

Kaon mixing beyond the standard model with physical masses

P. A. Boyle,^{1,2} F. Erben,³ J. M. Flynn,⁴ N. Garron,^{5,6} J. Kettle,² R. Mukherjee,⁴ and J. T. Tsang^{3,*}
(RBC and UKQCD Collaborations)¹Brookhaven National Laboratory, Upton, NY 11973, USA²Higgs Centre for Theoretical Physics, School of Physics & Astronomy, The University of Edinburgh, Edinburgh EH9 3FD, UK³CERN, Theoretical Physics Department, Geneva, Switzerland⁴School of Physics and Astronomy, University of Southampton, Southampton SO17 1BJ, UK⁵School of Mathematics, Computer Science and Engineering, Liverpool Hope University, Hope Park, Liverpool L16 9JD, UK⁶Theoretical Physics Division, Department of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, UK

We present non-perturbative results for beyond the standard model kaon mixing matrix elements in the isospin symmetric limit ($m_u = m_d$) of QCD, including a complete estimate of all dominant sources of systematic error. Our results are obtained from numerical simulations of lattice QCD with $N_f = 2 + 1$ flavours of dynamical domain wall fermions. For the first time, these quantities are simulated directly at the physical pion mass $m_\pi \sim 139$ MeV for two different lattice spacings. We include data at three lattice spacings in the range $a = 0.11 - 0.07$ fm and with pion masses ranging from the physical value up to 450 MeV. Compared to our earlier work, we have added both direct calculations at physical quark masses and a third lattice spacing making the removal of discretisation effects significantly more precise and eliminating the need for any significant mass extrapolation beyond the range of simulated data. We renormalise the lattice operators non-perturbatively using RI-SMOM off-shell schemes. These schemes eliminate the need to model and subtract non-perturbative pion poles that arises in the RI-MOM scheme and, since the calculations are performed with domain wall fermions, the unphysical mixing between chirality sectors is suppressed. Our results for the bag parameters in the $\overline{\text{MS}}$ scheme at 3 GeV are $B_K \equiv \mathcal{B}_1 = 0.5240(17)(54)$, $\mathcal{B}_2 = 0.4794(25)(35)$, $\mathcal{B}_3 = 0.746(13)(17)$, $\mathcal{B}_4 = 0.897(02)(10)$ and $\mathcal{B}_5 = 0.6882(78)(94)$, where the first error is from lattice uncertainties and the second is the uncertainty due to the perturbative matching to $\overline{\text{MS}}$.

CONTENTS

I. Introduction	1
II. Background	3
III. Simulation details and ensemble properties	4
IV. Non-perturbative renormalisation	6
V. Chiral continuum fits and final results	9
VI. Conclusions	15
Acknowledgments	17
A. CIM and MIM ensembles	18
B. Correlation function fits	20
C. Renormalisation factors	21
D. Relations between basis conventions	24
E. Perturbative scaling	27
F. Further detailed numerical fit results	30
References	32

I. INTRODUCTION

A. Standard model kaon mixing

Neutral kaon mixing has long been an important area of study in standard model (SM) particle physics. Most famously, CP violation was first observed in the Nobel-prize-winning Christenson-Cronin-Fitch-Turlay experiment [1]. Interested readers are referred to Refs. [2–4] and references therein. Kaon mixing is mediated by a flavour changing neutral current interaction, which is absent at tree-level, whereby the neutral kaon oscillates with its antiparticle. The leading-order SM processes are the well-known W exchange box diagrams shown in Figure 1.

Typically, one separates the short and long distance contribution to this process using the operator product expansion (OPE). This isolates the non-perturbative matrix element which can be computed using lattice QCD, $\langle \bar{K}^0 | O_1 | K^0 \rangle$, of the (vector–axial) \times (vector–axial) four-quark left handed operator O_1 from the perturbatively computed Inami-Lim functions [5]. Conventionally, the hadronic contribution to the matrix element is parametrised by the kaon bag parameter and has been the subject of many lattice calculations [6–12]. It is now known at the percent-level and is reported in the FLAG [13] review with consistent results from multiple collaborations.

The SM operator matrix element is a factor in the expression for the dominant short-distance contribution to the indirect CP-violation parameter ε_K . Lattice computations of the matrix element have reached the point

* Corresponding author

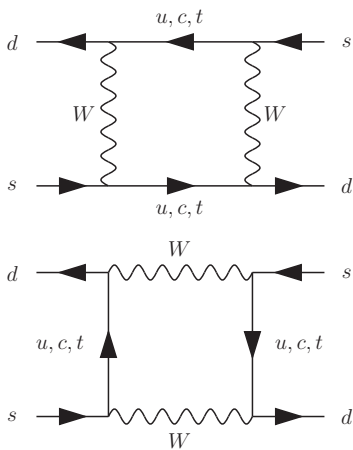


FIG. 1. W exchange box diagrams mediating neutral kaon mixing in the standard model.

where isospin and electromagnetic effects are as large as the total error quoted in the isospin-symmetric pure QCD theory. However, including these effects would need to be accompanied by reduced uncertainty in the Cabibbo-Kobayashi-Maskawa (CKM) quark-mixing matrix element $|V_{cb}|$, also appearing in the SM short-distance contribution to ε_K [13, 14].

For greater precision in the short-distance contribution to be meaningful, we would in addition need to include long-distance effects from bi-local V-A currents where two weak Hamilton insertions are connected by quark loops. Initial progress has been made in references [15–17]. The precision on the computation of ΔM_K , including all long-distance effects, is already at the 10% level [18]. A first full computation of the long-distance part of ε_K at the level of 40% precision has recently been reported in Ref. [19]. For recent broader reviews of the status and prospects of kaon physics we refer the interested reader to Refs. [20–22].

B. Beyond the standard model kaon mixing

Beyond the standard model (BSM), new mediating particles could contribute to neutral kaon mixing. These mediators are not restricted to the V-A Dirac structure of the W boson and new four-quark operators would be allowed in the effective Hamiltonian with Wilson coefficients suppressed by the new mass scale. Eight such four-quark operators are allowed, but for computing their $K^0\bar{K}^0$ matrix elements only five parity-even operators are needed, shown in Eq. (2.3), which are model independent and whose hadronic matrix elements can be calculated using lattice QCD. Any BSM contributions to the V-A structure would be very hard to distinguish from the SM signal in experiments. However the new colour-Dirac structure operators have no SM contribution at this order in the weak forces and thus new physics arising from

these operators would be easier to detect in experiment. In addition, the matrix elements of these BSM operators are enhanced in the chiral limit compared to the SM operator, as can be seen from their chiral expansions. Therefore our results can be combined with the experimental value of ε_K to constrain the parameter space of specific BSM theories and the scale of New Physics, see for example Ref. [23].

Lattice QCD appears as a natural candidate for computing the BSM operator matrix elements. However the mixing pattern of these four-quark operators makes this more challenging than the computation for B_K alone. We refer the interested reader to the pedagogical review [24]. Here we take advantage of the good chiral-flavour properties of the domain wall fermion formulation to constrain the mixing to be the same as in the continuum theory. In practice, this is only true up to lattice artefacts that are exponentially suppressed in the extent of the domain wall fifth dimension and that we must keep under control.

Early studies of BSM kaon mixing [25–27] were performed in the quenched approximation. They were followed by dynamical simulations with N_f quark flavours by several collaborations: RBC-UKQCD ($N_f = 2 + 1$) [28, 29], SWME ($N_f = 2 + 1$) [8, 9, 30], and ETM ($N_f = 2$) [12] and ($N_f = 2 + 1 + 1$) [11]. In contrast to results for the SM operator, there are tensions between the different collaborations’ results for some of the BSM operators, as shown in table I — in which we already anticipate the results of this work — and summarised in the FLAG report [13]. We note that a similar discrepancy is observed in neutral $B_{(s)}$ -mixing [13, 31].

In references [29, 33], it was proposed that the source of these tensions was the choice of the intermediate renormalisation scheme. Specifically, it was proposed that the symmetric momentum subtraction scheme RI-SMOM (which has non-exceptional kinematics) advocated by RBC-UKQCD has several beneficial features compared to the previously used RI-MOM (which has exceptional kinematics). This is likely due to the exceptional (divergent in the massless limit), infrared non-perturbative “pion pole” behaviour in the RI-MOM vertex functions, which must be correctly modelled and subtracted, while the mass is simultaneously taken to zero to establish a mass independent scheme. This behaviour is absent in the RI-SMOM scheme, giving greater theoretical control as it avoids the possibility of imperfect modelling of the non-perturbative pole systematically affecting the result. The results obtained from two RI-SMOM schemes are in agreement with each other and with the perturbatively renormalised results from the SWME collaboration [8, 9, 30], while the calculation with RI-MOM agreed with previous RBC-UKQCD [28] and ETM [11, 12] results which also used RI-MOM.

This paper improves upon our most recent RBC-UKQCD BSM kaon mixing calculation [29, 33] by adding a third lattice spacing and including two data points at the physical light quark mass. We present results in the isospin symmetric limit of pure $N_f = 2 + 1$ QCD with

TABLE I. Results from calculations of BSM bag parameters in $\overline{\text{MS}}(\mu = 3 \text{ GeV})$ from RBC-UKQCD, SWME and ETM show tensions for \mathcal{B}_4 and \mathcal{B}_5 . The results obtained by ETM, which were renormalised via RI-MOM, agree with RBC-UKQCD's results obtained via RI-MOM. The SWME results, obtained via a 1 loop intermediate scheme agree with RBC-UKQCD's results obtained via RI-SMOM, for both γ_μ and \not{q} [32]. This suggests tensions arise from the implementation of intermediate schemes, in particular caused by RI-MOM exhibiting exceptional infrared behaviour which is absent in RI-SMOM. All results are shown in the SUSY basis.

	ETM12 [12]	ETM15 [11]	RBC-UKQCD12 [28]	SWME15 [9]	RBC-UKQCD16 [29]		THIS WORK
N_f	2	2+1+1	2+1	2+1	2+1	2+1	2+1
scheme	RI-MOM	RI-MOM	RI-MOM	1 loop	RI-SMOM	RI-MOM	RI-SMOM
\mathcal{B}_2	0.47(2)	0.46(3)(1)	0.43(5)	0.525(1)(23)	0.488(7)(17)	0.417(6)(2)	0.4794(25)(35)
\mathcal{B}_3	0.78(4)	0.79(5)(1)	0.75(9)	0.773(6)(35)	0.743(14)(65)	0.655(12)(44)	0.746(13)(17)
\mathcal{B}_4	0.76(3)	0.78(4)(3)	0.69(7)	0.981(3)(62)	0.920(12)(16)	0.745(9)(28)	0.897(02)(10)
\mathcal{B}_5	0.58(3)	0.49(4)(1)	0.47(6)	0.751(7)(68)	0.707(8)(44)	0.555(6)(53)	0.6882(78)(94)

sufficient precision that further work on this topic must address the strong and electromagnetic isospin breaking effects. The status of this work has been previously reported in Refs. [34, 35]. Finally, it is worth noting that a similar analysis performed in the pion sector allows to extract the matrix elements which could dominate the short-distance contribution to neutrino-less double beta decays, see for example Refs. [36, 37]. In particular, the renormalisation factors computed here could be employed for such a study.

II. BACKGROUND

A. Effective weak Hamiltonian and BSM basis

By integrating out heavy particles such as the W boson we can separate the long- and short-distance effects into matrix elements, $\langle \bar{K}^0 | O_i | K^0 \rangle$, and Wilson coefficients respectively. Beyond the standard model a generic effective weak $\Delta S = 2$ Hamiltonian can be constructed, in which the standard model operator, O_1 below, and seven additional four-quark operators appear

$$\mathcal{H}^{\Delta S=2} = \sum_{i=1}^5 C_i(\mu) O_i(\mu) + \sum_{i=1}^3 \tilde{C}_i(\mu) \tilde{O}_i(\mu), \quad (2.1)$$

where μ is a renormalisation scale. The Wilson coefficients $C_i(\mu)$ depend on the BSM physics, but the QCD matrix elements $\langle \bar{K}^0 | O_i | K^0 \rangle$ do not. The operators O_i in the so-called ‘‘SUSY basis’’ introduced in Ref. [38] are

$$\begin{aligned} O_1 &= \bar{s}_a \gamma_\mu (1 - \gamma_5) d_a \bar{s}_b \gamma_\mu (1 - \gamma_5) d_b, \\ O_2 &= \bar{s}_a (1 - \gamma_5) d_a \bar{s}_b (1 - \gamma_5) d_b, \\ O_3 &= \bar{s}_a (1 - \gamma_5) d_b \bar{s}_b (1 - \gamma_5) d_a, \\ O_4 &= \bar{s}_a (1 - \gamma_5) d_a \bar{s}_b (1 + \gamma_5) d_b, \\ O_5 &= \bar{s}_a (1 - \gamma_5) d_b \bar{s}_b (1 + \gamma_5) d_a, \end{aligned} \quad (2.2)$$

where a, b are colour indices. The $\tilde{O}_{1,2,3}$ are parity partners of $O_{1,2,3}$ obtained by swapping $1 - \gamma_5 \rightarrow 1 + \gamma_5$, while

$O_{4,5}$ are parity-even. Owing to parity invariance of QCD, only the parity even parts, denoted with a + superscript, contribute in the $\langle \bar{K}^0 | O_i | K^0 \rangle$ matrix elements:

$$\begin{aligned} O_1^+ &= \bar{s}_a \gamma_\mu d_a \bar{s}_b \gamma_\mu d_b + \bar{s}_a \gamma_\mu \gamma_5 d_a \bar{s}_b \gamma_\mu \gamma_5 d_b, \\ O_2^+ &= \bar{s}_a d_a \bar{s}_b d_b + \bar{s}_a \gamma_5 d_a \bar{s}_b \gamma_5 d_b, \\ O_3^+ &= \bar{s}_a d_b \bar{s}_b d_a + \bar{s}_a \gamma_5 d_b \bar{s}_b \gamma_5 d_a, \\ O_4^+ &= \bar{s}_a d_a \bar{s}_b d_b - \bar{s}_a \gamma_5 d_a \bar{s}_b \gamma_5 d_b, \\ O_5^+ &= \bar{s}_a d_b \bar{s}_b d_a - \bar{s}_a \gamma_5 d_b \bar{s}_b \gamma_5 d_a. \end{aligned} \quad (2.3)$$

In practice we find it convenient to work in a different basis, referred to as the ‘‘lattice’’ or ‘‘NPR’’ basis [33]. This comprises colour-unmixed operators obtained by Fierz transforming the equivalent colour-mixed operators, as detailed in section D of the appendix, with $Q_1 = O_1$ and

$$\begin{aligned} Q_2 &= \bar{s}_a \gamma_\mu (1 - \gamma_5) d_a \bar{s}_b \gamma_\mu (1 + \gamma_5) d_b, \\ Q_3 &= \bar{s}_a (1 - \gamma_5) d_a \bar{s}_b (1 + \gamma_5) d_b, \\ Q_4 &= \bar{s}_a (1 - \gamma_5) d_a \bar{s}_b (1 - \gamma_5) d_b, \\ Q_5 &= \frac{1}{4} \bar{s}_a \sigma_{\mu\nu} (1 - \gamma_5) d_a \bar{s}_b \sigma_{\mu\nu} (1 + \gamma_5) d_b. \end{aligned} \quad (2.4)$$

Again we need to consider only the parity-conserving parts which read

$$\begin{aligned} Q_1^+ &= \bar{s}_a \gamma_\mu d_a \bar{s}_b \gamma_\mu d_b + \bar{s}_a \gamma_\mu \gamma_5 d_a \bar{s}_b \gamma_\mu \gamma_5 d_b, \\ Q_2^+ &= \bar{s}_a \gamma_\mu d_a \bar{s}_b \gamma_\mu d_b - \bar{s}_a \gamma_\mu \gamma_5 d_a \bar{s}_b \gamma_\mu \gamma_5 d_b, \\ Q_3^+ &= \bar{s}_a d_a \bar{s}_b d_b - \bar{s}_a \gamma_5 d_a \bar{s}_b \gamma_5 d_b, \\ Q_4^+ &= \bar{s}_a d_a \bar{s}_b d_b + \bar{s}_a \gamma_5 d_a \bar{s}_b \gamma_5 d_b, \\ Q_5^+ &= \sum_{\nu > \mu} \bar{s}_a \gamma_\mu \gamma_\nu d_a \bar{s}_b \gamma_\mu \gamma_\nu d_b. \end{aligned} \quad (2.5)$$

We perform the lattice calculations and renormalisation in this basis and transform to the SUSY basis prior to performing the required chiral and continuum limit extrapolations.¹ Observe that under $\text{SU}(3)_L \times \text{SU}(3)_R$

¹ Unless stated otherwise all results in this paper are quoted in the SUSY basis.

quark flavour symmetry, $O_1^{(+)}$ transforms as $(27, 1)$, $O_{2,3}^{(+)}$ as $(6, \bar{6})$ and $O_{4,5}^{(+)}$ as $(8, 8)$, while $Q_1^{(+)}$ is $(27, 1)$, $Q_{2,3}^{(+)}$ are $(8, 8)$ and $Q_{4,5}^{(+)}$ are $(6, \bar{6})$.

B. Bag parameters

The conventional way to parameterise the hadronic matrix elements of the four-quark operators is through the so-called bag parameters, defined as the ratio of the matrix elements over their vacuum saturation approximation (VSA) value:

$$\mathcal{B}_i(\mu) = \frac{\langle \bar{K}^0 | O_i(\mu) | K^0 \rangle}{\langle \bar{K}^0 | O_i(\mu) | K^0 \rangle_{\text{VSA}}}. \quad (2.6)$$

For the standard model operator, $B_K \equiv \mathcal{B}_1$ is given by,

$$\mathcal{B}_1(\mu) = \frac{\langle \bar{K}^0 | O_1(\mu) | K^0 \rangle}{\frac{8}{3} m_K^2 f_K^2}, \quad (2.7)$$

where m_K is the mass of the kaon and f_K is the kaon decay constant defined by the coupling of the kaon to the renormalised axial-vector current \mathbb{A}_μ^R ,

$$\langle 0 | \mathbb{A}_\mu^R(x) | K(p) \rangle = i f_K p_\mu e^{-ip \cdot x}, \quad (2.8)$$

where p_μ is the 4-momentum of the kaon. The BSM bag parameters are,

$$\mathcal{B}_i(\mu) = \frac{(m_s(\mu) + m_d(\mu))^2}{N_i m_K^4 f_K^2} \langle \bar{K}^0 | O_i(\mu) | K^0 \rangle, \quad i > 1, \quad (2.9)$$

and the factors N_i in the SUSY basis are $N_{i>1}^{\text{SUSY}} = -\frac{5}{3}, \frac{1}{3}, 2, \frac{2}{3}$. The corresponding factors in the parity-even NPR basis are $N_{i>1}^{\text{NPR}} = -\frac{4}{3}, 2, -\frac{5}{3}, -1$. The VSA replaces the four-quark matrix elements with products of two-quark matrix elements. When ‘‘chirally enhanced’’ matrix elements of the pseudoscalar density $\bar{s}\gamma_5 d$ appear, matrix elements of axial vector and tensor currents are dropped. This leads to the appearance of the square of the ratio $(m_s(\mu) + m_d(\mu))/m_K^2 f_K$ in the \mathcal{B}_i for $i > 1$.

C. Ratios R_i of BSM to SM matrix elements

The bag parameters are not the only way to parameterise these four-quark operators; other quantities have been defined, for example in Refs. [26, 27, 39]. Here we choose to consider the simple ratios of the BSM to SM matrix elements

$$R_i(\mu) = \frac{\langle \bar{P} | O_i(\mu) | P \rangle}{\langle \bar{P} | O_1(\mu) | P \rangle}, \quad i = 2, \dots, 5. \quad (2.10)$$

There are some clear advantages: there is no explicit quark-mass dependence in the expression, so that the BSM matrix elements can be recovered from knowledge

of R_i , the SM bag parameter \mathcal{B}_1 and the experimentally measured kaon mass and decay constant (cf. Eq. (2.7)). Additionally, the similarity of the numerator and denominator leads to partial cancellation of systematic and statistical errors. Another approach originally proposed in Ref. [40] is to consider quantities in which the chiral logs cancel out (either at all orders or at NLO). This strategy has been employed for example in Refs. [11, 29, 39].

III. SIMULATION DETAILS AND ENSEMBLE PROPERTIES

A. Simulation parameters

We use RBC-UKQCD’s $N_f = 2 + 1$ gauge ensembles [6, 7, 32, 41–43] generated with the Iwasaki gauge action [44, 45] and a domain wall fermion (DWF) action with either Möbius (M) [46–48] or Shamir (S) [49] kernel. In our work the Möbius and Shamir kernels differ only in their approximation of the sign function, and agree in the limit $L_s \rightarrow \infty$, where L_s is the size of the fifth dimension [7]. We hence assume that the Möbius and Shamir kernels lie on the same scaling trajectory. The set of ensembles contains three lattice spacings in the range $a = 0.11 - 0.07$ fm, labelled C(oarse), M(edium) and F(ine), and includes two physical pion mass ensembles. The remaining ensembles have heavier pion masses ranging up to $m_\pi \approx 450$ MeV, which are used to guide the small chiral extrapolation on the finest ensemble.

On each ensemble, the light valence-quark mass (am_t^{uni}) was chosen identical to the light-quark mass in the sea. The strange valence quark mass (am_s^{val}) was simulated near its physical value (am_s^{phys}) which typically differs from the sea quark mass (am_s^{sea}). The main ensemble properties and the simulated masses are listed in Table II. Large parts of the data were generated using the Grid and Hadrons framework [50–52].

The lattice scale and the physical light and strange quark masses were set using the physical values of m_π , m_K and m_Ω [7] before the ensemble F1M was generated. This fit was repeated including the ensemble F1M in reference [43] where more details about this ensemble are described. We also introduce two new ensembles ‘‘C1M’’ and ‘‘M1M’’, which are the Möbius equivalents of the C1S and M1S. Since they share the same gauge coupling and Möbius scale as the C0M and M0M, respectively, they have the same lattice spacing and physical strange quark mass. More details about these ensembles are summarised in Appendix A.

B. Correlation functions

The quark propagators $S_F(y, x)$, where we write $x = (\mathbf{x}, t_x)$, are obtained by inverting the domain wall Dirac operators on \mathbb{Z}_2 noise wall sources $\eta(x)$ [53–55]. In order to improve the overlap with the ground state, these

TABLE II. Summary of the main parameters of the ensembles used in this work. In the ensemble name the first letter (C, M or F) stand for coarse, medium and fine, respectively. The last letter (M or S) stands for Möbius and Shamir kernels, respectively. The column N_{conf} denotes the number of de-correlated gauge field configurations, N_{src} the number of equivalent measurements per configuration. Ensembles labelled with ‘†’ only enter the analysis in order to constrain the chiral extrapolation of the renormalisation constants described in Section IV.

name	L/a	T/a	$a^{-1}[\text{GeV}]$	$m_\pi[\text{MeV}]$	$N_{\text{conf}} \times N_{\text{src}}$	am_l^{uni}	am_s^{sea}	am_s^{val}	am_s^{phys}
C0M	48	96	1.7295(38)	139	90×48	0.00078	0.0362	0.0358	0.03580(16)
C1S	24	64	1.7848(50)	340	100×32	0.005	0.04	0.03224	0.03224(18)
C2S	24	64	1.7848(50)	431	99×32	0.01	0.04	0.03224	0.03224(18)
M0M	64	128	2.3586(70)	139	82×64	0.000678	0.02661	0.0254	0.02539(17)
M1S	32	64	2.3833(86)	304	83×32	0.004	0.03	0.02477	0.02477(18)
M2S	32	64	2.3833(86)	361	76×32	0.006	0.03	0.02477	0.02477(18)
M3S	32	64	2.3833(86)	411	80×32	0.008	0.03	0.02477	0.02477(18)
F1M	48	96	2.708(10)	232	72×48	0.002144	0.02144	0.02144	0.02217(16)
C1M†	24	64	1.7295(38)	276	-	0.005	0.0362	-	0.03580(16)
M1M†	32	64	2.3586(70)	286	-	0.004	0.02661	-	0.02539(17)

sources are Gaussian-smearred following a Jacobi procedure, i.e.

$$\eta_\omega(\mathbf{x}, t_{\text{src}}) = \sum_{\mathbf{y}} \omega_{\text{src}}^S(\mathbf{x}, \mathbf{y}) \eta(\mathbf{y}, t_y) \delta_{t_y, t_{\text{src}}}, \quad (3.1)$$

where we omit spin-colour indices for simplicity. Further details about the smearing parameters defining ω can be found in reference [43].

