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Split Dimensional Regularization for the Coulomb Gauge

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Abstract

A new procedure for regularizing Feynman integrals in the noncovariant Coulomb gauge $\vec{\nabla}\cdot\vec{A}^a = 0$ is proposed for Yang-Mills theory. The procedure is based on a variant of dimensional regularization, called *split dimensional regularization*, which leads to internally consistent, ambiguity-free integrals, some of which turn out to be *nonlocal*. It is demonstrated that split dimensional regularization yields a one-loop Yang-Mills self-energy, $\Pi^{ab}_{\mu\nu}$, that is nontransverse, but local. Despite the noncovariant nature of the Coulomb gauge, ghosts are necessary in order to satisfy the appropriate Ward/BRS identity. The computed Coulomb-gauge Feynman integrals are applicable to both Abelian and non-Abelian gauge models.

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1 Introduction

The quantization of non-Abelian gauge theories in the noncovariant Coulomb gauge,

$$\vec{\nabla}\cdot\vec{A}^a = 0, \tag{1}$$

has perplexed theorists for decades [1]. Despite numerous analyses and ingenious attempts over the past 30 odd years [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21], the Coulomb gauge has remained an enigma, especially for non-Abelian gauge models [22, 23, 24, 25, 26, 27]. This assessment may come as somewhat of a surprise in light of the progress made for other ghost-free gauges, notably the light-cone gauge $n\cdot A^a = 0, n^2 = 0$ [28, 29, 30], and the temporal gauge $n\cdot A^a = 0, n^2 > 0$ [31, 32], n_μ being an arbitrary, fixed four-vector [1, 33].

Our understanding and technical know-how of these axial-type gauges make it particularly hard to understand why quantization and renormalization in the Coulomb gauge (also called the *radiation gauge*) should have been so elusive [34]. Could it really be that this gauge is endowed with characteristics that defy proper definition? To answer this question, and in view of the tremendous range of applicability of the Coulomb gauge in physics generally [1, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55], we have decided to take another look at this baffling gauge.

It almost goes without saying that the spurious singularities in the Coulomb gauge arise specifically from the three-dimensional factor $(\vec{q}^{\,2})^{-1}$ in the gauge propagator $G^{ab}_{\mu\nu}(q)$,

$$G^{ab}_{\mu\nu}(q) = \frac{-\mathrm{i}\delta^{ab}}{(2\pi)^4(q^2+\mathrm{i}\epsilon)}\left[g_{\mu\nu} - \left(\frac{n^2 q_\mu q_\nu - q\cdot n(q_\mu n_\nu + q_\nu n_\mu)}{-\vec{q}^{\,2}}\right)\right], \tag{2}$$
$$\epsilon > 0, \qquad \mu,\nu = 0,1,2,3, \qquad n_\mu = (1,0,0,0),$$

where $\mathrm{diag}(g_{\mu\nu}) = (+1,-1,-1,-1)$. Although we could express $(\vec{q}^{\,2})^{-1}$ in covariant form, i.e.

$$\frac{1}{\vec{q}^{\,2}} = \frac{1}{(q\cdot n)^2 - q^2}, \qquad q^2 = q_0^2 - \vec{q}^{\,2}, \tag{3}$$

we shall refrain from using the above notation, since it deflects attention from the crux of the problem, which is: how do we compute integrals such as

$$\int \frac{\mathrm{d}^4 q}{[(q+p)^2 + \mathrm{i}\epsilon]\vec{q}^{\,2}}, \tag{4}$$

where the 0-component of q is absent from at least one of the propagators:

$$\frac{1}{-\vec{q}^{\,2}} = \frac{1}{0q_0^2 - \vec{q}^{\,2}} \quad ? \tag{5}$$

To be clear, our goal is to find a prescription for $(\vec{q}^{\,2})^{-1}$ directly, rather than in the limiting form

$$\vec{q}^{\,2} = \lim_{\lambda\to 1}[\lambda(q\cdot n)^2 - q^2]. \tag{6}$$

Accordingly, the purpose of this article is three-fold:

1. To propose a new procedure, called *split dimensional regularization*, for computing Feynman integrals in the noncovariant Coulomb gauge.
2. To apply the new technique to the one-loop Yang-Mills self-energy $\Pi^{ab}_{\mu\nu}$.
3. To check the appropriate Ward/BRS identity, and hence the value of $\Pi^{ab}_{\mu\nu}$.

Our paper is organized thus. In Section 2 we summarize the Feynman rules and state the unintegrated expression for the gluon self-energy to one-loop order. The new procedure for evaluating Feynman integrals is explained in Section 3 and illustrated there by several examples. The computation of $\Pi^{ab}_{\mu\nu}$ is discussed in Section 4. In Section 5, we examine the ghost contributions and verify the appropriate Ward/BRS identity. The main features of our calculation are summarized in Section 6. Finally, we enumerate in the Appendix some of the integrals needed for the determination of $\Pi^{ab}_{\mu\nu}$.

