J5

University of Trondheim UNIT

Theoretical Physics Seminar in Trondheim

No 12 1995

High energy channeling electron-positron pair production by photons in a crystal

Haakon A. Olsen

Institute of Physics
University of Trondheim
N-7055 Dragvoll, Norway

Yuri P. Kunashenko

Nuclear Physics Institute
634055 Tomsk, P. O. Box 25, Russia

ISSN 0365-2459
November 1995

Abstract

The cross section for channeling electron-positron pair production in continuum states is obtained for a crystal string potential proportional to ρ^{-1} with ρ the distance to the crystal string. For two dimensional Sommerfeld-Maue-like electron and positron wave functions matrix elements and cross section are obtained for unpolarized photons. The fact that channeling continuum pair production can only occur when the photon is hitting the crystal string at a small, finite angle, is taken into account.

1. Introduction

The production of electron-positron pairs by photons in a crystal is a process which has been studied experimentally and theoretically for many years [1]. The theoretical studies have been confined to semi-classical calculations, wich seem to give useful results for experimental applications. The usefulness of approximate semi-classical methods must be considered in the light of the fact that exact calculations are difficult and in general complicated.

The present paper presents a quantum mechanical calculation of pair production in a crystal for a crystal potential proportional to $1/\rho$, with $\rho = \left(x^2 + y^2\right)^{1/2}$, the distance to the crystal string. With this potential the Dirac equation has high-energy two-dimentsional Sommerfeld-Maue-like solutions, as shown in the preceding paper [2] which can be used for exact calculations of the matrix elements. The calculation is similar to the calculation by H. A. Bethe and L. C. Maximon [3] of pair production on single atoms, although the present two-dimensional calculation proves to be more complicated. It is of course gratifying for the authors to note that matrix element calculations can be performed in two dimensions with similar methods to those used by A. Sommerfeld [4] and that the integrations give hypergeometric functions as in the three-dimentsional case which is not abvious prior to the calculation. These questions are discussed explicitly in Appendix 1.

2. The matrix elements

With a potential $V(\rho)$ which does not take into account the crystal structure along the crystal string, no momentum can be transfered in the string direction, which we take as the z-axis,

$$q_z = k_z - p_z^+ - p_z^- = 0. \tag{2.1}$$

where $\vec{q}$, is the momentum transfer and $\vec{k}, \vec{p}^{\,+}, \vec{p}^{\,-}$, are the momenta of the photon, positron and electron, respectively. Now if the photon momentum is parallel to the crystal string, $k_z = \omega$, momentum and energy balance cannot be maintained. In order to obtain pair production, then, $\vec{k}$ must have a transverse component, $\vec{k}_\perp$, giving $\vec{q}_\perp = \vec{k}_\perp - \vec{p}_\perp^{\,+} - \vec{p}_\perp^{\,-}$. Only photons hitting the crystal string at a small angle $\delta = k_\perp / \omega$ larger than

$$\delta_{min} = \frac{k_{\perp\, min}}{\omega} > \frac{m}{\sqrt{E_+ E_-}} \tag{2.2}$$

can produce pairs. The minimum value $k_{\perp\, min}$ is obtained from the useful high energy, small angle relation

$$\frac{\omega}{E_+}\left(p_\perp^+\right)^2 + \frac{\omega}{E_-}\left(p_\perp^-\right)^2 = k_\perp^2 - \frac{\omega^2}{E_+ E_-} m^2. \tag{2.3}$$

These considerations do not seem to have been considered to be of importance and taken into account in the published papers on semi-classical calculations.

The cross section for pair production is given by

$$d^4\sigma = \frac{1}{(2\pi)^4}\frac{\alpha}{\omega}|M|^2\delta^4(k - p_+ - p_- - q)d^3p_+d^3p_-d^3q \tag{2.4}$$

with α the fine structur constant and ω the photon energy. M is the matrix element

$$M = \int d^3x\overline{\psi}_-(\vec{r})\vec{\gamma}\cdot\vec{e}e^{ikr}\psi_+(\vec{r}) \tag{2.5}$$

with $\psi_\pm(\vec{r})$ the positron and electron wave functions, $\vec{e}$ the photon polarization and $\vec{\gamma}$ the Dirac vector matrix.

