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CHIRAL LAGRANGIAN WITH VECTOR AND AXIAL VECTOR MESON FOR $\pi^+ - \pi^0$ MASS DIFFERENCE

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ABSTRACT

It is shown that the simple expression for the forward virtual Compton scattering on a soft pion for the $\pi^+ - \pi^0$ electromagnetic mass difference calculation usually obtained from Current Algebra and Weinberg sum rules can be derived in a simple manner from a chiral Lagrangian with vector and axial vector mesons. We also discuss the relation between the chiral Lagrangian approach and the dispersion relation for the Compton amplitude.

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Recently there has been renewed interest in the electromagnetic mass difference of pseudoscalar mesons [1, 2] . In these works, the electromagnetic mass shift of pion and kaon are rederived using an effective chiral Lagrangian with vector and axial vector meson included. Using the electromagnetic form factor of the pion obtained from the vector meson dominance hypothesis and the relation between the coupling of the vector current to vector meson and axial vector current coupling to the axial vector meson given by the second Weinberg sum rule, the expression for the forward virtual Compton scattering amplitude is obtained and is found to be in agreement with the soft pion results obtained long ago by Das *et al.* [3]. This result has also been obtained long ago in a linear sigma model with vector and axial vector meson included as shown by Wick and Zumino [4] and by Lee and Nieh [5] where all the individual double pole terms in the vector meson propagator cancel out in the expression leaving only simple pole terms for the amplitude. In the work of Donoghue *et al.* [2], this cancellation is also obtained from a chiral Lagrangian in which the vector and axial vector meson fields are treated as antisymmetric tensor representation instead of the usual four-vector field operator [6]. In this paper, we feel it is useful to give a simple derivation of the soft pion result using the conventional Lagrangian for the vector and axial vector mesons. Furthermore, to insure the convergence of the $\pi^+ - \pi^0$ mass difference calculation, we need to construct a chiral Lagrangian similar to that given in Ref. [5] in which the vector and axial vector meson dominance as well as the first and the second Weinberg sum rules come out automatically. This non-linear gauged chiral Lagrangian has been given in a previous work [7] and will be used in this paper. Thus we shall start with a local left-right symmetric Lagrangian in which the two gauge bosons associated with a local $SU(3) \times SU(3)$ symmetry are the two hypothetical left-handed and right-handed vector bosons l_μ and r_μ respectively. We have in standard notation [5, 7]

$$\mathcal{L} = -\frac{1}{16}\mathrm{Tr}\left(l'^2_{\mu\nu} + r'^2_{\mu\nu}\right) + \frac{1}{8}m^2\left(l^2_\mu + r^2_\mu\right) + \frac{\alpha}{8f^2}\mathrm{Tr}(D_\mu M D_\mu M^\dagger) \tag{1}$$

where M is the meson coupling matrix which takes the standard exponential form [8],*i.e*

$$M = exp(2if\phi), \qquad f = f_\pi^{-1} \tag{2}$$

ϕ is the matrix of the pseudoscalar meson octet in $SU(3)$ space and $f_\pi = 131\mathrm{MeV}$, the usual pion decay constant measured in $\pi^+ \to l^+\nu$ decay. A covariant derivative for M in this gauged chiral model is then,

$$D_\mu M = \partial_\mu M - \frac{ig}{2}(l_\mu M - M r_\mu),$$
$$D_\mu M^\dagger = \partial_\mu M^\dagger + \frac{ig}{2}(M^\dagger l_\mu - r_\mu M^\dagger) \quad (3)$$

In this model, as discussed previously [7] the usual covariant field strength tensors $l_{\mu\nu}$ and $r_{\mu\nu}$ for the left-handed l_μ and the right-handed r_μ defined by

$$l_{\mu\nu} = \partial_\nu l_\mu - \partial_\mu l_\nu + \frac{ig}{2}[l_\mu, l_\nu],$$
$$r_{\mu\nu} = \partial_\nu r_\mu - \partial_\mu r_\nu + \frac{ig}{2}[r_\mu, r_\nu] \quad (4)$$

are replaced by $l'_{\mu\nu}$ and $r'_{\mu\nu}$ respectively in the modified gauge boson kinetic terms. We have

$$l'_{\mu\nu} = l_{\mu\nu} + \frac{i}{2g}\left[M D_\mu M^\dagger, M D_\mu M^\dagger\right],$$
$$r'_{\mu\nu} = r_{\mu\nu} + \frac{i}{2g}\left[M^\dagger D_\mu M, M^\dagger D_\mu M\right] \quad (5)$$