At the sink, we consider both the local (L) and smeared (S) case,

$$S_F^{L,S}(x, y) = \sum_{\mathbf{z}} \omega_{\text{snk}}^{L,S}(\mathbf{x}, \mathbf{z}) S(z, y) \delta_{t_x, t_z}, \quad (3.2)$$

where $\omega^L(\mathbf{x}, \mathbf{y}) = \delta_{\mathbf{x}, \mathbf{y}}$. From these propagators, we construct two-point functions which are defined by

$$\begin{aligned} C_{\Gamma_1, \Gamma_2}^{s_1, s_2}(t) &\equiv \sum_{\mathbf{x}} \left\langle \left(O_{\Gamma_2}^{s_2}(\mathbf{x}, t) \right) \left(O_{\Gamma_1}^{s_1}(\mathbf{0}, 0) \right)^\dagger \right\rangle \\ &= \sum_n \frac{(M_{\Gamma_2}^{s_2})_n (M_{\Gamma_1}^{s_1})_n^*}{2E_n} \left(e^{-E_n t} \pm e^{-E_n(T-t)} \right), \end{aligned} \quad (3.3)$$

where O_Γ^s is a bilinear with the flavour content of a kaon,

defined by

$$O_\Gamma^s(\mathbf{x}, t) = \left(\bar{q}_2(\mathbf{x}, t) \sum_{\mathbf{y}} \omega_s(\mathbf{x}, \mathbf{y}) \Gamma q_1(\mathbf{y}, t) \right). \quad (3.4)$$

The Dirac structure is represented by Γ_i . The hadronic matrix elements are denoted by $(M_\Gamma^s)_n = \langle X_n | (O_\Gamma^s)^\dagger | 0 \rangle$ (so that $(\bar{M}_\Gamma^s)_n = (M_\Gamma^s)_n^* = \langle 0 | O_\Gamma^s | X_n \rangle$) with the n^{th} excited meson states $|X_n\rangle$ with corresponding energy E_n . For the bilinear, we only consider pseudoscalars ($\Gamma = \gamma_5 \equiv P$) and the temporal component of the axial current ($\Gamma = \gamma_0 \gamma_5 \equiv A$). The smearing operator ω_s and the superscripts s_1, s_2 label the type of smearing. In our setup, we use local (L) and smeared (S) propagators at source and sink. At the source, all our quark fields are smeared, $s_1 = SS$. We also require the smearing at the sink to be the same for both the strange and the down quark, $s_2 \in \{SS, LL\}$. An exception to this is the ensemble F1M, where we keep both the source and sink local, $s_1 = s_2 = LL$.

For three-point correlation functions, in contrast to the two-point functions, we consider only pseudoscalar operators \mathbb{P} (\mathbb{P}) inducing the quantum numbers of a \bar{K} (K) at the source (sink). These operators are smeared ($s = SS$) on all ensembles apart from the F1M, where they are local ($s = LL$). For notational convenience we drop the smearing indices for the operators. Dropping around-the-world effects, we obtain

$$C_3^i(t, \Delta T) \equiv \left\langle \mathbb{P}(\Delta T) Q_i^+(t) \bar{\mathbb{P}}^\dagger(0) \right\rangle = \sum_{n, n'} \frac{(M_P^s)_n}{4E_n E_{n'}} \langle X_n | Q_i^+(t) | X_{n'} \rangle (M_P^s)_{n'}^* e^{-(\Delta T - t)E_n} e^{-tE_{n'}}. \quad (3.5)$$

$C_3^i(t, \Delta T)$ describes a three-point correlation function with a source at $t = 0$, sink at $t = \Delta T$ and a four-quark

operator insertion Q_i^+ at t .

By placing sources on every second time plane, we com-

pute the above correlation functions for $(T/a)/2$ time translations, where T/a is the integer number of time slices for a given ensemble. We time-translate and average equivalent measurements on a given configuration into a single effective measurement prior to any further analysis. This helps to reduce the variance of the measured correlation functions. The only exception is that we use all available measurements to estimate the correlation matrix, as outlined in Appendix B.

C. Combined fits to two-point and three-point functions

For each of the operators Q_i^+ , we extract the desired masses and matrix elements from a combined fit to several two-point and three-point functions. In particular, we jointly fit C_{PP}^{SL} , C_{PP}^{SS} and C_{PA}^{SL} (C_{PP}^{LL} and C_{PA}^{LL} on F1M) and $C_3^i(t; \Delta T)$ for multiple choices of ΔT , typically parameterising the ground state and the first excited state. From these fits we extract the main quantities of interest: the bare bag parameters $\mathcal{B}_i^{\text{bare}}$; and the ratios of operators R_i^{bare} . They are determined and quoted in the NPR basis (see Tables III and IV) but can subsequently be translated into the SUSY basis. For completeness we also quote the meson masses and bare decay constants at our simulation points for the pion and the kaon in Table XII in Appendix B.

We pursue two independent fit strategies and systematically vary the fit ranges of the two-point and three-point functions (including the choice of which source-sink separations enter the fit) until we see stability in all fit parameters. Figure 2 demonstrates this stability for the example of Q_2^+ on the C0M ensemble for the first strategy. The superimposed dashed lines and magenta bands in the first two panels correspond to the chosen fit if only the two-point functions are fitted. The green bands illustrate our preferred choice of fit. Each set of three data points corresponds to variations in the fit range for the three-point functions of $-1, 0, +1$ compared to the chosen fit. Finally, the different blocks correspond to our chosen fit; the same fit but only to the middle (first, last) half of the source-sink separations; additionally including an excited-to-excited matrix element; the same fit but for a varied choice of t_{min} for the two point functions which enter the fit. We find that the ground state fit results are insensitive to any of these choices. Further details are provided in Appendix B.

D. Valence strange quark correction

As is evident from Table II, the valence strange quark mass on the F1M ensembles is slightly mistuned from the physical strange quark mass value. We account for this effect by repeating the simulation at the physical strange quark mass on an eighth of the full statistics. We then

compute the appropriate correction factors as

$$\alpha_{R_i} \equiv \frac{R_i^{\text{phys}}}{R_i^{\text{uni}}} = \lim_{0 \ll t \ll \Delta T} \frac{R_i^{\text{eff}}(t, \Delta T)|_{m_s^{\text{phys}}}}{R_i^{\text{eff}}(t, \Delta T)|_{m_s^{\text{uni}}}}, \quad (3.6)$$

$$\alpha_{\mathcal{B}_i} \equiv \frac{\mathcal{B}_i^{\text{phys}}}{\mathcal{B}_i^{\text{uni}}} = \lim_{0 \ll t \ll \Delta T} \frac{\mathcal{B}_i^{\text{eff}}(t, \Delta T)|_{m_s^{\text{phys}}}}{\mathcal{B}_i^{\text{eff}}(t, \Delta T)|_{m_s^{\text{uni}}}},$$

where

$$R_i^{\text{eff}}(t, \Delta T) = \frac{C_3^i(t, \Delta T)}{C_3^1(t, \Delta T)}, \quad (3.7)$$

$$\mathcal{B}_i^{\text{eff}}(t, \Delta T) = \frac{C_3^i(t, \Delta T)}{N_i C_2(t) C_2(\Delta T - t)}.$$

We find that the effect of the $\sim 3\%$ mistuning of the strange quark valence mass leads to a $\sim 0.3 - 0.5\%$ correction for the bag parameters and a $\sim 3\%$ correction for the ratio of operators — see Table V. Given that the relative uncertainty of the correction factor is more than an order of magnitude smaller than that of the values it is applied to, we treat this correction factor as uncorrelated. Figure 3 illustrates this correction factor for the case of \mathcal{B}_5 and R_5 .

IV. NON-PERTURBATIVE RENORMALISATION

In order to obtain a well-defined value in the continuum limit, it is necessary to renormalise the matrix elements $\langle \bar{K}^0 | Q_i^+ | K^0 \rangle$. We determine the matrix of renormalisation factors Z_{ij} using the Rome-Southampton method [56] with non exceptional kinematics (RI-SMOM) [57]. At some renormalisation scale μ the renormalised matrix element is then given by

$$\langle \bar{K}^0 | Q_i^+ | K^0 \rangle^{\text{RI}}(\mu, a) = Z_{ij}^{\text{RI}}(\mu, a) \langle \bar{K}^0 | Q_j^+ | K^0 \rangle^{\text{bare}}(a). \quad (4.1)$$

Provided chiral symmetry breaking effects are negligible the matrix $Z_{ij}^{\text{RI}}(\mu, a)$ has a block diagonal structure — which is the case for the set-up at hand [33]. The scale μ should fall within the “Rome-Southampton window”

$$\Lambda_{\text{QCD}}^2 \ll \mu^2 \ll \left(\frac{\pi}{a}\right)^2, \quad (4.2)$$

in which the upper limit is relevant to control discretisation effects and the lower limit ensures accurate perturbative matching to $\overline{\text{MS}}$.

For the technical definitions, numerical values of the renormalisation constants and details of the extrapolation of the renormalisation constants to the massless limit, we refer the interested reader to Sections C1, C2 and C3 in Appendix C. There we also discuss two choices, called γ_μ and \not{q} , of the renormalisation conditions imposed to precisely define the renormalisation scheme.

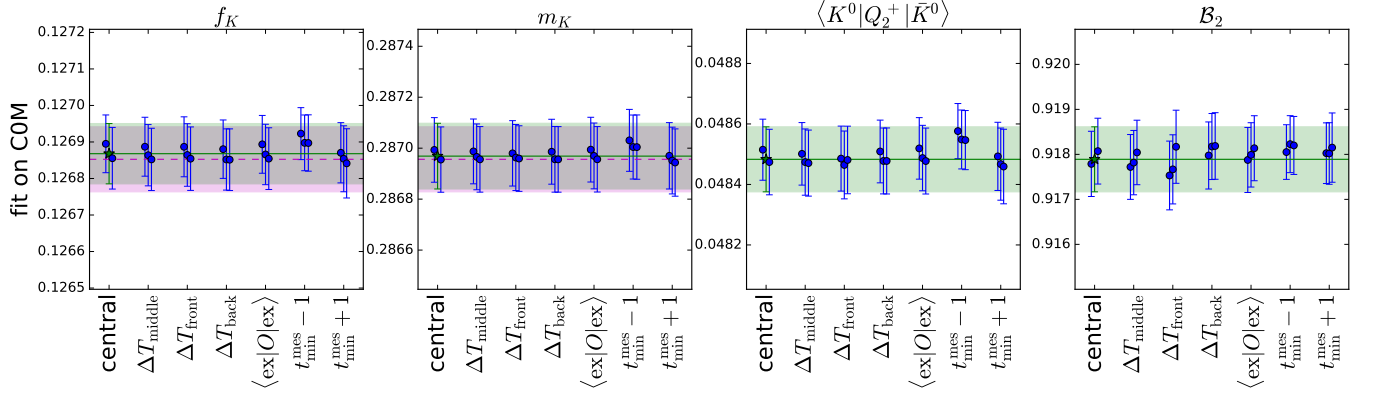


FIG. 2. Stability of correlation function fits, illustrated on the example of the C0M ensemble for Q_2^+ . All numbers are quoted in lattice units and the NPR basis.

TABLE III. Bare bag parameters on all ensembles quoted in the NPR basis.

$N_i \mathcal{B}_i^{\text{bare}} _{\text{NPR}}$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
C0M	1.5565(17)	-1.22385(96)	1.8631(15)	-0.96015(86)	-0.49446(43)
C1S	1.5692(28)	-1.2294(18)	1.8574(27)	-0.9907(17)	-0.51051(86)
C2S	1.5949(24)	-1.2366(16)	1.8586(23)	-1.0118(15)	-0.52241(74)
M0M	1.4890(21)	-1.2052(11)	1.8461(17)	-0.87841(86)	-0.44361(42)
M1S	1.5038(35)	-1.2066(26)	1.8374(41)	-0.9051(23)	-0.4577(11)
M2S	1.5101(24)	-1.2119(20)	1.8409(31)	-0.9133(17)	-0.46219(80)
M3S	1.5223(45)	-1.2129(30)	1.8380(47)	-0.9252(26)	-0.4684(13)
F1M	1.4776(34)	-1.1901(20)	1.8218(30)	-0.8691(14)	-0.43601(70)

TABLE IV. Bare ratio parameters on all ensembles quoted in the NPR basis.

$R_i^{\text{bare}} _{\text{NPR}}$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
C0M	-23.511(30)	35.788(48)	-18.439(28)	-9.494(15)
C1S	-20.587(55)	31.098(85)	-16.592(59)	-8.549(31)
C2S	-18.368(35)	27.604(56)	-15.001(43)	-7.747(23)
M0M	-28.144(39)	43.113(60)	-20.516(29)	-10.361(15)
M1S	-25.125(72)	38.25(11)	-18.848(72)	-9.531(36)
M2S	-23.505(51)	35.704(81)	-17.742(67)	-8.979(33)
M3S	-22.107(56)	33.501(88)	-16.866(65)	-8.538(33)
F1M	-28.621(80)	43.82(13)	-20.929(70)	-10.500(36)

A. Analysis of non-perturbative renormalisation data

In practice, our data covers many values in the range $2 \text{ GeV} \lesssim \mu \lesssim 3 \text{ GeV}$ (see Fig 17 in Appendix C2 for details). On the computationally most expensive ensembles C0M and M0M we simulate at $am_s^{\text{sea}}/2$ in the valence sector. On all other ensembles we have additional simulation points at am_l^{sea} and $2am_l^{\text{sea}}$. We use this data ensemble-by-ensemble to extrapolate the renormalisation constants to the massless limit in which the renormalisation constants are formally defined (see Fig 4). The

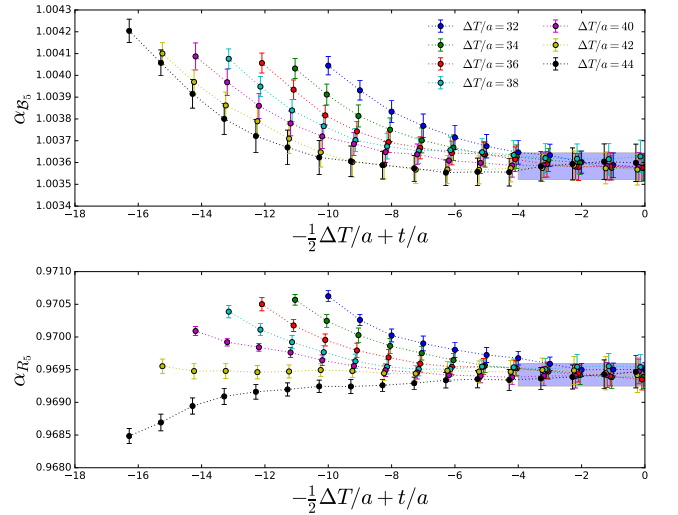


FIG. 3. Strange valence mistuning quark correction on the F1M ensemble for the example of B_5 (top) and R_5 (bottom).

extrapolation of the data on the physical pion mass ensembles is performed by applying the slope of the C1M (M1M) ensemble to the C0M (M0M) data.

In a subsequent step we extrapolate the results for each action to zero sea light quark mass as is illustrated in Fig-

TABLE V. Correction factors to be applied to the bare values of R_i and \mathcal{B}_i on the F1M ensemble in the NPR basis in order to correct the observables to the physical strange quark mass.

α_X	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$X = \mathcal{B}_i$	1.004983(99)	1.004231(66)	1.003036(65)	1.003193(71)	1.003583(61)
$X = R_i$		0.97005(18)	0.96890(18)	0.96904(17)	0.96942(17)

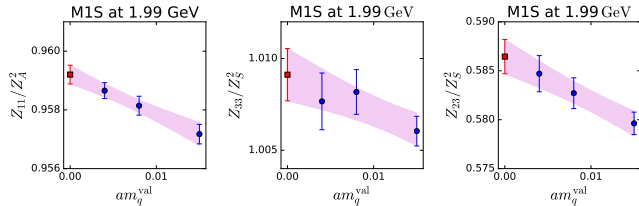


FIG. 4. Extrapolation of the renormalisation constants to massless valence-quark limit for the example of the (11), (33) and (23) elements of the M1S ensemble close to 2 GeV. Results are presented in the RI-SMOM (γ_μ, γ_μ) scheme in the NPR basis.

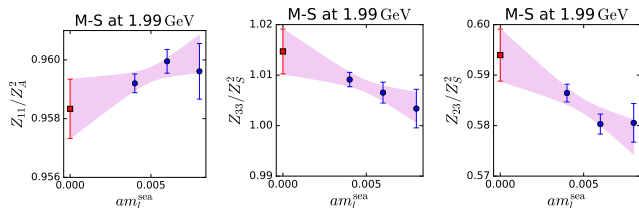


FIG. 5. Extrapolation of the renormalisation constants to the zero light-sea quark mass limit for the example of the (11), (33) and (23) elements of the M-S ensembles close to 2 GeV. Results are presented in the RI-SMOM (γ_μ, γ_μ) scheme in the NPR basis.

ure 5 for the M-S ensembles at a momentum point close to 2 GeV. Since for the case of the F1M ensemble we only have a single sea quark mass data point, in practice we first interpolate the results on all ensembles to a common renormalisation scale and then perform the sea-light quark mass to zero limit for each choice of distinct lattice spacing. For the F-M ensemble this is done by applying each of the four slopes (obtained from C-S, C-M, M-S and M-M) in turn and assigning a systematic uncertainty of half the spread of these results.

We list the chirally extrapolated renormalisation constants for each lattice spacing at $\mu = 2$ GeV in Table VI (results for other values of μ can be found in Tables XVIII - XIX in Appendix C3). Since these numbers contain information from multiple ensembles and the NPR calculations are based on a subset of the configurations, we propagate these small uncertainties in an uncorrelated fashion. To this end we add statistical and systematic uncertainties in quadrature and generate bootstrap samples for each of the Z_{ij} by drawing from a Gaussian distribution with the appropriate mean and width.

Finally, we use these values to renormalise the quanti-

ties of interest. In particular we find

$$\begin{aligned}
 R_i(\mu)^{\text{ren}} &= \frac{Z_{ij}}{Z_{11}} R_j(\mu, a)|_{\text{bare}}, \\
 \mathcal{B}_1(\mu)^{\text{ren}} &= \frac{Z_{11}}{Z_A^2} \mathcal{B}_1(\mu, a)|_{\text{bare}}, \\
 N_i \mathcal{B}_i(\mu)^{\text{ren}} &= \frac{Z_{ij}}{Z_P^2} N_j \mathcal{B}_j(\mu, a)|_{\text{bare}} \quad i = 2, \dots, 5.
 \end{aligned} \tag{4.3}$$

Here the N_i are the appropriate normalisation factors defined above and in practice we make use of the relation $Z_S \approx Z_P$ due to chiral symmetry.

B. Step-scaling

When performing the renormalisation we have the freedom to choose the renormalisation scale μ within the Rome-Southampton window of our ensembles, which includes $2 \text{ GeV} \lesssim \mu \lesssim 3 \text{ GeV}$. We note that higher scales are more susceptible to discretisation effects, whilst lower scales face larger errors when matching perturbatively to e.g. $\overline{\text{MS}}$.

We can scale the value of an operator renormalised at one scale to another with the use of a scale evolution matrix, $\sigma(\mu_2, \mu_1)$, in a procedure called step-scaling [58–60]. We define the continuum scale evolution matrix for the renormalisation of the four-quark operators as

$$\sigma(\mu_2, \mu_1) = \lim_{a^2 \rightarrow 0} Z(\mu_2, a) Z^{-1}(\mu_1, a), \tag{4.4}$$

where $Z(\mu, a)$ is the 5×5 block-diagonal matrix described above. Therefore it is possible to scale our operators, once renormalised and extrapolated to the continuum limit, from μ_1 to μ_2 . By renormalising at $\mu = 2$ GeV, where lattice artefacts are less significant, but step-scaling our results in RI-SMOM to $\mu = 3$ GeV before perturbatively matching to $\overline{\text{MS}}$ we also avoid the higher errors associated with the truncation of the perturbative series at lower scales. Since we have mapped out the region $2 \text{ GeV} \lesssim \mu \lesssim 3 \text{ GeV}$, we can further split (4.4) into multiple smaller steps $\Delta = (\mu_2 - \mu_1)/N$, i.e. we can compute the product

$$\prod_{k=0}^{N-1} \sigma(\mu_1 + k\Delta + \Delta, \mu_1 + k\Delta). \tag{4.5}$$

Alongside directly renormalising at 3 GeV, we can also renormalise the result at 2 GeV and step scale to 3 GeV

TABLE VI. Elements of $Z_{ij}/Z_{A/S}^2$ extrapolated to the massless limit. All results are provided in RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ at $\mu = 2.0$ GeV in the SUSY basis. The first parenthesis is the statistical error and the second is the systematic error. More detail is provided in App. C3.

a^{-1} [GeV]	1.7295(38)	1.7848(50)	2.3586(70)	2.3833(86)	2.708(10)
Z_{11}/Z_A^2	0.93258(26)(0)	0.93444(77)(1)	0.96021(51)(2)	0.9579(12)(0)	0.97120(69)(47)
Z_{22}/Z_S^2	1.0703(26)(0)	1.0788(37)(2)	1.1185(18)(0)	1.1237(64)(1)	1.1405(27)(18)
Z_{23}/Z_S^2	-0.06092(49)(8)	-0.0603(10)(0)	-0.04045(55)(5)	-0.0387(10)(0)	-0.03054(62)(27)
Z_{32}/Z_S^2	0.0343(31)(3)	0.0403(74)(1)	0.1135(47)(1)	0.120(11)(0)	0.1522(57)(6)
Z_{33}/Z_S^2	1.6094(53)(13)	1.643(13)(0)	1.9107(85)(9)	1.948(21)(0)	2.064(10)(6)
Z_{44}/Z_S^2	1.0113(21)(2)	1.0161(35)(2)	1.0134(11)(0)	1.0160(51)(0)	1.0122(21)(16)
Z_{45}/Z_S^2	-0.07270(48)(4)	-0.07384(98)(6)	-0.06217(46)(7)	-0.06268(98)(0)	-0.05774(49)(44)
Z_{54}/Z_S^2	-0.23507(99)(38)	-0.2417(31)(1)	-0.2851(20)(2)	-0.2940(46)(2)	-0.3119(22)(15)
Z_{55}/Z_S^2	1.4762(38)(9)	1.4994(96)(6)	1.6718(64)(5)	1.699(15)(0)	1.7670(74)(55)

in one step, or in multiple steps as described above. This allows us to probe the effect the scale of the renormalisation has. Details of the computation and numerical

values for the step-scaling matrices are provided in Section C4. The numerical values for the step-scaling matrices in the RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ -scheme and in the SUSY basis are given by

$$\sigma(3 \text{ GeV}, 2 \text{ GeV}) = \begin{bmatrix} 0.98021(53) & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.9194(22) & -0.0630(16) & 0.0 & 0.0 \\ 0.0 & -0.00284(35) & 0.6846(19) & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.9988(24) & 0.0784(25) \\ 0.0 & 0.0 & 0.0 & 0.00838(59) & 0.7542(24) \end{bmatrix}, \quad (4.6)$$

$$\sigma(3 \text{ GeV} \xleftarrow{\Delta=0.5 \text{ GeV}} 2 \text{ GeV}) = \begin{bmatrix} 0.98030(35) & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.9199(22) & -0.0634(15) & 0.0 & 0.0 \\ 0.0 & -0.00260(31) & 0.6863(17) & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.9990(25) & 0.0778(24) \\ 0.0 & 0.0 & 0.0 & 0.00860(44) & 0.7552(23) \end{bmatrix}. \quad (4.7)$$

V. CHIRAL CONTINUUM FITS AND FINAL RESULTS

A. Fit ansatz

To recover continuum results at physical quark masses we perform a simultaneous chiral-continuum limit fit.

