renormalized lattice field  $Z_\phi \phi(x)$  is represented by an effective field,

$$\phi_{\text{eff}}(x) = \phi_0(x) + a\phi_1(x) + a^2\phi_2(x) + \dots, \quad (12)$$

where the  $\phi_k(x)$  are linear combinations of composite, local fields with the appropriate dimension

and symmetries. For example, in the case of the axial current (4),  $\phi_1$  is given as a linear combination of the terms,

$$\begin{aligned} (\mathcal{O}_6)_\mu^a &= \bar{\psi} \gamma_5 \frac{1}{2} \tau^a \sigma_{\mu\nu} [D_\nu - \overleftarrow{D}_\nu] \psi, \\ (\mathcal{O}_7)_\mu^a &= \bar{\psi} \frac{1}{2} \tau^a \gamma_5 [D_\mu + \overleftarrow{D}_\mu] \psi, \\ (\mathcal{O}_8)_\mu^a &= m \bar{\psi} \gamma_\mu \gamma_5 \frac{1}{2} \tau^a \psi. \end{aligned} \quad (13)$$

The convergence of  $G_n(x_1, \dots, x_n)$  to its continuum limit can now be studied in the effective theory,

$$\begin{aligned} G_n(x_1, \dots, x_n) &= \langle \phi_0(x_1) \dots \phi_0(x_n) \rangle_{\text{con}} \\ &\quad - a \int d^4y \langle \phi_0(x_1) \dots \phi_0(x_n) \mathcal{L}_1(y) \rangle_{\text{con}} \\ &\quad + a \sum_{k=1}^n \langle \phi_0(x_1) \dots \phi_1(x_k) \dots \phi_0(x_n) \rangle_{\text{con}} \\ &\quad + \mathcal{O}(a^2), \end{aligned} \quad (14)$$

where the expectation values on the right hand side are to be taken in the continuum theory with action  $S_0$ .

### 2.3. Using the field equations

For most applications, it is sufficient to compute on-shell quantities such as particle masses, S-matrix elements and correlation functions at space-time arguments which are separated by a physical distance. It is then possible to make use of the field equations to reduce first the number of basis fields in the effective lagrangian  $\mathcal{L}_1$  and, in a second step, also in the  $\mathcal{O}(a)$  counterterm  $\phi_1$  of the effective composite fields.

If one uses the field equations in the lagrangian  $\mathcal{L}_1$  under the space-time integral in eq. (14), the errors made are contact terms that arise when  $y$  comes close to one of the arguments  $x_1, \dots, x_n$ . Taking into account the dimensions and symmetries, one easily verifies that these contact terms must have the same structure as the insertions of  $\phi_1$  in the last term of eq. (14). Using the field equations in  $\mathcal{L}_1$  therefore just means a redefinition of the coefficients in the counterterm  $\phi_1$ .

It turns out that one may eliminate two of the terms in eq. (10). A possible choice is to stay with the terms  $\mathcal{O}_1$ ,  $\mathcal{O}_3$  and  $\mathcal{O}_5$ , which yields the effective continuum action for on-shell quantities

to order  $a$ . Having made this choice one may apply the field equations once again to simplify the term  $\phi_1$  in the effective field as well. In the example of the axial current it is then possible to eliminate the term  $\mathcal{O}_6$  in eq. (13).

### 2.4. Improved lattice action and fields

The on-shell  $\mathcal{O}(a)$  improved lattice action is obtained by adding a counterterm to the unimproved lattice action such that the action  $S_1$  in the effective theory is cancelled in on-shell amplitudes. This can be achieved by adding lattice representatives of the terms  $\mathcal{O}_1$ ,  $\mathcal{O}_3$  and  $\mathcal{O}_5$  to the unimproved lattice lagrangian, with coefficients that are functions of the bare coupling,  $g_0$ , only. Leaving the discussion of suitable improvement conditions to sect. 4, we here note that the fields  $\mathcal{O}_3$  and  $\mathcal{O}_5$  already appear in the unimproved theory and thus merely lead to a reparametrization of the bare parameters  $g_0$  and  $m_0$ . In the following, we will not further consider these terms. For a discussion of their relevance we refer the reader to ref. [7].