2 Feynman Rules

The Lagrangian density for pure Yang-Mills theory in the Coulomb gauge,

$$\vec{\nabla}\cdot\vec{A}^a = 0, \qquad \vec{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \tag{7}$$

may be written in the form [56]

$$\mathcal{L}' = \mathcal{L} - \frac{1}{2\alpha}\left(\mathcal{F}^{ab}_{\mu} A^{b\mu}\right)^2, \qquad \alpha \equiv \text{gauge parameter}, \quad \alpha \to 0, \tag{8}$$

where

$$\mathcal{F}^{ab}_{\mu} \equiv \left(\partial_\mu - \frac{n\cdot\partial}{n^2} n_\mu\right)\delta^{ab}, \qquad \mu = 0, 1, 2, 3,$$

$$\mathcal{F}^{ab}_{\mu} A^{b\mu} = \vec{\nabla}\cdot\vec{A}^a, \qquad n_\mu \equiv (n_0, \vec{n}) = (1, \vec{0}), \qquad n^2 = n_0^2 = 1,$$

and

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}(F^a_{\mu\nu})^2 + (J^c_\mu + \overline{\omega}^a \mathcal{F}^{ac}_\mu)\mathcal{D}^{cb\mu}\omega^b - \frac{1}{2} g f^{abc} K^a \omega^b \omega^c, \\
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \\
\mathcal{D}^{ab}_\mu &= \delta^{ab}\partial_\mu + g f^{abc} A^c_\mu.
\end{aligned}
$$

Here, g is the gauge coupling constant, f^{abc} are group structure constants, and A^a_μ denotes a massless gauge field with $a = 1, \ldots, N^2 - 1$, for SU(N); $\omega^a, \overline{\omega}^a$ represent ghost, anti-ghost fields, respectively, while J^a_μ and K^a are external sources; the quantities $J^a_\mu, \omega^a, \overline{\omega}^a$ are anti-commuting. The action, $S = \int \mathrm{d}^4 x \mathcal{L}$, is invariant under the following Becchi-Rouet-Stora transformations [57]:

$$
\begin{aligned}
\delta A^a_\mu &= \lambda \mathcal{D}^{ab}_\mu \omega^b, \\
\delta \omega^a &= -\frac{1}{2}\lambda g f^{abc} \omega^b \omega^c, \\
\delta \overline{\omega}^a &= \frac{1}{\alpha}\lambda \mathcal{F}^{ab}_\mu A^{b\mu},
\end{aligned}
\tag{9}
$$

λ being an anti-commuting constant.

The Feynman rules may be summarized as follows. The *gauge boson propagator* in the Coulomb gauge has already been listed in Eq. (2) as [1]

$$
G^{ab}_{\mu\nu}(q) = \frac{-\mathrm{i}\delta^{ab}}{(2\pi)^4(q^2 + \mathrm{i}\epsilon)} \left[g_{\mu\nu} - \left(\frac{n^2 q_\mu q_\nu - q\cdot n(q_\mu n_\nu + q_\nu n_\mu)}{-\vec{q}^{\,2}} \right) \right], \tag{10}
$$

$\epsilon > 0$, with components

$$
\begin{aligned}
G^{ab}_{00} &= \frac{\mathrm{i}\delta^{ab}}{(2\pi)^4 \vec{q}^{\,2}}, \qquad G^{ab}_{i0} = G^{ab}_{0i} = 0, \qquad i = 1,2,3, \\
G^{ab}_{ij} &= \frac{-\mathrm{i}\delta^{ab}}{(2\pi)^4(q^2 + \mathrm{i}\epsilon)} \left(-\delta_{ij} + \frac{q_i q_j}{\vec{q}^{\,2}} \right), \qquad i,j = 1,2,3.
\end{aligned}
\tag{11}
$$

The *three-gluon vertex* [1, 30] reads

$$
\begin{aligned}
V^{abc}_{\mu\nu\rho}(p,q,r) &= g f^{abc} (2\pi)^4 \delta^4(p+q+r) \\
&\quad \cdot \Big[g_{\mu\nu}(p-q)_\rho + g_{\nu\rho}(q-r)_\mu + g_{\rho\mu}(r-p)_\nu \Big],
\end{aligned}
\tag{12}
$$

and the *scalar ghost propagator* (cf. Eq. (3.2) of [56]),

$$
G^{ab}_{\mathrm{ghost}} = \frac{\mathrm{i}\delta^{ab}}{(2\pi)^4 \vec{q}^{\,2}}. \tag{13}
$$

Figure 1: One-loop gluon self-energy diagram.