Our fit ansatz is based on NLO SU(2) chiral perturbation theory (χ_{PT}), covered in more detail in Ref. [29], and includes a chiral logarithm term. Furthermore our fit function is linear in a^2 and m_π^2 and the mistuning of the strange quark mass $\delta_{m_s}^{\text{sea}}$. It is given by

$$Y_i(a^2, m_\pi^2, m_s^{\text{sea}}) = Y_i^{\text{phys}} \left(1 + \alpha_i (a\Lambda)^2 + \beta_i \frac{m_\pi^2 - (m_\pi^{\text{phys}})^2}{(m_\pi^{\text{phys}})^2} + \gamma_i \delta_{m_s}^{\text{sea}} + L_i^Y(m_\pi) - L_i^Y(m_\pi^{\text{phys}}) \right). \quad (5.1)$$

where $Y \in \{\mathcal{B}, R\}$ is the quantity of interest. Λ is a typical QCD-scale and we take the isospin-averaged pion mass to be $m_\pi^{\text{phys}} = (2m_\pi^\pm + m_\pi^0)/3 \approx 138$ MeV [61].

Y_i^{phys} , α , β and γ are fit parameters, $\delta_{m_s}^{\text{sea}} = (m_s^{\text{sea}} - m_s^{\text{phys}})/m_s^{\text{phys}}$ parameterises the mistuning of the sea strange quark mass and the chiral logarithms are given

by $L_i^Y(m_\pi) = C_i^Y m_\pi^2 \log(m_\pi^2/\Lambda^2)/(16\pi^2 f_\pi^2)$. The coefficients C_i are known constants with numerical values $C_i^B = -0.5$ for $i = 1, 2, 3$ and $C_i^B = 0.5$ for $i = 4, 5$ and we take $f_\pi = 130.41(23)$ MeV [61]. For the ratios R_2 and R_3 the chiral logarithms vanish ($C_2^R = C_3^R = 0$) and finally $C_4^R = C_5^R = 1$.

As stated in section III the lattice spacings were determined in Ref. [7] (and updates thereof in Refs [32, 43]) from some of the same ensembles included in this work, hence a correlation between the data exists. However we perform an uncorrelated fit to decouple this work from the previous work. We propagate the error on the lattice spacing by generating bootstrap samples following a Gaussian distribution with width equal to the error on the lattice spacing. Given that the errors on the lattice spacings are of order 0.5% and all extrapolated quantities are dimensionless, we believe that neglecting these correlations has a negligible effect.

Our central fit results in the two intermediate schemes are obtained from a chiral-continuum limit fit at $\mu = 2$ GeV performed in the SUSY basis. These results are then step-scaled to 3 GeV (e.g. using the matrix provided in Eq. (4.6)) and perturbatively matched to $\overline{\text{MS}}$. In the following sections we will present the results of the chiral-continuum limit fits and assemble the full uncertainty budget relating to the lattice computation in the intermediate RI-SMOM schemes at 3 GeV. Only subsequently do we match these results perturbatively to $\overline{\text{MS}}$. This allows us to cleanly separate the uncertainties due to the perturbative matching to $\overline{\text{MS}}$ from those arising in the lattice calculation.

B. Results of the chiral-continuum limit fits

In this section we present the chiral-continuum limit fits in the RI-SMOM (γ_μ, γ_μ) scheme at $\mu = 2$ GeV and in the SUSY basis. We show these fits for the ratios R_i in Figures 6 and 7 and for the bag parameters \mathcal{B}_i in Figures 8 and 9. Since we find that the C2S ensemble — which is at the heaviest pion mass and the coarsest lattice spacing — is not always well described by the fit ansatz, we remove it from our central fits. The data is well described by the fit function (5.1) in all cases with acceptable p -values ($> 5\%$) for all fits presented.

C. Error budget

In the following we quantify all relevant sources of uncertainties. We consider variations to the data and the fit ansatz; variations of the renormalisation procedure; and uncertainties stemming from the perturbative matching. We quantify the uncertainties for the variations by considering

$$\delta_i^{\text{var}}(\mu) = \frac{|Y_i^{\text{central}} - Y_i^{\text{variation}}|}{\frac{1}{2}(Y_i^{\text{central}} + Y_i^{\text{variation}})}, \quad (5.2)$$

where $Y \in \{\mathcal{B}, R\}$.

1. Chiral extrapolation

The two precise data points at physical pion masses make the mass extrapolation element of the fit very benign. We quantify the associated uncertainty by varying the pion mass cut to the data, removing terms from our fit form and by repeating the fits using the alternative correlator fit results (see App. B3). For each of these variations we compute the associated δ (see Eq. (5.2)) which measures the shift in central value and list the corresponding values in Table VII. For all ratios and bag parameters this error is well below 1% and typically sub-statistical. For each observable we assign the maximum of those values as the systematic uncertainty associated to the chiral extrapolation listed as “chiral” in Table IX.

2. Discretisation effects

The good chiral symmetry of domain wall fermions constrains $O(a)$ and $O(a^3)$ terms to be small. The $O(a^2)$ effects are controlled and removed by our three lattice spacings present in the fit. Power counting suggests that $O(a^4)$ effects for hadronic physics scales with a 1.73 GeV coarsest inverse lattice spacing will remain small on all data points. However, the same is not necessarily true for hard, off-shell vertex functions where the momenta are chosen as the best compromise for a Rome-Southampton window. The leading unremoved discretisation effects are thus likely to come from the non-perturbative renormalisation, and may be probed by comparing different ways of renormalising our data. Our central chiral-continuum limit fit is based on data renormalised at $\mu = 2$ GeV which is then step-scaled to 3 GeV by the step-scaling function $\sigma(3 \text{ GeV}, 2 \text{ GeV})$ presented in Eq. (4.6) for the bag parameters. We compare the results obtained this way to using the alternative prescription to obtain the scaling function (cf. Eq. (4.5)) with $N = 2, 3$ and to performing the continuum limit to data renormalised directly at $\mu = 3$ GeV. We compute and report the associated values for δ_i in Table VIII.

For our main analysis we extract the bare matrix elements and renormalisation factors in the NPR basis, transform them to the SUSY basis and then perform the various analysis steps. Performing the entire analysis in the NPR basis and converting the final values to the SUSY basis causes a reshuffling of discretisation effects. The corresponding δ_i are presented in the column labelled SUSY \leftarrow NPR in Table VIII.

We take the maximum of these variations as estimate for the systematic uncertainties due to higher order discretisation effects, labelled “discr” in Table IX.

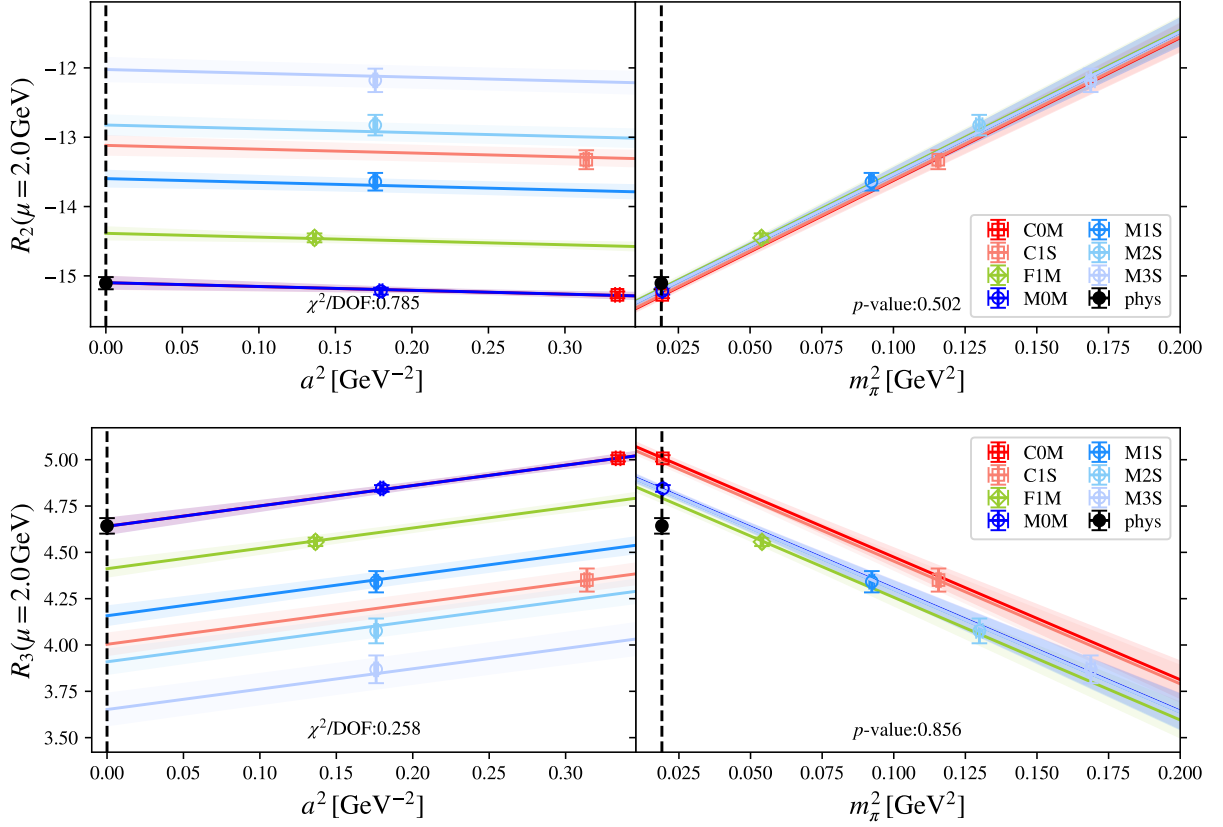


FIG. 6. Chiral-continuum limit fit to BSM ratio parameters R_2 (top) and R_3 (bottom) in the SUSY basis, renormalised in the RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ scheme.

TABLE VII. Chiral-continuum limit fit systematics depending on choice of ansatz at $\mu = 2.0$ GeV in RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ in the SUSY basis. The first column shows the central value with statistical uncertainty, whilst the remaining columns quantify variations arising from different choices in the data that enters the fit as well as the model to which the chiral dependences is fitted. The last column illustrates the effect of using the alternative choice of correlation function fits underlying the analysis.

	Central fit	no δ_{m_s}	no chiral logs	$m_\pi < 440$ MeV	$m_\pi < 370$ MeV	$m_\pi < 350$ MeV	alternate fit
R_2	-15.106(87)	0.22%	-	0.46%	0.05%	0.08%	0.45%
R_3	4.643(41)	0.42%	-	0.28%	0.10%	0.11%	0.21%
R_4	29.22(19)	0.51%	0.59%	0.58%	0.04%	0.06%	0.49%
R_5	7.965(62)	0.10%	0.47%	0.50%	0.00%	0.05%	0.13%
\mathcal{B}_1	0.5268(13)	0.10%	0.21%	0.49%	0.02%	0.06%	0.40%
\mathcal{B}_2	0.5596(23)	0.05%	0.17%	0.06%	0.01%	0.08%	0.02%
\mathcal{B}_3	0.856(11)	0.06%	0.28%	0.02%	0.07%	0.06%	0.00%
\mathcal{B}_4	0.9097(35)	0.03%	0.17%	0.08%	0.01%	0.02%	0.06%
\mathcal{B}_5	0.750(19)	0.02%	0.24%	0.18%	0.02%	0.03%	0.13%

3. Residual chiral symmetry breaking

Domain wall fermions provide a good approximation to chiral symmetry, however a small degree of residual chiral symmetry breaking is present in the data. Chiral symmetry restricts the allowed mixing pattern to be block-diagonal. For our central analysis we impose this, by setting the chirally forbidden elements of Z_{ij} to zero

which we refer to as “masking”. To test the effect residual chiral symmetry breaking has on our results, we repeat the entire analysis without masking. We find that the deviations are well below the percent level, indicating that our approximation to chiral symmetry is very well controlled. We report the associated systematic uncertainties in Table IX as “rcsb”.

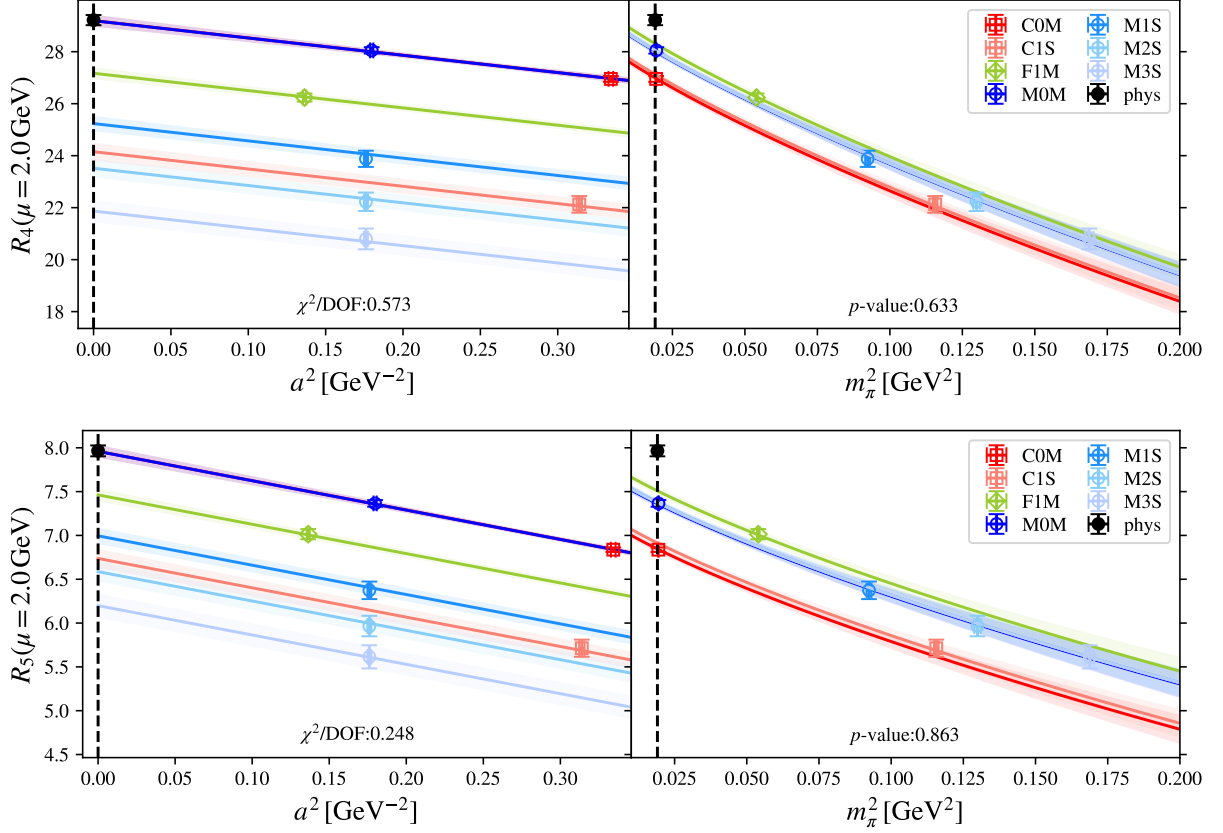


FIG. 7. Chiral-continuum limit fit to BSM ratio parameters R_4 (top) and R_5 (bottom) in the SUSY basis, renormalised in the RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ scheme.

TABLE VIII. Bag and ratio parameters at $\mu = 3$ GeV in RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ in the SUSY basis. Central value comes from performing the chiral-continuum limit fit at $\mu = 2$ GeV and non-perturbative scaling the result to $\mu = 3$ GeV using $\sigma(3 \text{ GeV}, 2 \text{ GeV})$. We also list variations where the continuum step-scaling is obtained in steps, or the data is renormalised directly at 3 GeV. The central value uses Z -factors with chirally vanishing elements removed (masked) from $(PA)^T$ before the inversion $Z = F((PA)^T)^{-1}$. We list the percent shift in the result in foregoing this step (no mask). We also compare with performing the entire analysis in the NPR basis and then rotating to the SUSY basis.

	$\sigma(3 \text{ GeV}, 2 \text{ GeV})$	$\sigma(3 \text{ GeV} \xleftarrow{\Delta=0.5} 2 \text{ GeV})$	$\sigma(3 \text{ GeV} \xleftarrow{\Delta=0.33} 2 \text{ GeV})$	NPR at 3 GeV	rcsb	SUSY \leftarrow NPR
R_2	-18.37(10)	0.12%	0.13%	0.17%	0.11%	0.02%
R_3	5.485(36)	0.18%	0.50%	0.37%	0.14%	0.15%
R_4	38.60(27)	0.02%	0.02%	0.48%	0.09%	0.01%
R_5	10.932(47)	0.11%	0.01%	0.97%	0.03%	1.22%
B_1	0.5164(14)	0.00%	0.01%	0.01%	0.04%	0.01%
B_2	0.5150(12)	0.04%	0.20%	0.45%	0.03%	0.05%
B_3	0.7624(52)	0.32%	0.24%	1.51%	0.06%	0.15%
B_4	0.9107(19)	0.02%	0.16%	0.02%	0.01%	0.02%
B_5	0.7792(79)	0.11%	0.24%	0.38%	0.00%	0.26%

4. Finite Volume Effects

Finite volume effects (FVEs) could be neglected in our previous studies, but at this level of precision we need to revisit this assumption. We estimate these effects using chiral perturbation theory and note that the finite volume

corrections appear with the same pre-factors C_i^Y as the chiral logarithms [40]. The FVEs are given by [62]

$$C_i^Y \frac{m_\pi^2}{(4\pi f_\pi)^2} \frac{12\sqrt{2\pi} \exp(-m_\pi L)}{(m_\pi L)^{3/2}}. \quad (5.3)$$

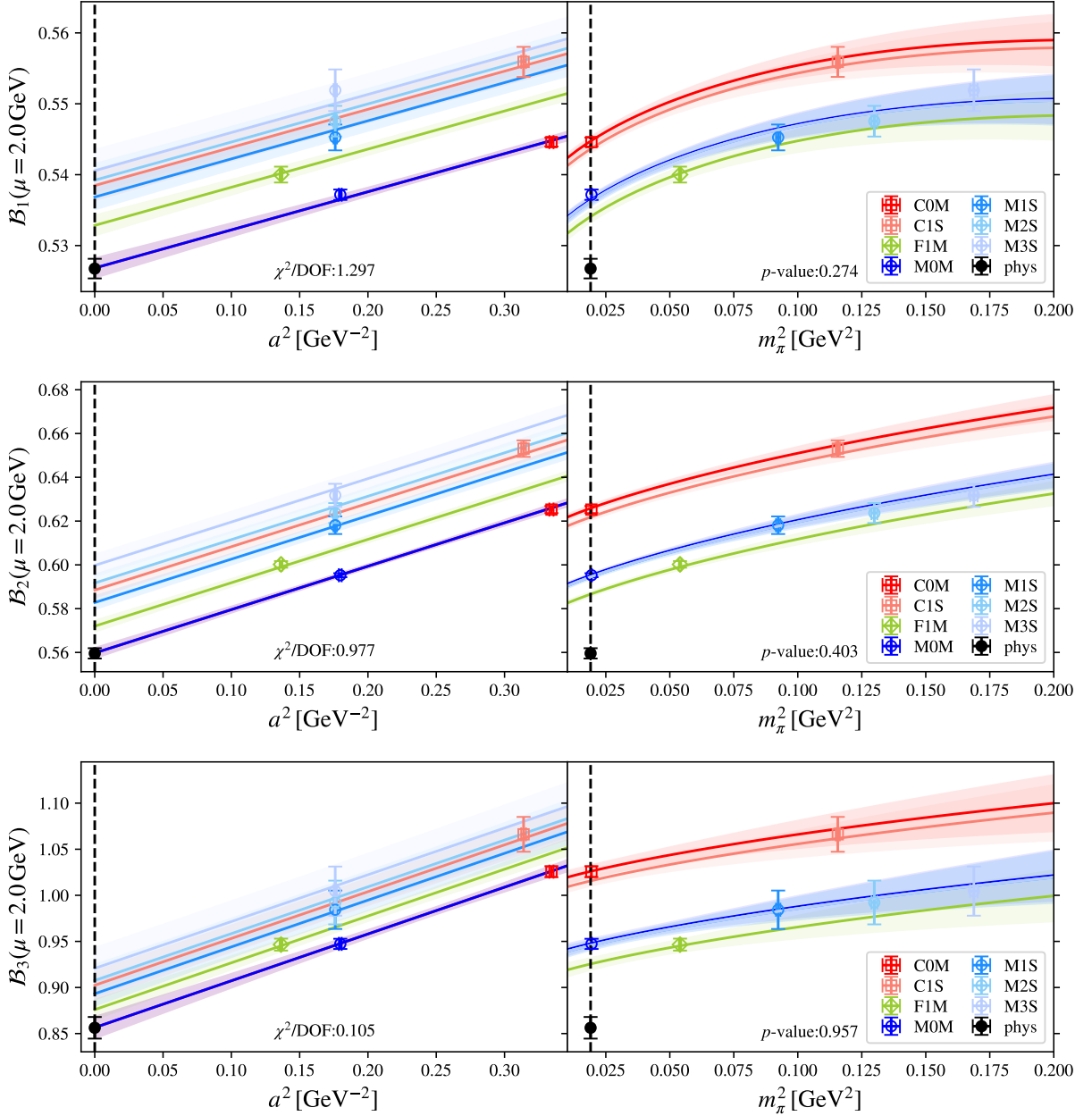


FIG. 8. Chiral-continuum limit fit to the standard model bag parameter \mathcal{B}_1 (top) and BSM bag parameters \mathcal{B}_2 (middle) and \mathcal{B}_3 (bottom) in the SUSY basis, renormalised in the $\text{RI-SMOM}^{(\gamma_\mu, \gamma_\mu)}$ scheme.

The leading order FVEs cancel in the ratios R_2 and R_3 . Numerically evaluating Eq. (5.3) for our ensembles, we find that the largest effect is observed on the M1S ensemble, where the estimate of finite size effects is 1.1 per-mille for the bag parameters and 2.1 per-mille for R_4 and R_5 . Noting that this is a sub-leading effect (c.f. Tab. IX) and that the FVEs on the ensembles which are most constraining for the fit (COM, M0M, F1M) are more than a factor three smaller than this, we conclude that FVEs remain negligible at our current level of precision.