We choose the standard discretization  $\widehat{F}_{\mu\nu}$  of the field tensor [7] and add the improvement term to the Wilson-Dirac operator (2),

$$D_{\text{impr}} = D + c_{\text{sw}} \frac{ia}{4} \sigma_{\mu\nu} \widehat{F}_{\mu\nu}. \quad (15)$$

With properly chosen coefficient  $c_{\text{sw}}(g_0)$  this yields the on-shell  $\mathcal{O}(a)$  improved lattice action which has first been proposed by Sheikholeslami and Wohlert [6].

In perturbation theory, the coefficient  $c_{\text{sw}}$  has been computed to one-loop order by Wohlert [11]. His result,  $c_{\text{sw}} = 1 + c_{\text{sw}}^{(1)} g_0^2 + \mathcal{O}(g_0^4)$ , with  $c_{\text{sw}}^{(1)} = 0.26590(7)$ , has recently been confirmed to three significant digits [8].

The  $\mathcal{O}(a)$  improved isospin currents and the axial density can be parametrized as follows,

$$\begin{aligned} (A_R)_\mu^a &= Z_A (1 + b_A a m_q) \{ A_\mu^a + a c_A \overleftarrow{\partial}_\mu P^a \}, \\ (V_R)_\mu^a &= Z_V (1 + b_V a m_q) \{ V_\mu^a + a c_V \overleftarrow{\partial}_\nu T_{\mu\nu}^a \}, \\ (P_R)^a &= Z_P (1 + b_P a m_q) P^a. \end{aligned} \quad (16)$$

Here all the fields on the right hand sides have been defined in subsection 2.1 except the tensor density  $T_{\mu\nu}^a = i \bar{\psi} \sigma_{\mu\nu} \frac{1}{2} \tau^a \psi$ . We have included the

normalization constants  $Z_{A,V,P}$  which have to be fixed by appropriate normalization conditions (cf. sect. 5). Again, the improvement coefficients  $b_{A,V,P}$  and  $c_{A,V}$  are functions of  $g_0$  only. At tree level of perturbation theory we have  $b_A = b_P = b_V = 1$  and  $c_A = c_V = 0$  [12,8]. For  $c_A$  also the one-loop result,

$$c_A = -0.00756(1) \times g_0^2 + O(g_0^4), \quad (17)$$

has recently been obtained [8].

### 2.5. The PCAC relation

We assume for the moment that on-shell  $O(a)$  improvement has been fully implemented, i.e. the improvement coefficients are assigned their correct values. If  $\mathcal{O}$  denotes a renormalized on-shell  $O(a)$  improved field localized in a region not containing  $x$ , we thus expect that the PCAC relation

$$\tilde{\partial}_\mu \langle (A_R)_\mu^a(x) \mathcal{O} \rangle = 2m_R \langle (P_R)^a(x) \mathcal{O} \rangle, \quad (18)$$

holds up to corrections of order  $a^2$ . At this point we note that the field  $\mathcal{O}$  need not be improved for this statement to be true. To see this we use again Symanzik's local effective theory and denote the  $O(a)$  correction term in  $\mathcal{O}_{\text{eff}}$  by  $\phi_1$ . Eq. (18) then receives an order  $a$  contribution

$$a \langle \{ \partial_\mu (A_R)_\mu^a(x) - 2m_R (P_R)^a(x) \} \phi_1 \rangle, \quad (19)$$

which is to be evaluated in the continuum theory. The PCAC relation holds exactly in this limit and the extra term (19) thus vanishes.

## 3. THE SCHRÖDINGER FUNCTIONAL

### 3.1. Definitions

The space-time lattice is now taken to be a discretized hyper-cylinder of length  $T$  and circumference  $L$ . In the spatial directions the quantum fields are  $L$ -periodic, whereas in the Euclidean time direction inhomogeneous Dirichlet boundary conditions are imposed as follows. The spatial components of the gauge field are required to satisfy

$$U(x, k)|_{x_0=0} = \exp(aC_k), \quad C_k = i\phi/L, \quad (20)$$

with  $\phi = \text{diag}(\phi_1, \phi_2, \phi_3)$ , and an analogous boundary condition with  $C'$  is imposed at  $x_0 = T$ .