It is convencient to factor out the z-dependent part of the matrix element, which is

$$|M_z|^2 = \left|\int_0^L e^{i(k_z - p_z^- - p_z^+)z}dz\right|^2 = 2\pi L\delta(q_z) \tag{2.6}$$

for large coherence lengths L. Integrating out the redundant coordinates in Eq. (2.4) we find the physical cross section per unit length of the crystal

$$d^4\sigma / L = \frac{1}{(2\pi)^3}\frac{\alpha}{\omega}|M_\perp|^2\frac{E_+E_-}{|p_z^+E_- - p_z^-E_+|}p_\perp^+dp_\perp^+d\varphi^+p_\perp^-dp_\perp^-d\varphi^- \tag{2.7}$$

with the transverse part of the matrix element

$$M_\perp = \overline{u}_-\int d^2\rho\overline{F}_-(\vec{\rho})\vec{\gamma}\cdot\vec{e}E^{iq_\perp\rho}F_+(\vec{\rho})u_+ \tag{2.8}$$

where $F_\pm(\vec{\rho})$ is obtained from Eq. (6.6) in I.

$$F_\pm(\vec{\rho}) = N_\pm\left(1 \pm \frac{i}{2F_\pm}\gamma_0\vec{\gamma}\cdot\nabla_\perp\right)F\left(id_\pm;\frac{1}{2};\mp i(p_\perp^+\rho + \vec{p}_\perp^+\vec{\rho})\right) \tag{2.9}$$

and $u_\pm$ are the free particle positron and electron spinors, and $d_\pm = d\frac{E_\pm}{p^\pm}$, $d = Z\alpha\frac{a}{b}c$ with the parameters defined in I. The electron and positron wave functions describing produced particles are accordingly asymtotically given by plane waves plus cylindrical ingoing waves.

When the sum over electron and positron polarizations and the average over photon polarizations are performed we find

$$\frac{1}{2}\sum_{pol}|M_\perp|^2 = \frac{(N_+N_-)^2}{2E_+E_-}\Big\{\big(E_+E_- + m^2 - (\vec{p}_+\cdot\hat{k})(\vec{p}_-\cdot\hat{k})\big)|I_1|^2$$
$$+\big(E_+E_- - m^2 + (\vec{p}_+\cdot\hat{k})(\vec{p}_-\cdot\hat{k})\big)\big(|\vec{I}_+|^2 + |\vec{I}_-|^2\big)$$
$$+2\,Re\Big[E_-I_1^*\big((\vec{I}_+\cdot\vec{p}_+) - (\vec{p}_+\cdot k)(\vec{I}_+\cdot\hat{k})\big) + E_+I^*\big((\vec{I}_-\cdot\vec{p}_-) - (\vec{p}_-\cdot\hat{k})(\vec{I}_-\cdot\hat{k})\big)\Big]\Big\} \quad (2.10)$$

where the integrals I_1, I_+ and I_- are given by

$$I_1 = \int d^2\rho F_-(\vec{\rho}_-)e^{iq_\perp\rho}F_+(\vec{\rho})$$
$$I_+ = \frac{i}{2E_+}\int d^2\rho F_-(\vec{\rho})e^{iq_\perp\rho}\nabla_\perp F_+(\vec{\rho}) \quad (2.11)$$
$$\vec{I}_- = \frac{i}{2E_-}\int d^2\rho(\nabla_\perp F_-(\vec{\rho}))e^{iq_\perp\rho}F_+(\vec{\rho}).$$

where

$$F_\pm(\vec{\rho}) = F\left(\mp id_\pm; \frac{1}{2}; i(p_\perp^\pm\rho + \vec{p}^\pm\rho)\right) \quad (2.12)$$