The *unintegrated* expression for the one-loop gluon self-energy (Figure 1), in four-dimensional Minkowski space, is then given by:

$$\begin{aligned}
\Pi^{ab}_{\mu\nu}(p) \;=\;& \\
& \frac{\mathrm{i}C^{ab}}{2}\int \mathrm{d}^4q \Big[g_{\mu\alpha}(q+2p)_\sigma - g_{\alpha\sigma}(2q+p)_\mu + g_{\sigma\mu}(q-p)_\alpha\Big]\frac{1}{(q+p)^2+\mathrm{i}\epsilon} \\
& \cdot\left[g^{\alpha\beta} - \left(\frac{n^2(q+p)^\alpha(q+p)^\beta - (q+p)\cdot n[(q+p)^\alpha n^\beta + (q+p)^\beta n^\alpha]}{-(\vec{q}+\vec{p})^2}\right)\right] \\
& \cdot\Big[g_{\beta\nu}(q+2p)_\rho + g_{\nu\rho}(q-p)_\beta - g_{\rho\beta}(2q+p)_\nu\Big] \\
& \cdot\frac{1}{q^2+\mathrm{i}\epsilon}\left[g^{\sigma\rho} - \left(\frac{n^2q^\sigma q^\rho - q\cdot n(q^\sigma n^\rho + q^\rho n^\sigma)}{-\vec{q}^{\,2}}\right)\right], \qquad \epsilon > 0, \qquad (14)
\end{aligned}$$

where we have defined $f^{acd}f^{bcd} \equiv \delta^{ab}C_{\rm YM}$, and $C^{ab} \equiv g^2 C_{\rm YM}\delta^{ab}/(4\pi^2)$. Expansion of the integrand of Eq. (14), followed by a Wick rotation to Euclidean space, gives rise to about 40 noncovariant integrals of the type

$$\int \frac{\mathrm{d}^4q\; f(q)}{q^2(\vec{q}+\vec{p})^2}, \quad \int \frac{\mathrm{d}^4q\; g(q)}{q^2(q+p)^2(\vec{q}+\vec{p})^2}, \quad \int \frac{\mathrm{d}^4q\; h(q)}{q^2(q+p)^2\vec{q}^{\,2}(\vec{q}+\vec{p})^2}, \;\ldots\; .$$

We describe the methodology for computing these Coulomb-gauge integrals in Section 3.

3 Procedure for Coulomb-gauge integrals

By a Coulomb-gauge integral we mean any Feynman integral containing one or more three-dimensional factors such as

$$\frac{1}{\vec{q}^{\,2}}, \qquad \frac{1}{(\vec{q}+\vec{p})^2}, \qquad \text{etc.}$$

These noncovariant propagators give rise to spurious singularities which necessarily complicate the integration. In this section, we propose a new method for evaluating Coulomb-gauge Feynman integrals. We shall illustrate our technique by calculating the integral J_0 in Euclidean space, where

$$J_0 \equiv \int \frac{\mathrm{d}^4 q\; q_4^2}{(2\pi)^4 q^2 (\vec{q}+\vec{p})^2}\,. \tag{15}$$

The integration proceeds in four steps:

1. It is convenient, although not essential, to begin with Feynman's formula

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x\, [xA + (1-x)B]^{-2}, \tag{16}$$

so that

$$J_0 = (2\pi)^{-4} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 q\, q_4^2}{[x q_4^2 + \vec{q}^{\,2} + 2\vec{q}\cdot\vec{p}(1-x) + (1-x)\vec{p}^{\,2}]^2}\,, \tag{17}$$

and then apply exponential parametrization to the denominator:

$$J_0 = (2\pi)^{-4} \int_0^1 \mathrm{d}x \int_0^\infty \mathrm{d}\alpha\, \alpha \mathrm{e}^{-\alpha G} \int \mathrm{d}^3\vec{q}\; \mathrm{e}^{-\alpha U} \int_{-\infty}^{\infty} \mathrm{d}q_4\, q_4^2 \mathrm{e}^{-\alpha V}, \tag{18}$$

with

$$G \equiv (1-x)\vec{p}^{\,2}, \qquad U \equiv \vec{q}^{\,2} + 2(1-x)\vec{q}\cdot\vec{p}, \qquad V \equiv x q_4^2.$$

Two points are worth emphasizing:

   (a) While V in this example is purely quadratic in q_4, in general V may also contain a term linear in q_4. Hence, it is necessary to complete the square in q_4 before proceeding with the integration.

   (b) In contrast to the covariant-gauge case, the coefficient of q_4^2 (in V) differs from that of $\vec{q}^{\,2}$ (in U).

2. The second step in the computation of the integral (15) is to introduce two *distinct* dimensional regularization parameters, ω and σ, for the $\vec{q}$- and q_4-integrals, respectively:

$$\mathrm{d}^3\vec{q} = \mathrm{d}^{2\omega}\vec{Q}\Big|_{\omega\to 3/2}\,; \qquad \mathrm{d}q_4 = \mathrm{d}^{2\sigma}S\Big|_{\sigma\to 1/2}\,, \tag{19}$$

with the limits $\omega \to \frac{3}{2}$ and $\sigma \to \frac{1}{2}$ to be taken after all integrations have been completed. In this context, the three-dimensional $\vec{p}$-vector is replaced by the 2ω-dimensional vector $\vec{P}$. Accordingly, Eq. (18) becomes

$$J = \lim_{\omega\to\frac{3}{2}} \lim_{\sigma\to\frac{1}{2}} \frac{1}{(2\pi)^{2\omega+2\sigma}} \int_0^1 \mathrm{d}x\, D, \tag{20}$$

with

$$D \equiv \int_0^\infty \mathrm{d}\alpha\, \alpha \mathrm{e}^{-\alpha H} \int \mathrm{d}^{2\omega}\vec{Q}\, \mathrm{e}^{-\alpha A} \int \mathrm{d}^{2\sigma}S\, S^2 \mathrm{e}^{-\alpha B},$$
$$H \equiv (1-x)\vec{P}^2, \qquad A \equiv \vec{Q}^2 + 2(1-x)\vec{Q}\cdot\vec{P}, \qquad B \equiv xS^2.$$

