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IMPACT OF SECOND-ORDER CHROMATICITY ON THE SCHOTTKY SPECTRA OF BUNCHED BEAM

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Abstract

Observation of Schottky signals provides information on important beam and machine parameters, such as transverse emittance, betatron tune, and first-order chromaticity. However, the so-far developed theory of Schottky spectra does not include the impact of the higher-order chromaticity, which can be non-negligible in the case of the Large Hadron Collider (LHC). In this contribution, we expand the theory of Schottky spectra to also take into account second-order chromaticity. Analytical results are compared with macro-particle simulations and the errors resulting from neglecting second-order chromaticity are assessed for the case of the LHC.

INTRODUCTION

The fundamental theory of transverse Schottky signals of bunched beams was presented in two classical references, written by D. Boussard [1] and S. Chattopadhyay [2]. According to these texts, the first three moments of the Schottky transverse sidebands, that is their cumulative power, center of mass, and root-mean-squared (RMS) width can be used to derive respectively the transverse emittance, betatron tune, and the first order chromaticity. In this contribution, we present an extension of the Schottky signal theory to the case of a non-zero second-order chromaticity. Such an extension is closer to the reality of the standard LHC operation [3], and especially when high values of the second-order chromaticity are introduced to stabilize transverse collective instabilities [4].

DERIVATION

Let us assume that in a given location of the machine the transverse displacement of a given particle i is given by

$$x_i(t) = \widehat{x_i} \cos\left(\phi_{\beta_i}(t)\right),$$

where $\widehat{x_i} = \sqrt{2\beta J_i}$ is the amplitude of betatron oscillations, β is the Courant-Snyder beta parameter, J_i is the transverse action, and $\phi_{\beta_i}(t)$ is the instantaneous betatron phase.

In the scope of this contribution, we will assume that the frequency of the betatron motion is given by a product of the instantaneous tune Q_i and the angular revolution frequency $\omega_0 \equiv 2\pi f_0$:

$$\omega_{\beta_i} = Q_i\omega_0 = \left[Q_0 + Q'\frac{\Delta p_i}{p_0} + \frac{Q''}{2!}\left(\frac{\Delta p_i}{p_0}\right)^2\right]\omega_0, \tag{1}$$

where Q_i is dependent on the instantaneous momentum deviation $\frac{\Delta p_i}{p_0}$, the nominal tune is denoted by Q_0, and Q', Q'' are respectively first and second order chromaticities. We shall also assume that the momentum deviation is a consequence of a harmonic synchrotron motion, and as follows from Ref. [5], can be expressed as:

$$\frac{\Delta p_i}{p_0} = -\frac{\widehat{\tau_i}\Omega_{s_i}}{\eta}\cos(\Omega_{s_i}t + \varphi_{s_i}),$$

where $\widehat{\tau_i}$, Ω_{s_i} and φ_{s_i} are respectively the amplitude, the angular frequency and the phase of the synchrotron motion and η is the slip factor, assumed to be positive above the transition energy.

The betatron phase can be calculated by integrating Eq. (1). It yields:

$$\begin{aligned}\phi_{\beta_i}(t) = \Bigg[&\left(Q_0 + \frac{Q''\widehat{\tau_i}^2\Omega_{s_i}^2}{4\eta^2}\right)t - \frac{Q'\widehat{\tau_i}}{\eta}\sin\left(\Omega_{s_i}t + \varphi_{s_i}\right)\\ &+ \frac{Q''\widehat{\tau_i}^2\Omega_{s_i}}{8\eta^2}\sin\left(2\Omega_{s_i}t + 2\varphi_{s_i}\right)\Bigg]\omega_0 + \varphi_{\beta_i},\end{aligned}$$

where φ_{β_i} is chosen to match the initial betatron phase.