With the projectors  $P_\pm = \frac{1}{2}(1 \pm \gamma_0)$ , the boundary conditions for the quark and antiquark fields read

$$P_+ \psi|_{x_0=0} = \rho, \quad P_- \psi|_{x_0=T} = \rho', \quad (21)$$

$$\bar{\psi} P_-|_{x_0=0} = \bar{\rho}, \quad \bar{\psi} P_+|_{x_0=T} = \bar{\rho}'. \quad (22)$$

The functional integral in this situation [15,16],

$$\mathcal{Z}[C', \bar{\rho}', \rho'; C, \bar{\rho}, \rho] = \int_{\text{fields}} e^{-S}, \quad (23)$$

is known as the QCD Schrödinger functional (SF). Concerning the (unimproved) action  $S$ , we note that its gauge field part has the same form as in infinite volume. The quark action is again given by eq. (1), provided one formally extends the quark and antiquark fields to Euclidean times  $x_0 < 0$  and  $x_0 > T$  by “padding” with zeros [7]. However, we will use a slightly more general covariant lattice derivative,

$$\nabla_\mu \psi(x) = \frac{1}{a} [\lambda_\mu U(x, \mu) \psi(x + a\hat{\mu}) - \psi(x)], \quad (24)$$

where  $\lambda_0 = 1$  and  $\lambda_k = \exp(ia\theta/L)$ . This modification of the covariant derivative is equivalent to demanding spatial periodicity of the quark fields up to the phase  $\exp(i\theta)$ . We thus have the angle  $\theta$  as an additional parameter that plays a rôle in the improvement condition for the coefficient  $c_A$  [9].

We are now prepared to define the expectation values of any product  $\mathcal{O}$  of fields by

$$\langle \mathcal{O} \rangle = \left\{ \frac{1}{\mathcal{Z}} \int_{\text{fields}} \mathcal{O} e^{-S} \right\}_{\bar{\rho}' = \rho' = \bar{\rho} = \rho = 0}. \quad (25)$$

Apart from the gauge field and the quark and anti-quark fields integrated over,  $\mathcal{O}$  may involve the “boundary fields” at time  $x_0 = 0$ ,

$$\zeta(\mathbf{x}) = \frac{\delta}{\delta \bar{\rho}(\mathbf{x})}, \quad \bar{\zeta}(\mathbf{x}) = -\frac{\delta}{\delta \rho(\mathbf{x})}, \quad (26)$$

and similarly the fields at  $x_0 = T$ . Note that the functional derivatives only act on the Boltzmann factor, because the functional measure is independent of the boundary values of the fields.

### 3.2. Continuum limit and improvement of the Schrödinger functional

Based on the work of Symanzik [13,14] and explicit calculations to one-loop order of perturbation theory [15,16] one expects that the SF is renormalized if the coupling constant and the quark masses are renormalized in the usual way and the quark boundary fields are scaled with a logarithmically divergent renormalization constant.

As in the case of the infinite volume theory discussed in subsect. 2.2, the cutoff dependence of the SF may be described by a local effective theory. An important difference is that the  $O(a)$  effective action  $S_1$  now includes a few terms localized at the space-time boundaries [7]. Such terms then also appear in the  $O(a)$  improved lattice action. However, by an argument similar to the one given at the end of subsect. 2.5, it can be shown that they only contribute at order  $a^2$  to the PCAC relation and the chiral Ward identity considered in sect. 5. In the calculations reported below, the inclusion of boundary counterterms is, therefore, not required.

## 4. IMPROVEMENT CONDITIONS

A non-perturbative determination of  $c_{\text{sw}}$  and  $c_A$  starts from considering two different lattice artefacts, i.e. combinations of observables that are known to vanish in the continuum limit of the theory. One then imposes the improvement conditions that these lattice artefacts vanish for any finite value of the lattice spacing. The specific values of  $c_{\text{sw}}$  and  $c_A$ , where this is achieved then define the  $O(a)$  improved theory for that value of  $a$ .