The last terms in Eq. (5) are needed to reproduce the KSFR relation for the physical $\rho\pi\pi$ coupling ($g_{\rho\pi\pi} = g = m_\rho/f_\rho$). The second term in Eq. (1) is the mass term. This term breaks local $SU(3) \times SU(3)$ symmetry and generate the left-handed $(V - A)$ and right-handed $(V + A)$ currents given by

$$V_\mu - A_\mu = \frac{m^2}{g} l_\mu,$$
$$V_\mu + A_\mu = \frac{m^2}{g} r_\mu \quad (6)$$

We now express l_μ and r_μ in terms of the vector meson ρ_μ and axial vector meson a_μ fields defined as

$$l_\mu = \xi(\rho_\mu - a_\mu + 2i\lambda p_\mu)\xi^\dagger,$$
$$r_\mu = \xi^\dagger(\rho_\mu + a_\mu - 2i\lambda p_\mu)\xi \quad (7)$$

where [10, 11]

$$v_\mu = \frac{1}{2}[\xi^\dagger, \partial_\mu \xi], \qquad p_\mu = \frac{1}{2}\{\xi^\dagger, \partial_\mu \xi\} \quad (8)$$

and ξ is defined by $M = \xi^2$. By definition, M transforms as an $(\bar{3}, 3)$ under global $SU(3)_L \times SU(3)_R$:

$$M \to L M R^\dagger, \qquad \xi \to L \xi U^\dagger = U \xi R^\dagger \quad (9)$$

where L and R are respectively elements of $SU(3)_L$ and $SU(3)_R$. U is a function of ξ. p_μ transforms as covariant derivative and v_μ transforms as gauge field under the non-linear chiral transformation U, *i.e*

$$p_\mu \rightarrow U p_\mu U^\dagger , \qquad v_\mu \rightarrow U v_\mu U^\dagger - \partial_\mu U U^\dagger \tag{10}$$

while the vector and axial vector meson transform as

$$\rho_\mu \rightarrow U \rho_\mu U^\dagger , \qquad a_\mu \rightarrow U a_\mu U^\dagger \tag{11}$$

Putting the expressions for l_μ and r_μ given by Eq. (7) into the Lagrangian Eq. (1), we obtain, for the $\pi - \rho - A_1$ system

$$\mathcal{L}_{\pi-\rho-A_1} = -\frac{1}{8}\mathrm{Tr}\left(\rho'^2_{\mu\nu} + a^2_{\mu\nu}\right) + \frac{1}{4}\left(m_\rho^2 \mathrm{Tr}\,\rho_\mu^2 + m_A^2 \mathrm{Tr}\,a_\mu^2\right) + \frac{1}{8f^2}\mathrm{Tr}\left(\partial_\mu M \partial_\mu M^\dagger\right) \tag{12}$$

with

$$m_\rho^2 = m^2 , \qquad m_A^2 = m^2/(1 - g^2/2f^2m^2) \tag{13}$$

and

$$\alpha = (1 - g^2/2f^2m^2)^{-1} , \qquad \lambda = -\frac{g}{2f^2m^2} \tag{14}$$

The modified vector meson term $\rho'_{\mu\nu}$ now contains the $\rho\pi\pi$ interactions and is given by

$$\rho'_{\mu\nu} = D_\nu \rho_\mu - D_\mu \rho_\nu + \frac{ig}{2}[\rho_\mu, \rho_\nu] + \frac{2i}{g}[p_\mu, p_\nu] \tag{15}$$

The axial vector meson term $a_{\mu\nu}$ is

$$a_{\mu\nu} = D_\nu a_\mu - D_\mu a_\nu + (1 + g\lambda)\left([p_\mu, \rho_\nu] + [p_\nu, \rho_\mu]\right) \tag{16}$$

with

$$\begin{aligned} D_\nu \rho_\mu &= \partial_\nu \rho_\mu + [v_\nu, \rho_\mu] , \\ D_\nu a_\mu &= \partial_\nu a_\mu + [v_\nu, a_\mu] \end{aligned} \tag{17}$$