5. Perturbative matching

The dominant source of uncertainty arises in the conversion of our results to $\overline{\text{MS}}$ where the matching is done in perturbation theory to one-loop. The truncation of the perturbative series leads to an uncertainty. We have defined two intermediate RI-SMOM schemes, differentiated by their projectors and use these to estimate the size of this error. We expect results in $\overline{\text{MS}}$ to be independent of the intermediate renormalisation scheme. We take our central value as the average between the results

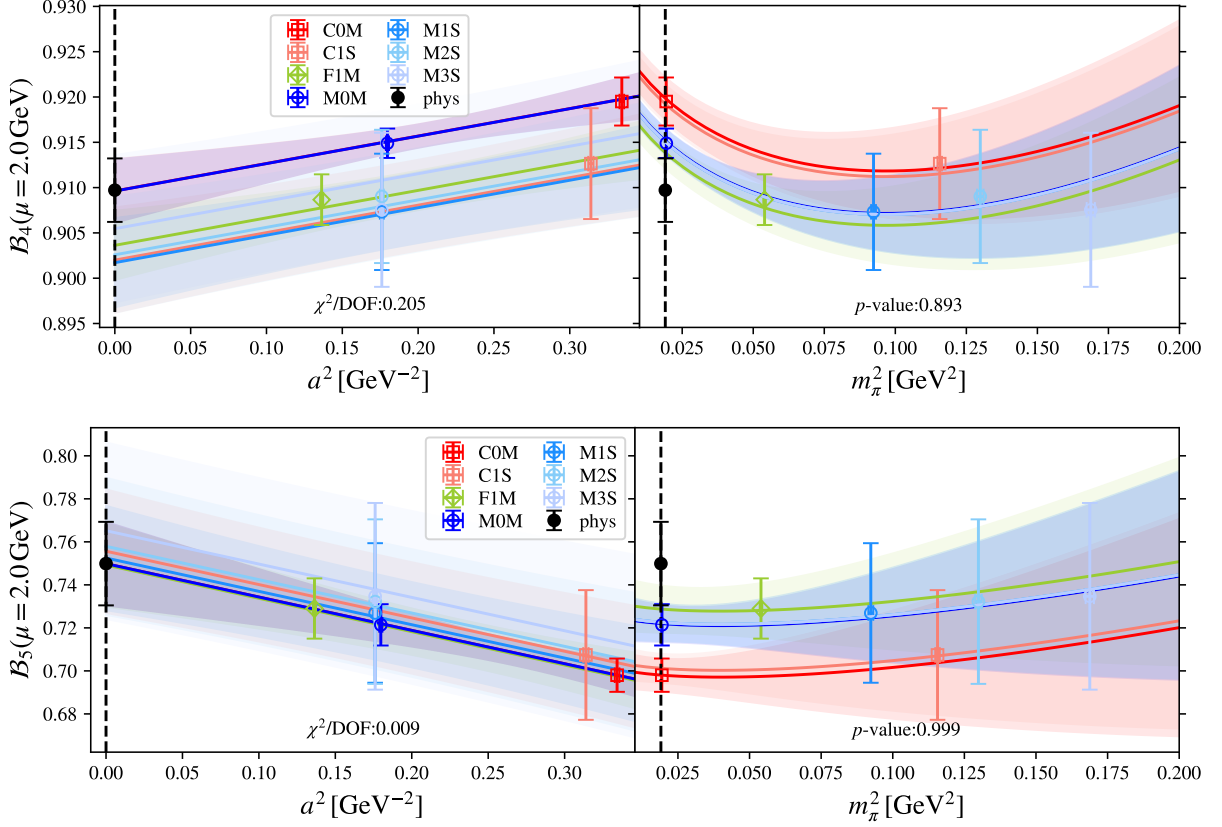


FIG. 9. Chiral-continuum limit fit to BSM bag parameters \mathcal{B}_4 and \mathcal{B}_5 in the SUSY basis, renormalised in the RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ scheme.

obtained from the two intermediate schemes and associate a truncation uncertainty of half their difference. For definiteness we assign the relative error from the (γ_μ, γ_μ) scheme to quantify the combined lattice uncertainty in our final results. The estimate of the perturbative truncation uncertainty is quoted as “PT” in the last column of Table IX.

D. Self consistency check

Having determined the R_i and the \mathcal{B}_i parameters we can perform a self-consistency check. Recalling the definitions in Eqs. (2.7), (2.9) and (2.10) we consider

$$\frac{N_i \mathcal{B}_i}{R_i} = \frac{8}{3} \frac{(m_s(\mu) + m_d(\mu))^2}{m_K^2} \mathcal{B}_1, \quad i = 2, \dots, 5. \quad (5.4)$$

The right hand side is independent of i and hence the ratios from each operator should give compatible results.

The black data points in Figure 10 display this comparison for the results at $\mu = 3$ GeV in RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ (top) and $\overline{\text{MS}}$ (bottom). The R_i and \mathcal{B}_i have notably different — and sometimes steep — approaches to the continuum limit. The good agreement between the different results gives us confidence that uncertainties in general and discretisation effects in particular have been well estimated.

We compare our $\overline{\text{MS}}$ results to the value obtained by evaluating the right hand side using external inputs. We use the isospin-symmetrised kaon mass $m_K = (m_{K^0} + m_{K^\pm})/2 = 496.144(9)$ MeV [61]. We take FLAG [13] values for the $N_f = 2 + 1 + 1$ [11, 63–68] and $N_f = 2 + 1$ [7, 9, 69–77] isospin-symmetrised light quark mass and strange quark mass in $\overline{\text{MS}}$ at $\mu = 2$ GeV, together with \hat{B}_K , the renormalisation group invariant (RGI) value for \mathcal{B}_1 ,

$$\begin{aligned} N_f = 2+1+1 : \quad & m_{ud} = 3.410(43) \text{ MeV} \quad m_s = 93.44(68) \text{ MeV} \quad \hat{B}_K = 0.717(24) , \\ N_f = 2+1 : \quad & m_{ud} = 3.364(41) \text{ MeV} \quad m_s = 92.03(88) \text{ MeV} \quad \hat{B}_K = 0.7625(97) . \end{aligned} \quad (5.5)$$

TABLE IX. Central values and combined systematic errors for ratio and bag parameters in the SUSY basis at $\mu = 3$ GeV in the two RI-SMOM schemes — (γ_μ, γ_μ) and (\not{q}, \not{q}) — as well as in $\overline{\text{MS}}$. For the RI-SMOM schemes we list the errors arising from statistics, chiral extrapolation, residual chiral symmetry breaking and discretisation effects and combine them into total uncertainties. For the $\overline{\text{MS}}$ values we list the separate conversions from (γ_μ, γ_μ) and (\not{q}, \not{q}) . The central values are defined as the average of those two numbers and the perturbative truncation error as half their difference. The lattice error is taken from the (γ_μ, γ_μ) scheme (see Table XXI for scheme-wise error budget).

scheme		R_2	R_3	R_4	R_5	\mathcal{B}_1	\mathcal{B}_2	\mathcal{B}_3	\mathcal{B}_4	\mathcal{B}_5
RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$	central	-18.37	5.485	38.60	10.93	0.5164	0.5150	0.762	0.9107	0.7792
	stat	0.59%	0.66%	0.72%	0.44%	0.28%	0.24%	0.69%	0.22%	1.02%
	chiral	0.22%	0.42%	0.59%	0.47%	0.21%	0.17%	0.28%	0.17%	0.24%
	rscsb	0.11%	0.14%	0.09%	0.03%	0.04%	0.03%	0.06%	0.01%	0.00%
	discr	0.17%	0.50%	0.48%	1.22%	0.01%	0.45%	1.51%	0.16%	0.38%
	total	0.66%	0.94%	1.05%	1.38%	0.35%	0.54%	1.68%	0.31%	1.11%
RI-SMOM $^{(\not{q}, \not{q})}$	central	-19.53	5.818	40.99	10.49	0.5342	0.5155	0.765	0.9137	0.7078
	stat	0.68%	0.90%	0.81%	0.83%	0.29%	0.42%	1.20%	0.36%	2.19%
	chiral	0.47%	0.77%	1.21%	1.23%	0.24%	0.27%	0.43%	0.28%	0.53%
	rscsb	0.28%	0.20%	0.23%	0.13%	0.08%	0.19%	0.28%	0.03%	0.01%
	discr	0.35%	0.66%	0.20%	2.25%	0.11%	0.63%	1.88%	0.19%	0.10%
	total	0.94%	1.37%	1.48%	2.69%	0.40%	0.83%	2.29%	0.49%	2.25%
$\overline{\text{MS}}$	(γ_μ, γ_μ)	-18.73	5.781	41.45	10.80	0.5185	0.4759	0.728	0.8862	0.6977
	(\not{q}, \not{q})	-19.07	6.059	42.43	10.49	0.5295	0.4829	0.764	0.9070	0.6788
	central	-18.90	5.920	41.94	10.64	0.5240	0.4794	0.746	0.8966	0.6882
	lattice	0.66%	0.96%	1.06%	1.40%	0.34%	0.52%	1.75%	0.32%	1.14%
	PT	0.91%	2.35%	1.17%	1.47%	1.05%	0.74%	2.40%	1.16%	1.38%
	total	1.12%	2.54%	1.57%	2.03%	1.10%	0.90%	2.97%	1.20%	1.79%

and run them to $\mu = 3$ GeV, allowing us to construct the right hand side of Eq. (5.4) (the conversion of the four-quark operators at a given scale to RGI operators is shown in Eq. (E21)). This is shown as the grey band in the lower plot in Figure 10.

Furthermore, we can use the constant value in both the RI-SMOM and $\overline{\text{MS}}$ schemes, combining it with our value for \mathcal{B}_1 , to predict the sum of the quark masses (see also the discussion in Ref. [27]). From our result for $i = 2$ we find:

$$\begin{aligned} (m_s + m_{ud})^{\text{RI}}(3 \text{ GeV}) &= 91.38(41) \text{ MeV}, \\ (m_s + m_{ud})^{\overline{\text{MS}}}(3 \text{ GeV}) &= 86.29(79) \text{ MeV}. \end{aligned} \quad (5.6)$$

We compare this to the corresponding FLAG values

$$\begin{aligned} N_f = 2+1+1 : (m_s + m_{ud})^{\overline{\text{MS}}}(3 \text{ GeV}) &= 88.18(63) \text{ MeV}, \\ N_f = 2+1 : (m_s + m_{ud})^{\overline{\text{MS}}}(3 \text{ GeV}) &= 86.34(79) \text{ MeV}. \end{aligned} \quad (5.7)$$

E. Comparison to our previous work

In Figure 11 we compare our results in the RI-SMOM scheme at $\mu = 3$ GeV to our previous determination [29]. The addition of two physical pion mass ensembles and a third lattice spacing helps to constrain the chiral and

continuum limit extrapolations respectively, yielding a significantly reduced uncertainty. Given the significantly different data set, we find good agreement between our previous result and this work.

F. Correlation between the different fit parameters

We provide the statistical correlation matrix between the \mathcal{B}_i , R_i and $\langle K | O_i^+ | \bar{K} \rangle$ as an ancillary hdf5 file. For completeness we include results for RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}(2 \text{ GeV})$, RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}(3 \text{ GeV})$, RI-SMOM $^{(\not{q}, \not{q})}(2 \text{ GeV})$ and RI-SMOM $^{(\not{q}, \not{q})}(3 \text{ GeV})$ as well as $\overline{\text{MS}} \leftarrow \text{RI-SMOM}^{(\gamma_\mu, \gamma_\mu)}(3 \text{ GeV})$ and $\overline{\text{MS}} \leftarrow \text{RI-SMOM}^{(\not{q}, \not{q})}(3 \text{ GeV})$.

VI. CONCLUSIONS

In this paper we have performed the first calculation of the non-standard model neutral kaon mixing matrix elements with data directly simulated with physical quark masses. Using an increased level of volume averaging, with many \mathbb{Z}_2 wall sources on each configuration, we have been able to obtain much reduced statistical errors compared to our previous publications, even with physical quark masses.

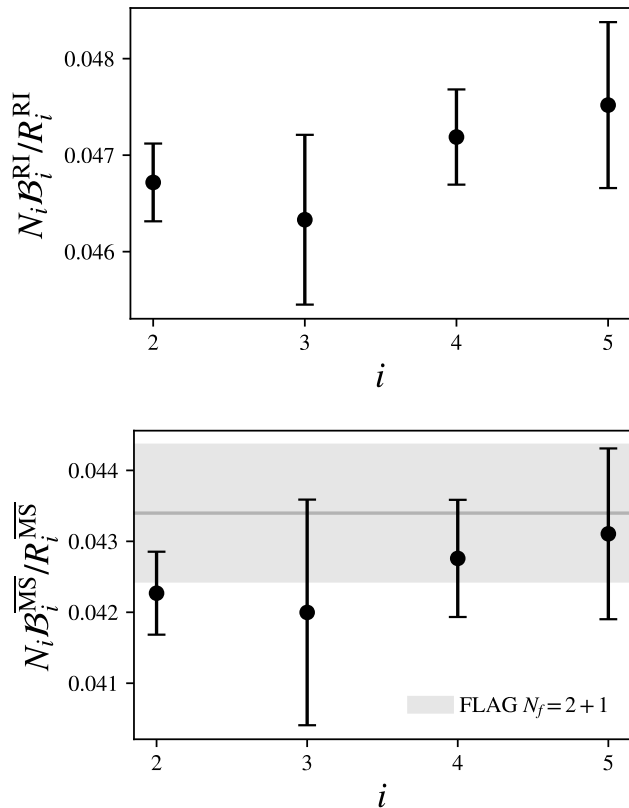


FIG. 10. Self-consistency check by forming the ratio Eq. (5.4) at $\mu = 3 \text{ GeV}$. The data points are from our calculations in the RI-SMOM $(\gamma_{\mu}, \gamma_{\mu})$ scheme (top) and in the $\overline{\text{MS}}$ scheme (bottom). For the $\overline{\text{MS}}$ plot we show the expected value using FLAG inputs as the grey horizontal band.

All sources of systematic uncertainties have been estimated. For each of the bag parameters and ratios of matrix elements a simultaneous fit has been performed to the mass and lattice spacing dependence. Direct simulation at physical quark masses leaves the mass dependence of this extrapolation a negligible systematic. With the inclusion of a third lattice spacing we can test the validity of a^2 scaling and find that in the range covered by our data it works well. We assess discretisation uncertainties by considering different renormalisation points and/or different ways of obtaining the non-perturbative scaling matrix. The self-consistency check of comparing ratios $N_i B_i / R_i$ increases our confidence that the discretisation effects have been well estimated, since those ratios approach the continuum limit in notably different ways.

The dominant systematic error comes from perturbative matching from the RI-SMOM scheme to $\overline{\text{MS}}$ at the 3 GeV renormalisation point. This key error was assessed by comparing two different intermediate RI-SMOM schemes after continuum extrapolation. If the matching were non-perturbative the intermediate scheme would be irrelevant, but with truncated, perturbative matching the results differ due to the truncation error.

The differences are of the order 1-3%.

Our final results in the $\overline{\text{MS}}$ scheme at 3 GeV where the first error is the RI-SMOM error and the second the uncertainty from the matching to $\overline{\text{MS}}$ are:

$$\begin{aligned}
 \mathcal{B}_1^{\overline{\text{MS}}} &= 0.5240(17)(54) \\
 \mathcal{B}_2^{\overline{\text{MS}}} &= 0.4794(25)(35) \\
 \mathcal{B}_3^{\overline{\text{MS}}} &= 0.746(13)(17) \\
 \mathcal{B}_4^{\overline{\text{MS}}} &= 0.897(02)(10) \\
 \mathcal{B}_5^{\overline{\text{MS}}} &= 0.6882(78)(94) \\
 R_2^{\overline{\text{MS}}} &= -18.90(12)(17) \\
 R_3^{\overline{\text{MS}}} &= 5.92(05)(13) \\
 R_4^{\overline{\text{MS}}} &= 41.94(44)(46) \\
 R_5^{\overline{\text{MS}}} &= 10.64(14)(15)
 \end{aligned} \tag{6.1}$$

We have also presented the value we obtained for the SM bag parameter B_K . We achieve good consistency with our collaborations most up-to-date result, $B_K^{\overline{\text{MS}} \leftarrow \text{SMOM}^{(g, g)}}(3 \text{ GeV}) = 0.530(11)$ [7]. A different fitting procedure in which the physical point data is overweighted is employed in Ref. [7], and while it also includes the coarse and medium ensembles included in this work, it included a different third lattice spacing with a heavier pion. Further, it combined additional coarser ensembles with a different gauge action in a global fit and reweighting factors to adjust the sea strange mass to the physical values. In this work, we instead leave the sea strange dependence as a fit parameter. Given the differences in the underlying correlator data and the various fitting procedures, the consistency of the results is reassuring. A comparison of our results for the RGI value \hat{B}_K with the literature is shown in Figure 15 where good agreement with the literature is seen.

The prospects for further improvements of this calculation are as follows: We believe that the RI-SMOM scheme results are sufficiently precise that there is no purpose in further reduction in the error within the isospin symmetric pure QCD approximation. Instead, strong isospin breaking and QED must be addressed if greater accuracy is required. For our final results a significant source of error stems from the perturbative matching to $\overline{\text{MS}}$. This could be addressed by raising the matching scale at which we convert operators. The convergence is logarithmic in the energy scale and this will not lead to a rapid improvement in the calculation. It would be better to accompany this with a two-loop calculation of the scheme change factors presented in Ref. [33]. The quadratic suppression in α_s would be more beneficial than an increase in the renormalisation scale towards the b -threshold.

Consequently we believe our results are a robust determination that in the short term may only be further improved with an additional loop in the perturbative

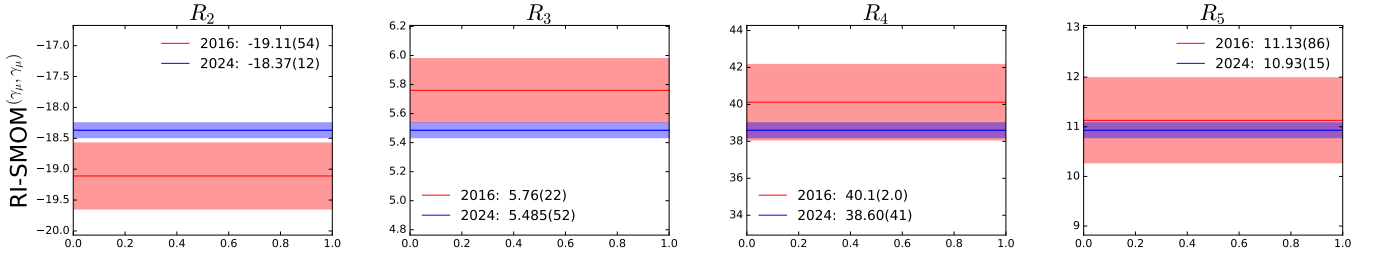


FIG. 11. Comparison of the R_i in RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ at $\mu = 3$ GeV to our previous work [29].

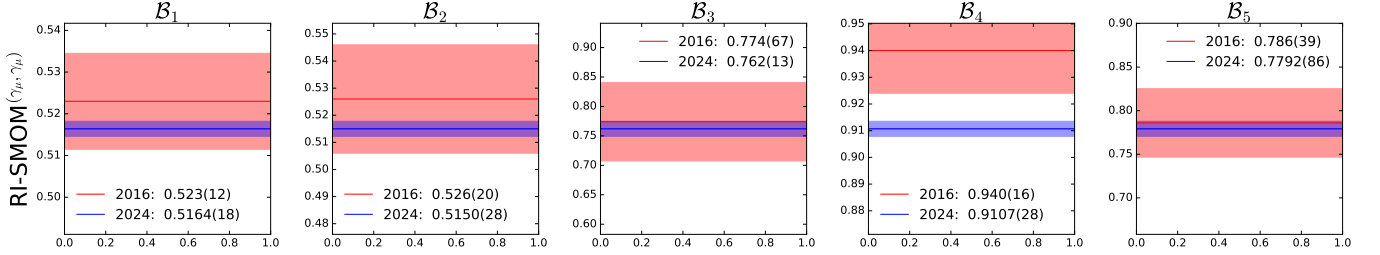


FIG. 12. Comparison of the B_i in RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ at $\mu = 3$ GeV to our previous work [29].

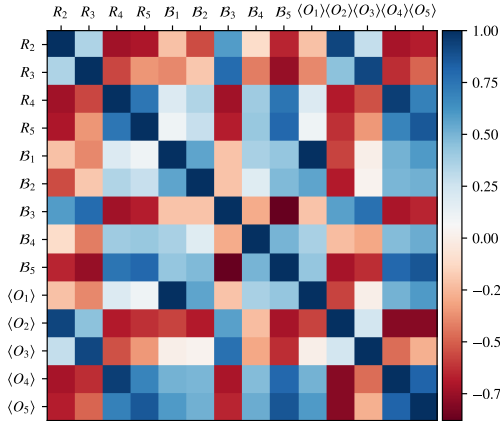


FIG. 13. Heat-map of the statistical correlation matrix between the B_i , R_i and the $\langle K | O_i^+ | \bar{K} \rangle$ in \overline{MS} at 3 GeV.

matching, or by the inclusion of isospin breaking effects.

This work is an important step towards the determination of the same observables in the $B_{(s)}^0 - \bar{B}_{(s)}^0$ systems which is also pursued by our collaboration [78, 79].

ACKNOWLEDGMENTS

We thank our colleagues in RBC and UKQCD for their contributions and discussion, in particular Nikolai Husung for helpful discussions on perturbative renormalisation, Fabian Joswig for important contributions to transitioning Hadrons' NPR code to GPU architectures, Antonin Portelli for developing and maintaining Hadrons, and Andreas Jüttner for contributions at an

early stage of this analysis.

This work used the DiRAC Blue Gene Q Shared Petaflop system and the DiRAC Extreme Scaling service at the University of Edinburgh, operated by the Edinburgh Parallel Computing Centre on behalf of the STFC DiRAC HPC Facility (www.dirac.ac.uk). This equipment was funded by BIS National E-infrastructure capital grant ST/K000411/1, STFC capital grant ST/H008845/1, and STFC DiRAC Operations grants ST/K005804/1 and ST/K005790/1. DiRAC is part of the National e-Infrastructure.

This work has received funding from an STFC funded studentship. F.E. has received funding from the European Union's Horizon Europe research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 101106913. R.M. is funded by a University of Southampton Presidential Scholarship. P.B. has been supported by US DOE Contract DESC0012704(BNL). N.G. acknowledges support from STFC grant ST/X000699/1.

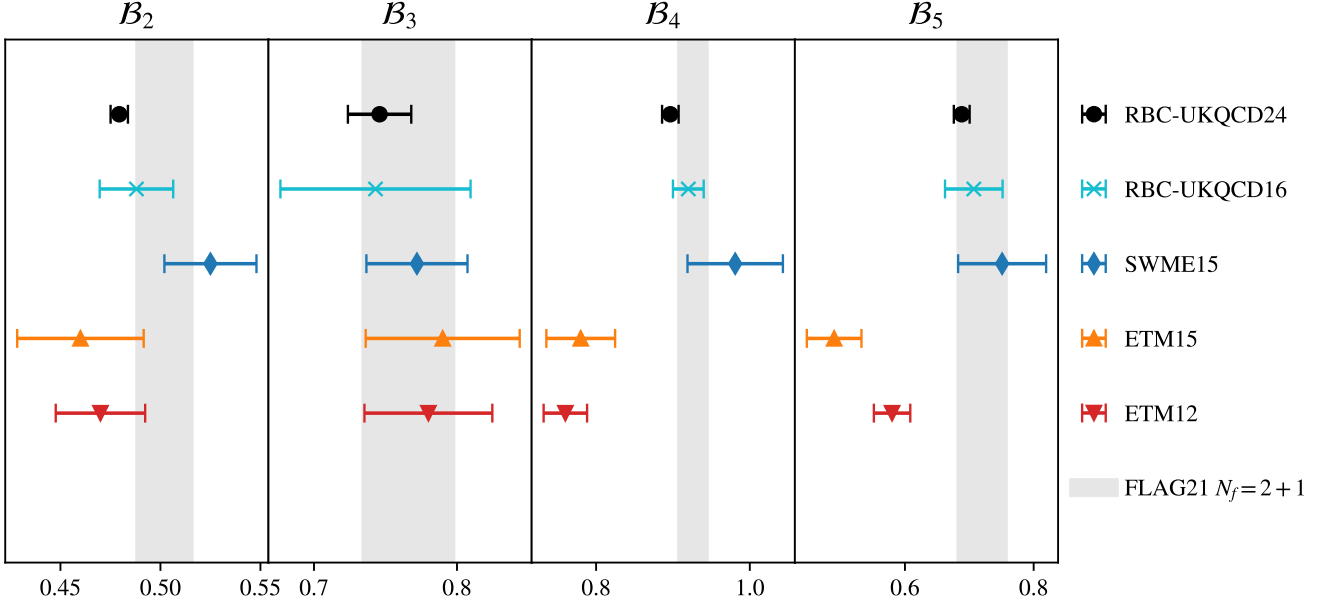


FIG. 14. Comparison of our results for the BSM bag parameters in $\overline{\text{MS}}$ at 3 GeV with previous results (RBC-UKQCD16 [29], SWME15 [9], ETM12 [12], ETM15 [11]).