Of course, there is a large freedom to choose improvement conditions and – on the non-perturbative level – the resulting values of  $c_{\text{sw}}$  and  $c_A$  depend on the choices made. The corresponding variation of  $c_{\text{sw}}$ ,  $c_A$  is of order  $a$ . This variation changes the effects of order  $a^2$  in physical observables computed after improvement. In principle this is irrelevant at the level of  $O(a)$  improvement. Nevertheless, one ought to choose improvement conditions where such terms have small coefficients. The improvement conditions derived from the SF can be studied in perturbation theory.

Such a study provided essential criteria for our detailed choice of improvement conditions. A further criterion is the statistical accuracy that can be obtained in the MC computations of the lattice artefacts.

### 4.1. Determination of $c_{\text{sw}}$ and $c_A$

Using the operator

$$\mathcal{O}^a = a^6 \sum_{\mathbf{y}, \mathbf{z}} \bar{\zeta}(\mathbf{y}) \gamma_5 \frac{\tau^a}{2} \zeta(\mathbf{z}), \quad (27)$$

we define the bare correlation functions

$$\begin{aligned} f_A(x_0) &= -\frac{1}{3} \langle A_0^a(x) \mathcal{O}^a \rangle, \\ f_P(x_0) &= -\frac{1}{3} \langle P^a(x) \mathcal{O}^a \rangle. \end{aligned} \quad (28)$$

The same correlation functions, but with the boundary values  $C$  and  $C'$  interchanged, will be referred to as  $g_A$  and  $g_P$ .

In terms of  $f_A$  and  $f_P$  the PCAC relation for the unrenormalized improved axial current and density may be written in the form

$$m = \frac{\tilde{\partial}_0 f_A(x_0) + c_A a \tilde{\partial}_0^* \partial_0 f_P(x_0)}{2 f_P(x_0)}. \quad (29)$$

We take this as the definition of the bare current quark mass  $m$ . The renormalized quark mass  $m_R$  appearing in eq. (18) is then given by

$$m_R = m \frac{Z_A(1 + b_A a m_q)}{Z_P(1 + b_P a m_q)} + O(a^2). \quad (30)$$

At fixed bare parameters,  $m_R$  and hence also the unrenormalized mass  $m$  should be independent of the kinematical parameters such as  $T$ ,  $L$  and  $x_0$ . This will be true up to corrections of order  $a^2$ , provided  $c_{\text{sw}}$  and  $c_A$  have been assigned their proper values. The coefficients may, therefore, be fixed by imposing the condition that  $m$  has exactly the same numerical value for three different choices of the kinematical parameters.

In detail we set  $\theta = 0$ ,  $T = 2L$ ,  $(\phi_1, \phi_2, \phi_3) = \frac{1}{6}(-\pi, 0, \pi)$  and  $(\phi'_1, \phi'_2, \phi'_3) = \frac{1}{6}(-5\pi, 2\pi, 3\pi)$ . Using the shorthand notation  $m = r_f(x_0) + c_A s_f(x_0)$  for eq. (29), and, correspondingly,  $m = r_g(x_0) + c_A s_g(x_0)$  for the same equation with  $g_A$  and  $g_P$ , our improvement conditions are

$$r_f(x_0) + c_A s_f(x_0) = r_g(x_0) + c_A s_g(x_0), \quad (31)$$

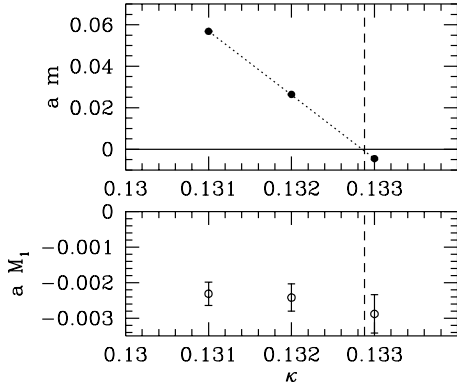


Figure 1. Quark mass dependence of  $M_1$  for  $\beta = 6.4$  and  $c_{\text{sw}} = 1.776625$ . The position of  $m = 0$  is marked by the dashed line. The hopping parameter is defined as  $\kappa = 1/(8 + 2am_0)$ .

at both  $x_0 = T/4$  and  $x_0 = 3T/4$ .