The expression for the axial vector meson mass in Eq. (13) can also be written in the form [5]

$$\left(\frac{m_\rho^2}{g}\right)^2 \left(\frac{1}{m_\rho^2} - \frac{1}{m_A^2}\right) = \frac{f_\pi^2}{2} , \qquad f_\pi = f^{-1} \simeq m_\pi \tag{18}$$

which is the familiar first Weinberg sum rule in the zero-width resonance approximation [9]. Infact, the 1st Weinberg sum rule reads,

$$\int_0^\infty \frac{\rho_V(m^2) - \rho_A(m^2)}{m^2} dm^2 = \frac{f_\pi^2}{2} \tag{19}$$

with

$$\rho_V(m^2) = g_\rho^2 \delta(m^2 - m_\rho^2), \qquad \rho_A(m^2) = g_A^2 \delta(m^2 - m_A^2) \tag{20}$$

and

$$g_\rho = g_A = \frac{m_\rho^2}{g} \tag{21}$$

as a consequence of the left-right symmetric chiral Lagrangian and is usually derived from the second Weinberg sum rule [3],

$$\int_0^\infty \left[\rho_V(m^2) - \rho_A(m^2)\right] dm^2 = 0 \tag{22}$$

As pointed out in a previous paper [7], the relation $m_A^2 = 2m_\rho^2$ follows if g satisfies

$$g^2 = \frac{m_\rho^2}{f_\pi^2} \tag{23}$$

The on-shell $g_{\rho\pi\pi}$ coupling constant is

$$g_{\rho\pi\pi} = g + (f^2 m_\rho^2/g)(1 - g^2/f^2 m^2) \tag{24}$$

which is equal to g and satisfies the KSFR relation by virtue of Eq. (23). Note that the predictions for g_ρ and g_A are consistent with the measured values from the leptonic decays of the ρ meson and from the $\tau \to \nu A_1$ decays [3, 12].

Having derived the two Weinberg sum rules from our vector and axial vector meson Lagrangian, we now introduce the electromagnetic interactions for the $\pi - \rho - A_1$ system using a minimal substitution [5]

$$l_\mu \to l_\mu + \frac{2e}{g} Q A_\mu, \qquad r_\mu \to r_\mu + \frac{2e}{g} Q A_\mu \tag{25}$$

where Q is the quark charge operator in the $SU(3)$ space. The first and third term in Eq. (1) are invariant under a local $SU(3) \times SU(3)$ transformation and are therefore not affected by this substitution. Thus all electromagnetic interactions appear in the gauge boson mass term and the total Lagrangian for the $\pi - \rho - A_1$ system Eq. (12) becomes

$$\begin{aligned} \mathcal{L}_{\text{total}} &= -\frac{1}{8}\mathrm{Tr}\,(\rho_{\mu\nu}'^2 + a_{\mu\nu}^2) + \frac{1}{4}\left(m_\rho^2 \mathrm{Tr}\,\rho_\mu^2(\text{new}) + \mathrm{m_A^2\,Tr\,a_\mu^2(new)}\right) \\ &+ \frac{1}{8f^2}\mathrm{Tr}\,(\partial_\mu M \partial_\mu M^\dagger) + L_1 + L_2 \end{aligned} \tag{26}$$

with

$$\rho_\mu(\text{new}) = \rho_\mu + \frac{e}{g}\mathrm{Q_v}A_\mu\,, \qquad \mathrm{a}_\mu(\text{new}) = \mathrm{a}_\mu + \frac{e}{g}\mathrm{Q_a}A_\mu \tag{27}$$

and

$$Q_v = \xi Q\xi^\dagger + \xi^\dagger Q\xi\,, \qquad Q_a = \xi Q\xi^\dagger - \xi^\dagger Q\xi \tag{28}$$