As first used by A. Sommerfeld [4] and later by A. T. Nordsieck [5] an integral

$$I_0 = \int d^2\rho\rho^{-1}F_-(\vec{\rho})e^{iq_\perp\rho-\varepsilon\rho}F_+(\vec{\rho}) \quad (2.13)$$

is defined which makes it possible to derive all integrals from I_0 by the use of the relation

$$\nabla_\perp(p_\perp\rho + \vec{p}_\perp\vec{\rho}) = (p_\perp/\rho)\nabla_{p_\perp}(p_\perp\rho + \vec{p}_\perp\vec{\rho}).$$

which gives

$$I_1 = -\frac{\partial}{\partial\varepsilon}I_0(\vec{p}_\perp^\pm,\varepsilon)\Big|_{\varepsilon=0}$$
$$\vec{I}_\pm = (ip_\perp^\pm/2E_\pm)\nabla_{p_\perp^\pm}I_0(\vec{p}_\perp^\pm,\varepsilon)\Big|_{\varepsilon=0} \quad (2.14)$$

The calculation of the integral I_0, following A. Sommerfeld's [4] method of integration, also used by Nordsieck [5] is performed in Appendics 1.

Performing the derivations in Eq. (2.14) we find the final results for the integrals

$$I_l = C\left\{\left(\frac{p_\perp^- d_-}{D_-} - \frac{p_\perp^+ d_+}{D_+}\right)F(x) + i\left[\mu^2\left(\frac{p_\perp^-}{D_-} + \frac{p_\perp^+}{D_+}\right) - p_\perp^+ - p_\perp^-\right]G(x)\right\}$$

$$\vec{I}_\pm = C\frac{p_\perp^\pm}{2E_\pm}\left\{\mp d_\pm \frac{\vec{q}_\perp}{D_\pm}F(x) + i\left[\left(\frac{\mu^2}{D_\pm} - 1\right)\vec{q}_\perp \mp \left(\vec{p}_\perp^\pm - \frac{p_\perp^\mp}{p_\perp^\pm}\vec{p}_\perp^\pm\right)\right]G(x)\right\}.$$

(2.15)

Here

$$C = \frac{4\pi}{q_\perp}\left(\frac{q_\perp^2}{D_+}\right)^{-id_+}\left(\frac{q_\perp^2}{D_-}\right)^{id_-}, D_\pm = q_\perp^2 + 2(\vec{q}_\perp p_\perp^\pm).$$

$$\mu^2 = \vec{k}_\perp^2 - (p_\perp^+ + p_\perp^-)^2$$

and

$$F(x) = {}_{21}F\left(-id_+, id_-; \frac{1}{2}; x\right)$$
$$G(x) = 2q_\perp^2 \frac{d_+ d_-}{D_+ D_-}\,{}_{21}F\left(1 - id_-, 1 + id_+; \frac{3}{2}; x\right) \tag{2.16}$$

with ${}_{21}F$ the hypergeometric function and

$$x = (2/D_+D_-)\{q_\perp^2(p_\perp^+ p_\perp^- - \vec{p}_\perp^+ \vec{p}_\perp^-) + 2(\vec{q}_\perp \cdot \vec{p}_\perp^+)(\vec{q}_\perp \cdot \vec{p}_\perp^-)\}. \tag{2.17}$$

3 The cross section

In the further calculation it is convenient to introduce the vectors $\vec{V}_+$ and $\vec{V}_-$

$$\vec{V}_\pm = \vec{p}_\perp^\pm - E_\pm(\vec{k}_\perp/\omega) \tag{3.1}$$

and note that

$$\vec{q}_\perp = \vec{k}_\perp - \vec{p}_\perp^+ - \vec{p}_\perp^- = -(\vec{V}_+ + \vec{V}_-). \tag{3.2}$$

Further

$$D_\pm = q_\perp^2 + 2(\vec{q}_\perp \vec{p}_\perp^\pm) = \frac{\omega}{E_\mp}(m^2 + V_\mp^2) \tag{3.3}$$

which shows the convenience of introducing in analogy to reference [3]

$$\xi = (m^2 + V_+^2)^{-1}, \quad \eta = (m^2 + V_-^2)^{-1} \tag{3.4}$$

With these notations equations in the previous chapter simplify considerably.