The transverse Schottky signal is given by the product of the transverse displacement and the intensity signal $I_i(t)$:

$$\begin{aligned}D^i(t) &= I_i(t)\cdot x_i(t)\\ &= f_0 q\sum_{n=-\infty}^{\infty} e^{jn\omega_0\left[t+\widehat{\tau_i}\sin\left(\Omega_{s_i}t+\varphi_{s_i}\right)\right]}\widehat{x_i}\cos\left(\phi_{\beta_i}(t)\right)\\ &= \frac{f_0 q\widehat{x_i}}{2}\sum_{\pm,n=-\infty}^{\infty} e^{jn\omega_0\left[t+\widehat{\tau_i}\sin\left(\Omega_{s_i}t+\varphi_{s_i}\right)\right]\pm j\phi_{\beta_i}(t)},\end{aligned}$$

where q is the charge of the particle, and $\pm$ denotes that the final result is a sum of two series, each having the opposite sign. Expanding $\phi_{\beta_i}(t)$, one obtains

$$\begin{aligned}D^i(t) = &\frac{f_0 q\widehat{x_i}}{2}\sum_{\pm,n=-\infty}^{\infty} e^{j(n\pm Q_0\pm\Delta Q(Q'',\widehat{\tau_i}))\omega_0 t}\\ &\times e^{j\chi_n^{\pm}(\widehat{\tau_i})\sin\left(\Omega_{s_i}t+\varphi_{s_i}\right)}e^{\pm j\Gamma(Q'',\widehat{\tau_i})\sin\left(2\Omega_{s_i}t+2\varphi_{s_i}\right)}e^{\pm j\varphi_{\beta_i}}.\end{aligned} \tag{2}$$

For readability, we have used the following notation:

$$\begin{cases}\chi_n^{\pm}(\widehat{\tau_i}) = \left(n\widehat{\tau_i} \mp \frac{Q'\widehat{\tau_i}}{\eta}\right)\omega_0,\\ \Delta Q(Q'',\widehat{\tau_i}) = \frac{Q''\widehat{\tau}_i^2\Omega_{s_i}^2}{4\eta^2},\\ \Gamma(Q'',\widehat{\tau_i}) = \frac{Q''\widehat{\tau_i}^2\Omega_{s_i}\omega_0}{8\eta^2}.\end{cases}$$

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By using the Jacobi-Anger expansion [6, Eq. 17.1.7], Eq. (2) can be transformed into

$$\frac{f_0 q \widehat{x_i}}{2} \sum_{\pm,n,p=-\infty}^{\infty} J_p \left[\chi_n^{\pm}\left(\widehat{\tau_i}\right)\right] e^{jt\left[\left(n\pm Q_0 \pm \Delta Q\left(Q'',\widehat{\tau_i}\right)\right)\omega_0 + p\Omega_{s_i}\right]} \times e^{\pm j\Gamma\left(Q'',\widehat{\tau_i}\right)\sin\left(2\Omega_{s_i}t+2\varphi_{s_i}\right)} e^{j\left(\pm\varphi_{\beta_i}+p\varphi_{s_i}\right)},$$

and then into

$$\frac{f_0 q \widehat{x_i}}{2} \sum_{\pm,n,p,k=-\infty}^{\infty} J_p\left(\chi_n^{\pm}\left(\widehat{\tau_i}\right)\right) J_k\left(\pm\Gamma\left(Q'',\widehat{\tau_i}\right)\right) \times e^{jt\left[\left(n\pm Q \pm \Delta Q\left(Q'',\widehat{\tau}\right)\right)\omega_0 + (p+2k)\Omega_{s_i}\right]} e^{j\left(\pm\varphi_{\beta_i}+(p+2k)\varphi_{s_i}\right)},$$

where J_m denotes the Bessel function of the first kind of order m. Rearranging this sum and using the substitution $d = p + 2k$, we obtain:

$$\begin{aligned} D^i(t) &= \frac{f_0 q \widehat{x_i}}{2} \sum_{\pm,n,d,k=-\infty}^{\infty} J_{d-2k}\left(\chi_n^{\pm}\left(\widehat{\tau_i}\right)\right) J_k\left(\pm\Gamma\left(Q'',\widehat{\tau_i}\right)\right) \\ &\quad \times e^{jt\left[\left(n\pm Q \pm \Delta Q\left(Q'',\widehat{\tau_i}\right)\right)\omega_0 + d\Omega_{s_i}\right]} e^{j\left(\pm\varphi_{\beta_i}+d\varphi_{s_i}\right)} \\ &= \frac{f_0 q \widehat{x_i}}{2} \sum_{\pm,n,d=-\infty}^{\infty} J_d\left(\chi_n^{\pm}\left(\widehat{\tau}\right), \pm\Gamma\left(Q'',\widehat{\tau_i}\right)\right) \\ &\quad \times e^{jt\left[\left(n\pm Q \pm \Delta Q\left(Q'',\widehat{\tau_i}\right)\right)\omega_0 + d\Omega_{s_i}\right]} e^{j\left(\pm\varphi_{\beta_i}+d\varphi_{s_i}\right)}, \end{aligned} \tag{3}$$

where $J_d(\cdot,\cdot)$ is a 2D Generalized Bessel Function (GBF), defined as in Ref. [7]. Among the fundamental properties of $J_d(\cdot,\cdot)$ one has $J_d(\cdot,0) = J_d(\cdot)$, hence for $Q'' = 0$ the formula above is in agreement with the previously developed theory of Schottky spectra [1, 2].

For the analysis, one is usually interested only in the spectral components close to a specific harmonic n of the revolution frequency. For the fractional nominal tune $Q_F < 0.5$, these components correspond not to the index n in the Eq. (3), but to indices $n - Q_I$ and $n + Q_I$, where Q_I is the integer part of the nominal betatron tune. Shifting the summing indices so that index n corresponds to the spectral region around $n\omega_0$, one obtains:

$$D^i(t) = \frac{f_0 q \widehat{x_i}}{2} \sum_{\pm,n,d=-\infty}^{\infty} J_d\left(\chi_{n\mp Q_I}^{\pm}\left(\widehat{\tau_i}\right), \pm\Gamma\left(Q'',\widehat{\tau_i}\right)\right) \times e^{jt\left[\left(n\pm Q_F \pm \Delta Q\left(Q'',\widehat{\tau_i}\right)\right)\omega_0 + d\Omega_{s_i}\right]} e^{j\left(\pm\varphi_{\beta_i}+d\varphi_{s_i}\right)}.$$

ANALYSIS OF THE MODIFIED SPECTRUM

The Power Spectral Density (PSD) of the multiparticle Schottky spectrum, around the n^{th} harmonic of the revolution frequency and for either the upper (+) or lower (−) transverse band is given by a sum over N single-particle contributions:

$$P_{\pm}(\omega) = \sum_{i=1}^{N} \frac{f_0^2 q^2 \widehat{x_i}^2 \pi}{2} \sum_{d=-\infty}^{\infty} J_d^2\left(\chi_{n\mp Q_I}^{\pm}\left(\widehat{\tau_i}\right), \pm\Gamma\left(Q'',\widehat{\tau_i}\right)\right) \times \delta\left(\omega - \left(n \pm Q_F \pm \Delta Q\left(Q'',\widehat{\tau_i}\right)\right)\omega_0 - d\Omega_{s_i}\right), \tag{4}$$

where cross-terms between distinct particles cancel due to the random, uniformly distributed betatron phases.

The presence of non-zero second-order chromaticity Q'' affects the Schottky spectrum in two ways. Firstly, it introduces an incoherent tune shift given by $\Delta Q\left(Q'',\widehat{\tau_i}\right)$. In addition, even at the single-particle level, the synchrotron satellites are not symmetric with respect to the central, $d = 0$, satellite. The second argument of the GBF, $\pm\Gamma\left(Q'',\widehat{\tau_i}\right)$, determines the magnitude of this asymmetry. We shall study this effect for typical LHC proton beam parameters, listed in Table 1, assuming a Gaussian bunch profile.