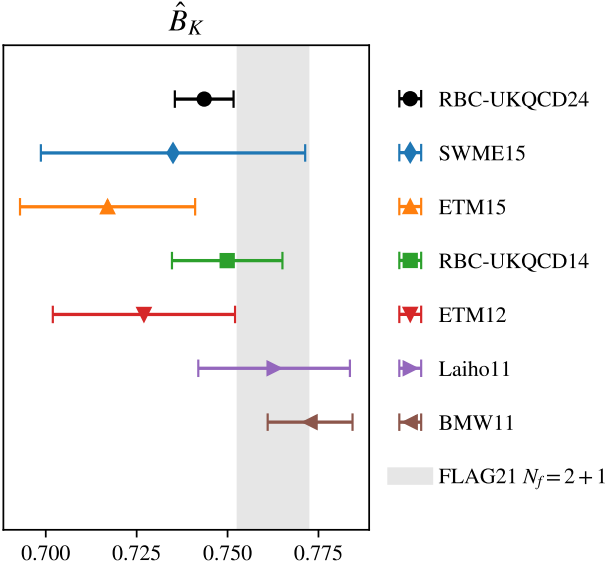


FIG. 15. Comparison of our results for the RGI SM bag parameter \hat{B}_K with previous results.

Appendix A: C1M and M1M ensembles

RBC-UKQCD 2+1f configurations initially used the standard domain wall fermion action with $L_s = 16$ [80], but when substantially lighter quark masses at physical values were introduced [7], it was desirable to reduce the level of residual chiral symmetry breaking and the Möbius domain wall fermion framework [48] was adopted

with fixed independent Möbius parameter $b = 1.5$ and $c = 0.5$. We take $H_W = \gamma_5 D_W$ as the hermitian Wilson operator at negative domain wall mass, and with this choice of b and c this yields a kernel H_M entering the overlap sign function that is identical to the kernel H_T of standard domain wall fermions in the large L_s limit, with:

$$H_M = \frac{2H_W}{2 + D_W} = 2H_T, \quad (\text{A1})$$

entering the four dimensional effective overlap action via an approximate sign function which (if exact) removes the rescaling factor:

$$D_{ov} = \frac{1+m}{2} + \frac{1-m}{2} \gamma_5 \epsilon(H_M). \quad (\text{A2})$$

Here the sign function approximation $\epsilon(H_M)$ is the tanh approximation to the exact sign function,

$$\begin{aligned} \epsilon(H_M) &= \frac{(1+H_M)^{L_s} - (1-H_M)^{L_s}}{(1+H_M)^{L_s} + (1-H_M)^{L_s}} \\ &= \tanh(L_s \tanh^{-1} H_M). \end{aligned} \quad (\text{A3})$$

The four dimensional effective action of two actions coincide exactly when the extent of the fifth dimension is infinite, and at finite L_s differ only by terms that are exponentially suppressed in the fifth dimension extent, a scale which can be measured by the residual mass measured as the defect of the Möbius chiral Ward identity [7]. Indeed on these two new ensembles, we obtain residual masses which are very close to those of the corresponding physical pion mass ensembles [7] as can be seen from Table XI.

Ensemble	C1M	M1M
Volume	$24^3 \times 64$	$32^3 \times 64$
β	2.13	2.25
b	1.5	1.5
c	0.5	0.5
L_s	24	12
m_l	0.005	0.004
m_s	0.0362	0.02661
m_h	{0.02, 0.2, 0.6}	{0.02, 0.2, 0.6}
N_{Traj}	1990	1950
Fermion steps	12	10
Sexton-Weingarten ratio	8	8
Integrator	Force Gradient	Force Gradient

TABLE X. Simulation and HMC parameters for the C1M and M1M ensembles.

Previously RBC-UKQCD have directly combined calculations on ensembles with the Shamir formulation of domain wall fermions at heavier quark masses with physical quark mass ensembles using the about Möbius action. This cost saving measure is reasonable at the percent scale level accuracy sought at the time, with residual chiral symmetry breaking effects being of lower magnitude and around 3×10^{-3} in the worst-case coarsest ensemble. In these initial ensembles, since the input quark mass that corresponds to the physical strange quark mass is determined by simulation, the strange quark mass could only be approximately tuned in advance of their generation and so differed from the physical value. However, longer term there is great simplicity in generating new ensembles so that comparisons can be made and extrapolations performed with fewer variables changing (volume, residual chiral symmetry breaking, strange quark masses), while increased computing power made this now relatively easily affordable.

To an extent chiral effective approaches and other methods can both estimate finite volume effects to be small and allow to correct for them. We therefore continue to use smaller volumes $24^3 \times 64$ for C1M and $32^3 \times 64$ for M1M, but took the opportunity to eliminate two of these three confounding effects that arise when using the older ensembles. The first is to keep the same residual chiral symmetry breaking approximation to the overlap operator, and the second is to retune the dynamical strange quark mass to its physical value on each ensemble, matching that used in the physical point ensembles C0M and M0M.

The new ensembles were generated using the Grid library [50], with simulation parameters and intermediate two flavor determinant Hasenbusch masses [81] that are given in Table X. The exact one flavor algorithm [82] was used for the strange quark using the implementation in Grid [83].

For M1M, a two level nested integrator was used in the HMC with the Force Gradient integrator [84, 85],

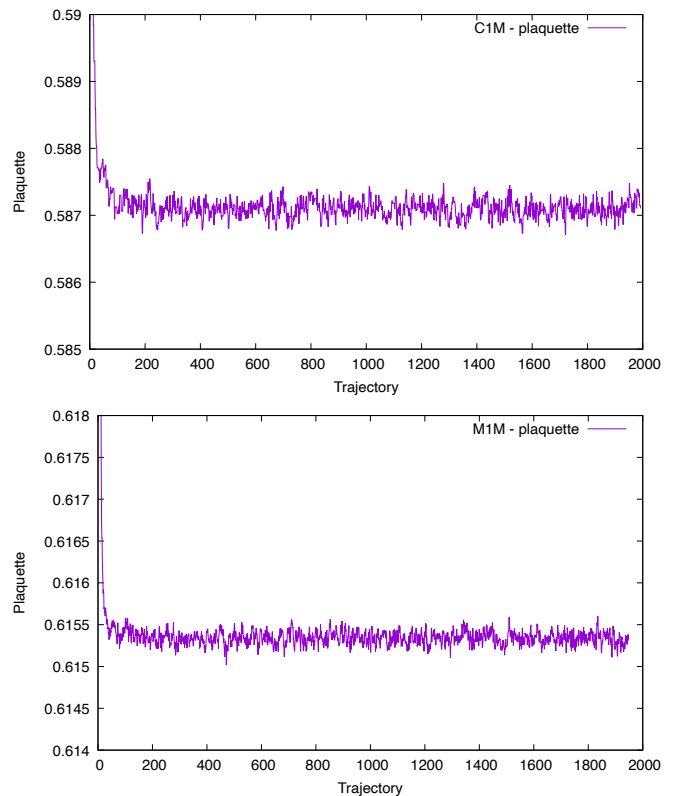


FIG. 16. Plaquette molecular dynamics time history for the C1M (top) and M1M (bottom) ensembles.

Ensemble	Plaquette	Residual mass
C0M	0.5871119(25)	0.0006102(40)
C1M	0.587091(7)	0.000604(11)
M0M	0.6153342(21)	0.0003116(23)
M1M	0.615337(4)	0.0003063(53)

TABLE XI. Average plaquettes for the new M1M and C1M ensembles are more than adequately consistent with the (more precise) values obtained on the corresponding large volume and physical quark mass ensembles, given the volume and mass differ.

all pseudofermion action fragments taking ten steps with timestep $\delta t = \frac{1}{10}$, and eight gauge steps for each fermion step. For C1M, a two level nested integrator was used in the HMC with the Force Gradient integrator, all pseudofermion action fragments taking twelve steps with timestep $\delta t = \frac{1}{12}$, and eight gauge steps for each fermion step.

The first 200 trajectories were discarded from a hot start before measurements. The plaquette histories are shown in Figure 16, and the average values are more than adequately consistent with those obtained more precisely in a larger volume at physical quark mass, Table XI, given that the quark mass and volume differ.

Appendix B: Correlation function fits

This section discusses the choices required in the correlation function fits. In Section B 1 we discuss the construction of the covariance matrix used in these fits. To mitigate any bias stemming from fit range choices, two of the authors independently did the analysis of the correlation function fits. This resulted in two different fit strategies which will be outlined in Sections B 2 and B 3.

1. Correlated fits to data

In all of this paper, our fits are correlated frequentist minimisations of the χ^2 -function

$$\chi^2 = \sum_{i,j} (f(a; x_i) - y_i) \text{cov}(y_i, y_j)^{-1} (f(a; x_j) - y_j), \quad (\text{B1})$$

where y_i are our data, $f(a; x_i)$ is the model with parameters a that is being fitted and $\text{cov}(y_i, y_j)$ is the covariance matrix of the data. Resampling N statistical estimators into N_{boot} bootstrap samples \tilde{y}_i and denoting the mean of y_i by \bar{y}_i , we define the covariance matrix $\text{cov}(y_i, y_j)$ as

$$\text{cov}(y_i, y_j) = \frac{1}{N_{\text{boot}}} \sum_{k=1}^{N_{\text{boot}}} ((\tilde{y}_i)_k - \bar{y}_i)((\tilde{y}_j)_k - \bar{y}_j). \quad (\text{B2})$$

We relate the covariance matrix to the standard deviation σ via $\sigma_i = \sqrt{\text{cov}(y_i, y_i)}$. Furthermore we define the *normalised covariance matrix* or *correlation matrix* $\text{cor}(y_i, y_j)$ as

$$\text{cor}(y_i, y_j) = \text{diag}(1/\sigma_i) \text{cov}(y_i, y_j) \text{diag}(1/\sigma_j). \quad (\text{B3})$$

Since we jointly fit multiple two-point and three-point functions, it is important to be able to accurately invert the covariance matrix that appears in the χ^2 -function.

On a given ensemble, we have $N_{\text{meas}} = N_{\text{conf}} \times N_{\text{src}}$ estimators for the y_i (compare Table II). When estimating the correlation matrix and the standard deviations, we therefore need to choose whether we treat measurements on different time translations on the same configuration as independent or whether we bin them into an effective measurement. From these two choices we obtain $(\sigma_i, \text{cor}(y_i, y_j))^{\text{unbinned}}$ and $(\sigma_i, \text{cor}(y_i, y_j))^{\text{binned}}$ based on $N_{\text{conf}} \times N_{\text{src}}$ measurements and N_{conf} effective measurements, respectively. We now construct the covariance matrix we use in the fit to the correlation functions as

$$\text{cov}(y_i, y_j) = \text{diag}(\sigma_i^{\text{binned}}) \text{cor}(y_i, y_j)^{\text{unbinned}} \text{diag}(\sigma_j^{\text{binned}}). \quad (\text{B4})$$

This has the benefit that it resolves the correlations but estimates the statistical uncertainties without any assumption of independence for measurements from different source positions on the same configurations.

TABLE XII. Masses and bare decay constants of the pion and kaons for all of the ensembles used in this work. The ensembles labelled with ‘†’ only enter the analysis in order to constrain the chiral extrapolation of the renormalisation constants described in Section IV.

ens	am_π	af_π^{bare}	am_K	af_K^{bare}
C0M	0.08048(10)	0.10654(12)	0.28696(13)	0.126852(89)
C1S	0.19052(40)	0.11902(27)	0.30630(39)	0.13201(22)
C2S	0.24159(38)	0.12743(20)	0.32518(35)	0.13737(18)
M0M	0.059078(74)	0.074620(86)	0.21065(10)	0.089081(60)
M1S	0.12750(35)	0.08292(28)	0.22491(36)	0.09379(20)
M2S	0.15123(36)	0.08680(22)	0.23208(35)	0.09578(17)
M3S	0.17238(42)	0.09023(25)	0.23994(40)	0.09775(20)
F1M	0.08581(16)	0.06768(15)	0.18810(19)	0.07821(15)
C1M†	0.15987(50)	0.11659(60)	0.30560(51)	0.13261(52)
M1M†	0.12116(52)	0.07943(39)	0.22778(62)	0.09193(32)

2. Correlation function fits - strategy and stability

We jointly fit several two-point and three-point functions directly to their functional forms given by Eqs. (3.3) and (3.5). In a first step, we start by only fitting the two-point functions. We determine fit ranges $t_{\text{min}}^{2\text{pt}}$ and $t_{\text{max}}^{2\text{pt}}$ for the two-point functions which produce stable ground and first excited state results for masses and overlap factors. For each four-quark operator Q_i^+ we then perform a joint fit to the same two-point functions but also the corresponding three-point functions $C_3^i(t, \Delta T)$ for several values of ΔT . As a first step, we keep $t_{\text{min}}^{2\text{pt}}$, $t_{\text{max}}^{2\text{pt}}$ from above for the two-point functions. For the three-point functions we use the same $t_{\text{min}}^{3\text{pt}}$ irrespective of ΔT . This is determined by choosing an integer δ to set $t_{\text{min}}^{3\text{pt}} = t_{\text{min}}^{2\text{pt}} + \delta$ and $t_{\text{max}}^{3\text{pt}} = \Delta T - t_{\text{min}}^{3\text{pt}}$. Typically we have $\delta \in \{0, 1\}$. We then vary δ by ± 1 , vary the choice of which values of ΔT enter the fit and vary $t_{\text{min}}^{2\text{pt}}$ by ± 1 . We adjust these choices until we see stability in all fit parameters.

For completeness we summarise the meson masses $m_P = E_0$ and the bare decay constants f_P^{bare} for the pion and the kaon in Table XII.

3. Alternative strategy

We define the ratios of two-point functions (3.3) and three-point functions (3.5)

$$r^1(t, \Delta T) = \frac{C_3^1(t, \Delta T)}{C_{PA}(t)C_{AP}(\Delta T - t)}, \quad (\text{B5})$$

$$r^i(t, \Delta T) = \frac{C_3^i(t, \Delta T)}{C_{PP}(t)C_{PP}(\Delta T - t)}, \quad i > 1$$

which are constructed to asymptotically approach the bag parameters

$$r^i(t, \Delta T) \xrightarrow{0 \ll t \ll \Delta T \ll T} N_i \mathcal{B}_i. \quad (\text{B6})$$

For simplicity we omit the smearing labels s_1, s_2 , which are chosen to ensure that only local matrix elements remain. Expanding numerator and denominator of the first line of Eq. (B5) using Eqs.(3.3) and (3.5) taking into ac-

count the ground state ($|0\rangle$) and first excited state ($|1\rangle$) contributions ($n = 0, 1$) (but neglecting the excited-to-excited matrix elements) yields

$$r_1(t, \Delta T) = \frac{\langle 0|\mathcal{O}_1|0\rangle}{M_{A,0}^2} \left[1 + X_1(t, \Delta T)e^{-\Delta E\Delta T/2} + Y_1(t, \Delta T)e^{-\Delta E\Delta T} \right]. \quad (\text{B7})$$

where we defined

$$\begin{aligned} X_1(t, \Delta T) &= 2 \frac{M_{P,1}E_0}{M_{P,0}E_1} \cosh[\Delta E(t - \Delta T/2)] \left(\frac{\langle 0|\mathcal{O}_1|1\rangle}{\langle 0|\mathcal{O}_1|0\rangle} - \frac{M_{A,1}}{M_{A,0}} \right), \\ Y_1(t, \Delta T) &= -4 \frac{M_{P,1}^2 E_0^2}{M_{P,0}^2 E_1^2} \cosh^2[\Delta E(t - \Delta T/2)] \frac{\langle 0|\mathcal{O}_1|1\rangle}{\langle 0|\mathcal{O}_1|0\rangle} \frac{M_{A,1}}{M_{A,0}}, \end{aligned} \quad (\text{B8})$$

and $\Delta E = E_1 - E_0$. The expression for r^i ($i > 1$) is very similar. Defining a summed version of the ratio

$$r^i(t_c, \Delta T) \equiv \sum_{t=t_c}^{\Delta T-t_c} r^i(t, \Delta T) \quad (\text{B9})$$

and using the identity

$$\sum_{t=t_c}^{\Delta T-t_c} \cosh[\Delta E(t - \Delta T/2)] = \frac{\sinh[\Delta E/2(\Delta T - 2t_c + 1)]}{\sinh[\Delta E/2]} \quad (\text{B10})$$

the summed ratio can be expressed as

$$r^1(t_c, \Delta T) = \frac{\langle 0|\mathcal{O}_1|0\rangle}{M_{A,0}^2} \left[\hat{t} + 2 \frac{M_{P,1}E_0}{M_{P,0}E_1} e^{-\Delta E\Delta T/2} \frac{\sinh[\Delta E/2\hat{t}]}{\sinh[\Delta E/2]} \left(\frac{\langle 0|\mathcal{O}_1|1\rangle}{\langle 0|\mathcal{O}_1|0\rangle} - \frac{M_{A,1}}{M_{A,0}} \right) \right], \quad (\text{B11})$$

where $\hat{t} \equiv \Delta T - 2t_c + 1$. For a given operator \mathcal{O}_i and value of t_c we then jointly fit the correlation functions and $C_{PP}^{SL}(t)$, $C_{PA}^{SL}(t)$, $C_{PP}^{SS}(t)$ and ratios $r^i(t_c, \Delta T)$ (and the LL equivalent for F1M). The fit ranges of $t \in [t_{\min}^{2\text{pt}}, t_{\max}^{2\text{pt}}]$ and $\Delta T \in [\Delta T_{\min}^{3\text{pt}}, \Delta T_{\max}^{3\text{pt}}]$ are chosen such that all fit parameters remain stable when these ranges are varied by small amounts.

Appendix C: Renormalisation factors

1. Definitions

We closely follow Ref. [32] in which the reader will find more details. The renormalisation factors are defined

by imposing the renormalisation condition that the projected renormalised amputated-vertex Green's function, in the Landau gauge, for some chosen external momenta, is equal to its tree level value (denoted by F). Using the SMOM kinematics

$$(p_1 - p_2)^2 = p_1^2 = p_2^2 = \mu^2, \quad (\text{C1})$$

the Z -factors are defined in the massless limit ($m_q \rightarrow 0$) and extracted by imposing

$$\lim_{m_q \rightarrow 0} P_k \left[\frac{Z_{ij}^{\text{RI-SMOM}}(\mu, a)}{(Z_q^{\text{RI-SMOM}}(\mu, a))^2} \Pi_j^{\text{bare}}(a, p_1, p_2) \right]_{\text{SMOM}} = P_k[\Pi_i^{(0)}], \quad (\text{C2})$$

and the tree-level value $F_{ik} \equiv P_k[\Pi_i^{(0)}]$ is obtained by replacing the propagators by the identity in colour-Dirac space. Explicitly, for a given four-quark operator Q_i , the vertex functions is defined as (with $\tilde{x}_i = x_i - x$)

$$\Pi_i^{\text{bare}}(a, p_1, p_2)^{\delta\tilde{\gamma};\tilde{\beta}\tilde{\alpha}} = \langle \bar{G}(p_2)^{-1} \rangle^{\delta\tilde{\delta}} \langle G(p_1)^{-1} \rangle^{\gamma\tilde{\gamma}} \langle \bar{G}(p_2)^{-1} \rangle^{\tilde{\beta}\tilde{\beta}} \langle G(p_1)^{-1} \rangle^{\alpha\tilde{\alpha}} (M_i^{\text{bare}})^{\delta\gamma;\beta\alpha}(a, p_1, p_2), \quad (\text{C3})$$

where

$$\begin{aligned}
(M_i^{\text{bare}})^{\delta\gamma;\beta\alpha}(q^2) &= \sum_{x,x_1,\dots,x_4} \langle 0 | s^\delta(x_4) \bar{d}^\gamma(x_3) [Q_i(x)] s^\beta(x_2) \bar{d}^\alpha(x_1) | 0 \rangle e^{-ip_1 \cdot \tilde{x}_1 + ip_2 \cdot \tilde{x}_2 - ip_1 \cdot \tilde{x}_3 + ip_2 \cdot \tilde{x}_4}, \\
&= 2 \sum_x \left(\langle [\bar{G}_x(p_2) \Gamma^1 G_x(p_1)]^{\delta\gamma} [\bar{G}_x(p_2) \Gamma^2 G_x(p_1)]^{\beta\alpha} \right. \\
&\quad \left. - \langle [\bar{G}_x(p_2) \Gamma^1 G_x(p_1)]^{\delta\alpha} [\bar{G}_x(p_2) \Gamma^2 G_x(p_1)]^{\beta\gamma} \rangle \right).
\end{aligned} \tag{C4}$$

Here $G_x(p)$ represents an incoming quark with momentum p and $\bar{G}_x(-p)$ an outgoing quark with momentum p . In addition we have also introduced the inverse of the ‘‘full momentum’’ propagators

$$G(p) = \sum_x G_x(p) \quad \text{and} \quad \bar{G}(p) = \sum_x \bar{G}_x(p). \tag{C5}$$

In Eq. (C4), the Dirac structure of the four-quark operator Q_i is encoded in $\Gamma^1 \times \Gamma^2$.

