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Time Evolution of $K^0 - \bar{K}^0$ System in Spectral Formulation

Marek Nowakowski
INFN–Laboratori Nazionali di Frascati, P.O. Box 13, I-00044 Frascati (Roma), Italy

Abstract

We reanalyse the time evolution of the $K^0 - \bar{K}^0$ system in the language of certain spectral function whose Fourier transforms give the time dependent survival and transition amplitudes. Approximating the spectral function by an one-pole ansatz we give insight into the limitation of the validity of one-pole approximation, not only for small/large time scales, but also for intermediate times where new effects, albeit small, are possible. It will be shown that the same validity restrictions apply to the known formulae of Weisskopf-Wigner approximation as well. The present analysis can also be applied to the effect of vacuum regeneration of K_L and K_S, a possibility pointed out by Khalfin. As a result of this possibilty new contributions to the well known oscillatory terms will enter the time dependent transition probabilities. These new terms are not associated with small/large time behaviour of the amplitudes and therefore their magnitude is apriori unknown. It will be shown that the order of magnitude of this new effect is very small and, in principle, its exact determination lies outside the scope of the one-pole ansatz.

1 Introduction

The present paper reconsiders an old subject of quantum mechanical time evolution of the $K^0 - \bar{K}^0$ system. Instead of applying the well known Weisskopf-Wigner (WW) approach [1] to the $K^0 - \bar{K}^0$ system [2] we examine the time evolution in the spectral formalism which is often employed for unstable quantum mechanical systems [3]. In this formulation the Fourier transform of a spectral density function gives the time dependent transitions and survival amplitudes. The reasons to pick up once again the old subject of time development are twofold. Since the WW approach is an approximation it is rather useful to have yet another, different formalism which either confirms the WW results (within a certain accuracy) or is capable of displaying new (howsoever small) effects. Due to some peculiarities of the $K^0 - \bar{K}^0$ system one might indeed suspect that the limitations of the applicability of the WW approximations are, in principle, different as compared with other quantum mechanical systems (see below). In view of the planned high precision experiments in this system it is then not unreasonable to reconsider this subject. Secondly, the more specific reason for this reanalysis is a result by Khalfin on the possibilty of vacuum regeneration [4] of K_S and K_L [5], [6], [7]. The latter would induce new terms in the time development formulae which lie outside the usual WW approximation and whose size needs to be estimated. We do it here by using a more refined analysis which is based on a consistency check of the one-pole ansatz.

The $K^0 - \bar{K}^0$ complex is one of the most important test grounds of basic symmetry properties of nature, like CP- and eventually CPT-(non)conservation [8], [9], [10], [11]. It has also been realized that the $K^0 - \bar{K}^0$ system can be used as a sensitive probe of one of the fundamental aspect of the theory of nature, namely Quantum Mechanics [10], [11]. This and the fact that the $K^0 - \bar{K}^0$ system is till now the only system to show experimental evidence of CP-violation makes it clear why this specific subject has always played an outstanding role in particle physics. Since the discovery of CP-violation in 1964 [12] an enormous number of papers has been devoted to this subject, but even today it is an alive area and both, the experiment and the theory, try to infer more information towards a better understanding of CP-violation. From the theoretical side the basic framework to calculate ϵ_K and ϵ'_K in the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ Standard Model and its various extensions is well understood [13]. We have also a good theoretical understanding of (rare) kaon decays [14].

Independent of any specific theory a precision experiment of Quantum Theory in the $K^0 - \bar{K}^0$ system is desirable. It is worth stressing that many ongoing and suggested tests, as well as their refutals, of CP-, T-, [15], CPT-symmetry [16] and QM [17] have directly to do with the time evolution of the system. This brings us to the quantum mechanical time development which is indeed, beside the theoretical determination of the system parameters ϵ_K and ϵ'_K, the second pillar of the $K^0 - \bar{K}^0$ system. Keeping in mind that any possible violation of CPT and QM is forced to

be rather small it is quite important to examine the nature of new effects the time development might hide beyond the WW approximation (the WW approach is an approximation, though a rather good one). To understand why deviations from WW are expected let us recall a well founded theorem which confirms deviations from the exponential decay law $\exp(-\Gamma t)$ for very small (the region of 'quantum Zeno' effect [18]) and very large times [19]. It is also known that the exponential decay law can be derived consistently up to terms of order Γ/M [20] which in the case of interest is

$$\frac{\Gamma_X}{m_X} \sim 10^{-15}, \quad X = K_S, K_L \tag{1.1}$$

That the situation in the $K^0 - \bar{K}^0$ system might be different can be seen from the following reasoning. First, due to mixing the mass difference $m_L - m_S$ will enter the transition probabilities like $|\langle K^0|\bar{K}^0(t)\rangle|^2$ etc. We then find, in addition to (1.1), other dimensionless quantities like

$$\frac{\Gamma_S}{m_L - m_S} \sim \mathcal{O}(1), \quad \frac{m_L - m_S}{m_L} \sim 10^{-15}$$
$$\frac{\Gamma_L}{m_L - m_S} \sim 10^{-3} \tag{1.2}$$

Of course no new effects will be present which go hand in hand with the first ratio. It is the third dimensionless ratio in (1.2) which is intruguing and which is small enough to be dropped in the first approximation, but on the other hand not small enough to be neglected completely.

The second reason why the $K^0 - \bar{K}^0$ system differs from a 'normal' unstable quantum mechanical system is that the K_S and K_L, defined as usual, are not orthogonal to each other due to the presence of CP-violation in the mixing. We mention for completeness that this peculiar property led to speculations about a EPR-like paradox [21] in $K^0 - \bar{K}^0$ system. This result has, however, been criticized by several authors [22] on account of measurement theory and wave function collapse. It is therefore not unimportant to state here that the effect discussed in the present paper, in spite of being counter-intuitive, does not violate any known principle of nature (like causality). For a more detailed discussion on the issue of non-orthogonality of K_S and K_L we refer the reader to the papers [23]. Based on this non-orthogonality it has been proved, in the formalism of spectral functions ρ_S and ρ_L (suitable also otherwise for any unstable quantum mechanical system) that the *vacuum* (in contrast to similar phenomena in *matter*) regeneration probability of $K_S \leftrightarrow K_L$ is *non-zero* unless there is no CP-violation through mixing in the $K^0 - \bar{K}^0$ system [6], [7], [24]. To estimate this effect one needs a reasonable ansatz for ρ_S and ρ_L. It has been known for some time that the one-pole approximation serves very well the purpose (the main part of the paper will be concerned with the question up to which accuracy we can trust the one-pole ansatz). In [6] and [7] the one-pole approximation has been used with the result that the *new* terms in the transition probability $|\langle K^0|\bar{K}^0(t)\rangle|^2$ etc. are of

the order of $\Gamma_L/(m_L - m_S)$. It will be shown below that a consistent treatment of the spectral formalism in general and the one-pole approximation in specific yields a quite different picture as far as the size of this 'new' effect is concerned. Indeed in [6] and [7] not all the information available in the formalism is used. This, in our opinion, leads to the overestimate of the effect. In detail the following will be shown below

(i) Taking into account all available information on the spectral functions ρ_S and ρ_L we investigate the consistency of the one-pole approximation and find that it is valid up to terms of order Γ_X/m_X, $(m_L - m_S)/m_L$. It will be argued that such corrections do arise *not only* for very large and very small time scales [28].

(ii) Through this consistency check we can determine all parameters of the one-pole approximation needed for the time evolution formulae (again up to accuracy of Γ_X/m_X, $(m_L - m_S)/m_L$) in terms of known quantities.

(iii) This makes it possible to derive time evolution formulae like $|\langle K^0|\bar{K}^0(t)\rangle|^2$ etc. in the spectral formalism and with the one-pole ansatz (in the accuracy mentioned above) without any further assumptions. The result of a lengthy calculation is that all formulae agree with the corresponding expressions derived within the WW approach.

(iv) In consequence this result shows explicitly that the vacuum regeneration probability must be of the order Γ_X/m_X, $(m_L - m_S)/m_L$. This, however, does not mean that that such an effect is associated with small/large time behaviour of the amplitudes.

Chiu and Sudarshan [24] treat the same subject and use the solvable Friedrichs-Lee model to show that, in general, the conclusions of ref. [6] and [7] on the vacuum regeneration are indeed correct. However, their numerical estimate is much more modest and agrees with the conclusion of the present paper. It will be shown below that one can arrive at these conclusions by a careful analysis of the one-pole ansatz which gives us also accuracy restrictions of the WW formulae.

The paper is organized as follows. In section 2 we collect all essential and quite general formulae for the time development. In section 3 we present two of Khalfin's results. Section 3 investigates the one-pole ansatz and its consistency. In section 4 all the forgoing results will be gathered to derive the time evolution of the system. In section 5 we present our conclusions.

2 Basic Formulae

Out of the Weisskopf-Wigner approximation we will essentially need only the part which has to do with the eigenvectors of the effective, non-hermitian Hamiltonian

which is the result of two approximations made in the Schrödinger equation [25], [26]. This part defines the K_S and K_L states in the usual way

$$|K_S\rangle = p|K^0\rangle + q|\bar{K}^0\rangle, \quad |K_L\rangle = p|K^0\rangle - q|\bar{K}^0\rangle \tag{2.1}$$

$$\langle K_S|K_S\rangle = \langle K_L|K_L\rangle = |p|^2 + |q|^2 = 1 \tag{2.2}$$

$$\langle K_S|K_L\rangle = \langle K_L|K_S\rangle = |p|^2 - |q|^2 \neq 0 \tag{2.3}$$

The equality $\langle K_S|K_L\rangle = \langle K_L|K_S\rangle$ in eq.(2.3) is imposed by CPT-invariance which we will assume to hold throughout the paper. The presence of CP-violation in the mixing is reflected by $|p|^2 - |q|^2 \neq 0$ which enforces the states K_S and K_L to be non-orthogonal to each other.