Table 1: Typical LHC and Schottky Monitor Parameters

Revolution frequency	$\omega_0 = 2\pi \times 11\,245.5\,\text{Hz}$
Schottky harmonic	$n = 427\,725$
Synchrotron frequency	$\Omega_s = 2\pi \times 50\,\text{Hz}$
Bunch length (RMS)	$\sigma_\tau = 0.3\,\text{ns}$
Tune	$Q_0 = 64.28$
Chromaticity	$Q' = 10$
Slippage factor	$\eta = 3.182 \times 10^{-4}$

The Schottky spectra for three different values of the second-order chromaticity are presented in Fig. 1. Using the theory presented in the last section, we have calculated the expected Schottky spectra using the matrix formalism introduced in Ref. [8]. As a benchmark, we present also the spectra calculated from the results of PyHEADTAIL macroparticle simulations as discussed in Ref. [9].

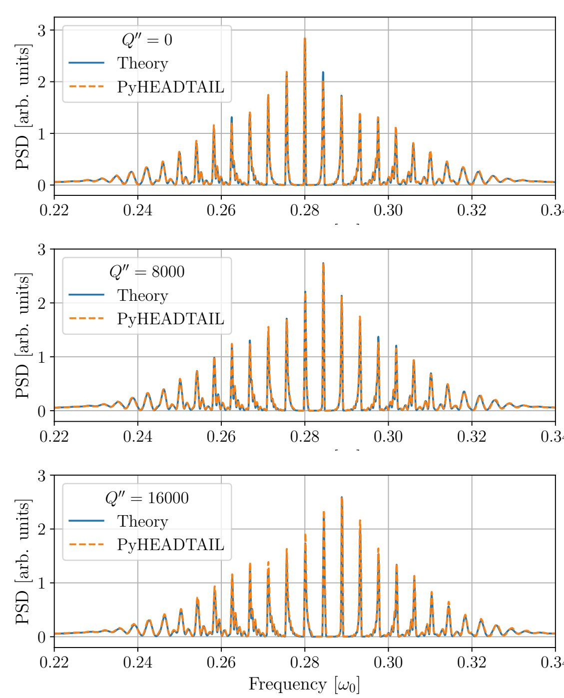

Figure 1: Upper transverse sidebands for different values of the second order chromaticities, using the parameters from Table 1.

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If we were only considering the theory of first-order chromaticity, the shape of the transverse sidebands could have been used to determine the values of the transverse emittance, tune and chromaticity. In such a case, and if $P^{\text{total}}_{\pm}$, $\mu_\pm$ and $\sigma_\pm$ are, respectively, the cumulative power, mean and standard deviation of the upper (+) and lower (−) sidebands treated as distributions over frequency, we would get [1, 10]:

$$\varepsilon = \frac{2P^{\text{total}}_{-}}{N f_0^2 q^2 \beta} = \frac{2P^{\text{total}}_{+}}{N f_0^2 q^2 \beta}, \tag{5}$$

$$Q_F = \frac{\mu_+ - n\omega_0}{\omega_0} = \frac{n\omega_0 - \mu_-}{\omega_0}, \tag{6}$$

$$Q' = \eta \left(n \frac{\sigma_- - \sigma_+}{\sigma_- + \sigma_+} - Q_I \right). \tag{7}$$

We can now question whether these formulae remain correct in the case of non-zero Q''. We shall use the following properties of the GBF:

$$\sum_{d=-\infty}^{\infty} J_d^2 (x, y) = 1, \tag{8}$$

$$\sum_{d=-\infty}^{\infty} d J_d^2 (x, y) = 0, \tag{9}$$

$$\sum_{d=-\infty}^{\infty} d^2 J_d^2 (x, y) = \frac{x^2}{2} + 2y^2. \tag{10}$$

The first two equalities are given explicitly in Ref. [7], while the third one was derived by us following a similar method.

The cumulative power in each sideband can be calculated by direct integration of Eq. (4), with the use of Eq. (8):

$$P^{\text{total}}_{\pm} = \int_{-\infty}^{\infty} \frac{P_\pm(\omega)}{2\pi}\, d\omega = \frac{f_0^2 q^2 \sum_{i=1}^{N} \widehat{x_i}^2}{4} = \frac{N f_0^2 q^2 \beta \varepsilon}{2}.$$

As a consequence, Eq. (5) remains valid.