In this work we use two different sets of projectors, $P^{(\gamma_\mu)}$ and $P^{(\not{q})}$, as defined in Ref. [32]. The $P^{(\gamma_\mu)}$ follow the same structure as the four-quark operators. We split them in three groups according to their chiral-flavour properties. For the standard model, we have

$$\left[P_1^{(\gamma_\mu)} \right]_{\beta\alpha;\delta\gamma}^{ba;dc} = [(\gamma^\mu)_{\beta\alpha}(\gamma^\mu)_{\delta\gamma} + (\gamma^\mu \gamma^5)_{\beta\alpha}(\gamma^\mu \gamma^5)_{\delta\gamma}] \delta^{ba} \delta^{dc}. \tag{C6}$$

For the (8, 8) doublet we define

$$\begin{aligned}
\left[P_2^{(\gamma_\mu)} \right]_{\beta\alpha;\delta\gamma}^{ba;dc} &= [(\gamma^\mu)_{\beta\alpha}(\gamma^\mu)_{\delta\gamma} - (\gamma^\mu \gamma^5)_{\beta\alpha}(\gamma^\mu \gamma^5)_{\delta\gamma}] \delta^{ba} \delta^{dc}, \\
\left[P_3^{(\gamma_\mu)} \right]_{\beta\alpha;\delta\gamma}^{ba;dc} &= [\delta_{\beta\alpha} \delta_{\delta\gamma} - (\gamma^5)_{\beta\alpha}(\gamma^5)_{\delta\gamma}] \delta^{ba} \delta^{dc},
\end{aligned} \tag{C7}$$

and similarly for the $(\bar{6}, 6)$:

$$\begin{aligned}
\left[P_4^{(\gamma_\mu)} \right]_{\beta\alpha;\delta\gamma}^{ba;dc} &= [\delta_{\beta\alpha} \delta_{\delta\gamma} + (\gamma^5)_{\beta\alpha}(\gamma^5)_{\delta\gamma}] \delta^{ba} \delta^{dc}, \\
\left[P_5^{(\gamma_\mu)} \right]_{\beta\alpha;\delta\gamma}^{ba;dc} &= \left[\sum_{\nu > \mu} (\gamma^\mu \gamma^\nu)_{\beta\alpha} (\gamma^\mu \gamma^\nu)_{\delta\gamma} \right] \delta^{ba} \delta^{dc}.
\end{aligned} \tag{C8}$$

To define the $P^{(\not{q})}$ projectors we replace γ_μ by \not{q}/q^2 in the previous equations [6]. When there is no explicit γ_μ , we take advantage of some Fierz identities, which relate the colour-unmixed four-quark operators of certain Dirac structure to colour-mixed four-quark operators of a different Dirac structure [86]. Following Refs. [33, 87] we define

$$\begin{aligned}
\left[P_1^{(\not{q})} \right]_{\beta\alpha;\delta\gamma}^{ba;dc} &= \frac{1}{q^2} [(\not{q})_{\beta\alpha}(\not{q})_{\delta\gamma} + (\not{q} \gamma^5)_{\beta\alpha}(\not{q} \gamma^5)_{\delta\gamma}] \delta^{ba} \delta^{dc}, \\
\left[P_2^{(\not{q})} \right]_{\beta\alpha;\delta\gamma}^{ba;dc} &= \frac{1}{q^2} [(\not{q})_{\beta\alpha}(\not{q})_{\delta\gamma} - (\not{q} \gamma^5)_{\beta\alpha}(\not{q} \gamma^5)_{\delta\gamma}] \delta^{ba} \delta^{dc}, \\
\left[P_3^{(\not{q})} \right]_{\beta\alpha;\delta\gamma}^{ba;dc} &= \frac{1}{q^2} [(\not{q})_{\beta\alpha}(\not{q})_{\delta\gamma} - (\not{q} \gamma^5)_{\beta\alpha}(\not{q} \gamma^5)_{\delta\gamma}] \delta^{bc} \delta^{da}, \\
\left[P_4^{(\not{q})} \right]_{\beta\alpha;\delta\gamma}^{ba;dc} &= \frac{1}{p_1^2 p_2^2 - (p_1 \cdot p_2)^2} \left[(p_1^\mu (\sigma^{\mu\nu} P_L) p_2^\nu)_{\beta\alpha} (p_1^\rho (\sigma^{\rho\sigma} P_L) p_2^\sigma)_{\delta\gamma} \right] \delta^{bc} \delta^{da}, \\
\left[P_5^{(\not{q})} \right]_{\beta\alpha;\delta\gamma}^{ba;dc} &= \frac{1}{p_1^2 p_2^2 - (p_1 \cdot p_2)^2} \left[(p_1^\mu (\sigma^{\mu\nu} P_L) p_2^\nu)_{\beta\alpha} (p_1^\rho (\sigma^{\rho\sigma} P_L) p_2^\sigma)_{\delta\gamma} \right] \delta^{ba} \delta^{dc},
\end{aligned} \tag{C9}$$

where we used the standard definition $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$.

In order to eliminate the explicit Z_q -dependence in Eq.(C2), it is customary to divide the (amputated-projected) vertex function of the four-quark operators by the one of a bilinear operator. Here we choose the axial-vector current and we denote Π_A, P_A, F_A the corre-

sponding vertex function, projector and tree level value, respectively. Finally, the choice of projectors completes the definition of the non-perturbative scheme. Using \mathcal{A} to indicate γ_μ or \not{q} we define the renormalisation factors $Z_{ij}^{(\mathcal{A},\mathcal{A})}/Z_A^2$ as²

² We can also define $Z_{ij}^{(\mathcal{A},\mathcal{B})}$ with $\mathcal{A} \neq \mathcal{B}$, but in this work we

consider only (γ_μ, γ_μ) and (\not{q}, \not{q})

$$\frac{Z_{ij}^{(A,A)}(\mu, a)}{Z_A^2(a)} \times \lim_{m_q \rightarrow 0} \frac{P_k^{(A)}[\Pi_j^{\text{bare}}(a, p_1, p_2)]}{(P_A^{(A)}[\Pi_A^{\text{bare}}(a, p_1, p_2)])^2} \Big|_{\text{SMOM}} = \frac{F_{ik}^{(A)}}{(F_A^{(A)})^2}. \quad (\text{C10})$$

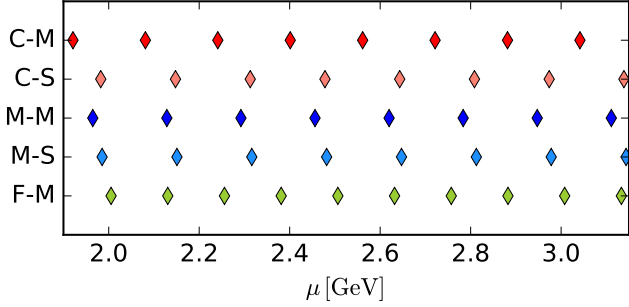


FIG. 17. Simulation values of the scale μ on the various ensembles.

Our conventions are such that from Eq. (C10) we can define $Z_{B_1} \equiv Z_{B_K} = Z_{11}/Z_A^2$. Finally, to obtain the bag parameters we also need to renormalise the quark mass. Making use of $Z_m = 1/Z_S$ we impose

$$\frac{Z_A^{(A)}(\mu, a)}{Z_S(\mu, a)} \times \lim_{m_q \rightarrow 0} \frac{P_A^{(A)}[\Pi_A^{\text{bare}}(a, p_1, p_2)]}{P_S[\Pi_S^{\text{bare}}(a, p_1, p_2)]} \Big|_{\text{SMOM}} = \frac{F_A^{(A)}}{F_S}. \quad (\text{C11})$$

The use of two schemes provides a way of estimating systematic errors in the renormalisation by examining the spread of the results. The vertex function's external momenta p_i^μ are chosen to ensure non-exceptional SMOM kinematics given in Eq. (C1). Using a combination of Fourier momenta and partially twisted boundary conditions, the behaviour of the renormalisation factors as a function of the momentum scale μ is mapped out in the range $\mu \in [2 - 3]$ GeV. In the original exceptional kinematics (used in RI-MOM), infrared effects fall only as p^{-2} and pion pole subtraction is required to tame these. Using non-exceptional kinematics (used in RI-SMOM), the infrared effects are far more suppressed falling with p^{-6} as has been shown in Ref. [88]. Alternative renormalisation schemes such as the massive RI-SMOM [89, 90] and interpolating MOM-schemes [91] are currently being explored by the collaboration. Their application to Kaon mixing is left for future studies.

2. Numerical results for the renormalisation factors ensemble by ensemble

On each ensemble we simulate the NPR data points at a range of choices for the renormalisation scale μ . These are depicted in Figure 17. We furthermore repeat the simulation at multiple quark masses, in particular at am_l^{sea} , $2am_l^{\text{sea}}$ and $am_s^{\text{sea}}/2$ with the exception of the most expensive COM and M0M ensembles where we only

simulate at $am_s^{\text{sea}}/2$.

We follow the procedure outlined in Section C 1 to calculate the four-quark matrix elements' renormalisation constants Z_{ij}/Z_A^2 . We then normalise these to build the appropriate renormalisation constants listed in Eq. (4.3). In order to ensure reproducibility, we provide numerical values of the renormalisation constants at some choice of lattice momenta on the lightest pion mass ensemble for each distinct lattice spacing in Tables XIII-XVII.

3. Extrapolation of the renormalisation factors to the massless limit

Formally the renormalisation constants are defined in the massless (zero quark mass) limit. In order to perform this limit lattice-spacing-by-lattice-spacing we proceed as follows. We first extrapolate the valence quark mass to zero ensemble-by-ensemble as described in Sec. IV A. We then interpolate the renormalisation constants on all ensembles to a fixed scale μ . We either perform a linear fit to the two closest simulated values of μ or a quadratic fit to the three closest points. We then perform a chiral extrapolation in $(am_\pi)^2$ to all ensembles that share an identical lattice spacing (C1S and C2S; C0M and C1M; M1S, M2S and M3S; M0M and M1M). Since we only have data on a single ensemble (F1M) for the finest lattice spacing, we apply each of the slopes with m_π^2 in turn. On F-M, we assign the central value to be the mean of these four extrapolated results and take half the spread of the results as a systematic uncertainty associated to the chiral extrapolation. We provide numerical values of these chirally extrapolated renormalisation constants for each of the lattice spacings at $\mu = 2$ GeV (Table VI), $\mu = 2.5$ GeV (Table XVIII), and $\mu = 3$ GeV (Table XIX).

4. Step-scaling matrices

We choose a lower scale μ for non-perturbative renormalisation to reduce cutoff effects, and a higher scale μ' for matching to $\overline{\text{MS}}$. This distance is bridged using non-perturbative running

$$O_i^{\overline{\text{MS}}}(\mu') = R_{ij}^{\overline{\text{MS}} \leftarrow \text{RI}}(\mu') \sigma_{jk}(\mu', \mu) O_k^{\text{RI}}(\mu), \quad (\text{C12})$$

O_i being any of the quantities in Eq. (4.3). The matching factors $R_{ij}^{\overline{\text{MS}} \leftarrow \text{RI}}$ are computed in next-to-leading order perturbation theory and presented in Ref. [33] for both schemes. The non-perturbative scale evolution matrix is

TABLE XIII. Values of Z_{ij}/Z_A^2 for chirally non-vanishing matrix elements for a subset of the simulated momenta (in lattice units) on the COM ensemble for $am_q^{\text{val}} = 0.0181$. All values are given in the RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ scheme and in the SUSY basis.

$a\mu$	1.11072	1.3884	1.66608	1.94376
Z_{11}/Z_A^2	0.93224(13)	0.923561(68)	0.915757(54)	0.907465(34)
Z_{22}/Z_A^2	0.74937(13)	0.82424(16)	0.877220(37)	0.917178(57)
Z_{23}/Z_A^2	-0.041062(66)	-0.056518(65)	-0.071844(45)	-0.087583(32)
Z_{32}/Z_A^2	0.03145(13)	-0.010829(52)	-0.043507(40)	-0.071655(36)
Z_{33}/Z_A^2	1.154966(63)	1.106912(37)	1.079801(38)	1.063261(10)
Z_{44}/Z_A^2	0.70331(20)	0.79997(14)	0.867628(12)	0.918149(54)
Z_{45}/Z_A^2	-0.050481(64)	-0.060964(52)	-0.073113(41)	-0.086769(28)
Z_{54}/Z_A^2	-0.171645(50)	-0.149960(37)	-0.139921(39)	-0.137573(37)
Z_{55}/Z_A^2	1.051566(66)	1.049023(33)	1.049000(28)	1.050999(11)
Z_A/Z_S	1.197636(71)	1.129805(84)	1.088466(26)	1.061227(33)

TABLE XIV. Same as Table XIII but for the C1S ensemble for $am_q^{\text{val}} = 0.005$.

$a\mu$	1.11072	1.3884	1.66608	1.94376
Z_{11}/Z_A^2	0.93303(13)	0.924165(55)	0.916271(66)	0.907958(58)
Z_{22}/Z_A^2	0.75249(61)	0.82630(28)	0.87856(11)	0.91832(12)
Z_{23}/Z_A^2	-0.04129(15)	-0.056775(74)	-0.071886(82)	-0.087656(88)
Z_{32}/Z_A^2	0.03059(21)	-0.01137(10)	-0.043656(74)	-0.071733(95)
Z_{33}/Z_A^2	1.15504(28)	1.10673(11)	1.079562(84)	1.063174(48)
Z_{44}/Z_A^2	0.70746(75)	0.80253(30)	0.86921(11)	0.91939(13)
Z_{45}/Z_A^2	-0.05062(20)	-0.06120(10)	-0.073071(83)	-0.086787(86)
Z_{54}/Z_A^2	-0.17065(21)	-0.149293(88)	-0.139280(88)	-0.137090(68)
Z_{55}/Z_A^2	1.05273(21)	1.049571(82)	1.049163(54)	1.051141(43)
Z_A/Z_S	1.19361(86)	1.12820(21)	1.08753(10)	1.060548(76)

given by

$$\begin{aligned} \sigma_{ij}(\mu', \mu) &= \lim_{a \rightarrow 0} \sigma_{ij}(\mu', \mu, a) \\ &= \lim_{a \rightarrow 0} Z_{ik}(\mu', a) Z_{kj}(\mu, a)^{-1}. \end{aligned} \quad (\text{C13})$$

We perform this continuum limit including the chirally extrapolated renormalisation constants from all lattice

spacings as a fit linear in a^2 . In the few cases where the quality of fit does not lead to an acceptable p -value, we rescale the uncertainty by $\sqrt{\chi^2/\text{d.o.f.}}$. In Figure 18 we compare our non-perturbative step-scaling results to leading and next-to-leading order perturbation theory. For completeness we also provide the step-scaling matrices, σ_R , for the R_i

$$\sigma_R(3 \text{ GeV}, 2 \text{ GeV}) = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.2153(28) & -0.08396(60) & 0.0 & 0.0 \\ 0.0 & -0.00426(52) & 0.90868(42) & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.3186(42) & 0.1018(15) \\ 0.0 & 0.0 & 0.0 & 0.00976(68) & 0.99984(59) \end{bmatrix}, \quad (\text{C14})$$

$$\sigma_R(3 \text{ GeV} \xleftarrow{\Delta=0.5} 2 \text{ GeV}) = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.2149(25) & -0.08315(50) & 0.0 & 0.0 \\ 0.0 & -0.00374(43) & 0.90889(40) & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.3190(35) & 0.1023(14) \\ 0.0 & 0.0 & 0.0 & 0.01083(44) & 0.99940(59) \end{bmatrix}. \quad (\text{C15})$$

Appendix D: Relations between basis conventions

We distinguish between operators O_i in the ‘‘SUSY’’ basis, defined in Eq. (2.2), and operators Q_i in the

‘‘NPR’’ or ‘‘lattice’’ basis, in Eq. (2.4). The SUSY basis

TABLE XV. Same as Table XIII but for the M0M ensemble for $am_q^{\text{val}} = 0.0133$.

$a\mu$	0.83304	1.0413	1.24956	1.3884
Z_{11}/Z_A^2	0.958988(97)	0.948999(54)	0.941580(40)	0.937357(31)
Z_{22}/Z_A^2	0.701561(98)	0.774046(79)	0.826662(61)	0.854299(47)
Z_{23}/Z_A^2	-0.023850(59)	-0.034541(40)	-0.044346(40)	-0.050758(38)
Z_{32}/Z_A^2	0.075307(47)	0.034965(33)	0.005582(30)	-0.010480(23)
Z_{33}/Z_A^2	1.212886(87)	1.149630(45)	1.111072(35)	1.093473(27)
Z_{44}/Z_A^2	0.63286(11)	0.727410(91)	0.796468(58)	0.832571(45)
Z_{45}/Z_A^2	-0.038240(46)	-0.043612(32)	-0.049903(29)	-0.054545(30)
Z_{54}/Z_A^2	-0.182166(54)	-0.154917(30)	-0.137278(21)	-0.129566(23)
Z_{55}/Z_A^2	1.057128(68)	1.050256(31)	1.045950(24)	1.044326(22)
Z_A/Z_S	1.26232(13)	1.181992(83)	1.131563(41)	1.107811(22)

TABLE XVI. Same as Table XIII but for the M1S ensemble for $am_q^{\text{val}} = 0.004$.

$a\mu$	0.83304	1.0413	1.24956	1.45782
Z_{11}/Z_A^2	0.95866(27)	0.94835(17)	0.94111(13)	0.93495(10)
Z_{22}/Z_A^2	0.70006(63)	0.77412(33)	0.82662(14)	0.86644(13)
Z_{23}/Z_A^2	-0.02394(22)	-0.034736(81)	-0.044604(78)	-0.054232(65)
Z_{32}/Z_A^2	0.07552(21)	0.03452(10)	0.005247(63)	-0.018170(55)
Z_{33}/Z_A^2	1.21448(44)	1.14967(17)	1.111076(76)	1.086321(33)
Z_{44}/Z_A^2	0.63178(68)	0.72810(36)	0.79677(14)	0.84859(14)
Z_{45}/Z_A^2	-0.03853(25)	-0.043920(89)	-0.050118(52)	-0.057240(60)
Z_{54}/Z_A^2	-0.18333(20)	-0.15511(13)	-0.137472(66)	-0.126927(60)
Z_{55}/Z_A^2	1.05863(24)	1.05060(10)	1.046175(54)	1.044027(36)
Z_A/Z_S	1.2629(14)	1.18148(46)	1.13157(10)	1.09803(10)

contains both colour-unmixed and colour-mixed operators, while the NPR basis comprises only colour-unmixed operators, more convenient for lattice computations. For the $K^0\bar{K}^0$ matrix elements of these operators we need

only the parity-even parts, O_i^+ or Q_i^+ , shown in Eqs. (2.3) and (2.5). Since we work with the Q_i^+ on the lattice, we quote here the matrix T which relates the Q_i^+ to the O_i^+ :

$$O^+ = TQ^+ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix} Q^+ \quad \text{or} \quad \begin{pmatrix} O_1^+ \\ O_2^+ \\ O_3^+ \\ O_4^+ \\ O_5^+ \end{pmatrix} = \begin{pmatrix} Q_1^+ \\ Q_4^+ \\ (Q_5^+ - Q_4^+)/2 \\ Q_3^+ \\ -Q_2^+/2 \end{pmatrix}. \quad (\text{D1})$$

We perform our non-perturbative calculations of the matrix elements and renormalisation constants in the NPR basis and subsequently convert to the SUSY basis. Matrix elements of the renormalised Q_i^+ in some scheme X at scale μ are given by,

$$\langle K | Q_i^+ | \bar{K} \rangle^X(\mu) = Z_{ij}^X(\mu) \langle K | Q_{j,\text{bare}}^+ | \bar{K} \rangle. \quad (\text{D2})$$

When chiral symmetry is maintained, $Z_{ij}^X(\mu)$ is block-diagonal and the matrix elements of the renormalised operators O_i^+ in the SUSY basis in scheme X at scale μ are related to matrix elements of the bare operators in the NPR basis by,

TABLE XVII. Same as Table XIII but for the F1M ensemble for $am_q^{\text{val}} = 0.0021$.

$a\mu$	0.74048	0.87932	1.01816	1.43468
Z_{11}/Z_A^2	0.97098(29)	0.96182(11)	0.955127(87)	0.941078(33)
Z_{22}/Z_A^2	1.06173(45)	1.05453(10)	1.049270(35)	1.041636(15)
Z_{23}/Z_A^2	0.37422(52)	0.32971(35)	0.29528(14)	0.239419(67)
Z_{32}/Z_A^2	0.01748(19)	0.018812(88)	0.020455(43)	0.026601(18)
Z_{33}/Z_A^2	0.61015(69)	0.68079(75)	0.73890(33)	0.85407(11)
Z_{44}/Z_A^2	0.69653(77)	0.75380(76)	0.80124(37)	0.89759(12)
Z_{45}/Z_A^2	-0.00933(14)	-0.01280(13)	-0.015992(62)	-0.025037(17)
Z_{54}/Z_A^2	-0.36177(54)	-0.31116(28)	-0.27401(16)	-0.215021(71)
Z_{55}/Z_A^2	1.23079(76)	1.17174(16)	1.130128(97)	1.058518(33)
Z_A/Z_S	1.28592(66)	1.22090(96)	1.17291(37)	1.093643(58)

TABLE XVIII. Same as Table VI but at mass scale $\mu = 2.5$ GeV.

a^{-1} [GeV]	1.7295(38)	1.7848(50)	2.3586(70)	2.3833(86)	2.708(10)
Z_{11}/Z_A^2	0.92273(29)(1)	0.92491(52)(0)	0.94913(20)(0)	0.94752(86)(2)	0.95950(43)(33)
Z_{22}/Z_S^2	1.05048(77)(4)	1.0519(13)(0)	1.08128(96)(1)	1.0830(21)(0)	1.0964(11)(6)
Z_{23}/Z_S^2	-0.07511(48)(3)	-0.07280(82)(2)	-0.04932(29)(0)	-0.04900(66)(1)	-0.04007(40)(17)
Z_{32}/Z_S^2	-0.0228(16)(1)	-0.0162(45)(0)	0.0447(28)(1)	0.0465(63)(1)	0.0746(37)(0)
Z_{33}/Z_S^2	1.3816(30)(7)	1.4014(83)(2)	1.5913(50)(6)	1.603(11)(0)	1.6981(67)(17)
Z_{44}/Z_S^2	1.02458(44)(3)	1.02303(97)(4)	1.01873(49)(7)	1.0201(14)(0)	1.01595(76)(71)
Z_{45}/Z_S^2	-0.07967(51)(2)	-0.07835(69)(1)	-0.06134(25)(1)	-0.06193(46)(0)	-0.05599(23)(24)
Z_{54}/Z_S^2	-0.18458(58)(19)	-0.1880(18)(0)	-0.2130(11)(1)	-0.2162(26)(1)	-0.2305(15)(5)
Z_{55}/Z_S^2	1.3170(21)(5)	1.3310(60)(1)	1.4581(36)(3)	1.4673(80)(5)	1.5267(46)(17)

TABLE XIX. Same as Table VI but at mass scale $\mu = 3.0$ GeV.

a^{-1} [GeV]	1.7295(38)	1.7848(50)	2.3586(70)	2.3833(86)	2.708(10)
Z_{11}/Z_A^2	0.91427(17)(0)	0.91641(55)(0)	0.94123(17)(1)	0.94044(67)(0)	0.95157(31)(27)
Z_{22}/Z_S^2	1.03795(18)(4)	1.03874(73)(2)	1.05744(45)(5)	1.0596(10)(0)	1.06947(54)(36)
Z_{23}/Z_S^2	-0.08818(26)(2)	-0.08564(98)(0)	-0.05786(22)(0)	-0.05804(54)(0)	-0.04741(25)(25)
Z_{32}/Z_S^2	-0.0589(10)(0)	-0.0532(31)(0)	0.0036(18)(0)	0.0052(44)(0)	0.0300(24)(1)
Z_{33}/Z_S^2	1.2568(19)(2)	1.2711(59)(2)	1.4093(34)(4)	1.4206(79)(1)	1.4919(44)(11)
Z_{44}/Z_S^2	1.03009(18)(0)	1.02856(45)(0)	1.02098(20)(0)	1.02223(60)(0)	1.01811(29)(38)
Z_{45}/Z_S^2	-0.08895(23)(2)	-0.08683(80)(0)	-0.06453(14)(1)	-0.06486(36)(0)	-0.05716(16)(22)
Z_{54}/Z_S^2	-0.16199(30)(9)	-0.1636(11)(0)	-0.17286(72)(9)	-0.1757(17)(0)	-0.1842(10)(3)
Z_{55}/Z_S^2	1.2268(12)(1)	1.2368(41)(1)	1.3299(24)(2)	1.3389(57)(1)	1.3852(31)(10)

TABLE XX. Chirally-allowed elements of the non-perturbative scaling matrix $\sigma(3 \text{ GeV}, 2 \text{ GeV})$ using chirally extrapolated Z -factors.

a^{-1} [GeV]	1.7848(50)	1.7295(38)	2.3833(86)	2.3586(70)	2.708(10)
σ_{22}	1.1866(15)	1.18862(99)	1.2055(27)	1.1980(16)	1.2047(16)
σ_{23}	-0.020560(36)	-0.02201(14)	-0.01408(33)	-0.01294(22)	-0.01165(18)
σ_{32}	-0.09661(26)	-0.09870(19)	-0.09318(55)	-0.09034(41)	-0.08994(17)
σ_{33}	0.94800(88)	0.95149(38)	0.92958(89)	0.93208(35)	0.92586(49)
σ_{44}	1.2441(30)	1.2447(15)	1.2844(46)	1.2748(21)	1.2910(23)
σ_{45}	-0.00982(21)	-0.01235(22)	-0.00146(54)	-0.00159(23)	0.00067(30)
σ_{54}	0.0439(12)	0.04063(45)	0.0711(17)	0.06801(83)	0.0774(10)
σ_{55}	1.017341(75)	1.018409(16)	1.00827(44)	1.00994(39)	1.00838(48)

$$\begin{pmatrix} \langle K | O_1^+ | \bar{K} \rangle \\ \langle K | O_2^+ | \bar{K} \rangle \\ \langle K | O_3^+ | \bar{K} \rangle \\ \langle K | O_4^+ | \bar{K} \rangle \\ \langle K | O_5^+ | \bar{K} \rangle \end{pmatrix} = \begin{pmatrix} Z_{11}^X & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z_{44}^X & Z_{45}^X \\ 0 & 0 & 0 & \frac{-Z_{44}^X + Z_{54}^X}{2} & \frac{-Z_{45}^X + Z_{55}^X}{2} \\ 0 & Z_{32}^X & Z_{33}^X & 0 & 0 \\ 0 & -\frac{Z_{22}^X}{2} & -\frac{Z_{23}^X}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \langle K | Q_{1,\text{bare}}^+ | \bar{K} \rangle \\ \langle K | Q_{2,\text{bare}}^+ | \bar{K} \rangle \\ \langle K | Q_{3,\text{bare}}^+ | \bar{K} \rangle \\ \langle K | Q_{4,\text{bare}}^+ | \bar{K} \rangle \\ \langle K | Q_{5,\text{bare}}^+ | \bar{K} \rangle \end{pmatrix}. \quad (\text{D3})$$

The NPR-basis operators are written in colour-unmixed form. They are related by Fierz transformations to the same operators written in colour-mixed form by $Q_i^{\text{mix}} = F_{ij} Q_j^{\text{unmixed}}$, where

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{3}{2} & \frac{1}{2} \end{pmatrix}. \quad (\text{D4})$$

The matrix F is used when working out the matrix T which transforms from the NPR basis to the SUSY basis.