In order to determine  $c_{\text{sw}}$  we eliminate  $c_A$  in eq. (31) and consider the combination

$$M_1 \stackrel{\text{def}}{=} r_f(\frac{3}{4}T) - r_g(\frac{3}{4}T) + \gamma[s_f(\frac{3}{4}T) - s_g(\frac{3}{4}T)], \quad (32)$$

where

$$\gamma = [r_f(\frac{1}{4}T) - r_g(\frac{1}{4}T)]/[s_g(\frac{1}{4}T) - s_f(\frac{1}{4}T)]. \quad (33)$$

The equation  $M_1 = 0$  may then be used to determine  $c_{\text{sw}}$  for each value of  $g_0$ . In practice we suppress the  $O(a)$  uncertainties in  $c_{\text{sw}}$  by a power of  $g_0^2$  by equating  $M_1$  to its tree level value,  $M_1^{(0)}|_{a/L}$ , in the improved theory (at the finite value of  $a/L$ ). We further fixed  $a/L = 1/8$ , where both a good statistical accuracy was achieved and the  $O(a)$  uncertainties of  $c_{\text{sw}}$  are only around 2% in a perturbative estimate.

In general we evaluated  $M_1$  for vanishing mass,

$$m = r_f(T/2) + \gamma s_f(T/2).$$

As demonstrated in fig. 1, finding the value of  $M_1$  at  $m = 0$  is an easy task, since simulations of the SF are possible for both positive and negative values of the quark mass [2]. At each of our nine values of  $g_0$ , this is done for at least three values of  $c_{\text{sw}}$  and  $M_1 = M_1^{(0)}|_{a/L}$  is solved for  $c_{\text{sw}}$  by linear interpolation of  $M_1$  in  $c_{\text{sw}}$  (cf. fig. 2).

In the range  $0 \leq g_0^2 \leq 1$ , the results for  $c_{\text{sw}}$  are well represented by [9]

$$c_{\text{sw}}(g_0) = \frac{1 - 0.656g_0^2 - 0.152g_0^4 - 0.054g_0^6}{1 - 0.922g_0^2}. \quad (34)$$

In fig. 3 we compare our results to one-loop bare perturbation theory and also to tadpole improved perturbation theory [17], for which we have used

$$c_{\text{sw}} = u_0^{-3} \left[ 1 + (c_{\text{sw}}^{(1)} - 1/4)\tilde{g}^2 \right], \quad (35)$$

where  $\tilde{g}^2 = g_0^2/u_0^4$  [18]. Here  $u_0^4$  is the average plaquette in infinite volume.

In a similar way [9] we obtained ( $0 \leq g_0^2 \leq 1$ )

$$c_A(g_0) = -0.00756 \times g_0^2 \frac{1 - 0.748g_0^2}{1 - 0.977g_0^2}. \quad (36)$$

Note that the correction term to the axial current is rather small and both eq. (34) and eq. (36) deviate substantially from the one-loop results except for values of  $g_0^2$  as small as  $g_0^2 \leq 1/2$ .

## 4.2. The chiral region

As chiral symmetry is violated by lattice artefacts, there is no precise definition of a chiral point  $\kappa = \kappa_c \equiv 1/(8 + 2am_c)$  for any finite value of the lattice spacing [7]. Rather  $\kappa_c$  has an intrinsic uncertainty which is reduced from  $O(a^2)$  to  $O(a^3)$  by non-perturbative improvement.

Here, we define  $\kappa_c$  as the value of  $\kappa$ , where  $m$ , eq. (29), vanishes for  $C = C' = \theta = 0, T = 2L, x_0 = T/2$ . To study the  $O(a^3)$  effects we can then still vary the resolution  $a/L$ .