Since the CP-violation in the $K^0 - \bar{K}^0$ system (or equivalently the non-orthogonality of K_S and K_L) will play an important role we define for the sake of a short notation

$$\Delta_K \equiv |p|^2 - |q|^2 \tag{2.4}$$

In the context of the present paper it will be convenient to look upon the WW approximation as consisting of two parts: one is the definition of K_L and K_S as given in (2.1) and the other is the time development. At first glance such a separation might look artificial. Consider, however, the following. We know that the time evolution in the WW approach is not complete since we know that there will be corrections to the WW formulae for very small/large times. It is then not unreasonable to ask if in the full solution of the problem (2.1) gets also modified. Indeed in the first step one can assume the validity of (2.1) [27], but try to extend the formulae for the time evolution beyond WW. If in such an approach we derive a counter-intuitive result like the aforementioned vacuum regeneration of K_L and K_S then we are left with two possibilities. One is to question the starting assumption (2.1), the other is of course to accept the physical relevance of the new effect. Assuming the presence of CP-violation in the mixing of $K^0 \leftrightarrow \bar{K}^0$ and implementing therein CPT-contraints it is, however, hard to imagine a definition of K_L and K_S which is different from (2.1), up to possible contributions from continuum states which we neglect (for a different point of view where in the context of a generalized quantum mechanical vector space the physical states are orthogonal see [23] and references therein).

Given a full, hermitian Hamiltonian H according to general principles of Quantum Mechanics the time evolution for K^0 and $\bar{K}^0$ can be summarized as follows

$$\begin{aligned} P_{K_\alpha K_\beta}(t) = \langle K_\alpha|e^{-iHt}|K_\beta\rangle = \langle K_\alpha|K_\beta(t)\rangle \\ |K_\alpha(t)\rangle = e^{-iHt}|K_\alpha\rangle \\ K_\alpha = K^0, \bar{K}^0, \end{aligned} \tag{2.5}$$

Due to the non-orthogonality of K_S and K_L there is a difference between the treatment of the time evolution of K^0, $\bar{K}^0$ and K_S, K_L. For the former the $P_{K_\alpha K_\beta}(t)$ are

expansion coefficients in

$$
\begin{aligned}
|K^0(t)\rangle &= P_{K^0K^0}(t)|K^0\rangle + P_{\bar{K}^0K^0}(t)|\bar{K}^0\rangle \\
|\bar{K}^0(t)\rangle &= P_{\bar{K}^0\bar{K}^0}(t)|\bar{K}^0\rangle + P_{K^0\bar{K}^0}(t)|K^0\rangle
\end{aligned}
\tag{2.6}
$$

which according to the orthogonality of K^0 and $\bar{K}^0$ and in agreement with the first equation in (2.5) are identical to $\langle K_\alpha|K_\beta(t)\rangle$ for $K_\alpha = K^0, \bar{K}^0$. Since the quantum mechanical principle $|A(t)\rangle = \exp(-iHt)|A\rangle$ is valid for any state $|A\rangle$ we can use eqs.(2.1)-(2.3) and eq.(2.5) to derive the following time dependence of K_S and K_L

$$
\begin{aligned}
|K_S(t)\rangle &= p\left[P_{K^0K^0}(t)|K^0\rangle + P_{\bar{K}^0K^0}(t)|\bar{K}^0\rangle\right] + q\left[P_{\bar{K}^0\bar{K}^0}(t)|\bar{K}^0\rangle + P_{K^0\bar{K}^0}(t)|K^0\rangle\right] \\
|K_L(t)\rangle &= p\left[P_{K^0K^0}(t)|K^0\rangle + P_{\bar{K}^0K^0}(t)|\bar{K}^0\rangle\right] - q\left[P_{\bar{K}^0\bar{K}^0}(t)|\bar{K}^0\rangle + P_{K^0\bar{K}^0}(t)|K^0\rangle\right]
\end{aligned}
\tag{2.7}
$$

Note that in this section we are keeping all formulae as general as possible, in accordance with the general principles of Quantum Mechanics. In analogy to eq.(2.6) and again in full generality we can also define expansion coefficients $P_{K_SK_S}(t)$, $P_{K_LK_L}(t)$, $P_{K_LK_S}(t)$ and $P_{K_SK_L}(t)$ through

$$
\begin{aligned}
|K_S(t)\rangle &= P_{K_SK_S}(t)|K_S\rangle + P_{K_LK_S}(t)|K_L\rangle \\
|K_L(t)\rangle &= P_{K_LK_L}(t)|K_L\rangle + P_{K_SK_L}(t)|K_S\rangle
\end{aligned}
\tag{2.8}
$$

Clearly the time dependent functions $P_{K_SK_L}(t)$ and $P_{K_LK_S}(t)$, absent in the WW approximation, would be, unless identical to zero, responsible for vacuum regeneration of $K_S \leftrightarrow K_L$. Using already the following CPT-constraint (being at same time a quite model-indendent test for CPT conservation [7])

$$
P_{K^0K^0}(t) = P_{\bar{K}^0\bar{K}^0}(t) \tag{2.9}
$$

the $P_{K_SK_S}(t)$ etc can be easily obtained from (2.7) by using the inverse transformation of eq.(2.1). The result is

$$
\begin{aligned}
P'_{K_SK_S}(t) - P_{K_LK_L}(t) &= \frac{q}{p}P_{K^0\bar{K}^0}(t) + \frac{p}{q}P_{\bar{K}^0K^0}(t) \\
P_{K_SK_S}(t) + P_{K_LK_L}(t) &= P_{K^0K^0}(t) + P_{\bar{K}^0\bar{K}^0}(t) = 2P_{K^0K^0}(t) \\
P_{K_LK_S}(t) = -P_{K_SK_L}(t) &= \frac{1}{2}\left\{\frac{q}{p}P_{K^0\bar{K}^0}(t) - \frac{p}{q}P_{\bar{K}^0K^0}(t)\right\}
\end{aligned}
\tag{2.10}
$$

Trivially eqs.(2.8) imply a relation between the expansion coefficients $P_{K_SK_S}(t)$ etc and the corresponding matrix elements $\langle K_S|K_S(t)\rangle$ etc.

$$
\begin{aligned}
\langle K_S|K_S(t)\rangle &= P_{K_SK_S}(t) + P_{K_LK_S}(t)\Delta_K \\
\langle K_S|K_L(t)\rangle &= P_{K_LK_L}(t)\Delta_K - P_{K_LK_S}(t) \\
\langle K_L|K_L(t)\rangle &= P_{K_LK_L}(t) - P_{K_LK_S}(t)\Delta_K \\
\langle K_L|K_S(t)\rangle &= P_{K_SK_S}(t)\Delta_K + P_{K_LK_S}(t)
\end{aligned}
\tag{2.11}
$$

This explicitly displays the above mentioned difference between the K^0, $\bar{K}^0$ and the K_S, K_L cases. The matrix element e.g. $\langle K_L|K_S(t)\rangle$ is not equal to the corresponding coefficient $P_{K_LK_S}(t)$. Only if we impose $\Delta_K = 0$ is this equality guaranteed. Hence this property, $\langle K_L|K_S(t)\rangle \neq P_{K_LK_S}(t)$, has nothing to do with the generality of our formulae, but in general with the fact that $\Delta_K \neq 0$.

Based on a S-matrix formalism developped by Sachs [29] for the $K^0 - \bar{K}^0$ system it has been shown in [30] that $|\langle K_S|K_L\rangle|^2$ cannot be interpreted as a probablity. A safer interpretation is to say that the overlap of the corresponding wave functions is non-zero. These interpretatory issues do not, however, affect the results presented in section 3.

Let us now come to the main point of the paper. The question which will be addressed in the next sections is whether [4]

$$P_{K_LK_S}(t) = -P_{K_SK_L}(t) = 0 \ \ or \ \ \neq 0 \tag{2.12}$$

As discussed in the introduction it has been proved in [6], [7] (confirmed in [24]) that indeed the second possibilty must be true unless there is no CP-violation in the mixing, i.e. $\Delta_K = 0$. We will describe this result in more detail in the next section. Before doing so let us state explicitly that in the WW approximation we have $P_{K_LK_S}(t) = -P_{K_SK_L}(t) = 0$ and that the K_S and K_L have the simple time evolution

$$\begin{aligned} P_{K_SK_S}(t)|_{WW} &= e^{-im_St}e^{-\frac{1}{2}\Gamma_St} \\ P_{K_LK_L}(t)|_{WW} &= e^{-im_Lt}e^{-\frac{1}{2}\Gamma_Lt} \end{aligned} \tag{2.13}$$

as would have been expected for physical, unstable particle states (which do not mix). As discussed above even in the WW approximation we have

$$\langle K_L|K_S(t)\rangle|_{WW} \neq 0, \ \ \langle K_S|K_L(t)\rangle|_{WW} \neq 0 \tag{2.14}$$

This simple exercise tells us that after all it might not be so surprising to find a non-zero value of $P_{K_LK_S}(t) = -P_{K_SK_L}(t)$ in the expansion (2.8). It is also useful to derive two further relations which will be the basis of the discussion in the next sections. The first one follows immediately from eq.(2.11) and reads

$$\langle K_S|K_L(t)\rangle + \langle K_L|K_S(t)\rangle = \Delta_K\left[\langle K_L|K_L(t)\rangle + \langle K_S|K_S(t)\rangle\right] \tag{2.15}$$

This expression will lead in the next section to a relation between the spectral density functions ρ_S and ρ_L. This in turn will yield a couple of consistency equation when the spectral functions are approximated by a one-pole ansatz. To obtain the second relation we have to essentially invert the formulae (2.10) and express the $P_{K^0K^0}(t)$ etc matrix elements through the expansion coeffients $P_{K_SK_S}(t)$ etc.

$$P_{K^0\bar{K}^0}(t) = \frac{p}{q}\left\{\frac{1}{2}\left[P_{K_SK_S}(t) - P_{K_LK_L}(t)\right] + P_{K_LK_S}(t)\right\} \tag{2.16}$$

$$P_{\bar{K}^0 K^0}(t) = \frac{q}{p}\left\{\frac{1}{2}\left[P_{K_S K_S}(t) - P_{K_L K_L}(t)\right] - P_{K_L K_S}(t)\right\} \tag{2.17}$$

$$P_{K^0 K^0}(t) = P_{\bar{K}^0 \bar{K}^0}(t) = \frac{1}{2}\left[P_{K_S K_S}(t) + P_{K_L K_L}(t)\right] \tag{2.18}$$

Setting therein $P_{K_L K_S}(t) = 0$ we get

$$\frac{P_{K^0 \bar{K}^0}(t)}{P_{\bar{K}^0 K^0}(t)} = \frac{p^2}{q^2} = const \tag{2.19}$$

This last equation will, when rewritten in the spectral language, lead to $\Delta_K = 0$. Hence the conclusion that $P_{K_S K_L}(t) \neq 0$.