Eq. (2.10) becomes for high energies and small angles

$$\frac{1}{2}\sum_{pol}|M_\perp|^2 = \frac{|N_+N_-|^2}{2E_+E_-}\left\{\frac{1}{2E_+E_-}\left(E_+^2V_-^2 + E_-^2V_+^2 + \omega^2 m^2\right)|I_l|^2 + 2E_+E_-\left(\left|\vec{I}_+\right|^2 + \left|\vec{I}_-\right|^2\right)\right.$$
$$\left. + Re\left[I_l^*\left(E_+\left(\vec{I}_-\cdot\vec{V}_-\right) + E_-\left(\vec{I}_+\cdot\vec{V}_+\right)\right)\right]\right\}. \tag{3.5}$$

Likewise the integrals Eq. (2.15) simplify

$$I_l = C\left\{d\frac{E_+E_-}{\omega}(\xi-\eta)F(x) + i\left[\frac{\mu^2}{\omega}\left(p_\perp^- E_+\xi + p_\perp^+\xi_-\zeta\right) - p - p_\perp^-\right]G(x)\right\}$$
$$\vec{I}_+ = C\frac{p_\perp^+}{2E_+}\left\{-d\frac{\vec{q}_\perp}{p_\perp^+}\frac{E_+E_-}{\omega}\eta F(x) + i\left[\left(\frac{\mu^2 E}{\omega}\eta - 1\right)\vec{q}_\perp - \left(\vec{p}_\perp^- - \frac{p_\perp^-}{p_\perp^+}\vec{p}_\perp^+\right)\right]G(x)\right\} \tag{3.6}$$
$$\vec{I}_- = C\frac{p_\perp^-}{2E_-}\left\{-d\frac{q_\perp}{p_\perp^-}\frac{E_+E_-}{\omega}\xi F(x) + i\left[\left(\frac{\mu^2 E_+}{\omega}\xi - 1\right)\vec{q}_\perp - \left(\vec{p}_\perp^+ - \frac{p_\perp^+}{p_\perp^-}\vec{p}_\perp^-\right)\right]G(x)\right\},$$

where $d = d_\pm \rho_\perp^\pm / E_\pm$, while x can be written as

$$x = 4p_\perp^+ p_\perp^- \frac{E_+E_-}{\omega^2}\xi\eta\left(\vec{V}_+ + \vec{V}_-\right)^2 cos^2\left(\frac{\phi_+ + \phi_-}{2}\right). \tag{3.7}$$

where ϕ_+ and ϕ_- are the angles in the $\vec{\rho}$ plane given by

$$\hat{q}_\perp \hat{p}_\perp^\pm = cos\phi_\pm .$$

Alternatively x may be expressed as

$$x = 4p_\perp^+ p_\perp^- \frac{E_+E_-}{\omega^2}\xi\eta\left[k_\perp cos\left(\frac{\varphi_+ + \varphi_-}{2}\right) - \left(p_\perp^+ + p_\perp^-\right)cos\left(\frac{\varphi_+ - \varphi_-}{2}\right)\right]^2 \tag{3.8}$$

where the angles φ_+ and φ_- refer to the fixed vector $\vec{k}_\perp$,

$$\hat{k}_\perp \hat{p}_\perp^\pm = cos\varphi_\pm .$$

In order to obtain the cross section, we want the polarization independent matrix element squared written in terms of *F(x)* and *G(x)*. We define the coefficients *f, g,* and *h,* rewriting Eq. (3.5) in the form

$$\frac{1}{2}\sum_{pol}|M_\perp|^2 = \frac{|N_+N_-|^2}{2E_+E_-}|C|^2\left\{f|F(x)|^2 + g|G(x)|^2 + h\,Im\left[F^*(x)G(x)\right]\right\}. \tag{3.9}$$