Close to  $\kappa_c$  the mass  $m$  is a linear function of  $\kappa$  to a high degree of accuracy. The point

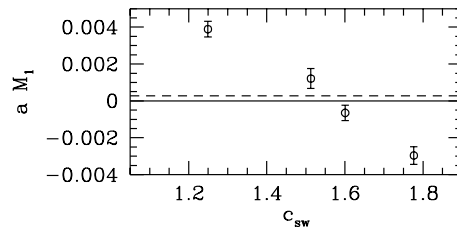


Figure 2. Determination of  $c_{\text{sw}}$  at  $\beta = 6.4$ . The dashed line denotes the tree level value  $M_1^{(0)}|_{a/L}$ .

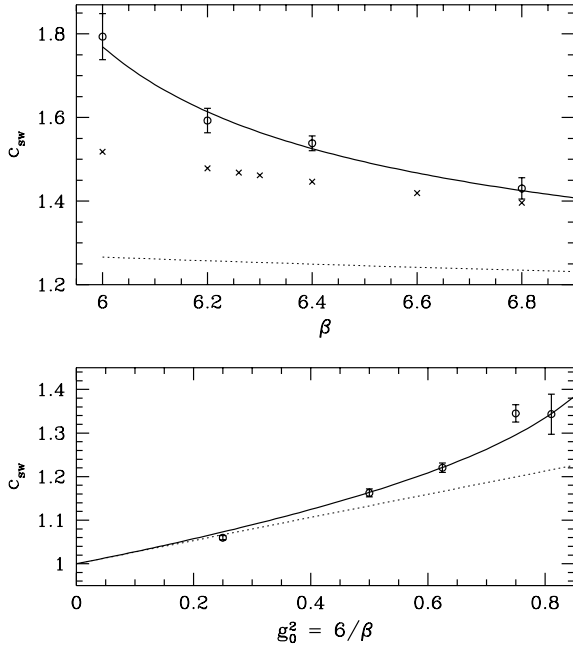


Figure 3. Non-perturbative improvement coefficient  $c_{sw}$ . Results from one-loop bare and tadpole improved perturbation theory are denoted by dotted lines and crosses, respectively.

$\kappa_c$  is therefore easily found by linear interpolation (or, for  $\beta \leq 6.2$ , slight extrapolation). For  $\beta > 6.8$ , the results (fig.4) for  $a/L = 1/8$  and  $a/L = 1/16$  agree on the level of their statistical accuracy which is better than  $10^{-5}$ . For lower  $\beta$  small but significant  $O(a^3)$  effects are seen.

Note that the pion mass (in infinite volume) will not vanish exactly at  $\kappa = \kappa_c$  since there is no exact chiral symmetry for finite values of  $a$ . However, for  $\kappa = \kappa_c$ , as defined here, the pion mass is expected to vanish up to a small lattice artefact of order  $O(a^2)$ .

## 5. CURRENT NORMALIZATION

### 5.1. Chiral Ward identities

For zero quark masses chiral symmetry is expected to become exact in the continuum limit. It has therefore been proposed to fix the normalization constants  $Z_A$  and  $Z_V$  of the isospin currents by imposing the continuum chiral Ward identities also at finite values of the cutoff [19–21].