3 Spectral Formulation

For the readers convenience we will reproduce in this section two results which have been drived in [5] and [6], [24]. The first result is a relation between ρ_S and ρ_L which in section 4 will play an important role in checking the consistency of the one-pole ansatz. The second result concerns the vacuum regeneration of K_L and K_S. We refer the reader to the original references for a more detailed discussions on the derivation of the two results.

What we called spectral formalism for unstable quantum mechanical systems is based on two observations. The first one is simply the completeness of the eigenvectors $|q\rangle$ of a hermitian quantum mechanical Hamiltonian. We can then write an unstable state $|\lambda, t\rangle$ (which is never an eigenstate of the Hamiltonian) as

$$|\lambda, t\rangle = \sum_q |q, t\rangle\langle q|\lambda\rangle \tag{3.1}$$

The second observation is the reasonable assumption that the unstable state has only projections on continuum states in which it decays. Denoting from now on the continous eigenvalue of a Hamiltonian by m we can write the survival amplitude $A(t)$ (or, as in case of $K^0 \leftrightarrow \bar{K}^0$ oscillations, transition amplitude) as

$$A(t) = \int_{Spec(H)} dm\, e^{-imt}\rho(m) \tag{3.2}$$

where the integration extends over the whole spectrum of the Hamiltonian and $\rho(m)$ is

$$\rho(m) = |\langle m|\lambda\rangle|^2 \tag{3.3}$$

Of course the spectrum of any sensible Hamiltonian should be bounded from below. The ground state (vacuum) can be then normalized to have zero energy eigenvalue. The integration range in (3.2) is in this case from 0 to ∞. Despite this cut-off in the integral (3.2) imposed on us by physical requirements we stress that $A(t)$ and $\rho(m)$

are still Fourier-transforms of each other. This is guaranteed by the Dirichlet-Jordan (see e.g. [31]) conditions for Fourier integrals which under certain conditions (which we assume here to be fullfiled) allow us to introduce a finite number of discontinuities in the Fourier integrals [32]. With the following Breit-Wigner ansatz (see [20])

$$\rho_{BW}(m) = \frac{\Gamma}{2\pi} \frac{1}{(m-m_0)^2 + \frac{\Gamma^2}{4}} \tag{3.4}$$

we obtain then for the survival amplitude

$$A_{BW}(t) = \int_{-\infty}^{\infty} dm\, e^{-imt} \rho_{BW}(m) = e^{-im_0 t}\, e^{-\frac{1}{2}\Gamma|t|} \tag{3.5}$$

which gives for the survival probability the well known exponential decay law, $P_{BW}(t) = |A(t)|^2 = \exp(-\Gamma t)$. Despite of what has been said about the integration range above we have integrated in (3.5) over $(-\infty, \infty)$ for reasons which will be evident in section 5. There it will become apparent that taking the integral from $-\infty$ to ∞ is in some sense equivalent to neglecting terms of order Γ/M (where M is the mass). The existence of a ground state in $Spec(H)$ indroduces non-exponential corrections (and non-oscillatory terms in $P_{K^0K^0}(t)$ etc.) which, however, using the simple ansatz (3.4) cannot be trusted [20]. We will discuss this ansatz further in section 4.

We can now apply the above formalism to the case of K_S and K_L by introducing a hermitian Hamiltonian with, as before, continuous spectrum of the decay products which we label by indices α, β etc.

$$H|\phi_\alpha\rangle = m|\phi_\alpha\rangle, \;\; \langle\phi_\beta(m')|\phi_\alpha(m)\rangle = \delta_{\alpha\beta}\delta(m'-m) \tag{3.6}$$

The unstable states K_S and K_L are then written in accordance with (3.1) as superpositions of the eigenkets.

$$\begin{aligned} |K_S\rangle &= \int_0^\infty dm \sum_\alpha \rho_{S,\alpha}(m)|\phi_\alpha\rangle \\ |K_L\rangle &= \int_0^\infty dm \sum_\beta \rho_{L,\beta}(m)|\phi_\beta\rangle \end{aligned} \tag{3.7}$$

Note that this can be done for any unstable state. Therefore, strictly speaking, equations (3.7) are as such not the definitions of $|K_S\rangle$ and $|K_L\rangle$. The latter are still defined as linear superposition of $|K^0\rangle$ and $|\bar{K}^0\rangle$ in eq.(2.1).

In what follows we convert the general formulae of section 2 into the langauge of spectral functions $\rho(m)$. To do so we first write down the matrix elements from eq.(2.11). Using (3.6) and (3.7) they are given by

$$\begin{aligned} \langle K_S|K_S(t)\rangle &= \int_0^\infty dm \sum_\alpha |\rho_{S,\alpha}(m)|^2 e^{-imt} \\ \langle K_L|K_L(t)\rangle &= \int_0^\infty dm \sum_\beta |\rho_{L,\beta}(m)|^2 e^{-imt} \end{aligned}$$

$$\langle K_S|K_L(t)\rangle = \int_0^\infty dm \sum_\gamma \rho^*_{S,\gamma}(m)\rho_{L,\gamma}(m)e^{-imt}$$
$$\langle K_L|K_S(t)\rangle = \int_0^\infty dm \sum_\sigma \rho^*_{L,\sigma}(m)\rho_{S,\sigma}(m)e^{-imt} \tag{3.8}$$

Eq.(2.15) can be then recast in the following form

$$\int_0^\infty dm \sum_\alpha \left[\rho^*_{L,\alpha}(m)\rho_{S,\alpha}(m) + \rho^*_{S,\alpha}(m)\rho_{L,\alpha}(m)\right] e^{-imt}$$
$$= \Delta_K \int_0^\infty dm \sum_\beta \left[|\rho_{L,\beta}(m)|^2 + |\rho_{S,\beta}(m)|^2\right] e^{-imt} \tag{3.9}$$

Taking the inverse Fourier transform of (3.9) we arrive at

$$\sum_\alpha \left[\rho^*_{L,\alpha}(m)\rho_{S,\alpha}(m) + \rho^*_{S,\alpha}(m)\rho_{L,\alpha}(m)\right] = \Delta_K \sum_\beta \left[|\rho_{L,\beta}(m)|^2 + |\rho_{S,\beta}(m)|^2\right] \tag{3.10}$$

which is valid for $m \in (0,\infty)$. This equation is one of the main results [5] which will play an important role in the subsequent discussion. It tells us that the spectral functions $\rho_{S,\alpha}$ and $\rho_{L,\alpha}$ are inter-related with each other and any reasonable ansatz which approximates these functions should be such that eq.(3.10) is true at least to certain accuracy. Indeed an ansatz for $\rho_{S,\alpha}$ and $\rho_{L,\alpha}$ similar to (3.4) does not fulfill these requirements in full generality and in section 4 we address this question in more detail. Note also that since eq.(3.10) is an equation in the variable m we might expect that given a certain ansatz for the spectral functions we get more than one consistency equations from it.

To obtain the second main result of [6], [7], [24] it is necessary to derive corresponding spectral expression for $P_{K^0K^0}(t)$ etc. From (2.11), (2.16)-(2.18), (2.15) (alternatively (3.10)) and (3.8)we see that

$$P_{K^0K^0}(t) = P_{\bar{K}^0\bar{K}^0}(t) = \int_0^\infty dm\, \rho_{K^0K^0}(m)e^{-imt}$$
$$= \frac{1}{2}\int_0^\infty \sum_\alpha \Big\{|\rho_{S,\alpha}(m)|^2 + |\rho_{L,\alpha}(m)|^2\Big\}e^{-imt} \tag{3.11}$$

$$P_{K^0\bar{K}^0}(t) = \int_0^\infty dm\, \rho_{K^0\bar{K}^0}(m)e^{-imt} = \frac{1}{4p^*q}\int_0^\infty dm \sum_\beta \Big\{|\rho_{S,\beta}(m)|^2 - |\rho_{L,\beta}(m)|^2$$
$$- \rho^*_{S,\beta}(m)\rho_{L,\beta}(m) + \rho^*_{L,\beta}(m)\rho_{S,\beta}(m)\Big\}e^{-imt} \tag{3.12}$$

$$P_{\bar{K}^0K^0}(t) = \int_0^\infty dm\, \rho_{\bar{K}^0K^0}(m)e^{-imt} = \frac{1}{4pq^*}\int_0^\infty dm \sum_\sigma \Big\{|\rho_{S,\sigma}(m)|^2 - |\rho_{L,\sigma}(m)|^2$$
$$+ \rho^*_{S,\sigma}(m)\rho_{L,\sigma}(m) - \rho^*_{L,\sigma}(m)\rho_{S,\sigma}(m)\Big\}e^{-imt} \tag{3.13}$$