After some algebra we find

$$f = d^2\frac{E_+E_-}{\omega^2}\left\{\frac{1}{2}\omega^2\left(V_+^2+V_-^2\right)\xi\eta+\left(E_+^2+E_-^2\right)\vec{V}_+\cdot\vec{V}_-\xi\eta - E_+E_-\left(V_+^2\xi^2+V_-^2\eta^2\right)\right\} \qquad (3.10a)$$

$$g = \frac{\mu^4}{\omega^2}\left\{\frac{m^2\omega^2}{2E_+E_-}\left(p_\perp^+E_-\eta+p_\perp^-E_+\xi\right)^2 - p_\perp^+p_\perp^-\left(E_+^2+E_-^2\right)\vec{V}_+\cdot\vec{V}_-\xi\eta+\frac{E_+^2+E_-^2}{2E_+E_-}\left(p_\perp^{+2}E_-^2V_-^2\eta^2+p_\perp^{-2}E_+^2V_+^2\xi^2\right)\right\}$$
$$+\frac{\mu^2}{\omega}\left(p_\perp^+E_-\eta+p_\perp^-E_+\xi\right)\left\{-\left(p_\perp^++p_\perp^-\right)\frac{m^2\omega^2}{E_+E_-}+\frac{1}{\omega E_+E_-}\left(p_\perp^+E_--p_\perp^-E_+\right)\left(E_+^2\vec{V}_-\cdot\vec{k}_\perp-E_-^2\vec{V}_+\cdot\vec{k}_\perp\right)\right\}$$
$$+\frac{1}{2}\left(p_\perp^++p_\perp^-\right)^2\frac{m^2\omega^2}{E_+E_-}+\frac{E_+^2+E_-^2}{2E_+E_-}\left(p_\perp^+E_--p_\perp^-E_+\right)^2\frac{k_\perp^2}{\omega^2} \qquad (3.10b)$$

$$h = -\mu^2\left\{\frac{E_+^2+E_-^2}{\omega^2}\left(p_\perp^-E_+V_+^2\xi^2-p_\perp^+E_-V_-^2\eta^2\right)+m^2\left(p_\perp^-E_+\xi^2-p_\perp^+E_-\eta^2\right)+\frac{E_+E_-}{\omega}\xi\eta\vec{V}_+\cdot\vec{V}_-\left(\frac{E_+^2}{E_-}-\frac{E_-^2}{E_+}\right)\right.$$
$$\left.+\frac{E_+E_-}{\omega^2}\xi\eta\left(\frac{m^2\omega^2}{E_+E_-}+\vec{V}_+\cdot\vec{V}_-\right)\left(p_\perp^+E_--p_\perp^-E_+\right)\right\}+\left(p_\perp^++p_\perp^-\right)m^2\omega(\xi-\eta)+\frac{E_+^2+E_-^2}{\omega^2}\left(p_\perp^+E_--p_\perp^-E_+\right)\left(\vec{V}_+\cdot\vec{k}_\perp\xi+\vec{V}_-\cdot\vec{k}_\perp\eta\right). \qquad (3.10c)$$

Note that f and g are symmetric in $+\leftrightarrow-$, while h is antisymmetric.
The cross section Eq. (2.7), averaged over photon polarizations and summed over electron and positron poarizations, then becomes

$$d^4\sigma/L = \frac{1}{2\pi q_\perp^2}\frac{2\alpha}{\left|p_z^+E_--p_z^-E_+\right|}\frac{\left|N_+N_-\right|^2}{\omega}\left\{f\left|F(x)\right|^2+g\left|G(x)\right|^2+h\,Im\left[F^*(x)G(x)\right]\right\} \qquad (3.11)$$
$$\times p_\perp^+dp_\perp^+d\varphi^+p_\perp^-dp_\perp^-d\varphi^-$$

with f, g and h given in Eqs (3.10, a-c), and F(x) and G(x) in Eq. (2.16). The normalization factors are according to Eq. (6.8) in I

$$N_\pm = \left(cosh\,\pi d_\pm\right)^{-\frac{1}{2}}e^{\mp\pi d_\pm/2+i\lambda}$$

where λ is a phase. This gives

$$\left|N_+N_-\right|^2 = \frac{e^{-\pi d_+}}{cosh\,\pi d_+}\frac{e^{+\pi d_-}}{cosh\,\pi d_-} \qquad (3.12)$$

which in the high energy limit gives

$$\left|N_+N_-\right|^2 = 4e^{-2\pi d_+}.$$

The cross section is then given by

$$d^4\sigma / L = \frac{1}{2\pi q_\perp^2} \frac{2\alpha}{\omega\left|p_z^+ E_- - p_z^- E_+\right|} \frac{e^{-\pi d_+}}{\cosh\pi d_+} \frac{e^{\pi d_-}}{\cosh\pi d_-} \left\{ f\left|F(x)\right|^2 + g\left|G(x)\right|^2 + h\,Im\left[F^*(x)G(x)\right]\right\}$$

$$\times\, p_\perp^+ dp_\perp^+ p_\perp^- dp_\perp^- d\varphi_+ d\varphi_- \qquad (3.13)$$