3. Since $\vec{Q}^2$ and S^2 have unequal coefficients (see comment (b) in Step 1), we *re-scale* the 2σ-dimensional S-vector,

$$B = xS^2 = R^2, \qquad \mathrm{d}^{2\sigma}S = x^{-\sigma}\mathrm{d}^{2\sigma}R, \tag{21}$$

to obtain

$$\begin{aligned} D &= \int_0^\infty \mathrm{d}\alpha\, \alpha \mathrm{e}^{-\alpha H} \int \mathrm{d}^{2\omega}\vec{Q}\, \mathrm{e}^{-\alpha A} \int \frac{\mathrm{d}^{2\sigma}R\, R^2}{x^{1+\sigma}}\, \mathrm{e}^{-\alpha B}, \\ &= \frac{\sigma\pi^{\omega+\sigma}}{x^{1+\sigma}} \int_0^\infty \frac{\mathrm{d}\alpha}{\alpha^{\omega+\sigma}} \exp\left[-\alpha x(1-x)\vec{P}^2\right], \end{aligned} \tag{22}$$

since

$$\begin{aligned} \int \mathrm{d}^{2\omega}\vec{Q}\, \exp\Big(-\alpha[\vec{Q}^2 + 2(1-x)\vec{Q}\cdot\vec{P}]\Big) &= \pi^\omega \alpha^{-\omega} \exp\left[\alpha(1-x)^2\vec{P}^2\right], \\ \int \mathrm{d}^{2\sigma}R\, R^2 \exp(-\alpha R^2) &= \sigma\pi^\sigma \alpha^{-1-\sigma}. \end{aligned}$$

4. Performing the α-integration from Eq. (22), followed by the x-integration from Eq. (20), we find that

$$J = \lim_{\omega\to\frac{3}{2}} \lim_{\sigma\to\frac{1}{2}} \frac{\sigma\Gamma(1-\omega-\sigma)\Gamma(\omega-1)\Gamma(\omega+\sigma)}{(4\pi)^{\omega+\sigma}\Gamma(2\omega+\sigma-1)} (\vec{P}^2)^{\omega+\sigma-1}, \tag{23}$$

or, finally,

$$J_0 = -\frac{2}{3}\vec{p}^{\,2} I_1^*, \tag{24}$$

where I_1^* is defined appropriately by

$$I_1^* \equiv \text{divergent part of} \int \frac{\mathrm{d}^{2\omega}\vec{Q}}{(2\pi)^\omega} \int \frac{\mathrm{d}^{2\sigma}R}{(2\pi)^\sigma} \frac{1}{q^2(q+p)^2}, \tag{25}$$

$$= \text{divergent part of } \frac{\Gamma(2-\omega-\sigma)(p^2)^{\omega+\sigma-2}}{(4\pi)^{\omega+\sigma}}, \tag{26}$$

$$= \begin{cases} \dfrac{\mathrm{i}}{(4\pi)^{\omega+\sigma}(2-\omega-\sigma)} & \text{in Minkowski space,} \\[2ex] \dfrac{1}{(4\pi)^{\omega+\sigma}(2-\omega-\sigma)} & \text{in Euclidean space.} \end{cases} \tag{27}$$

The α- and x-integrations between Eqs. (22) and (23) require $\mathrm{Re}(\omega + \sigma) < 1$, and $\{\mathrm{Re}(\omega+\sigma) > 0,\ \mathrm{Re}\,\omega > 1\}$, respectively. Hence, there exists a region in the complex ω-plane where the α- and x-integrals are both defined. Performing the $\vec{Q}$- and R-integrations in this region, we then analytically continue the result to four-dimensional space ($\omega \to \frac{3}{2}$ and $\sigma \to \frac{1}{2}$, in either order). Notice that the value of J_0 in Eq. (24) depends on $\vec{p}^{\,2}$, rather than on p^2.

The evaluation of J_0 in the preceding example hinges decisively on the use of *two* complex regulating parameters ω and σ, a drastic departure from conventional dimensional regularization with its *single* regulating parameter ω. The conventional approach was actually applied to the same integral J_0 a couple of years ago by one of the present authors. Although the final result for J_0 looked quite reasonable, its validity was questioned by J. C. Taylor [58], who noted that the integrals over the α and x parameters were ill-defined.