Here $\rho_{K^0K^0}(m)$ etc. are simply defined by the right hand sides of the corresponding equations. As done at the end of the forgoing section if we now set $P_{K_LK_S}(t) = -P_{K_SK_L}(t) = 0$ we obtain the spectral version of (2.19). We write this relation in the following form

$$F(t) \equiv \int_0^\infty dm \left[\rho_{K^0\bar{K}^0}(m) - \frac{p^2}{q^2}\rho_{\bar{K}^0K^0}(m)\right] e^{-imt} = 0 \tag{3.14}$$

By virtue of the Paley-Wiener theorem it can be shown that (3.14) (i.e. $F(t) = 0$) is valid for $-\infty < t < \infty$. By observing from (3.12) and (3.13) that $\rho_{K^0\bar{K}^0} = \rho^*_{\bar{K}^0K^0}$ and taking again the inverse Fourier transform in (3.14) we get

$$\frac{p^2}{q^2} = \frac{\rho_{K^0\bar{K}^0}}{\rho^*_{K^0\bar{K}^0}} \tag{3.15}$$

This, however, immediately leads to

$$\Delta_K = |p|^2 - |q|^2 = 0 \tag{3.16}$$

Hence Khalfin's second result states that putting $P_{K_LK_S}(t) = -P_{K_SK_L}(t)$ to zero invariably implies that on consistency grounds there can be no CP-violation in the mixing provided the K_S and K_L states are defined as in eqs.(2.1). In other words since we know that CP-violation exists in the mixing of $K^0 - \bar{K}^0$ we have to allow for *vacuum* regeneration of K_S and K_L. Note that this conclusion does not depend on a particular choice of $\rho_{S,\alpha}$ and $\rho_{L,\alpha}$. This is quite an unexpected result which, using a different approach, has also been recently confirmed [24]. As far as the interpretation of this result is concerned we recall that the underlying assumption in the derivation was the usual definition of K_L and K_S as given in (2.1). We emphasize that assuming eq.(2.1), but extending the formulae for the time evolution beyond WW is not inconsistent from the formal point of view (in this context we should look upon the derivation as a mathematical theorem). From physical point of view the above result tells us then that either we accept (2.1) and the fact that the non-orthogonality of K_S and K_L makes this system different from any other known system (except for similar system with the same properties like $B^0 - \bar{B}^0$ or $D^0 - \bar{D}^0$) or we can suspect that (2.1) is, strictly speaking, not complete or ill-defined [23]. We will not dwell further on this possiblity here, and refer the reader to [23] and references therein where the issue of the orthogonality of K_L and K_S is discussed in more detail (see also the discussion below eq.(2.4).

In section 2 we have already mentioned that $|\langle K_S|K_L\rangle|^2$ cannot be interpreted as a probability [30]. Such problems are best investigated within the context of a S-matrix approach (combined with density matrix) developed by Sachs [29]. It would be a worthwhile task to examine the problemtic of Khalfin's effect by using this particular approach and eventually to find an interpretation of the coefficient $P_{K_LK_S}(t)$ in eq.(2.8) [33].

It is also worthwhile noting that the above result has been derived within the context of standard Quantum Mechanics and that CPT-symmetry has been implemented. Suggested tests of CPT and Quantum Mechanics based on terms which are in general forbidden by CPT or QM are then not affected by this result provided the chosen observables assume zero values in the limit of CPT conservation or in the context of QM. In particular for a Φ decay Bose statistics requires that

$$\Phi \to \frac{N}{\sqrt{2}} [K_S(q)K_L(-q) - K_L(q)K_S(-q)] \tag{3.17}$$

where q is the spatial momentum and N a normalization factor. This expression is the starting point of many discussions of tests of QM. We emphasize that orthogonality of the decay products is not required in (3.17). Since one can only distinguish the Kaons through their decays an often employed test of QM consists in looking at the two-times correlation amplitude $A(t_1, f_1; t_2, f_2)$ for two different decays modes f_1 and f_2 of the Kaons. Quantum Mechanics states that for $f_1 = f_2$ the correlation amplitude vanishes at equal times, $t_1 = t_2$. The general expression of the correlation amplitude is given by [7]

$$\begin{aligned}
A(t_1, f_1; t_2, f_2) = \frac{N}{\sqrt{2}} \cdot \Big\{ & \\
& \langle f_1|T|K_S\rangle\langle f_2|T|K_L\rangle P_{K_SK_S}(t_1)P_{K_LK_L}(t_2) - \langle f_1|T|K_L\rangle\langle f_2|T|K_S\rangle P_{K_SK_S}(t_2)P_{K_LK_L}(t_1) + \\
& \langle f_1|T|K_L\rangle\langle f_2|T|K_S\rangle P_{K_LK_S}(t_1)P_{K_SK_L}(t_2) - \langle f_1|T|K_S\rangle\langle f_2|T|K_L\rangle P_{K_LK_S}(t_2)P_{K_SK_L}(t_1) + \\
& \langle f_1|T|K_L\rangle\langle f_2|T|K_L\rangle [P_{K_LK_S}(t_1)P_{K_LK_L}(t_2) - P_{K_LK_L}(t_1)P_{K_LK_S}(t_2)] + \\
& \langle f_1|T|K_S\rangle\langle f_2|T|K_S\rangle [P_{K_SK_S}(t_1)P_{K_SK_L}(t_2) - P_{K_SK_L}(t_1)P_{K_SK_S}(t_2)] \Big\}
\end{aligned} \tag{3.18}$$

We easily see from this equation that $A(t, f; t, f) = 0$ i.e. QM is not violated by the extra terms $P_{K_LK_S}(t)$ and $P_{K_SK_L}(t)$. This means among other things that tests as for instance most of those suggested by Eberhard [34] are not affected by Khalfin's effect.

Any other tests which rely on standard WW expressions might, however, be affected. This will crucially depend on the size of this new effect. More importantly so as this effect has nothing to do with deviations of the exponential decay law for very small and very large time. The latter will become manifest in the formulae for time evolution in section 5.

It is therefore mandatory to try to estimate the size of this effect. A first step in this direction will be to make an ansatz for the spectral functions $\rho_{S,\alpha}$ and $\rho_{L,\alpha}$ and to check the consistency of this ansatz. Therefore we collect below all available general conditions which can shed some light on the spectral functions. From (2.1)-(2.3) we get

$$\int_0^\infty dm \sum_\alpha |\rho_{S,\alpha}(m)|^2 = \int_0^\infty dm \sum_\beta |\rho_{L,\beta}(m)|^2 = 1 \tag{3.19}$$

$$\int_0^\infty dm \sum_\sigma \Im m \left(\rho^*_{S,\sigma}(m)\rho_{L,\sigma}(m)\right) = 0 \tag{3.20}$$

$$\int_0^\infty dm \sum_\gamma \Re e \left(\rho^*_{S,\gamma}(m)\rho_{L,\gamma}(m)\right) = \Delta_K \tag{3.21}$$

Eqs.(3.20) and (3.21) follow from (2.3) and the fact that Δ_K is real. Together with (3.10) these equations is all the information on spectral functions $\rho_{S,\alpha}$ and $\rho_{L,\alpha}$ which is given to us. Any ansatz for the spectral functions has to respect these relations, up to a reasonable accuracy. We already mention that in [6] (see also [24] for a detailed discussion) essentially only eq.(3.19) is used. We also point out that once eq.(3.10) and (3.19) are assumed to hold eq.(3.21) follows.

4 One-Pole Approximation and its Consistency

We have seen that the Breit-Wigner ansatz led to the well known exponential decay law (up to corrections induced by the existence of the ground state). It is therefore reasonable to assume a similar form for the $\rho_{S,\alpha}$ and $\rho_{L,\alpha}$. More specifically we write

$$\begin{aligned}\rho_{S,\beta}(m) &= \sqrt{\frac{\Gamma_S}{2\pi}} \frac{A_{S,\beta}(K_S \to \beta)}{m - m_S + i\frac{\Gamma_S}{2}} \\ \rho_{L,\beta}(m) &= \sqrt{\frac{\Gamma_L}{2\pi}} \frac{A_{L,\beta}(K_L \to \beta)}{m - m_L + i\frac{\Gamma_L}{2}}\end{aligned} \tag{4.1}$$

where $A_{S,\alpha}$ and $A_{L,\alpha}$ are decay amplitudes. It is convenient to make the following definitions

$$\gamma_S \equiv \frac{\Gamma_S}{2}, \quad \gamma_L \equiv \frac{\Gamma_L}{2}, \quad \Delta m \equiv m_S - m_L \tag{4.2}$$

$$S \equiv \sum_\alpha |A_{S,\alpha}|^2, \quad L \equiv \sum_\alpha |A_{L,\alpha}|^2 \tag{4.3}$$

$$R \equiv \sum_\sigma \Re e \left(A^*_{S,\sigma} A_{L,\sigma}\right), \quad I \equiv \sum_\sigma \Im m \left(A^*_{S,\sigma} A_{L,\sigma}\right) \tag{4.4}$$

The quantities (4.3) and (4.4) are the only apriori unknown variables which, with the spectral functions given by (4.1), will enter e.g. equations like (3.11)-(3.13). As already mentioned at the end of the last section we have to insert (4.1) into the expressions (3.10) and (3.19)-(3.21) to examine the consistency of the one-pole approximation (4.1).