Using Eq. (13) and the identities

$$\begin{aligned}\mathrm{Tr}\,Q_v^2 &= \mathrm{Tr}\,(4Q^2 - [Q,M][Q,M^\dagger]),\\ \mathrm{Tr}\,Q_a^2 &= \mathrm{Tr}\,[Q,M][Q,M^\dagger]\end{aligned} \tag{29}$$

we get

$$\begin{aligned}L_1 &= \frac{ie}{4f^2}A_\mu \mathrm{Tr}\,Q[M,\partial_\mu M^\dagger],\\ L_2 &= -\frac{e^2}{8f^2}A_\mu^2 \mathrm{Tr}\,[Q,M][Q,M^\dagger]\end{aligned} \tag{30}$$

which together with the third term in Eq. (26) reproduces the minimal chiral Lagrangian for the pseudoscalar mesons in the presence of the electromagnetic interactions usually obtained by making a substitution

$$\partial_\mu M \;\rightarrow\; \partial_\mu M + ie[Q,M] \tag{31}$$

in the third term of Eq. (12). The electromagnetic interactions of the pseudoscalar mesons with the vector and axial vector mesons are now can be obtained in a simple manner from Eq. (26). They are given by the modified vector and axial vector mesons mass terms as given by the second term of Eq. (26). To derive the soft pion result for the pion Compton scattering, we shall redefine the vector and axial vector meson field so that the mass terms in Eq. (26) are gauge invariant under the electromagnetic gauge transformation. As can be seen from Eq. (27) an obvious choice is $\rho_\mu(\text{new})$ for the vector meson field and $a_\mu(\text{new})$ for the axial vector meson field. All electromagentic interactions are now transferred to the gauge boson kinetic terms, the $\rho_{\mu\nu}'^2$ and the $a_{\mu\nu}^2$ terms in Eq. (26). In terms of $\rho_\mu^2(\text{new})$ and $a_\mu^2(\text{new})$, our total Lagrangian $\mathcal{L}_{\text{total}}$ is then

$$\begin{aligned}\mathcal{L}_{\text{total}} &= -\frac{1}{8}\mathrm{Tr}\,(\rho_{\mu\nu}'^2 + a_{\mu\nu}^2) + \frac{1}{4}\left(m_\rho^2 \mathrm{Tr}\,\rho_\mu^2(\text{new}) + \mathrm{m_A^2}\,\mathrm{Tr}\,\mathrm{a}_\mu^2(\text{new})\right)\\ &\quad + \frac{1}{8f^2}\mathrm{Tr}\,(\partial_\mu M + ie[Q,M])(\partial_\mu M^\dagger + ie[Q,M^\dagger])\end{aligned} \tag{32}$$

where $\rho'_{\mu\nu}$ is now given by,

$$\rho'_{\mu\nu} = D_\nu\rho_\mu(\text{new}) - \mathrm{D}_\mu\rho_\nu(\text{new}) + \frac{\mathrm{ig}}{2}[\rho_\mu(\text{new}),\rho_\nu(\text{new})] + \frac{2\mathrm{i}}{\mathrm{g}}[\mathrm{p}_\mu,\mathrm{p}_\nu] - \frac{e}{g}Q_v F_{\mu\nu} + \text{other terms} \tag{33}$$

and for $a_{\mu\nu}$,

$$a_{\mu\nu} = D_\nu a_\mu(\text{new}) - \mathrm{D}_\mu \mathrm{a}_\nu(\text{new}) + (1+\mathrm{g}\lambda)\left([\mathrm{p}_\mu,\rho_\nu(\text{new})] + [\mathrm{p}_\nu,\rho_\mu(\text{new})]\right) - \frac{e}{g}\frac{m_\rho^2}{m_A^2}Q_a F_{\mu\nu} + \text{other terms} \tag{34}$$

where "other terms" contain the electromagnetic field A_μ and higher derivative term in the pseudoscalar meson field. For the derivation of our result in the soft pion limit, we need only terms without derivative of the pseudoscalar meson field and will therefore ignore those terms. Then the total contribution to the Compton amplitude can be read off automatically from the above Lagrangian. We find

$$T_{\mu\nu}(p,q) = 2e^2\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + 2e^2(q^2 g_{\mu\nu} - q_\mu q_\nu)\left(\frac{1}{(m_\rho^2 - q^2)} - \frac{m_\rho^2}{m_A^2(m_A^2 - q^2)}\right) \tag{35}$$

where the vector and axial vector meson contribution come from terms of the form $\mathrm{Tr}\,[Q,M][Q,M^\dagger]F_{\mu\nu}^2$ in the Lagrangian Eq. (32).