If we now assume that the momentum deviation and the betatron amplitudes are independent, this allows us to replace $\widehat{x_i}^2$ in Eq. (4) with $\text{Avg}\left[\widehat{x}^2\right]$. The mean frequencies of the modified sidebands are obtained through integration of the product of ω with the normalized transverse sideband, $\widetilde{P}_\pm(\omega) \equiv \dfrac{P_\pm(\omega)}{2\pi P^{\text{total}}_{\pm}}$, using Eqs. (4), (8) and (9):

$$\mu_\pm = \int_{-\infty}^{\infty} \omega \widetilde{P}_\pm(\omega)\, d\omega = \sum_{i=1}^{N} \frac{(n \pm Q_F \pm \Delta Q(Q'', \widehat{\tau}_i))\, \omega_0}{N}.$$

Calculating the tune as in Eq. (6), one obtains

$$\frac{\mu_+ - n\omega_0}{\omega_0} = \sum_{i=1}^{N} \frac{Q_F + \Delta Q(Q'', \widehat{\tau}_i)}{N} = \text{Avg}\,[Q]\,,$$

that is the average value of the betatron tune including the second-order chromaticity shift.

The variances of the modified sidebands are obtained through a direct integration of $(\omega - \mu_\pm)^2\, \widetilde{P_\pm}(\omega)$, using Eqs. (4) and (8) to (10):

$$\sigma_\pm^2 = \int_{-\infty}^{\infty} (\omega - \mu_\pm)^2\, \widetilde{P_\pm}(\omega)\; d\omega = \frac{1}{2}\text{Avg}\left[\Omega_s^2 \left(\chi^{\pm}_{n\mp Q_I}(\widehat{\tau})\right)^2\right]$$
$$+ \frac{Q''^2 \omega_0^2}{32\eta^4}\left(3\,\text{Avg}\left[\Omega_s^4 \widehat{\tau}^4\right] - 2\,\text{Avg}^2\left[\Omega_s^2 \widehat{\tau}^2\right]\right).$$

Compared to the previous results that assumed $Q'' = 0$ [10], the presence of the non-zero Q'' has introduced two additional terms to the second moment of the transverse sideband. Both of these terms are proportional to the square of the second-order chromaticity. Strictly speaking, the presence of these terms would essentially mean that we cannot use Eq. (7) for calculating Q', however, it can be shown that, for the typical LHC beam and Schottky monitor parameters (see Table 1), the effect of these terms is negligible. In Fig. 2 we can see the average tune shift, as well as the error resulting from using Eq. (7), as a function of Q''. The typical range of Q'' in the LHC does not exceed a few thousand units, while the maximal values presented in Fig. 2 correspond to high Q'' proposed in Ref. [4] as an alternative to the Landau damping using octupole magnets.

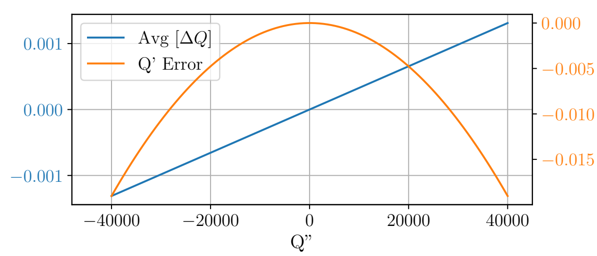

Figure 2: Effect of Q'' on the Schottky-based parameter estimates.

CONCLUSION

In this contribution, we have theoretically studied the impact of the second-order chromaticity on the Schottky spectra of bunched beams. The modified Schottky spectra can still be described with a concise formula, Eq. (4). Analytical results have been benchmarked against macroparticle simulations, that confirm their validity. In the case of the LHC, the impact of Q'' is not significant and does not pose a threat to the previously established diagnostic techniques. This, however, is not necessarily the case for other accelerators, especially if the value of the second-order chromaticity is significantly larger, or the measured Schottky spectra are taken around a lower harmonic of the revolution frequency.

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