We start with eq.(3.19) where the integral can be easily performed. The result is

$$S = 1 + \frac{\gamma_S}{\pi m_S} + \mathcal{O}((\gamma_S/m_S)^2), \quad L = 1 + \frac{\gamma_L}{\pi m_L} + \mathcal{O}((\gamma_L/m_L)^2) \tag{4.5}$$

For reasons explained below we will keep, up to a certain point, terms of order Γ_X/m_X. Since (3.10) contains the variable m pluggging the one-pole approximation (4.1) in

(3.10) we get a polynomial in the variable m of degree two which should be identically zero. Therefore coefficient of each power in m should be also zero. Instead of one equation we have three consistency conditions.

$$m^2\left[2\sqrt{\gamma_S\gamma_L}\cdot R-\Delta_K(\gamma_S\cdot S+\gamma_L\cdot L)\right]=0$$

$$m\left[-2\sqrt{\gamma_S\gamma_L}(m_L+m_S)\cdot R-2\sqrt{\gamma_S\gamma_L}(\gamma_S-\gamma_L)\cdot I+2\Delta_K(\gamma_S m_L\cdot S+\gamma_L m_S\cdot L)\right]=0$$

$$\delta_{SL}\equiv\Delta_K\left[\gamma_S\cdot S({m_L}^2+{\gamma_L}^2)+\gamma_L\cdot L({m_S}^2+{\gamma_S}^2)\right]-2\sqrt{\gamma_S\gamma_L}(\gamma_S\gamma_L+m_Sm_L)\cdot R$$
$$+2\sqrt{\gamma_S\gamma_L}(\gamma_Lm_S-\gamma_Sm_L)\cdot I=0 \tag{4.6}$$

From the first two we easily get

$$R=\frac{\Delta_K}{2\sqrt{\gamma_S\gamma_L}}(\gamma_S\cdot S+\gamma_L\cdot L) \tag{4.7}$$

$$I=\frac{\Delta_K}{2\sqrt{\gamma_S\gamma_L}}\frac{\Delta m}{\gamma_L-\gamma_S}(\gamma_S\cdot S-\gamma_L\cdot L) \tag{4.8}$$

whereas the last condition in (4.6) needs a more detailed treatment. The reason why we did not neglect till now terms of order Γ_X/m_X is now apparent. Namely, in zeroth order of Γ_X/m_X we obtain

$$\delta_{SL}|_{S=L=1}=0 \tag{4.9}$$

Hence to estimate how badly δ_{SL} deviates from zero it is necessary to include the next order of Γ_X/m_X. In this order using (4.5) δ_{SL} reads

$$\delta_{SL}=\frac{\Delta_K}{\pi}\left(\frac{\gamma_L}{m_L}\right)\frac{\gamma_S}{\gamma_S-\gamma_L}\left[(\gamma_L-\gamma_S)^2+\Delta m^2\right]\left[(\gamma_S-\gamma_L)-\gamma_S\frac{\Delta m}{m_L}\right]$$
$$\sim\frac{\Delta_K}{\pi}\Delta m^3\left(\frac{\gamma_L}{m_L}\right) \tag{4.10}$$

For the order of magnitude estimate in (4.10) we have used $\Gamma_S/\Delta m\sim\mathcal{O}(1)$. Strictly speaking this amounts to saying that the ansatz (4.1) is not consistent. Note, however, the following. The smallest mass scale parameter which appears in calculations involving the $K^0-\bar{K}^0$ system is Δm. δ_{SL} in (4.6) has the canonical dimension 3. What eq.(4.10) then tells us is that as compared to the third power of the smallest mass scale δ_{SL} is zero, up to corrections of order Γ_X/m_X. Therefore to this accuracy everything is consistent so far. Clearly, by assuming $\Delta_K=0$ we obtain $R=I=0$.

The reader will have noticed that in making estimates like in eq.(4.10) we are relying on measured parameters of the $K^0-\bar{K}^0$ system. In order not to lose track of the main point we will not examine simultaneuosly the systems $B^0-\bar{B}^0$ and $D^0-\bar{D}^0$. There the smallest mass scale parameter is not Δm but the corresponding difference in the widths $\Delta\Gamma$. The investigation of the consistency of (4.1) will be then slightly different in those systems. The general (hypothetical) case as well as cases of physical interest other than the $K^0-\bar{K}^0$ system will be treated elsewhere [35].

Using only eq.(3.10) and the normalization condition (3.19) we have already pinned down the S, L, R and I in terms of known quantities like widths, masses and Δ_K. The equations (3.10) and (3.19)-(3.21) represent therefore an over-determined system. In contrast to situations discussed at the end of this section this is equivalent to a consistency check.

On account of the validity of eq.(3.10), proved for terms up to Γ_X/m_X, eq.(3.21) is bound to hold. We are therefore left with one more condition, namlely (3.20). We will discuss the calculation in connection with (3.20) in some more detail since part of the steps will enter also the formulae of time evolution in section 5. The calculation will become more transparent by writing down explictly the product $\sum_\beta \rho^*_{S,\beta}(m)\rho_{L,\beta}(m)$ with the spectral functions given by (4.1).

$$\sum_\beta \rho^*_{S,\beta}(m)\rho_{L,\beta}(m)|_{BW} = \frac{\sqrt{\gamma_S\gamma_L}}{\pi\left[(m-m_L)^2+{\gamma_S}^2\right]\left[(m-m_L)^2+{\gamma_L}^2\right]} \cdot\Big\{(a_R m^2+b_R m+c_R)+i(a_I m^2+b_I m+c_I)\Big\} \tag{4.11}$$

with

$$\begin{aligned} a_I = I,\quad b_I &= (\gamma_S-\gamma_L)\cdot R-(m_S+m_L)\cdot I \\ c_I &= (\gamma_L m_S-\gamma_S m_L)\cdot R+(m_L m_S+\gamma_S\gamma_L)\cdot I \end{aligned} \tag{4.12}$$

and similar expressions for a_R, b_R and c_R. Next the partial fraction decomposition

$$\frac{a_I m^2+b_I m+c_I}{\left[(m-m_L)^2+{\gamma_S}^2\right]\left[(m-m_L)^2+{\gamma_L}^2\right]} = \frac{C_I m+D_I}{(m-m_S)^2+{\gamma_S}^2}+\frac{E_I m+F_I}{(m-m_L)^2+{\gamma_L}^2} \tag{4.13}$$

leads as usually to a linear system for coefficients C_I, D_I, E_I and F_I

$$\begin{aligned} E_I &= -C_I \\ C_I\Delta m+D'_I+F'_I &= a_I \\ C_I\left[\left({m_L}^2+{\gamma_L}^2\right)-\left({m_S}^2+{\gamma_S}^2\right)\right]-2D'_I m_L-2F'_I m_S &= b_I \\ D'_I\left({m_L}^2+{\gamma_L}^2\right)+F'_I\left({m_S}^2+{\gamma_S}^2\right)+C_I\left[m_L\left({m_S}^2+{\gamma_S}^2\right)-m_S\left({m_L}^2+{\gamma_L}^2\right)\right] &= c_I \end{aligned} \tag{4.14}$$

where we have used the redefinitions

$$D'_I \equiv D_I+C_I m_S,\quad F'_I \equiv F_I-C_I m_L \tag{4.15}$$

This system plays a double role in our discussion. It appears here as a middle step in the consistency check and is a necessary ingredient in the calculation of the time dependent transition amplitudes in the next section. Hence we feel that it is of enough

importance to give the explicit solution of this system in appendix A. To perform the integral in (3.20) we need also

$$\Lambda(R,I) \equiv \int_0^\infty dm \, \frac{a_I m^2 + b_I m + c_I}{[(m-m_L)^2+\gamma_S{}^2]\,[(m-m_L)^2+\gamma_L{}^2]} = \\ -C_I \frac{\Delta m}{m_L} + \frac{D_I + C_I m_S}{\gamma_S}\left(\pi - \frac{\gamma_S}{m_S}\right) + \frac{F_I - C_I m_L}{\gamma_L}\left(\pi - \frac{\gamma_L}{m_L}\right) \\ + \mathcal{O}((\Gamma_X/m_X)^2) + \mathcal{O}((\Delta m/m_L)^2) \tag{4.16}$$

such that the condition (3.20) reduces to

$$\Lambda(R,I) = 0 \tag{4.17}$$

Taking the solutions for C_I, D_I' and F_I' in terms of R and I (see appendix A) and inserting them into (4.17) a lengthy calculation yields

$$R \cdot \Delta m \left[\Delta m^2 + (\gamma_S - \gamma_L)^2\right]\left[2\pi + \frac{\gamma_S + \gamma_L}{m_L}\right] \\ + I \cdot (\gamma_S + \gamma_L)\left[\Delta m^2 + (\gamma_S - \gamma_L)^2\right]\left[2\pi - \frac{\Delta m}{m_L}\frac{\Delta m}{\gamma_S + \gamma_L}\right] = 0 \tag{4.18}$$