The next example will serve to illustrate the *nonlocality* of certain Coulomb-gauge integrals. Consider the integral I, containing two covariant propagators, and one noncovariant propagator:

$$\begin{aligned} I &\equiv \int^{\text{Mink.}} \frac{\mathrm{d}^4 q}{(2\pi)^4 (q^2 + \mathrm{i}\epsilon)[(q+p)^2 + \mathrm{i}\epsilon](\vec{q}+\vec{p}\,)^2}\,, \qquad \epsilon > 0, \\ &= \mathrm{i}\int^{\text{Eucl.}} \frac{\mathrm{d}^4 q}{(2\pi)^4 q^2 (q+p)^2 (\vec{q}+\vec{p}\,)^2}\,, \qquad q^2 = q_4^2 + \vec{q}^{\,2}. \end{aligned} \tag{28}$$

Recalling the formula

$$\frac{1}{ABC} = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}z\, z \int_0^\infty \mathrm{d}\alpha\, \alpha^2 \exp\Big(-\alpha[C + z(B-C) + zx(A-B)]\Big), \tag{29}$$

we may write Eq. (28) initially as

$$I = \frac{\mathrm{i}}{(2\pi)^4} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}z\, z \int_0^\infty \mathrm{d}\alpha\, \alpha^2 \mathrm{e}^{-\alpha G} \int \mathrm{d}^3\vec{q}\; \mathrm{e}^{-\alpha U} \int_{-\infty}^{\infty} \mathrm{d}q_4\, \mathrm{e}^{-\alpha V}, \tag{30}$$

with

$$G \equiv (1 - zx)\vec{p}^{\,2} + z(1-x)p_4^2,$$

$$U \equiv \vec{q}^{\,2} + 2(1-zx)\vec{q}\cdot\vec{p}, \qquad V \equiv z q_4^2 + 2z(1-x)p_4 q_4,$$

and then complete the square in q_4 (see comment (a) in Step 1), so that

$$\int_{-\infty}^{\infty} \mathrm{d}q_4\, \mathrm{e}^{-\alpha V} = \exp\left[\alpha z (1-x)^2 p_4^2\right] \int_{-\infty}^{\infty} \mathrm{d}Q_4\, \exp\left(-\alpha z Q_4^2\right). \tag{31}$$

The next step is to define the $\vec{q}$- and q_4-integrals over 2ω- and 2σ-space, respectively:

$$\mathrm{d}^3\vec{q} = \mathrm{d}^{2\omega}\vec{Q}\Big|_{\omega\to 3/2}\,; \qquad \mathrm{d}Q_4 = \mathrm{d}^{2\sigma}S\Big|_{\sigma\to 1/2}\,, \tag{32}$$

in which case Eq. (30) is replaced by:

$$I = \lim_{\omega\to\frac{3}{2}}\lim_{\sigma\to\frac{1}{2}} \frac{\mathrm{i}}{(2\pi)^{2\omega+2\sigma}} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}z\, D, \tag{33}$$

with

$$\begin{aligned}
D &\equiv z\int_0^\infty \mathrm{d}\alpha\,\alpha^2 \mathrm{e}^{-\alpha H}\int \mathrm{d}^{2\omega}\vec{Q}\,\mathrm{e}^{-\alpha A}\int \mathrm{d}^{2\sigma}S\,\mathrm{e}^{-\alpha B},\\
H &\equiv (1-zx)\vec{p}^{\,2} - zx(1-x)p_4^2,\\
A &\equiv \vec{Q}^2 + 2(1-zx)\vec{Q}\cdot\vec{p}, \qquad B \equiv zS^2.
\end{aligned}$$

Executing Step 3 now by re-scaling the S-vector according to

$$zS^2 = R^2, \qquad \mathrm{d}^{2\sigma}S = z^{-\sigma}\mathrm{d}^{2\sigma}R, \tag{34}$$

and integrating over $\mathrm{d}^{2\omega}\vec{Q}$, $\mathrm{d}^{2\sigma}R$, and then $\mathrm{d}\alpha$, we readily obtain

$$\begin{aligned}
D &= \frac{\pi^{\omega+\sigma}}{z^{\sigma-1}}\int_0^\infty \frac{\mathrm{d}\alpha}{\alpha^{\omega+\sigma-2}}\exp\Big(-\alpha zx[(1-x)p_4^2 + (1-zx)\vec{p}^{\,2}]\Big),\\
&= \frac{\pi^{\omega+\sigma}}{z^{\sigma-1}}\frac{\Gamma(3-\omega-\sigma)}{(zx\,p^2)^{3-\omega-\sigma}}\left[1 - x\left(\frac{p_4^2 + z\vec{p}^{\,2}}{p^2}\right)\right]^{\omega+\sigma-3},
\end{aligned} \tag{35}$$

where the same lower case $\vec{p}$ has been used for convenience for both the three-vector $\vec{p}$ and the corresponding 2ω-dimensional vector. In order to complete the remaining integrations from Eq. (33), we first expand the square brackets in Eq. (35), and note that only the *first* term contributes to the divergent part of I. Hence,