Appendix 1

In order to calculate the integral I_0 Eq. (2.13) we follow the method of A. Sommerfeld [4] who calculated the corresponding integral in three dimensions. The function I_0

$$I_0 = \int d\rho d\varphi F\left(id_-;\frac{1}{2};i\left(p_\perp^-\rho + \vec{p}^- \cdot \vec{\rho}\right)\right) e^{i\vec{q}_\perp\vec{\rho}-\varepsilon\rho} F\left(-id_+;\frac{1}{2};i\left(p_\perp^+\rho + \vec{p}^+ \cdot \vec{\rho}\right)\right) \qquad (A.1)$$

becomes when the integral representation of the Kummer function[1]

$$F(a;c;x) = B\int_0^1 e^{xt} t^{a-1}(1-t)^{c-b-1}\,dt \qquad (A.2)$$

is introduced

$$I_0 = B_+B_-\int_0^1 dt\,t^{-id_+-1}(1-t)^{-\frac{1}{2}+id_+}\int_0^1 du\,u^{id_--1}(1-u)^{-\frac{1}{2}-id_-}$$

$$\times\int d\rho d\varphi\, e^{i\vec{q}_\perp\vec{\rho}-\varepsilon\rho+i\left(p_\perp^+\rho+\vec{p}^+\cdot\vec{\rho}\right)t+i\left(p_\perp^-\rho+\vec{p}^-\cdot\vec{\rho}\right)u}.$$

where $B_\pm = \dfrac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(\mp id_\pm)\Gamma\left(\frac{1}{2}\pm id_\pm\right)}$ (A.3)

The ρ and φ integrations give

$$I_0 = 2\pi B_+B_-\int_0^1 dt\,t^{-id_+-1}(1-t)^{-\frac{1}{2}+id_+}\int_0^1 du\,u^{id_--1}(1-u)^{-\frac{1}{2}-id_-}\left(A^2+B^2\right)^{-\frac{1}{2}} \qquad (A.4)$$

where

$$A = \varepsilon - i\left(tp_\perp^+ + up_\perp^-\right), \qquad \vec{B} = \vec{q}_\perp + t\vec{p}_\perp^+ + u\vec{p}^-.$$

It is important that the quadratic terms in $A^2 + B^2$ cancel giving a linear function in t and u,

[1] As demonstrated by A. Sommerfeld, the integration path can alternatively be described by a loop in the complex plane encircling the points 0 and 1

$$\left(A^2+B^2\right)^{-\frac{1}{2}}=(a-bu)^{-\frac{1}{2}}=\frac{1}{\sqrt{a}}\left(1-\frac{b}{a}u\right)^{-\frac{1}{2}} \tag{A.5}$$

with

$$a=q_\perp^2+2\left(\vec{q}_\perp\cdot\vec{p}_\perp^- - i\varepsilon p_\perp^+\right)t$$

and

$$b=2\left(p_\perp^+p_\perp^- - \vec{p}_\perp^+\cdot\vec{p}_\perp^-\right)t+2\left(i\varepsilon p_\perp-\vec{q}_\perp\cdot\vec{p}_\perp\right),$$

where we have neglected ε^2 terms, since they do not contribute to I_I. Introduction of (A.5) in (A.4) gives a hypergeometric function

$$\int_0^1 du\,u^{id_- -1}(1-u)^{-\frac{1}{2}-id_-}\left(1-\frac{b}{a}u\right)^{-\frac{1}{2}}=B_-^{-1}\,{}_{21}F\left(\frac{1}{2},id_-;\frac{1}{2};\frac{b}{a}\right)=B_-^{-1}\left(1-\frac{b}{a}\right)^{-id_-},$$

and this is a crucial point in the integration, the index 1/2 in the Kummer function matches the power 1/2 from the spatial integration reducing the hypergeometric function to a simple function which makes it possible to obtain a hypergeometric function as a result of the final t-integration. In the three-dimensional case of A. Sommerfeld the crucial index is 1.