In performing this calculation it is not advisable to make too strong approximations right from the beginning. This is due to some cancellations which can occur. It is now trivial to compare eq.(4.18) with (4.7) and (4.8). In a simplified form eq.(4.18) is

$$\frac{I}{R} \simeq -\frac{\Delta m}{\gamma_S + \gamma_L} + \mathcal{O}(\Gamma_X/m_X) + \mathcal{O}(\Delta m/m_L) \tag{4.19}$$

which agrees with (4.7) and (4.8) when taking the approximation $S = L \simeq 1$ [36]. The obvious conclusion here is that the one-pole ansatz (4.1) indeed passes the consistency check which has been imposed on us by a set of equations in section 3. This check revealed that (4.1) is valid up to terms of zeroth order in (Γ_X/m_X), $(\Delta m/m_L)$. We emphasize that this is not a trivial check. To see this let us investigate the situation where we put by hand $\Delta_K = 0$. In this case we would obtain an homogeneous linear system whose only solution is $R = I = 0$. No information on the accuracy of (4.1) would follow from this. On the other hand keeping $\Delta_K \neq 0$ but dropping eq.(3.10) from the analysis we would end up with four equations ((3.19)-(3.21) for the four unkowns S, L, R, and I. Again no conclusion on the accuracy could have been reached. This displays once again the different nature of the $K^0 - \bar{K}^0$ system and also the usefulness of (3.10). As far as the *size* of one possible correction term ($\sim \Gamma_X/m_X$) is concerned the alert reader might object that this has been known all along as corrections to the exponential decay law. This is only partly true. As we have tried to argue above the presence of CP-violation alters the picture completely as only in this case equations (3.10) and (3.19)-(3.21) are an overdetermined system. In this context

we remark that: 1. a consistency check has to be performed in any case as (4.1) could have been inconsistent for totally different reasons and 2. it is probably safer not to rely on restrictions obtained in the framework of a CP-conserving theory. Corrections of the order $\mathcal{O}(\Gamma_X/m_X)$ are of course expected to the *exponential decay law*, but the result here is more general as it explicitly states that corrections to *oscillatory terms* in $P_{K^0K^0}(t)$ etc. coming from exact (unkown) spectral functions $\rho_{S,\alpha}$ and $\rho_{L,\alpha}$ will be of the same order. Both these corrections are totally different in nature since corrections to $\exp(-\Gamma t)$ are associated with the small/large time behaviour of the amplitudes whereas corrections to oscillatory terms might also arise for intermediate time scales. Indeed Khalfin's result on vacuum regeneration of K_S and K_L discussed in section 3 induces corrections of the latter type (see section 5). The nature of such corrections steming from beyond (4.1) cannot be then apriori known and an analysis is required. That this analysis revealed $\mathcal{O}(\Gamma_X/m_X)$ and $\mathcal{O}(\Delta m/m_L)$ as limits of applicability of (4.1) means also that we can trust terms of order $\mathcal{O}(\Gamma_L/\Delta m)$, should such terms indeed appear along the line of further calculations. From now one we use

$$S = L \simeq 1 \tag{4.20}$$

unless otherwise stated.

We close this section by observing that the sum of (4.7) and (4.8) with $S = L \simeq 1$ is nothing else but the well known Bell-Steinberger unitarity relation [37], namely

$$\Delta_K \left(\gamma_S + \gamma_L - i\Delta m\right) = 2\sqrt{\gamma_S \gamma_L} \sum_\beta A^*_{S,\beta} A_{L,\beta} \tag{4.21}$$

The reason it appears here in a slightly different form (compare e.g. with [38]) is the different normalization of the amplitudes. The fact that our eqs.(4.7) and (4.8) lead directly to the Bell-Steinberger relation (4.21) shows that the results presented here are self-consistent. Recently corrections to (4.21) of the order $\mathcal{O}(\Delta m/m_L)$ have been calculated (see the second reference in [23]). As shown above such corrections are indeed expected. Finally we note that for the analysis in this section it is immaterial whether or not $P_{K_L K_S}(t)$ is zero.

5 Time Development

Having convinced ourselves that the one-pole approximation (4.1) is consistent if we exclude terms of order $\mathcal{O}(\Gamma_X/m_X)$ and $\mathcal{O}(\Delta m/m_L)$ we can proceed to calculate the matrix elements (3.11)-(3.13). With equations (4.7), (4.8) and (4.20) we have all necessary information to do so. We mentioned in section 3 that the ground state in $Spec(H)$ induces corrections to the exponential decay law (3.5). Since this also implies the integration domain $(0, \infty)$ in (3.11)-(3.13) we should handle such terms with care and make sure that all 'new' terms induced by the lower integration limit

are indeed strictly of non-oscillatory type in (3.11)-(3.13). This is also important as we want to find out if Khalfin's effect is correlated with small/large time scales. The relevant integrals have been calculated analytically in appendix B. We can infer from the expressions in appendix B that such terms contain the exponential integral function Ei [39]. We can safely neglect the terms with Ei as it should be clear that the simple ansatz (4.1) cannot account for the correctness of such non-oscillatory terms.

Let us now have a closer look at (3.11). In the one-pole approximation (4.1) $P_{K^0K^0}(t)$ can be conveniently written as (see also (B.9) in appendix B)

$$P_{K^0K^0}(t) = P_{\bar{K}^0\bar{K}^0}(t) = \frac{1}{2\pi}\Bigg\{e^{-im_S t}\left(-\int_0^{-m_S/\gamma_S} dy\,\frac{e^{-i\gamma_S ty}}{y^2+1} + \int_0^{\infty} dy\,\frac{e^{-i\gamma_S ty}}{y^2+1}\right) + [S \to L]\Bigg\} \tag{5.1}$$

We see that we have to calculate integrals of the following type

$$K^{(n)}(a) \equiv \int_0^{\infty} dx\,\frac{x^n}{x^2+1}e^{-iax}$$
$$J^{(n)}(a,\eta) \equiv \int_0^{\eta} dx\,\frac{x^n}{x^2+1}e^{-iax} \tag{5.2}$$

Collecting only oscillatory terms from the integrals in appendix B we obtain the same expression as in WW-approximation (this of course is not a surprise recalling that our concern here is the last equation in (2.10) where only $P_{K^0\bar{K}^0}(t)$ and $P_{\bar{K}^0K^0}(t)$ play a role)

$$P_{K^0K^0}(t) = P_{\bar{K}^0\bar{K}^0}(t) = \frac{1}{2}\left\{e^{-im_S t}e^{-\gamma_S t} + e^{-im_L t}e^{-\gamma_L t}\right\} + N_{K^0K^0}(t) \tag{5.3}$$

where $N_{K^0K^0}(t)$ denotes all non-oscilllatory terms present in the integral. $N_{K^0K^0}(t)$ can, in principle, be extracted from equations (B.1)-(B.5) but as we said before we cannot trust such terms to be the correct non-oscillatory corrections.

One more comment is order. Putting γ_S/m_S to zero the sum of the two integrals in (5.1) can be compactly written as

$$\int_{-\infty}^{\infty} dy\,\frac{e^{-i\mu y}}{a^2+y^2} = \frac{\pi}{a}e^{-\mu a} \tag{5.4}$$

which of course means that

$$N_{K^0K^0}(t) \to 0 \;\; as \;\; \frac{\Gamma_{S/L}}{m_{S/L}} \to 0 \tag{5.5}$$

in agreement with what we said at the beginning of section 3 (see discussion below eq.(3.5)).

Similarly the integration in (3.12) and (3.13) can be done analytically (see (B.10) in appendix B) and the result reads

$$P_{K^0\bar{K}^0}(t) = \frac{1}{4p^*q}\left\{e^{-im_S t}e^{-\gamma_S t}[1+\kappa_S] - e^{-im_L t}e^{-\gamma_L t}[1+\kappa_L]\right\} + N_{K^0\bar{K}^0}(t)$$
$$P_{\bar{K}^0K^0}(t) = \frac{1}{4pq^*}\left\{e^{-im_S t}e^{-\gamma_S t}[1-\kappa_S] - e^{-im_L t}e^{-\gamma_L t}[1-\kappa_L]\right\} + N_{\bar{K}^0K^0}(t) \quad (5.6)$$

where $N_{K^0\bar{K}^0}(t)$ and $N_{\bar{K}^0K^0}(t)$ are again non-oscillatory terms containing the exponential integral function Ei and $\kappa_{S/L}$ are given by

$$\kappa_S = -2i\frac{\sqrt{\gamma_S\gamma_L}}{\gamma_S}\left[D_I' - i\gamma_S C_I\right]$$
$$\kappa_L = +2i\frac{\sqrt{\gamma_S\gamma_L}}{\gamma_L}\left[F_I' + i\gamma_L C_I\right] \quad (5.7)$$

The parameter C_I, D_I' and F_I' are defined as solutions of the linear system (4.14). Equation (5.6) together with (2.10) shows that Khalfin's effect depends crucially on the size of the quantities $\kappa_{S/L}$. We could, in principle, calculate these quantities taking the solutions C_I, D_I' and F_I' from appendix A. There is, however, a more elegant way by going back to (4.14). This linear system fixes C_I, D_I' and F_I' in terms of R and I (eqs.(4.7)-(4.8)) the latter being kept at that stage in section 4 arbitrary i.e. in any order of Γ_X/m_X. But we know now that we are allowed to keep only the zeroth order of Γ_X/m_X. Then R, I taken together with (4.20) and a redefinition of the form

$$\begin{pmatrix}\tilde{C}_I\\ \tilde{D}_I\\ \tilde{F}_I\end{pmatrix} = 2\frac{\sqrt{\gamma_S\gamma_L}}{\Delta_K}\begin{pmatrix}C_I\\ D_I'\\ F_I'\end{pmatrix} + \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}\tilde{a}_I\\ \tilde{b}_I\\ \tilde{c}_I\end{pmatrix} = 2\frac{\sqrt{\gamma_S\gamma_L}}{\Delta_K}\begin{pmatrix}a_I\\ b_I\\ c_I\end{pmatrix} \quad (5.8)$$

convert (4.14) into a homogeneous linear system in the limit $\Gamma_X/m_X \to 0$

$$\begin{pmatrix}-\tilde{a}_I & 1 & 1\\ -\tilde{b}_I & -2m_L & -2m_S\\ -\tilde{c}_I & {m_L}^2+{\gamma_L}^2 & {m_S}^2+{\gamma_S}^2\end{pmatrix}\begin{pmatrix}\tilde{C}_I\\ \tilde{D}_I\\ \tilde{F}_I\end{pmatrix} = 0 \quad (5.9)$$