The first term in Eq. (35) is the usual Born term with the pion electromagnetic form factor taken to be off-shell. In the limit of $p \to 0$, the vector meson contribution to the off-shell form factor is of the order $O(p\cdot q)$ and can therefore be neglected. Thus the off-shell pion form factor can be taken to be point-like and the first term in Eq. (35) is the Born term in the soft pion limit. More precisely, from the total Lagrangian in Eq. (32), for an inital momentum p and a final momentum l, the pion form factor takes the following form

$$F_\pi(p,l) = (p+l)_\mu + [(p+l)_\mu q^2 - (p-l)_\mu(p^2 - l^2)]/(m_\rho^2 - q^2) \tag{36}$$

In obtaining the above result, we have used the relation $g^2 = m_\rho^2/f_\pi^2$ given in Eq. (23). Thus, for $l = p+q$, as in the Born term contribution, $p^2 - l^2 = -q^2 + 2p\cdot q$ so that the q_μ term cancels out in the form factor. The vector meson pole term thus produces only terms at least first power of the pion momenta and does not contribute in the soft pion limit. This agrees with previous results in Ref.[2].

The second term Eq. (35) is the vector and axial vector meson contributions. The vector meson contribution comes from the term $\mathrm{Tr}\,Q_v^2 F_{\mu\nu}^2$ in the soft pion limit. It exhibits a simple vector meson pole behaviour in the q^2 variable. The axial vector meson contribution comes from the term $\mathrm{Tr}\,Q_a^2 F_{\mu\nu}^2$ which is of opposite sign to the vector meson contribution, as can be seen from Eq. (29). This is an axial vector meson pole dominance term with a simple pole behaviour in the variable $s = (p+q)^2$. The form factor for the $A_1 - \pi - \gamma$ vertex, like the pion electromagnetic form factor, because of chiral symmetry, gets contribution from vector meson terms of the order $O(p \cdot q)$ as the $A_1 - \rho - \pi$ coupling is at least first derivative in the pion field as can be seen from our chiral Lagrangian Eq. (32). Thus in the soft pion limit there is no vector meson double pole terms in the axial vector meson contribution to the pion Compton amplitude.

Putting the total amplitude in the following form,

$$\begin{aligned} T_{\mu\nu}(p,q) &= 2e^2\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + 4e^2\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \\ &\times \left[-\left(1 - \frac{m_\rho^2}{m_A^2}\right) + \left(\frac{m_\rho^2}{(m_\rho^2 - q^2)} - \frac{m_\rho^2}{(m_A^2 - q^2)}\right)\right] \end{aligned} \tag{37}$$

and using Eq. (13), we find that the Born term cancels out the first term in the square bracket so that our final result for $T_{\mu\nu}(p,q)$ is

$$T_{\mu\nu}(p,q) = 4e^2\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) m_\rho^2 \left[\frac{1}{(m_\rho^2 - q^2)} - \frac{1}{(m_A^2 - q^2)}\right] \tag{38}$$

which is the soft pion result obtaineed by Das *et al.* [3].

Thus in a simple and transparent manner, we have obtained the soft pion result of Das *et al.*. Our chiral Lagrangian leads automatically to the first and second Weinberg sum rules and thereby guaranteeing the convergence of the $\pi^+ - \pi^0$ mass difference calculation. Once we understand the absence of double pole terms in the pion Compton amplitude, we can compare our derivation with the dispersion relation approach. Without using a chiral Lagrangian, one could just include the on-shell pion form factor and the $A_1 - \pi - \gamma$ on-shell form factor in the dispersion relations for the forward Compton amplitude [13] and obtain the erroneous result that the double pole terms must be present. Infact, in the work of Chanda *et al.* [14], the cancellation of the double pole terms imposed on the pion Compton amplitude by the soft pion theorem has been invoked to obtain dispersion sum rules for the coupling constants which are however found to be difficult to satisfy experimentally. In our appoach, the absence of the double pole terms is simply a consequence of chiral symmetry. We note also that the unsubtracted dispersion relation for the $\Delta I = 2$ amplitude can be made consistent with the soft pion result by including also the contact term from the vector meson pole term in a modified Born term. Then it would be more convenient to use the dispersion relation approach to calculate the $\pi^+ - \pi^0$ mass difference since terms of $O(p^2)$ can also be analysed in a straightforward manner.

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