The integral is now

$$I_0=2\pi B_+\int_0^1 dt\,t^{-id_+-1}(1-t)^{-\frac{1}{2}+id_+}a^{-\frac{1}{2}+id_+}(a-b)^{-id_-}\quad .$$

The integral has, considered as a loop integrand, the four branch points:

$$t_1=0,\qquad t_2=1\qquad t_3=-q_\perp^2/2\left(\vec{q}_\perp\cdot\vec{p}_\perp^+ - i\varepsilon p_\perp^+\right)\qquad (a=0)$$

and $\quad t_4=\dfrac{q_\perp^2+2\left(\vec{q}_\perp\cdot\vec{p}^- - i\varepsilon p_\perp^-\right)}{2\left[p_\perp^+p_\perp^- - \vec{p}_\perp^+\cdot\vec{p}_\perp^- - \vec{q}_\perp\cdot\vec{p}_\perp^+ + \varepsilon p_\perp^+\right]}\qquad (a-b=0)$

while the integrand vanishes as t^{-2} at infinity.

The change of variable, conserving the limits (0,1)

$$t=\frac{t_3v}{v-1+t_3},$$

changes the integral into

$$I_0=B_+C\int_0^1 dv\,v^{-id_+-1}(1-v)^{-\frac{1}{2}+id_+}(1-xv)^{-id} \tag{A.6}$$

with

$$x = 2\frac{q^2\left(p_\perp^+ p_\perp^- - \vec{p}_\perp^+ \vec{p}_\perp^-\right) + 2\left(\vec{q}_\perp \vec{p}_\perp^+\right)\left(\vec{q}_\perp \vec{p}_\perp^-\right) - 2i\varepsilon\left(p_\perp^+ \vec{q}_\perp \vec{p}_\perp^- + p_\perp^-(\vec{q}_\perp \vec{p})\right)}{\left(D^+ - 2i\varepsilon p_\perp^+\right)\left(D^- - 2i\varepsilon p_\perp^-\right)}$$

and

$$C = \frac{4\pi}{q_\perp}\left(\frac{q_\perp^2}{D_+ - 2i\varepsilon p_\perp^+}\right)^{-id_+}\left(\frac{q_\perp^2}{D_- - 2i\varepsilon p_\perp^-}\right)^{id_-} \tag{A.7}$$

where we always neglect ε^2-terms.

The integrand in terms of the new variable has the branch points $v_1 = 0$, $v_2 = 1$, $v_3 = 1/x$ and $v_4 = \infty$, and the integral I_0 is a hypergeometric function

$$I_0 = 2\pi K \, {}_2F_1\left(-id_+, id_-; \frac{1}{2}; x\right)$$

Following the prescriptions in Eq. (2.14) and remembering that

$$\frac{d}{dx}\, {}_2F_1\left(-id_+, id_-; \frac{1}{2}; x\right) = 2d_+d_- \, {}_2F_1\left(1 - id_+, 1 + id_-; \frac{3}{2}; x\right)$$

one finds the integrals I_1 and I_2 given in Eq (2.15).

References

[1] For a review see J. C. Kimball and N. Cue, Phys. Rep. 125, 69 (1985).

[2] Yuri Kunashenko and Haakon A. Olsen: Dirac states of relativistic electrons and positrons channeled in a crystal, preceding paper. In the following refered to as I.

[3] H. A Bethe and L. C. Maximon, Phys. Rev 93 , 768 (1954). See also H. A. Olsen, L. C. Maximon and H. Wergeland, Phys. Rev. 106, 27, (1957); H. A. Olsen and L. C. Maximon, Phys. Rev. 114, 887 (1959).

[4] A. Sommerfeld, Atombau und Spektrallinien II, Fridr. Vieweg & Sohn, Braunschweig, 1960.

[5] A. T. Norsieck, Phys. Rev. 93, 784 (1954)