Since the determinant of the cofficient matrix in (5.9) is non-zero [40] we get only a trivial solution

$$\tilde{C}_I = \tilde{D}_I = \tilde{F}_I = 0 \quad (5.10)$$

This immediately implies that

$$\kappa_S = \kappa_L = \Delta_K + \mathcal{O}(\Gamma_X/m_X) + \mathcal{O}(\Delta m/m_L) \quad (5.11)$$

Equipped with this simple result eq.(5.6) take the familar form

$$P_{K^0\bar{K}^0}(t) = \frac{p}{2q}\left\{e^{-im_S t}e^{-\gamma_S t} - e^{-im_L t}e^{-\gamma_L t}\right\} + non-osc.terms$$
$$P_{\bar{K}^0K^0}(t) = \frac{q}{2p}\left\{e^{-im_S t}e^{-\gamma_S t} - e^{-im_L t}e^{-\gamma_L t}\right\} + non-osc.terms \quad (5.12)$$

Up to non-oscillatory terms these equations are equivalent to the WW-expressions. What we have shown is that indeed corrections to oscillatory terms due to Khalfin's general result will appear in (5.12), but they are necessarily of the order $\mathcal{O}(\Gamma_{_X}/m_{_X})$, $\mathcal{O}(\Delta m/m_{_L})$. This follows from the fact that the one-pole approximation is trustable only up to such terms. In the calculation with the one-pole ansatz any term whose order of magnitude is much bigger than $\mathcal{O}(\Gamma_{_X}/m_{_X})$, $\mathcal{O}(\Delta m/m_{_L})$, like $\Gamma_{_L}/\Delta m$, would be then still acceptable. But such a term does not show up along the line of the calculation. It should also be appreciated that such corrections have nothing to do with small/large time behaviour of the transition amplitudes (i.e. they are not interrelated to the usual corrections to the exponential decay law). This is evident from the way $\kappa_{_{S/L}}$ enters (5.6). Since the existence of the vaccum regeneration depends crucially on the assumption $\Delta_K \neq 0$, one could expect that the size of the effect is also proportional to Δ_K.

Finally the answer to the question we have put forward in the form of equation (2.12) can also be given by a simple equation, namely

$$P_{K_{_L}K_{_S}}(t) = -P_{K_{_S}K_{_L}}(t) = 0 + \mathcal{O}(\Gamma_{_X}/m_{_X}) + \mathcal{O}(\Delta m/m_{_L}) \tag{5.13}$$

Had we not Khalfin's theorem discussed in section 3, it would be compeletely legitimate to assume $P_{K_{_L}K_{_S}}(t)$ to be strictly zero. Our result agrees with the conclusion of ref. [24] reached there in a different way. In view of the smallness of the effect (i.e. the effect is essentially not measureable) many tests of Quantum Mechanics [41] which make use of the WW-approximation are safe. In other words an experimentally verified signal which hints towards some new physics cannot be attributed to Khalfin's effect. We postpone any further discussion to the next section where we will give a summary. In the end we note that in [6] (see also [24]) the size of the new effect has been estimated, also in the context of the one-pole ansatz, to be 4×10^{-4} (see appendix C). The difference between the approach in [6] and ours is essentially our consistent treatment of the one-pole approximation in section 4 (in this context see also appendix C)

The reader will have noticed that our conclusion regarding the size of the effect depends of course on the ansatz we make for the spectral function. Therefore. strictly speaking, a new reasonable ansatz might, in principle, change this conclusion. Such a new ansatz which modifies mainly the tail behaviour of the spectral function has been proposed in [6] and [7]. One should, however, keep in mind that the one-pole ansatz gives an overall consistent picture (this was one of the main objectives of the present paper).

6 Conclusions

It is satisfactory to arrive after lengthy calculations at familar expressions of the Weisskopf-Wigner approximation. More so as our starting point was completely

different from the WW-approach. This not only gives us more confidence in the WW-approximation whose equations, as we know, are of utmost importance for the $K^0 - \bar{K}^0$ system, but has also the virtue that one is able to derive the limitations of the WW-approximation for the oscillatory as well as for the exponential terms. We have emphasized that corrections to the oscillatory terms are different in nature from corrections arising from small/large time behaviour of the amplitudes. It turned out, however, that both such corrections must be of the order $\mathcal{O}(\Gamma_{_X}/m_{_X})$, $\mathcal{O}(\Delta m/m_{_L})$. This is apriori not evident due to the specifics of the $K^0 - \bar{K}^0$ system where beside $\Gamma_{_X}/m_{_X}$ quantities like $\Gamma_{_L}/\Delta m$ do appear. Let us recapitulate the steps which have led to our result. We have presented two of Khalfin's theorems. One was eq.(3.10) which played a crucial role in our analysis. Actually without this equation no conclusion on the validity of the one-pole approximation could have been reached. The other one was the result on the existence of K_S and K_L vacuum regeneration, an effect usually associated with interactions of K_S and K_L in matter. Although this result is quite 'exotic' the author of the present paper could not find a loop-hole in the arguments which led to this result. The vacuum regeneration of K_S and K_L goes against what one would intuitively expect and what one is normally used to. Note, however, the this 'intuition' is based on quantum mechanical systems where the unstable states have zero overlap. $|K_S\rangle$ and $|K_L\rangle$ have non-zero overlap, a singled-out property which is then responsible for counter-intuitive effects. The proof of Khalfin's result relies on well established formalism of Quantum Mechnics (eqs.(3.1)-(3.3)) and seems therefore hard to dispute once we assume that $|K_S\rangle$ and $|K_L\rangle$ are given as in (2.1). To estimate the size of such an effect we had to perform a consistency check of the one-pole approximation (4.1). The outcome of this check provided us with limits of the applicability of (4.1) and the determination of apriori unkown variables (combinations of decay amplitudes). Indeed the difference between the present paper and the result obtained in [6] can be traced back to exactly this point. In a subsequent step we have derived the time evolution of the system starting from the equations (3.11)-(3.13). The formulae so obtained agreed with expressions from the WW-formalism. This in turn implied that the effect of vacuum regeneration of K_S and K_L is necessary small and of the order of $\mathcal{O}(\Gamma_{_X}/m_{_X})$, $\mathcal{O}(\Delta m/m_{_L})$, provided we accept the one-pole approximation as an reasonable ansatz. It was one of the objectives of the present paper to show that the one-pole ansatz gives a consistent overall picture and should therefore be trusted within a certain accuracy.

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Appendix A

We list here the solutions of the linear system (4.14). Since the expressions are lengthy it is convenient to use the following notational abbreviations

$$\begin{aligned} X_+ &= {\gamma_S}^2 + {\gamma_L}^2, \;\; X_- = {\gamma_S}^2 - {\gamma_L}^2 \\ Z &= m_L {\gamma_S}^2 - m_S {\gamma_L}^2 \\ Y_I &= \Delta m^4 + 2\Delta m^2 X_+ + X_-^2 \end{aligned} \tag{A.1}$$

The solutions in terms of a_I, b_I and c_I defined in eq.(4.12) then read

$$\begin{aligned} F_I' \cdot Y_I =& a_I\Big[\Delta m^2 {m_L}^2 - ({m_L}^2 + {\gamma_L}^2)X_- - \Delta m m_L X_+ + (m_L + m_S)Z\Big] + \\ & b_I\Big[\Delta m^2 m_L + Z - \Delta m \gamma_L\Big] + \\ & c_I\Big[\Delta m^2 + X_-\Big] \end{aligned} \tag{A.2}$$

$$\begin{aligned} D_I' \cdot Y_I =& a_I\Big[\Delta m^2 {m_S}^2 + ({m_S}^2 + {\gamma_S}^2)X_- + \Delta m m_S X_+ - (m_L + m_S)Z\Big] + \\ & b_I\Big[\Delta m^2 m_S - Z + \Delta m \gamma_S\Big] + \\ & c_I\Big[\Delta m^2 - X_-\Big] \end{aligned} \tag{A.3}$$

$$\begin{aligned} C_I \cdot Y_I =& a_I\Big[\Delta m^3 - \Delta m({m_S}^2 + {m_L}^2 - {\gamma_S}^2 - {\gamma_L}^2) - (m_S + m_L)X_-\Big] - \\ & b_I[\Delta m(m_S + m_L) + X_-] - \\ & c_I 2\Delta m \end{aligned} \tag{A.4}$$

Appendix B

This appendix contains the relevant integrals appearing in (3.11)-(3.13) with $\rho_{S,\alpha}$ and $\rho_{L,\alpha}$ approximated by (4.1). The integrals $K^{(n)}(a)$ and $J^{(n)}(a,\eta)$ are defined in (5.2). We have [39]

$$K^{(0)}(a) = \int_0^\infty dx \, \frac{1}{x^2+1} e^{-iax} = \frac{\pi}{2} e^{-a} - \frac{i}{2}\left[e^{-a} Ei(a) - e^{a} Ei(-a)\right] \tag{B.1}$$

where Ei are transzendental functions called exponential integral functions. Any other integral $K^{(n)}$ with $n > 0$ can be obtained from (B.1) by differentiating (B.1) with respect to the variable a. For instance

$$K^{(1)}(a) = \int_0^\infty dx \, \frac{x}{x^2+1} e^{-iax} = -i\frac{\pi}{2} e^{-a} - \frac{1}{2}\left[e^{-a} Ei(a) + e^{a} Ei(-a)\right] \tag{B.2}$$

The integral $J^{(n)}$ are more complicated. To obtain $J^{(0)}$ we have used the Fourier indentity

$$\int_0^\eta f(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \frac{e^{i\eta y}-1}{iy}\int_{\infty}^{\infty} e^{-iy\xi}f(\xi)d\xi \tag{B.3}$$