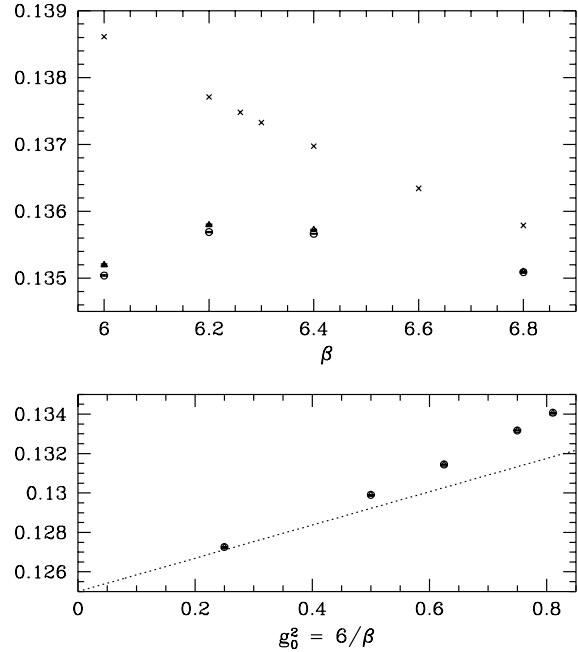


Figure 4.  $\kappa_c$  determined for  $L/a = 8$  (circles), and  $L/a = 16$  (triangles). The dotted line is the one-loop result [11,22,8]. Crosses represent tadpole improved perturbation theory.

In the case of the axial current the relevant Ward identity can be written in the form

$$\int_{\partial R} d\sigma_\mu(x) \epsilon^{abc} \langle A_\mu^a(x) A_\nu^b(y) \mathcal{Q}^c \rangle = 2i \langle V_\nu^c(y) \mathcal{Q}^c \rangle \quad (37)$$

where the integral is taken over the boundary of the space-time region  $R$  containing the point  $y$  and  $\mathcal{Q}^a$  is a source located outside  $R$ . In view of on-shell improvement it is important to note that all space-time arguments in eq.(37) are at non-zero distance from one another.

For the source field in eq.(37) we choose  $\mathcal{Q}^a = \epsilon^{abc} \mathcal{O}^b \mathcal{O}^c$  with  $\mathcal{O}^a$  as given in eq.(27), and  $\mathcal{O}'^a$  defined similarly using the primed fields.

We define the following correlation functions, using the unrenormalized improved currents  $(A_I)_\mu^a = A_\mu^a + ac_A \partial_\mu P^a$  and  $(V_I)_\mu^a = V_\mu^a + ac_V \partial_\nu T_{\mu\nu}$ ,

$$f_{AA}^I(x_0, y_0) = -\frac{a^6}{6L^6} \sum_{\mathbf{x}, \mathbf{y}} \epsilon^{abc} \epsilon^{cde} \times \langle \mathcal{O}'^d (A_I)_0^a(x) (A_I)_0^b(y) \mathcal{O}^e \rangle, \quad (38)$$

$$f_V^I(x_0) = \frac{a^3}{6L^6} \sum_{\mathbf{x}} i\epsilon^{abc} \langle \mathcal{O}'^a (V_1)_0^b(x) \mathcal{O}^c \rangle, \quad (39)$$

$$f_1 = -\frac{1}{3L^6} \langle \mathcal{O}'^a \mathcal{O}^a \rangle. \quad (40)$$

At  $m_q = 0$ , and using the correct values of  $c_{\text{sw}}$  and  $c_A$ , a lattice version of the chiral Ward identity (37) is

$$Z_A^2 f_{AA}^I(x_0, y_0) = f_1 + O(a^2), \quad x_0 > y_0. \quad (41)$$

Compared to eq. (37) we have set  $\nu = 0$  and included an additional summation over  $\mathbf{y}$ , thus obtaining the isospin charge. In deriving eq. (41) we have used the fact that the action of the latter on the chosen matrix elements can be evaluated due to the exact isospin symmetry on the lattice.

Since the isospin symmetry remains unbroken for non-zero quark mass, one need not restrict the normalization condition of the vector current to the case  $m_q = 0$ . Using similar arguments as in the derivation of eq. (41) one obtains

$$Z_V(1 + b_V am_q) f_V^I(x_0) = f_1 + O(a^2). \quad (42)$$

Note that the improvement coefficient  $c_V$  is not needed here, because the tensor density does not contribute to the isospin charge.

## 5.2. Numerical evaluation

Starting from eqs. (41, 42) we impose the following normalization conditions on the axial and vector currents at vanishing quark mass,  $m_q = 0$ ,

$$Z_A^2 f_{AA}^I(\frac{2}{3}T, \frac{1}{3}T) = f_1, \quad (43)$$

$$Z_V f_V^I(\frac{1}{2}T) = f_1, \quad (44)$$

where we have set  $C = C' = \theta = 0$ . It is clear that any other choice of the parameters will lead to the same result up to effects of the order  $a^2$ . We will use these effects as well as those from varying  $L, T$  and the insertion points  $x_0, y_0$  as an estimate of systematic errors on the normalization constants.