$$I = \lim_{\omega\to\frac{3}{2}}\lim_{\sigma\to\frac{1}{2}} \frac{\mathrm{i}\Gamma(3-\omega-\sigma)}{(4\pi)^{\omega+\sigma}(p^2)^{3-\omega-\sigma}(\omega+\sigma-2)(\omega-1)}\,, \tag{36}$$

or, finally,

$$I \equiv \mathrm{div}\int^{\text{Mink.}} \frac{\mathrm{d}^4q}{(2\pi)^4(q^2+\mathrm{i}\epsilon)[(q+p)^2+\mathrm{i}\epsilon](\vec{q}+\vec{p})^2} = -\frac{2}{p^2}I_1^*\,, \tag{37}$$

where I_1^* is defined in Eq. (25). Similarly, one may show that

$$\mathrm{div}\int^{\text{Mink.}} \frac{\mathrm{d}^4q}{(2\pi)^4(q^2+\mathrm{i}\epsilon)[(q+p)^2+\mathrm{i}\epsilon]\vec{q}^{\,2}} = -\frac{2}{p^2}I_1^*\,. \tag{38}$$

The appearance of *nonlocal* Feynman integrals, such as Eqs. (37) and (38), is both necessary and sufficient for the internal consistency of one-loop integrals in the Coulomb gauge. Nor is it entirely unexpected, considering the noncovariant nature of that gauge. After all, we have known for some time that axial gauges likewise lead not only to nonlocal Feynman integrals, but also to a nonlocal Yang-Mills self-energy [1, 29, 33].

4 The self-energy $\Pi^{ab}_{\mu\nu}$

Computations in the Coulomb gauge never seem particularly enjoyable or uplifting. Too many trivial things can and do go wrong, and the compilation of Feynman integrals seems to take forever. Needless to say, we were more than relieved to see the various results converge to manageable form. For technical reasons, we have chosen to evaluate the Yang-Mills self-energy $\Pi^{ab}_{\mu\nu}$, Eq. (14), in Euclidean space. Here is our final result for $\Pi^{ab}_{\mu\nu}(p)$, written covariantly in Minkowski space:

$$\begin{aligned}\Pi^{ab}_{\mu\nu}(p) \;&=\; C^{ab}\left[\frac{11}{3}(p^2 g_{\mu\nu} - p_\mu p_\nu) - \frac{8}{3}(p^2 g_{\mu\nu} - p_\mu p_\nu)\right.\\ &\left.\qquad - \frac{4}{3}\frac{p\cdot n}{n^2}(p_\mu n_\nu + p_\nu n_\mu) + \frac{8}{3}\frac{p^2 n_\mu n_\nu}{n^2}\right] I_1^*, \end{aligned} \tag{39}$$

where $n_\mu = (1,0,0,0)$, $C^{ab} = g^2 C_{\rm YM}\delta^{ab}/(4\pi^2)$, and I_1^* is defined in Eq. (25). This result for the Yang-Mills self-energy possesses some remarkable features:

1. $\Pi^{ab}_{\mu\nu}(p)$ is *nontransverse* in the Coulomb gauge.

2. Despite the appearance of *nonlocal integrals* at intermediate stages of the computation, $\Pi^{ab}_{\mu\nu}(p)$ is a *local* function of the external momentum p_μ.

3. Ghosts play an essential role, despite the "*ghost-free*" nature of the Coulomb gauge. (See Section 5.)

4. Apart from the complex parameters σ and ω, defining *split dimensional regularization*, no additional parameters are needed to evaluate $\Pi^{ab}_{\mu\nu}(p)$.

5. All one-loop integrals in the Coulomb gauge are ambiguity-free; they are consistent, at least in the context of split dimensional regularization,

with the values of the following integrals:

$$\int \frac{\mathrm{d}^{2\omega+2\sigma}q\ f(q)}{q^2\vec{q}^{\,2}} = \int \frac{\mathrm{d}^{2\omega+2\sigma}q\ f(q)}{\vec{q}^{\,2}(\vec{q}+\vec{p})^2} = \int \frac{\mathrm{d}^{2\omega+2\sigma}q\ f(q)}{(q+p)^2(\vec{q}+\vec{p})^2} = 0, \quad (40)$$

where $f(q)$ is any polynomial in the components of q. The latter integrals are the analogues of tadpole-like integrals which are known to appear in axial gauges, for example [1]

$$\int \frac{\mathrm{d}^{2\omega}q}{(q\cdot n)^2} = \int \frac{\mathrm{d}^{2\omega}q}{(q\cdot n)q^2} = \int \frac{\mathrm{d}^{2\omega}q}{(q\cdot n)((q-p)\cdot n)} = 0, \qquad \text{etc.} \quad (41)$$

5 Verification of the Ward identity

It has been known for some time [56, 57, 59, 60, 61, 62, 63, 64] that ghosts play a crucial role in the renormalization of non-Abelian theories, regardless whether the applied gauge is covariant or "ghost-free", i.e., noncovariant. This conclusion holds not only for the ghost-free gauges of the axial kind, such as the planar gauge and the light-cone gauge, but also for our Coulomb gauge. In this section, we shall examine the role played by ghosts in obtaining the correct Ward/BRS identity for $\Pi^{ab}_{\mu\nu}(p)$.