Appendix E: Perturbative scaling

We compare our non-perturbative scaling results with perturbation theory in Figure 18 and hence include here our notation and definitions for the perturbative computations to next-to-leading order. Full details are provided in a Mathematica [92] notebook as supplementary information. This is well-covered ground [93–95], but by explicitly including both g^2 and $g^2 \log(g)$ terms at NLO we are able to avoid having to take a limit to compute the scaling matrix at NLO.

The (matrix) operator renormalisation constants and scaling matrix are related by

$$Z(\mu') = \sigma(\mu', \mu) Z(\mu). \quad (\text{E1})$$

The anomalous dimension matrix, γ is defined by

$$\mu \frac{dZ}{d\mu} = -\gamma Z. \quad (\text{E2})$$

Using $\mu dg/d\mu = \beta(g)$, we have

$$\sigma(\mu', \mu) = T_g \exp \left(\int_{g(\mu)}^{g(\mu')} dg' \frac{-\gamma(g)}{\beta(g)} \right), \quad (\text{E3})$$

where T_g denotes g -ordering. We let

$$\bar{a} = \frac{\alpha}{4\pi} = \frac{g^2}{16\pi^2} \quad (\text{E4})$$

and expand

$$\begin{aligned} \beta(g) &= -\beta^0 g \bar{a} - \beta^1 g \bar{a}^2 + \dots \\ \gamma(g) &= \gamma^0 \bar{a} + \gamma^1 \bar{a}^2 + \dots \end{aligned} \quad (\text{E5})$$

so that

$$\int_{g(\mu)}^{g(\mu')} dg' \frac{-\gamma(g')}{\beta(g')} = \int_{\bar{a}(\mu)}^{\bar{a}(\mu')} \frac{d\bar{a}'}{\bar{a}'} \frac{(\gamma^0 + \gamma^1 \bar{a}' + \dots)}{2(\beta^0 + \beta^1 \bar{a}' + \dots)}. \quad (\text{E6})$$

With our conventions, $\beta^0 = 11 - 2N_f/3$ for N_f flavours. We work with the operators Q_i^+ defined in Eq. (2.5). They are related to the positive parity parts of the basis $Q_{\text{BMU}} = \{Q_1^{\text{VLL}}, Q_1^{\text{LR}}, Q_2^{\text{LR}}, Q_1^{\text{SLL}}, Q_2^{\text{SLL}}\}$ used in Ref. [94] by $Q^+ = R \cdot Q_{\text{BMU}}^+$ with $R = \text{diag}(4, 4, 4, 4, 1)$. This changes the off-diagonal elements of the anomalous dimension matrices in the bottom-right 2×2 block. In Ref. [95], the anomalous dimension is defined with the opposite sign and expanded as $\gamma^{\text{PPP}} = -g^2(\gamma^{0,\text{PPP}} + \gamma^{1,\text{PPP}} g^2 + \dots)$, which means that $\gamma^{0,\text{PPP}} = \gamma^0/(16\pi^2)$.

1. LO scaling

If we diagonalise γ^0 ,

$$V^{-1} \gamma^0 V = \gamma_D^0, \quad (\text{E7})$$

where γ_D^0 is the diagonal matrix of eigenvalues, then the scaling matrix at leading order is

$$\sigma^0(\mu', \mu) = V \left(\frac{a(\mu')}{a(\mu)} \right)^{\gamma_D^0/2\beta^0} V^{-1}. \quad (\text{E8})$$

To match notation from Ref. [94], we write

$$(V^{-1} \gamma^0 V)_{ij} = 2\beta^0 a_i \delta_{ij} \quad \text{where} \quad a = \frac{\gamma_D^0}{2\beta^0}. \quad (\text{E9})$$

Under $SU(3)_L \times SU(3)_R$ flavour, Q_1^+ is (27, 1) and renormalises multiplicatively, $Q_{2,3}^+$ are (8, 8) and mix, and $Q_{4,5}^+$ are (6, $\bar{6}$) and also mix. When we diagonalise γ^0 we permute the eigenvalues and eigenvectors to preserve the block-diagonal structure of the 5×5 anomalous dimension matrix. Our choice leads to

$$\gamma_D^0 = \text{diag} \left(4, 2, -16, \frac{2}{3}(1 + \sqrt{241}), \frac{2}{3}(1 - \sqrt{241}) \right). \quad (\text{E10})$$

2. NLO scaling

Beyond leading order we write

$$\sigma(\mu', \mu) \equiv K(\mu') \sigma^0(\mu', \mu) K^{-1}(\mu). \quad (\text{E11})$$

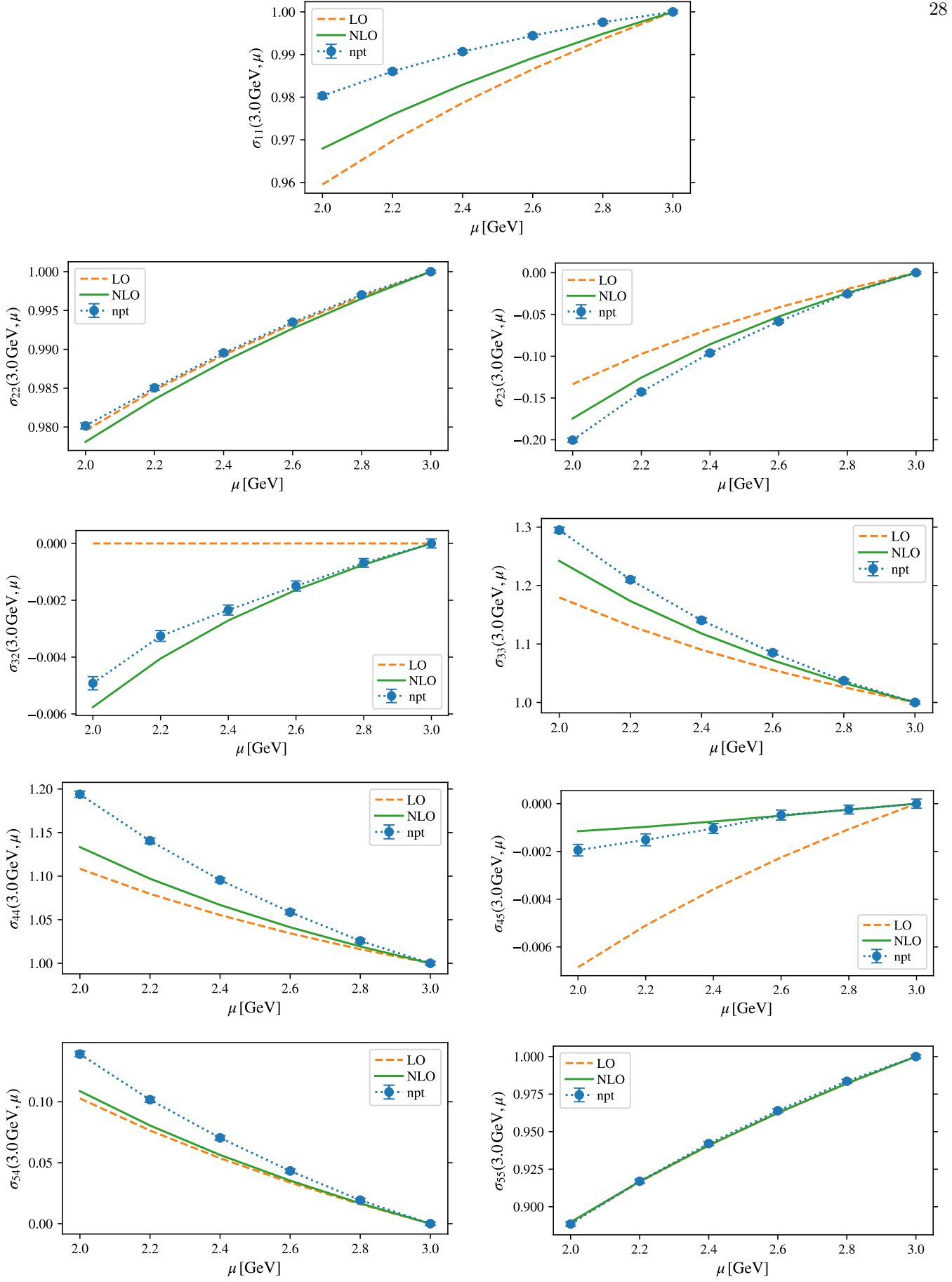


FIG. 18. Comparison of the scale evolution matrix $\sigma(3 \text{ GeV}, \mu)$ (see Eq.(4.4)) in the RI-SMOM (γ_μ, γ_μ) scheme and NPR basis evaluated non-perturbatively (blue circles), perturbatively at leading order (orange dashed lines) and next-to-leading order (green solid lines).

The equation satisfied by K is

$$\frac{\partial K}{\partial g} - \frac{1}{g} \left[\frac{\gamma^0}{\beta^0}, K \right] = - \left(\frac{\gamma(g)}{\beta(g)} + \frac{\gamma^0}{\beta^0 g} \right) K, \quad (\text{E12})$$

where $K = K(\mu)$ and $g = g(\mu)$. Expanding to NLO in the coupling and writing

$$K = 1 + \frac{g^2}{16\pi^2} J + \frac{g^2}{16\pi^2} \log(g) L, \quad (\text{E13})$$

we find that J and L satisfy

$$\begin{aligned} L &= \left[\frac{\gamma^0}{2\beta^0}, L \right] \\ J + \frac{1}{2}L &= \left[\frac{\gamma^0}{2\beta^0}, J \right] + \frac{1}{2\beta^0} \gamma^1 - \frac{\beta^1}{2(\beta^0)^2} \gamma^0. \end{aligned} \quad (\text{E14})$$

Using the matrix V from equation (E7) which diagonalises γ^0 , we define

$$\begin{aligned} G &= V^{-1} \gamma^1 V \\ S &= V^{-1} J V \\ T &= V^{-1} L V \end{aligned} \quad (\text{E15})$$

and recalling the a_i from equation (E9), we find

$$T_{ij} = (a_i - a_j) T_{ij}. \quad (\text{E16})$$

This shows that T_{ij} vanishes, *except* when $a_i - a_j = 1$. We also have

$$S_{ij} + \frac{1}{2} T_{ij} - (a_i - a_j) S_{ij} = \frac{1}{2\beta^0} G_{ij} - \frac{\beta^1}{\beta^0} a_i \delta_{ij}. \quad (\text{E17})$$

If $a_i - a_j = 1$ then S_{ij} drops out of this equation and its value is arbitrary (we choose to make it zero), but then T_{ij} is nonzero from the previous equation and its value is determined by this equation. If $a_i - a_j \neq 1$, then $T_{ij} = 0$ and the equation determines S_{ij} . Once we know S and T we can find J and L and hence determine the K matrix and the NLO expression for the scaling matrix σ .

The leading order anomalous dimension matrix for the 4-quark operators has $a_2 - a_3 = 1$ when the number of flavours is $N_f = 3$. This means that $T_{23} \neq 0$ and there is a $g^2 \log g$ term in K_{23} . Including the $g^2 \log g$ term allows us to avoid expanding the solution for the scaling matrix around $N_f = 3$. We checked that either method gives the same result for σ . For our check we shifted $a_2 \rightarrow a_2 + \delta$, computed the scaling matrix σ and took the limit $\delta \rightarrow 0$. As an additional check we also did the limiting procedure by shifting $\beta^0 \rightarrow 11 - 2(3 + \delta)/3$ when constructing $a = \gamma_D^0 / 2\beta^0$ in equation (E9) and again found the same result. We also learn and checked that we can add an arbitrary shift to S_{23} or J_{23} without changing the result for σ .

From equations (E14) we know that the only nonzero element of T in any scheme is $T_{23} = G_{23}/\beta^0$ and hence in any scheme,

$$L_{23} = \frac{(V^{-1} \gamma^1 V)_{23}}{\beta^0 (V^{-1})_{22} V_{33}} = -\frac{40}{27} = -1.48148. \quad (\text{E18})$$

We checked this for $\overline{\text{MS}}$, RI-MOM, RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ and RI-SMOM $^{(q, q)}$.

For γ^1 in $\overline{\text{MS}}$ we used Ref. [94] and for RI-MOM we used Ref. [95] (that paper does not give the 11 element; for this we used the value in Ref. [6]). For the RI-SMOM schemes we used $\gamma^{\overline{\text{MS}}, 1}$ together with the conversion factors Δr from RI-SMOM to $\overline{\text{MS}}$ in Ref. [33] and applied Eq. (E27) below.

To evaluate the perturbative scaling numerically, we used the 5-loop expression for the running strong coupling to evolve its value from the Z -mass to the charm mass, with quark-flavour thresholds at $\bar{m}_b(\bar{m}_b)$ and $\bar{m}_c(\bar{m}_c)$, the $\overline{\text{MS}}$ bottom and charm masses evaluated at their own scales [96–107]. We then evaluated $\alpha_s(\mu)$ for $2 \text{ GeV} \leq \mu \leq 3 \text{ GeV}$ for three flavours, corresponding to our 2 + 1 flavour simulations. Our inputs were [61]

$$\begin{aligned} \alpha_s(M_Z) &= 0.1180, \\ M_Z &= 91.1876 \text{ GeV}, \\ \bar{m}_b(\bar{m}_b) &= 4.18 \text{ GeV}, \\ \bar{m}_c(\bar{m}_c) &= 1.28 \text{ GeV}, \end{aligned} \quad (\text{E19})$$

and we determine

$$\alpha_s(2 \text{ GeV}) = 0.293347, \quad \alpha_s(3 \text{ GeV}) = 0.243580. \quad (\text{E20})$$

Changing from operators $Q^+(\mu)$ renormalised in some scheme at scale μ to RGI operators \hat{Q}^+ , is done by [95]:

$$\hat{Q}^+ = [\alpha(\mu)]^{-\gamma^0/2\beta^0} K^{-1}(\mu) Q^+(\mu). \quad (\text{E21})$$

The 11 element of this relation converts the kaon bag parameter $B_K(\mu)$ to the RGI \hat{B}_K (see for example the discussion in the 2019 FLAG review [108]). To compute RGI \hat{B}_i 's for the BSM operators, we would in addition have to take into account the quark-mass combination $m_s(\mu) + m_d(\mu)$ appearing in the $B_i(\mu)$'s definition.

3. Logarithmic term in NLO scaling expression

Here we will let $g' = g(\mu')$, $\alpha' = \alpha(\mu')$ and $g = g(\mu)$, $\alpha = \alpha(\mu)$. From equations (E11) and (E13), the $\log(g)$ terms in $\sigma(\mu', \mu)$ at NLO are

$$\begin{aligned} &\frac{1}{4\pi} (\alpha' \log g' L \sigma^0(\mu', \mu) - \alpha \log g \sigma^0(\mu', \mu) L) \\ &= \frac{1}{4\pi} V \left(\alpha' \log g' T \left(\frac{\alpha'}{\alpha} \right)^a - \alpha \log g \left(\frac{\alpha'}{\alpha} \right)^a T \right) V^{-1}. \end{aligned} \quad (\text{E22})$$

From above, the only nonzero element of T is T_{23} , so the term in parentheses becomes

$$\begin{aligned} T_{23} \left(\alpha' \log g' \left(\frac{\alpha'}{\alpha} \right)^{a_3} - \alpha \log g \left(\frac{\alpha'}{\alpha} \right)^{a_2} \right) \\ = T_{23} \left(\frac{\alpha'}{\alpha} \right)^{1/9} \alpha \log \left(\frac{g'}{g} \right) \end{aligned} \quad (\text{E23})$$

where we have used $a_2 = 1/9$ and $a_3 = -8/9$. Switching back to L and expressing everything in terms of α' and α we find

$$\text{NLO logs in } \sigma(\mu', \mu) = -\frac{5}{(27\pi)} \left(\frac{\alpha'}{\alpha}\right)^{1/9} \alpha \log\left(\frac{\alpha'}{\alpha}\right). \quad (\text{E24})$$

We checked explicitly for $\overline{\text{MS}}$, RI-MOM, RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ and RI-SMOM $^{(\not{d}, \not{d})}$ that the same term arises when we calculate the NLO scaling by the δ -shift and limit procedure.

4. Scheme conversion

Following Ref. [33] we define the matrix R to convert from renormalisation scheme A to scheme B by

$$Z^B = R^{B \leftarrow A} Z^A = (1 - \bar{a} \Delta r^{B \leftarrow A}) Z^A, \quad (\text{E25})$$

where $\bar{a} = g^2/16\pi^2$. From $\beta(g) dZ^B/dg = -\gamma^B(g) Z^B$ we find

$$\gamma^A = (R^{B \leftarrow A})^{-1} \gamma^B R^{B \leftarrow A} + \beta(R^{B \leftarrow A})^{-1} \frac{dR^{B \leftarrow A}}{dg}. \quad (\text{E26})$$

Expanding to $O(\bar{a}^2)$ and noting that γ^0 is universal, we can relate the NLO anomalous dimensions in schemes A and B using the 1-loop conversion factor $\Delta r^{B \leftarrow A}$,

$$\gamma^{A,1} = \gamma^{B,1} - [\gamma^0, \Delta r^{B \leftarrow A}] + 2\beta^0 \Delta r^{B \leftarrow A}. \quad (\text{E27})$$

In particular we can determine the NLO anomalous dimensions where A is RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ or RI-SMOM $^{(\not{d}, \not{d})}$ respectively, from the $\overline{\text{MS}}$ NLO anomalous dimension and $\Delta r^{\overline{\text{MS}} \leftarrow A}$

$$\gamma^{A,1} = \gamma^{\overline{\text{MS}},1} - [\gamma^0, \Delta r^{\overline{\text{MS}} \leftarrow A}] + 2\beta^0 \Delta r^{\overline{\text{MS}} \leftarrow A}. \quad (\text{E28})$$

By combining the scheme conversion equation (E27) with equations (E14) for J and L , we can show that

$$J^A - J^{\overline{\text{MS}}} = \Delta r^{\overline{\text{MS}} \leftarrow A} \quad \text{except for the 23 element} \quad (\text{E29})$$

and that $L^A = L^{\overline{\text{MS}}}$. We checked equation (E29) when A is RI-MOM, RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ or RI-SMOM $^{(\not{d}, \not{d})}$.

Appendix F: Further detailed numerical fit results

Our central value for the \mathcal{B}_i and R_i in $\overline{\text{MS}}$ at $\mu = 3 \text{ GeV}$ is taken as the mean of the conversion from RI-SMOM $^{(\gamma_\mu, \gamma_\mu)}$ and RI-SMOM $^{(\not{d}, \not{d})}$ to $\overline{\text{MS}}$. For completeness we also separately list these conversions with their full uncertainty budget in Table XXI.