Statistical errors on  $Z_A$  and  $Z_V$  were found to be much smaller than those on  $f_{AA}^I, f_V^I$  and  $f_1$ , due to the strong cancellation of correlations in the ratios  $f_1/f_{AA}^I$  and  $f_1/f_V^I$ . A detailed analysis of systematic errors at  $\beta = 6.4$  revealed

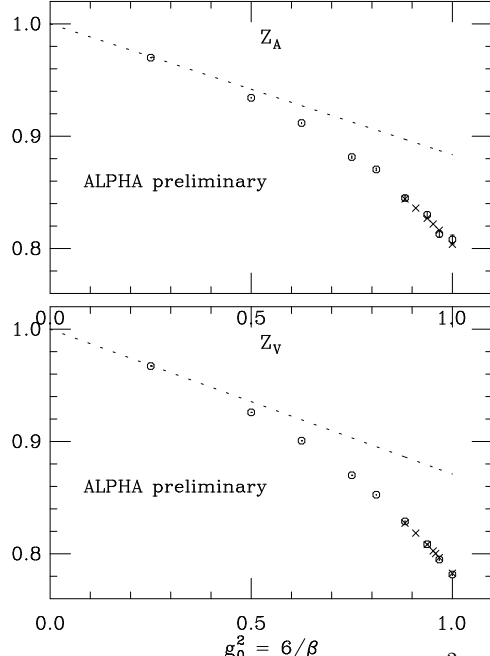


Figure 5.  $Z_A$  and  $Z_V$  plotted against  $g_0^2$ . Results from one-loop bare and tadpole improved perturbation theory are denoted by dotted lines and crosses, respectively.

that the spread of results obtained from varying  $C, C', \theta, L, T$  and the insertion points was mostly well within the statistical errors. The total error was estimated to be at the 1% level for  $Z_V$  and  $Z_A$ .

Fig. 5 shows the measured data for  $Z_V$  and  $Z_A$  obtained for  $L/a = 8, T/a = 18$ . They are compared to one-loop perturbation theory, for which we used the results of ref. [22] which have been confirmed by one of us. The figure shows that contact with one-loop lattice perturbation theory can be established for  $g_0^2 \lesssim 0.5$ . At larger couplings there is also very good agreement between our data and one-loop tadpole improved perturbation theory.

In the case of the vector current one may extend the normalization condition to non-vanishing quark masses, such that the ratio  $f_V^I(T/2)/f_1$  yields the combination  $Z_V(1 + b_V am_q)$ , from which one can extract the improvement coefficient  $b_V$ . A preliminary analysis shows



that  $b_V$  is about 40% above its tree-level value of  $b_V = 1$  for  $6.0 \leq \beta \leq 6.4$ .

This procedure can, however, not be applied to the axial current, since the mass dependence of its normalization is obscured by the physical mass dependence arising from the axial density in the Ward identity.

## 6. CONCLUSIONS

We have been able to implement on-shell  $O(a)$  improvement non-perturbatively in (quenched) lattice QCD. The improvement coefficients  $c_{sw}$ ,  $c_A$ , the critical mass  $m_c$  and the current normalization constants  $Z_A$  and  $Z_V$  have been determined for bare couplings in the range  $0 \leq g_0 \leq 1$ . In all cases contact with bare perturbation theory could be made at couplings around  $g_0^2 \approx 0.25 - 0.5$ . The convergence of perturbation theory appears to be speeded up significantly by tadpole improvement as shown by the crosses in figures 3–5. However, for values of  $\beta \leq 6.8$ , which is the range most relevant for present MC computations, the quality of tadpole improved perturbation theory is rather non-universal. While excellent agreement between perturbation theory and our results for  $Z_A, Z_V$  is found, substantial deviations are observed in the case of  $c_{sw}$ ,  $c_A$  and  $\kappa_c$ .

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