Here we quote only the result

$$J^{(0)}(a,\eta) = \int_0^\eta dx \frac{1}{x^2+1}e^{-iax} = -\frac{1}{2i}\Big\{-isgn(\eta)e^{-a} + e^{-a}Ei(a(1-i\eta)) \\ - e^{a}Ei(-a(1+i\eta)) - e^{-a}Ei(a) + e^{a}Ei(-a)\Big\} \tag{B.4}$$

where $sgn(\eta)$ is the sign of η. Again $J^{(n)}$, $n > 0$ can be obtained from (B.4) by differentiation of (B.4) with respect to a

$$J^{(1)}(a,\eta) = \int_0^\eta dx \frac{x}{x^2+1}e^{-iax} = -\frac{1}{2}\Big\{isgn(\eta)e^{-a} - e^{-a}Ei(a(1-i\eta)) \\ - e^{a}Ei(-a(1+i\eta)) + e^{a}Ei(-a) + e^{-a}Ei(a)\Big\} \tag{B.5}$$

The reason why we have to distinguish between the signs of η has to do with the following property of the exponential integral function Ei [39]

$$Ei(x \mp i0) = Ei(x) \pm i\pi, \quad x > 0 \tag{B.6}$$

One can check (B.4) by using the integral representation

$$Ei(\pm xy) = \pm e^{\pm xy}\int_0^\infty dt \frac{e^{-xt}}{y \mp t}, \quad \Re e(y) > 0,\, x > 0 \tag{B.7}$$

and differentiating both sides of (B.4) with respect to η. We also mention here the connection of $Ei(x)$ with the incomplete beta function $\Gamma(\alpha, x)$ [39] through

$$\Gamma(0,x) = -Ei(-x) \tag{B.8}$$

Finally the integrals (B.1)-(B.5) enter (3.11)-(3.13) through the expressions

$$\int_0^\infty dm \sum_\alpha |\rho_{S,\alpha}(m)|^2 e^{-imt} = \frac{1}{\pi}e^{-im_S t}\left[-J^{(0)}(\gamma_S t, -m_S/\gamma_S) + K^{(0)}(\gamma_S t)\right] \tag{B.9}$$

and

$$\int_0^\infty dm \sum_\beta \Im m\left(\rho_{S,\beta}(m)\rho^*_{L,\beta}(m)\right)e^{-imt} = \\ -\frac{\sqrt{\gamma_S\gamma_L}}{\pi}\int_0^\infty dm \frac{a_I m^2 + b_I m + c_I}{[(m-m_L)^2+{\gamma_S}^2]\,[(m-m_L)^2+{\gamma_L}^2]}e^{-imt} =$$

$$
\begin{aligned}
&- \frac{\sqrt{\gamma_S \gamma_L}}{\pi}\Bigg\{ \frac{e^{-im_S}}{\gamma_S}\Bigg[D_I' \cdot \Big(-J^{(0)}(\gamma_S t, -m_S/\gamma_S) + K^{(0)}(\gamma_S t)\Big) \\
&\qquad + \gamma_S C_I \cdot \Big(-J^{(1)}(\gamma_S t, -m_S/\gamma_S) + K^{(1)}(\gamma_S t)\Big)\Bigg] \\
&\quad + \frac{e^{-im_L}}{\gamma_L}\Bigg[F_I' \cdot \Big(-J^{(0)}(\gamma_L t, -m_L/\gamma_L) + K^{(0)}(\gamma_L t)\Big) \\
&\qquad - \gamma_L C_I \cdot \Big(-J^{(1)}(\gamma_L t, -m_L/\gamma_L) + K^{(1)}(\gamma_L t)\Big)\Bigg]\Bigg\}
\end{aligned}
\tag{B.10}
$$

Appendix C

In the main text we have strictly applied the spectral formalism to the $K^0 - \bar{K}^0$ system. Especially our integration domain over the variable m has been kept consistently in the range $(0, \infty)$ in determining the parameters L, S, R, I (see eqs.(3.19)-(3.21) and (4.3)-(4.4)) as well as in the formulae for time development in section 5. We have done it mainly for reasons of consistency to show explicitly that the spectral formalism, in the range of validity of the one-pole approximation, gives us the well known WW expressions up to non-oscillatory terms. It is, however, instructive to investigate the case when the range of integration is changed to $(-\infty, \infty)$ dropping thereby non-oscillatory terms right from the beginning. This can serve us as an explicit check of our calculation. To do so we take the expression for $P_{K^0\bar{K}^0}(t)$ as derived in [6], [7] (see also [24]) by changing the integration range in (3.12) to $(-\infty, \infty)$ [42].

$$
P_{K^0\bar{K}^0}(t) = \frac{1}{4p^*q}\Big\{ e^{-im_S t} e^{-\Gamma_S t}(1+\kappa) - e^{-im_L t} e^{-\Gamma_L t}(1+\kappa^*)\Big\} \tag{C.1}
$$

where

$$
\kappa \simeq \frac{2i\sqrt{\Gamma_S \Gamma_L}}{-\Delta m + i(\Gamma_S + \Gamma_L)} \sum_\alpha A_{S,\alpha} A^*_{L,\alpha} \tag{C.2}
$$

Note that a purely real κ would lead in eq.(C.1) to the WW expression up to a normalization factor. Assuming already $|\kappa| << 1$ it follows from (C.1)

$$
\begin{aligned}
|P_{K^0\bar{K}^0}(t)|^2 \simeq \frac{1}{16|p|^2|q|^2}\Big\{ & e^{-2\Gamma_S t} + e^{-2\Gamma_L t} \\
& - 2e^{-(\Gamma_S+\Gamma_L)t}[\cos \Delta m t + 2|\kappa| \sin\delta_\kappa \sin \Delta m t]\Big\}
\end{aligned}
\tag{C.3}
$$

where δ_κ is the phase of κ.

To evaluate κ let us first us the approach advocated in [6] and [7] where the sum $\sum_\alpha A_{S,\alpha} A^*_{L,\alpha}$ is saturated by the main decay of K_S, $K_S \to 2\pi$. The relevant

amplitudes are then $A_S(K_S \to 2\pi)$ and $A_L(K_L \to 2\pi) \simeq \epsilon_K A_S(K_S \to 2\pi)$ where ϵ_K is related to p and q from eq.(2.1) by

$$p = \frac{1+\epsilon_K}{\sqrt{2(1+|\epsilon_K|^2)}}, \quad q = \frac{1-\epsilon_K}{\sqrt{2(1+|\epsilon_K|^2)}} \tag{C.4}$$

We can then write

$$\sum_\alpha A_{S,\alpha} A^*_{L,\alpha} \simeq |A_S(K_S \to 2\pi)|^2 \epsilon^*_K \simeq \epsilon^*_K \tag{C.5}$$

The last equality follows from the fact that out of the consistency conditions (3.19)-(3.21) we are still using the normalization condition (3.21)(but not the other conditions). Only then we can recover the WW formulae for $P_{K^0\bar{K}^0}(t) = P_{\bar{K}^0 K^0}(t)$ where no new effect is expected. We have then $\sum_\alpha |A_{S,\alpha}|^2 \simeq |A_S(K_S \to 2\pi)|^2 \simeq 1$. Using experimental data (central values) and identifying in this case $2\Gamma_{S/L}$ with the mean life times we obtain

$$|2\kappa \sin\delta_\kappa| \sim 10^{-6} \tag{C.6}$$

i.e. in the context of the saturation method we obtain a rather sizeable result. Before commenting further on this estimate we first note that in the above calculation (we mean here specifically the calculation which makes use of (C.5)) it is not advisable to use too strong approximation like e.g. $\epsilon_K \simeq |\epsilon_K| e^{i\pi/4}$ and $\Delta m/\Gamma_S \simeq 1$. This would lead to a real κ even in this case.

Let us now show that the estimate (C.6) is essentially due to the saturation method expressed through eq.(C.5). Indeed using our solution from section 4 we can write

$$\sum_\alpha A_{S,\alpha} A^*_{L,\alpha} = R - iI \tag{C.7}$$

where R and I are given explicitly in eqs.(4.7)-(4.8) (with the replacement $\gamma_{S/L} \to \Gamma_{S/L}$ [42]) Recall that the solution for R and I have been obtained in section 4 through eqs.(3.19)-(3.21) which must be true for any ansatz, at least at a certain level of accuracy. Insofar we indeed should rely on the solutions for R and I given there, in the context of the one-pole approximation.

From (C.7) we now easily find

$$\kappa = \Delta_K \tag{C.8}$$

in full agreement with (5.11). This once again confirms our previous conclusion about the size of Khalfin's effect in the context of the one-pole approximation (see eqs.(5.12) and (5.13)). It also shows that the truncation of the sum $\sum_\alpha A_{S,\alpha} A^*_{L,\alpha}$ as done in (C.5) can lead to an overestimate of the effect (eq.(C.6)).

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[38] see e.g. L. Maiani in [10]

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[40] Demanding the determinant to be zero gives $\tilde{b}_{_I} - \tilde{a}_{_I}\tilde{c}_{_I} = 0$ and in the suitable accuracy ${\gamma_{_S}}^2 - {\gamma_{_L}}^2 \simeq 0$ or $({\gamma_{_S}}^2 - {\gamma_{_L}}^2)/\Delta m^2 \simeq -2(m_{_S} + m_{_L})/\Delta m$. Clearly both these relations are only hypothetical and not valid in the $K^0 - \bar{K}^0$ system.

[41] P. H. Eberhard, CERN-Report, CERN-72-1 (1972) (unpulished); see also W. C Carithers et al., Phys. Rev. **D14** (1976) 290

[42] In appendix C we use the notation of [6] and [7] where the definition of the one-pole approximation differs slightly from eq.(4.1). This definition can be obtained from eq.(4.1) by setting $\Gamma_{s/L}/2 \to \Gamma_{s/L}$.