Referring to Section 2 for the various definitions of S, $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{F}^{ab}_{\mu}$, etc., we recall that the action S satisfies the Becchi-Rouet-Stora identity [57, 65, 66]

$$\begin{aligned}\sigma S \ = \ \int \mathrm{d}^4x \Bigg[& \frac{\delta S}{\delta A^a_\mu(x)}\frac{\delta}{\delta J^a_\mu(x)} + \frac{\delta S}{\delta J^a_\mu(x)}\frac{\delta}{\delta A^a_\mu(x)} + \\ & \frac{\delta S}{\delta \omega^a(x)}\frac{\delta}{\delta K^a(x)} + \frac{\delta S}{\delta K^a(x)}\frac{\delta}{\delta \omega^a(x)} \Bigg] S = 0, \end{aligned} \quad (42)$$

and the ghost equation

$$\frac{\delta S}{\delta \overline{\omega}^a(x)} - \mathcal{F}^{ab}_{\mu}\frac{\delta S}{\delta J^b_\mu(x)} = 0, \quad (43)$$

σ being the Slavnov-Taylor operator, $\sigma^2 = 0$. It is advantageous to work with the vertex generating functional Γ for one-particle-irreducible Green functions with the gauge-fixing term omitted. The one-loop divergent parts D of the generating functional Γ must then obey the BRS identity [30, 56, 62]

$$\begin{aligned}\sigma D \ = \ \int \mathrm{d}^4x \Bigg[& \frac{\delta S}{\delta A^a_\mu(x)}\frac{\delta}{\delta J^a_\mu(x)} + \frac{\delta S}{\delta J^a_\mu(x)}\frac{\delta}{\delta A^a_\mu(x)} + \\ & \frac{\delta S}{\delta \omega^a(x)}\frac{\delta}{\delta K^a(x)} + \frac{\delta S}{\delta K^a(x)}\frac{\delta}{\delta \omega^a(x)} \Bigg] D = 0. \end{aligned} \quad (44)$$

Figure 2: Ghost-loop needed for the Ward identity (46).

Differentiation of Eq. (44) with respect to $A^b_\nu(y)$ and $\omega^c(z)$ yields eventually [67]

$$\frac{\delta^2(\sigma D)}{\delta\omega^c(z)\delta A^b_\nu(y)} = \int \mathrm{d}^4x \left[\frac{\delta^2 S}{\delta\omega^c(z)\delta J^a_\mu(x)}\frac{\delta^2 D}{\delta A^a_\mu(x)\delta A^b_\nu(y)} + \frac{\delta^2 S}{\delta A^b_\nu(y)\delta A^a_\mu(x)}\frac{\delta^2 D}{\delta\omega^c(z)\delta J^a_\mu(x)}\right]_{A,J,K,\omega=0} = 0. \quad (45)$$

Interpreting the functional derivatives [59], and Fourier-transforming to momentum space, we obtain from Eq. (45) the following Ward identity in Minkowski space:

$$p^\mu \Pi^{ab}_{\mu\nu}(p) + (g_{\mu\nu}p^2 - p_\mu p_\nu)H^{ab\mu}(p) = 0, \quad (46)$$

or, graphically,

$$p^\mu \ \times \ (\text{Figure 1}) \ + \ (g_{\mu\nu}p^2 - p_\mu p_\nu) \ \times \ (\text{Figure 2}) \ = 0. \quad (47)$$

It remains to evaluate the ghost contribution $H^{ab\mu}(p)$, corresponding to Figure 2, and then to check whether the computed values for $H^{ab\mu}(p)$, together with $\Pi^{ab}_{\mu\nu}(p)$ from Eq. (39), respect the Ward/BRS identity (46).

In order to compute $H^{ab\mu}(p)$, we employ the gluon propagator in Eq. (10), the ghost propagator in Eq. (13), the J^a-A^e-ω^d vertex factor $-gf^{aed}$, and the A^e-$\overline{\omega}^d$-ω^c vertex factor $(p_\mu - n\cdot p n_\mu)gf^{dce}$ [56]. Hence,

$$\begin{aligned} & H^{ab\mu}(p) \\ = & \ (-\mathrm{i}^2)C^{ab}\int \frac{\mathrm{d}^4q \ (p_\beta - n\cdot p n_\beta)}{(q^2+\mathrm{i}\epsilon)(\vec{q}+\vec{p})^2}\left[g^{\mu\beta} - \left(\frac{q^\mu q^\beta - q\cdot n(q^\mu n^\beta + q^\beta n^\mu)}{-\vec{q}^2}\right)\right], \\ = & \ \frac{4}{3}C^{ab}\left(p^\mu - \frac{p\cdot n}{n^2}n^\mu\right) I_1^*, \qquad n_\mu = (1,0,0,0), \end{aligned} \quad (48)$$

which agrees with reference [68]. We see that the respective values for $\Pi^{ab}_{\mu\nu}(p)$ in Eq. (39), and $H^{ab\mu}(p)$ in Eq. (48), do indeed satisfy the Ward/BRS identity (46).

6 Conclusion

In this article we have suggested a new procedure, called *split dimensional regularization*, for regularizing Feynman integrals in the Coulomb gauge $\vec{\nabla}\cdot\vec{A}^a = 0$. The principal feature of this procedure is the use of *two* complex parameters, ω and σ, which permit us to control more effectively the respective divergences arising from the $d^3\vec{q}$- and dq_4-integrations. The method leads to ambiguity-free and internally consistent integrals which may be either local or *nonlocal*, and are characterized by pole terms proportional to $\Gamma(2-\omega-\sigma)$, rather than $\Gamma(2-\omega)$ (as in conventional dimensional regularization [69, 70, 71]). No additional parameters, apart from ω and σ, are needed to evaluate these integrals.

To test the method of split dimensional regularization at the one-loop level, we calculated the Yang-Mills self-energy $\Pi^{ab}_{\mu\nu}(p)$. The latter turned out to be nontransverse, but *local*, despite the appearance of *nonlocal integrals* at intermediate stages of the computation. A further check was provided by the Ward/BRS identity, Eq. (46), which consists of the self-energy $\Pi^{ab}_{\mu\nu}(p)$ in Eq. (39), and the ghost-loop contribution given in Eq. (48). The fact that both contributions together respect the Ward identity underscores once again the significance of ghosts, even in the case of the so-called “ghost-free” gauges, such as the Coulomb gauge.

Although the present results seem encouraging, it is too early to predict whether or not the method of split dimensional regularization is destined to survive into the 21st century as a viable prescription for the Coulomb gauge. Clearly, more calculations are needed, particularly at two and three loops, before split dimensional regularization can be placed on a firm mathematical footing, similar to the successful n^*_μ-prescription for axial gauges.

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Table 1: Divergent parts of some Coulomb-gauge integrals in Euclidean space, as $\omega \to \frac{3}{2}$ and $\sigma \to \frac{1}{2}$. $E_{ijk} \equiv p_i\delta_{jk} + p_j\delta_{ki} + p_k\delta_{ij}$; $i, j, k = 1, 2, 3$. All entries are implicitly multiplied by I_1^* (see Eq. (26)).

A	$\int \frac{\mathrm{d}^{2\omega}\vec{q}\,\mathrm{d}^{2\sigma}q_4}{(2\pi)^{2\omega+2\sigma}}\frac{A}{B}$			
1	2	$-2/p^2$	$-2/\vec{p}^{\,2}$	$-4/(\vec{p}^{\,2}p^2)$
q_i	$-\frac{4}{3}p_i$	0	0	$2p_i/(\vec{p}^{\,2}p^2)$
q_4	0	0	0	$2p_4/(\vec{p}^{\,2}p^2)$
q_iq_j	$\frac{16}{15}p_ip_j - \frac{2}{15}\vec{p}^{\,2}\delta_{ij}$	$\frac{1}{3}\delta_{ij}$	$\frac{2}{3}\delta_{ij}$	$-2p_ip_j/(\vec{p}^{\,2}p^2)$
q_iq_4	0	0	0	$-2p_ip_4/(\vec{p}^{\,2}p^2)$
q_4^2	$-\frac{2}{3}\vec{p}^{\,2}$	1	-2	$-2p_4^2/(\vec{p}^{\,2}p^2)$
$q_iq_jq_k$	—	$-\frac{1}{10}E_{ijk}$	$-\frac{4}{15}E_{ijk}$	$2p_ip_jp_k/(\vec{p}^{\,2}p^2)$
$q_iq_jq_4$	—	$-\frac{1}{6}p_4\delta_{ij}$	0	$2p_ip_jp_4/(\vec{p}^{\,2}p^2)$
$q_iq_4^2$	—	$-\frac{1}{6}p_i$	$\frac{4}{3}p_i$	$2p_ip_4^2/(\vec{p}^{\,2}p^2)$
B {	$q^2(\vec{q}+\vec{p})^2$	$q^2(q+p)^2\vec{q}^{\,2}$	$q^2(\vec{q}+\vec{p})^2\vec{q}^{\,2}$	$q^2(q+p)^2(\vec{q}+\vec{p})^2\vec{q}^{\,2}$

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Appendix

Table 1 shows about half of the integrals needed in the evaluation of $\Pi^{ab}_{\mu\nu}(p)$ and $H^{ab\mu}(p)$. The others may be obtained by means of the transformation $p \to -p$, followed by $q \to q + p$, applied to all components of p and q in A, B, and the body of the table. See also Eq. (40).

The integrals in Table 1 were calculated using the efficient technique described in reference [72]. Briefly, the most complex B was first parametrized in accordance with the four-factor analog of Eq. (29). Integration over $d^{2\omega}\vec{q}$ and $d^{2\sigma}q_4$ was then carried out for the $A = 1$ case, and the result differentiated repeatedly to obtain momentum integrals for the other eight A's. Finally, parameter integrations tailored to various different B's were applied to each of the momentum integrals.

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