TABLE XXI. Central values and combined systematic errors for ratio and bag parameters at $\mu = 3 \text{ GeV}$ in $\overline{\text{MS}}$ after converting from the two RI-SMOM schemes — (γ_μ, γ_μ) and (\not{q}, \not{q}) , in the SUSY basis. We list the errors arising from statistics, chiral extrapolation, residual chiral symmetry breaking, and discretisation and combine it into total uncertainties.

scheme		R_2	R_3	R_4	R_5	\mathcal{B}_1	\mathcal{B}_2	\mathcal{B}_3	\mathcal{B}_4	\mathcal{B}_5
$\overline{\text{MS}} \leftarrow \text{RI-SMOM}^{(\gamma_\mu, \gamma_\mu)}$	central	-18.73	5.781	41.45	10.80	0.5185	0.4759	0.728	0.8862	0.6977
	stat	0.60%	0.69%	0.72%	0.43%	0.28%	0.24%	0.72%	0.21%	1.02%
	chiral	0.21%	0.42%	0.61%	0.46%	0.20%	0.17%	0.29%	0.17%	0.25%
	rscb	0.10%	0.15%	0.09%	0.03%	0.04%	0.03%	0.06%	0.01%	0.00%
	discr	0.16%	0.53%	0.49%	1.23%	0.01%	0.44%	1.61%	0.16%	0.38%
	total	0.66%	0.98%	1.07%	1.38%	0.35%	0.53%	1.79%	0.32%	1.12%
$\overline{\text{MS}} \leftarrow \text{RI-SMOM}^{(\not{q}, \not{q})}$	central	-19.07	6.059	42.43	10.49	0.5295	0.4829	0.764	0.9070	0.6788
	stat	0.68%	0.92%	0.81%	0.83%	0.29%	0.43%	1.24%	0.36%	2.21%
	chiral	0.48%	0.78%	1.25%	1.26%	0.24%	0.27%	0.44%	0.29%	0.51%
	rscb	0.29%	0.21%	0.23%	0.13%	0.08%	0.19%	0.29%	0.03%	0.01%
	discr	0.34%	0.65%	0.20%	2.30%	0.10%	0.64%	1.92%	0.19%	0.10%
	total	0.95%	1.39%	1.52%	2.75%	0.40%	0.83%	2.34%	0.50%	2.27%

- [1] J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, "Evidence for the 2π decay of the k_2^0 meson," *Phys. Rev. Lett.* **13**, 138–140 (1964).
- [2] Andrzej Buras, *Gauge Theory of Weak Decays* (Cambridge University Press, 2020).
- [3] Ikaros I. Bigi and A. I. Sanda, *CP violation*, Vol. 9 (Cambridge University Press, 2009).
- [4] Gustavo C. Branco, Luis Lavoura, and Joao P. Silva, *CP Violation*, Vol. 103 (Oxford University Press, 1999).
- [5] T. Inami and C. S. Lim, "Effects of Superheavy Quarks and Leptons in Low-Energy Weak Processes $k(L) \rightarrow \mu$ anti- μ , $K^+ \rightarrow \pi^+$ Neutrino anti-neutrino and $K^0 \leftrightarrow$ anti- K^0 ," *Prog. Theor. Phys.* **65**, 297 (1981), [Erratum: *Prog.Theor.Phys.* **65**, 1772 (1981)].
- [6] Y. Aoki, R. Arthur, T. Blum, P.A. Boyle, D. Brommel, *et al.*, "Continuum Limit of B_K from 2+1 Flavor Domain Wall QCD," *Phys.Rev.* **D84**, 014503 (2011), [arXiv:1012.4178 \[hep-lat\]](#).
- [7] T. Blum *et al.* (RBC, UKQCD), "Domain wall QCD with physical quark masses," *Phys. Rev. D* **93**, 074505 (2016), [arXiv:1411.7017 \[hep-lat\]](#).
- [8] Taegil Bae *et al.* (SWME), "Neutral kaon mixing from new physics: matrix elements in $N_f = 2 + 1$ lattice QCD," *Phys. Rev.* **D88**, 071503 (2013), [arXiv:1309.2040 \[hep-lat\]](#).
- [9] Benjamin J. Choi *et al.* (SWME), "Kaon BSM B-parameters using improved staggered fermions from $N_f = 2+1$ unquenched QCD," *Phys. Rev. D* **93**, 014511 (2016), [arXiv:1509.00592 \[hep-lat\]](#).
- [10] Taegil Bae *et al.* (SWME), "Improved determination of BK with staggered quarks," *Phys. Rev.* **D89**, 074504 (2014), [arXiv:1402.0048 \[hep-lat\]](#).
- [11] N. Carrasco, P. Dimopoulos, R. Frezzotti, V. Lubicz, G. C Rossi, S. Simula, and C. Tarantino (ETM), " $\Delta S=2$ and $\Delta C=2$ bag parameters in the standard model and beyond from $N_f=2+1+1$ twisted-mass lattice QCD," *Phys. Rev. D* **92**, 034516 (2015), [arXiv:1505.06639 \[hep-lat\]](#).
- [12] V. Bertone *et al.* (ETM), "Kaon Mixing Beyond the SM from $N_f=2$ tmQCD and model independent constraints from the UTA," *JHEP* **03**, 089 (2013), [Erratum: *JHEP*07,143(2013)], [arXiv:1207.1287 \[hep-lat\]](#).
- [13] Y. Aoki *et al.* (Flavour Lattice Averaging Group (FLAG)), "FLAG Review 2021," *Eur. Phys. J. C* **82**, 869 (2022), [arXiv:2111.09849 \[hep-lat\]](#).
- [14] Gerhard Buchalla, Andrzej J. Buras, and Markus E. Lautenbacher, "Weak decays beyond leading logarithms," *Rev. Mod. Phys.* **68**, 1125–1144 (1996), [arXiv:hep-ph/9512380](#).
- [15] Norman H. Christ, "Computing the long-distance contribution to the kaon mixing parameter ϵ_K ," *Proceedings, 29th International Symposium on Lattice field theory (Lattice 2011): Squaw Valley, Lake Tahoe, USA, July 10-16, 2011*, PoS **LATTICE2011**, 277 (2011), [arXiv:1201.2065 \[hep-lat\]](#).
- [16] Norman H. Christ and Ziyuan Bai, "Computing the long-distance contributions to ϵ_K ," *Proceedings, 33rd International Symposium on Lattice Field Theory (Lattice 2015): Kobe, Japan, July 14-18, 2015*, PoS **LATTICE2015**, 342 (2016).
- [17] Ziyuan Bai, "Long distance part of ϵ_K from lattice QCD," *Proceedings, 34th International Symposium on Lattice Field Theory (Lattice 2016): Southampton, UK, July 24-30, 2016*, PoS **LATTICE2016**, 309 (2017), [arXiv:1611.06601 \[hep-lat\]](#).
- [18] Bigeng Wang, "Calculating Δm_K with lattice QCD," *PoS LATTICE2021*, 141 (2022), [arXiv:2301.01387 \[hep-lat\]](#).
- [19] Ziyuan Bai, Norman H. Christ, Joseph M. Karpie, Christopher T. Sachrajda, Amarjit Soni, and Bigeng Wang, "Long-distance contribution to ϵ_K from lattice QCD," *Phys. Rev. D* **109**, 054501 (2024), [arXiv:2309.01193 \[hep-lat\]](#).
- [20] G. Anzivino *et al.*, "Workshop summary – Kaons@CERN 2023," in *Kaons@CERN 2023* (2023) [arXiv:2311.02923 \[hep-ph\]](#).
- [21] Andrzej J. Buras, "Kaon Theory: 50 Years Later," (2023) [arXiv:2307.15737 \[hep-ph\]](#).
- [22] Evgueni Goudzovski *et al.*, "Weak Decays of Strange and Light Quarks," (2022), [arXiv:2209.07156 \[hep-ex\]](#).
- [23] V. Bertone *et al.* (ETM), "Kaon Mixing Beyond the SM from $N_f=2$ tmQCD and model independent constraints from the UTA," *JHEP* **03**, 089 (2013), [Erratum: *JHEP* 07, 143 (2013)], [arXiv:1207.1287 \[hep-lat\]](#).
- [24] Laurent Lellouch, "Flavor physics and lattice quantum chromodynamics," in *Les Houches Summer School: Session 93: Modern perspectives in lattice QCD: Quantum field theory and high performance computing* (2011) pp. 629–698, [arXiv:1104.5484 \[hep-lat\]](#).
- [25] C. R. Allton, L. Conti, A. Donini, V. Gimenez, Leonardo Giusti, G. Martinelli, M. Talevi, and A. Vladikas, "B parameters for Delta S = 2 supersymmetric operators," *Phys. Lett.* **B453**, 30–39 (1999), [arXiv:hep-lat/9806016 \[hep-lat\]](#).
- [26] A. Donini, V. Gimenez, Leonardo Giusti, and G. Martinelli, "Renormalization group invariant matrix elements of Delta S = 2 and Delta I = 3/2 four fermion operators without quark masses," *Phys. Lett.* **B470**, 233–242 (1999), [arXiv:hep-lat/9910017 \[hep-lat\]](#).
- [27] Ronald Babich, Nicolas Garron, Christian Hoelbling, Joseph Howard, Laurent Lellouch, and Claudio Rebbi, "K0 - anti-0 mixing beyond the standard model and CP-violating electroweak penguins in quenched QCD with exact chiral symmetry," *Phys. Rev.* **D74**, 073009 (2006), [arXiv:hep-lat/0605016 \[hep-lat\]](#).
- [28] P. A. Boyle, N. Garron, and R. J. Hudspith (RBC, UKQCD), "Neutral kaon mixing beyond the standard model with $N_f = 2 + 1$ chiral fermions," *Phys. Rev.* **D86**, 054028 (2012), [arXiv:1206.5737 \[hep-lat\]](#).
- [29] Nicolas Garron, Renwick J. Hudspith, and Andrew T. Lytle (RBC/UKQCD), "Neutral Kaon Mixing Beyond the Standard Model with $N_f = 2 + 1$ Chiral Fermions Part 1: Bare Matrix Elements and Physical Results," *JHEP* **11**, 001 (2016), [arXiv:1609.03334 \[hep-lat\]](#).
- [30] Jaehoon Leem *et al.* (SWME), "Calculation of BSM Kaon B-parameters using Staggered Quarks," *Proceedings, 32nd International Symposium on Lattice Field Theory (Lattice 2014): Brookhaven, NY, USA, June 23-28, 2014*, PoS **LATTICE2014**, 370 (2014), [arXiv:1411.1501 \[hep-lat\]](#).

- [31] J. Tobias Tsang and Michele Della Morte, “B-physics from lattice gauge theory,” (2023), [10.1140/epjs/s11734-023-01011-3](https://arxiv.org/abs/2310.02705), [arXiv:2310.02705](https://arxiv.org/abs/2310.02705) [hep-lat].
- [32] Peter A. Boyle, Luigi Del Debbio, Andreas Juttner, Ava Khamseh, Francesco Sanfilippo, and Justus Tobias Tsang, “The decay constants f_D and f_{D_s} in the continuum limit of $N_f = 2 + 1$ domain wall lattice QCD,” *JHEP* **12**, 008 (2017), [arXiv:1701.02644](https://arxiv.org/abs/1701.02644) [hep-lat].
- [33] Peter A. Boyle, Nicolas Garron, Renwick J. Hudspith, Christoph Lehner, and Andrew T. Lytle (RBC, UKQCD), “Neutral kaon mixing beyond the Standard Model with $n_f = 2 + 1$ chiral fermions. Part 2: non perturbative renormalisation of the $\Delta F = 2$ four-quark operators,” *JHEP* **10**, 054 (2017), [arXiv:1708.03552](https://arxiv.org/abs/1708.03552) [hep-lat].
- [34] Peter Boyle, Nicolas Garron, Julia Kettle, Ava Khamseh, and Justus Tobias Tsang, “BSM Kaon Mixing at the Physical Point,” *EPJ Web Conf.* **175**, 13010 (2018), [arXiv:1710.09176](https://arxiv.org/abs/1710.09176) [hep-lat].
- [35] Peter Boyle, Nicolas Garron, Renwick James Hudspith, Andreas Juttner, Julia Kettle, Ava Khamseh, and Justus Tobias Tsang, “Beyond the Standard Model Kaon Mixing with Physical Masses,” *PoS LATTICE2018*, 285 (2019), [arXiv:1812.04981](https://arxiv.org/abs/1812.04981) [hep-lat].
- [36] A. Nicholson *et al.*, “Heavy physics contributions to neutrinoless double beta decay from QCD,” *Phys. Rev. Lett.* **121**, 172501 (2018), [arXiv:1805.02634](https://arxiv.org/abs/1805.02634) [nucl-th].
- [37] William Detmold, William I. Jay, David J. Murphy, Patrick R. Oare, and Phiala E. Shanahan, “Neutrinoless double beta decay from lattice QCD: The short-distance $\pi \rightarrow \pi + e^- e^-$ amplitude,” *Phys. Rev. D* **107**, 094501 (2023), [arXiv:2208.05322](https://arxiv.org/abs/2208.05322) [hep-lat].
- [38] F. Gabbiani, E. Gabrielli, A. Masiero, and L. Silvestrini, “A Complete analysis of FCNC and CP constraints in general SUSY extensions of the standard model,” *Nucl. Phys.* **B477**, 321–352 (1996), [arXiv:hep-ph/9604387](https://arxiv.org/abs/hep-ph/9604387) [hep-ph].
- [39] Taegil Bae *et al.* (SWME), “Neutral kaon mixing from new physics: matrix elements in $N_f = 2 + 1$ lattice QCD,” *Phys. Rev. D* **88**, 071503 (2013), [arXiv:1309.2040](https://arxiv.org/abs/1309.2040) [hep-lat].
- [40] Damir Becirevic and Giovanni Villadoro, “Remarks on the hadronic matrix elements relevant to the SUSY K_0 - anti- K_0 mixing amplitude,” *Phys. Rev. D* **70**, 094036 (2004), [arXiv:hep-lat/0408029](https://arxiv.org/abs/hep-lat/0408029).
- [41] C. Allton *et al.* (RBC-UKQCD Collaboration), “Physical Results from 2+1 Flavor Domain Wall QCD and SU(2) Chiral Perturbation Theory,” *Phys.Rev.* **D78**, 114509 (2008), [arXiv:0804.0473](https://arxiv.org/abs/0804.0473) [hep-lat].
- [42] Y. Aoki *et al.* (RBC Collaboration, UKQCD Collaboration), “Continuum Limit Physics from 2+1 Flavor Domain Wall QCD,” *Phys.Rev.* **D83**, 074508 (2011), [arXiv:1011.0892](https://arxiv.org/abs/1011.0892) [hep-lat].
- [43] Peter A. Boyle, Luigi Del Debbio, Nicolas Garron, Andreas Juttner, Amarjit Soni, Justus Tobias Tsang, and Oliver Witzel (RBC/UKQCD), “SU(3)-breaking ratios for $D_{(s)}$ and $B_{(s)}$ mesons,” (2018), [arXiv:1812.08791](https://arxiv.org/abs/1812.08791) [hep-lat].
- [44] Y. Iwasaki and T. Yoshie, “Renormalization Group Improved Action for SU(3) Lattice Gauge Theory and the String Tension,” *Phys. Lett.* **143B**, 449–452 (1984).
- [45] Y. Iwasaki, “Renormalization Group Analysis of Lattice Theories and Improved Lattice Action: Two-Dimensional Nonlinear O(N) Sigma Model,” *Nucl. Phys.* **B258**, 141–156 (1985).
- [46] Richard C. Brower, Hartmut Neff, and Kostas Orginos, “Mobius fermions: Improved domain wall chiral fermions,” *Nucl.Phys.Proc.Suppl.* **140**, 686–688 (2005), [arXiv:hep-lat/0409118](https://arxiv.org/abs/hep-lat/0409118) [hep-lat].
- [47] R.C. Brower, H. Neff, and K. Orginos, “Mobius fermions,” *Nucl.Phys.Proc.Suppl.* **153**, 191–198 (2006), [arXiv:hep-lat/0511031](https://arxiv.org/abs/hep-lat/0511031) [hep-lat].
- [48] Richard C. Brower, Hartmut Neff, and Kostas Orginos, “The Möbius domain wall fermion algorithm,” *Comput. Phys. Commun.* **220**, 1–19 (2017), [arXiv:1206.5214](https://arxiv.org/abs/1206.5214) [hep-lat].
- [49] Yigal Shamir, “Chiral fermions from lattice boundaries,” *Nucl.Phys.* **B406**, 90–106 (1993), [arXiv:hep-lat/9303005](https://arxiv.org/abs/hep-lat/9303005) [hep-lat].
- [50] Peter A. Boyle, Guido Cossu, Azusa Yamaguchi, and Antonin Portelli, “Grid: A next generation data parallel C++ QCD library,” *PoS LATTICE2015*, 023 (2016).
- [51] Azusa Yamaguchi, Peter Boyle, Guido Cossu, Gianluca Filaci, Christoph Lehner, and Antonin Portelli, “Grid: OneCode and FourAPIs,” *PoS LATTICE2021*, 035 (2022), [arXiv:2203.06777](https://arxiv.org/abs/2203.06777) [hep-lat].
- [52] Antonin Portelli, Nelson Lachini, Felix Erben, Michael Marshall, Fabian Joswig, Raoul Hodgson, Fionn Ó hÓgáin, Vera Gülpers, Peter Boyle, Nils Asmussen, Ryan Hill, Alessandro Barone, James Richings, Ryan Abbott, Simon Bürger, and Joseph Lee, “*aportelli/hadrons: Hadrons v1.4*,” (2023).
- [53] M. Foster and Christopher Michael (UKQCD), “Quark mass dependence of hadron masses from lattice QCD,” *Phys. Rev. D* **59**, 074503 (1999), [arXiv:hep-lat/9810021](https://arxiv.org/abs/hep-lat/9810021).
- [54] C. McNeile and Christopher Michael (UKQCD), “Decay width of light quark hybrid meson from the lattice,” *Phys. Rev. D* **73**, 074506 (2006), [arXiv:hep-lat/0603007](https://arxiv.org/abs/hep-lat/0603007).
- [55] P. A. Boyle, A. Jüttner, C. Kelly, and R. D. Kenway, “Use of stochastic sources for the lattice determination of light quark physics,” *JHEP* **08**, 086 (2008), [arXiv:0804.1501](https://arxiv.org/abs/0804.1501) [hep-lat].
- [56] G. Martinelli, C. Pittori, Christopher T. Sachrajda, M. Testa, and A. Vladikas, “A General method for nonperturbative renormalization of lattice operators,” *Nucl.Phys.* **B445**, 81–108 (1995), [arXiv:hep-lat/9411010](https://arxiv.org/abs/hep-lat/9411010) [hep-lat].
- [57] C. Sturm, Y. Aoki, N.H. Christ, T. Izubuchi, C.T.C. Sachrajda, *et al.*, “Renormalization of quark bilinear operators in a momentum-subtraction scheme with a nonexceptional subtraction point,” *Phys.Rev.* **D80**, 014501 (2009), [arXiv:0901.2599](https://arxiv.org/abs/0901.2599) [hep-ph].
- [58] R. Arthur and P.A. Boyle (RBC Collaboration, UKQCD Collaboration), “Step Scaling with off-shell renormalisation,” *Phys.Rev.* **D83**, 114511 (2011), [arXiv:1006.0422](https://arxiv.org/abs/1006.0422) [hep-lat].
- [59] R. Arthur, P.A. Boyle, N. Garron, C. Kelly, and A.T. Lytle (RBC Collaboration, UKQCD Collaboration), “Opening the Rome-Southampton window for operator mixing matrices,” *Phys.Rev.* **D85**, 014501 (2012), [arXiv:1109.1223](https://arxiv.org/abs/1109.1223) [hep-lat].
- [60] P.A. Boyle, N. Garron, and A.T. Lytle (RBC-UKQCD Collaboration), “Non-perturbative running and renormalization of kaon four-quark operators with $n_f=2+1$ domain-wall fermions,” *PoS LATTICE2011*, 227 (2011), [arXiv:1112.0537](https://arxiv.org/abs/1112.0537) [hep-lat].

- [61] R. L. Workman *et al.* (Particle Data Group), “Review of Particle Physics,” *PTEP* **2022**, 083C01 (2022).
- [62] Damir Becirevic and Giovanni Villadoro, “Impact of the finite volume effects on the chiral behavior of $f(K)$ and $B(K)$,” *Phys. Rev. D* **69**, 054010 (2004), arXiv:hep-lat/0311028.
- [63] C. Alexandrou *et al.* (Extended Twisted Mass), “Quark masses using twisted-mass fermion gauge ensembles,” *Phys. Rev. D* **104**, 074515 (2021), arXiv:2104.13408 [hep-lat].
- [64] A. Bazavov *et al.* (Fermilab Lattice, MILC, TUMQCD), “Up-, down-, strange-, charm-, and bottom-quark masses from four-flavor lattice QCD,” *Phys. Rev. D* **98**, 054517 (2018), arXiv:1802.04248 [hep-lat].
- [65] N. Carrasco *et al.* (European Twisted Mass), “Up, down, strange and charm quark masses with $N_f = 2+1+1$ twisted mass lattice QCD,” *Nucl. Phys. B* **887**, 19–68 (2014), arXiv:1403.4504 [hep-lat].
- [66] D. Giusti, V. Lubicz, C. Tarantino, G. Martinelli, F. Sanfilippo, S. Simula, and N. Tantalo, “Leading isospin-breaking corrections to pion, kaon and charmed-meson masses with Twisted-Mass fermions,” *Phys. Rev. D* **95**, 114504 (2017), arXiv:1704.06561 [hep-lat].
- [67] A. T. Lytle, C. T. H. Davies, D. Hatton, G. P. Lepage, and C. Sturm (HPQCD), “Determination of quark masses from $n_f = 4$ lattice QCD and the RI-SMOM intermediate scheme,” *Phys. Rev. D* **98**, 014513 (2018), arXiv:1805.06225 [hep-lat].
- [68] Bipasha Chakraborty, C. T. H. Davies, B. Galloway, P. Knecht, J. Koponen, G. C. Donald, R. J. Dowdall, G. P. Lepage, and C. McNeile, “High-precision quark masses and QCD coupling from $n_f = 4$ lattice QCD,” *Phys. Rev. D* **91**, 054508 (2015), arXiv:1408.4169 [hep-lat].
- [69] Mattia Bruno, Isabel Campos, Patrick Fritsch, Jonna Koponen, Carlos Pena, David Preti, Alberto Ramos, and Anastassios Vladikas (ALPHA), “Light quark masses in $N_f = 2 + 1$ lattice QCD with Wilson fermions,” *Eur. Phys. J. C* **80**, 169 (2020), arXiv:1911.08025 [hep-lat].
- [70] S. Durr, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, T. Kurth, L. Lellouch, T. Lippert, K. K. Szabo, and G. Vulvert, “Lattice QCD at the physical point: Simulation and analysis details,” *JHEP* **08**, 148 (2011), arXiv:1011.2711 [hep-lat].
- [71] S. Durr, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, T. Kurth, L. Lellouch, T. Lippert, K. K. Szabo, and G. Vulvert, “Lattice QCD at the physical point: light quark masses,” *Phys. Lett. B* **701**, 265–268 (2011), arXiv:1011.2403 [hep-lat].
- [72] A. Bazavov *et al.*, “Staggered chiral perturbation theory in the two-flavor case and SU(2) analysis of the MILC data,” *PoS LATTICE2010*, 083 (2010), arXiv:1011.1792 [hep-lat].
- [73] M. Bruno, I. Campos, J. Koponen, Carlos Pena, David Preti, Alberto Ramos, and Anastassios Vladikas (ALPHA), “Light and strange quark masses from $N_f = 2 + 1$ simulations with Wilson fermions,” *PoS LATTICE2018*, 220 (2019), arXiv:1903.04094 [hep-lat].
- [74] A. Bazavov *et al.* (MILC), “MILC results for light pseudoscalars,” *PoS CD09*, 007 (2009), arXiv:0910.2966 [hep-ph].
- [75] C. McNeile, C. T. H. Davies, E. Follana, K. Hornbostel, and G. P. Lepage, “High-Precision c and b Masses, and QCD Coupling from Current-Current Correlators in Lattice and Continuum QCD,” *Phys. Rev. D* **82**, 034512 (2010), arXiv:1004.4285 [hep-lat].
- [76] S. Durr *et al.* (BMW), “Precision computation of the kaon bag parameter,” *Phys. Lett. B* **705**, 477–481 (2011), arXiv:1106.3230 [hep-lat].
- [77] Jack Laiho and Ruth S. Van de Water, “Pseudoscalar decay constants, light-quark masses, and B_K from mixed-action lattice QCD,” *PoS LATTICE2011*, 293 (2011), arXiv:1112.4861 [hep-lat].
- [78] Justus Tobias Tsang, “Neutral meson mixing in the $B_{(s)}^0$ sector from Lattice QCD,” *PoS CKM2021*, 107 (2023), arXiv:2204.01259 [hep-lat].
- [79] Peter Boyle, Felix Erben, Andreas Jüttner, Takashi Kaneko, Michael Marshall, Antonin Portelli, Oliver Witzel, Luigi Del Debbio, and Justus Tobias Tsang, “BSM $B - \bar{B}$ mixing on JLQCD and RBC/UKQCD $N_f = 2 + 1$ DWF ensembles,” *PoS LATTICE2021*, 224 (2022), arXiv:2111.11287 [hep-lat].
- [80] Y. Aoki *et al.* (RBC, UKQCD), “Continuum Limit Physics from 2+1 Flavor Domain Wall QCD,” *Phys. Rev. D* **83**, 074508 (2011), arXiv:1011.0892 [hep-lat].
- [81] C. Urbach, K. Jansen, A. Shindler, and U. Wenger, “HMC algorithm with multiple time scale integration and mass preconditioning,” *Comput. Phys. Commun.* **174**, 87–98 (2006), arXiv:hep-lat/0506011.
- [82] Yu-Chih Chen and Ting-Wai Chiu (TWQCD), “Exact Pseudofermion Action for Monte Carlo Simulation of Domain-Wall Fermion,” *Phys. Lett. B* **738**, 55–60 (2014), arXiv:1403.1683 [hep-lat].
- [83] C. Jung, C. Kelly, R. D. Mawhinney, and D. J. Murphy, “Domain Wall Fermion QCD with the Exact One Flavor Algorithm,” *Phys. Rev. D* **97**, 054503 (2018), arXiv:1706.05843 [hep-lat].
- [84] M. A. Clark, Balint Joo, A. D. Kennedy, and P. J. Silva, “Improving dynamical lattice QCD simulations through integrator tuning using Poisson brackets and a force-gradient integrator,” *Phys. Rev. D* **84**, 071502 (2011), arXiv:1108.1828 [hep-lat].
- [85] Hantao Yin and Robert D. Mawhinney, “Improving DWF Simulations: the Force Gradient Integrator and the Möbius Accelerated DWF Solver,” *PoS LATTICE2011*, 051 (2011), arXiv:1111.5059 [hep-lat].
- [86] Markus Fierz, “Zur Fermischen Theorie des β -Zerfalls,” *Z. Phys.* **104**, 553–565 (1937).
- [87] Nicolas Garron, “Fierz transformations and renormalization schemes for fourquark operators,” *EPJ Web Conf.* **175**, 10005 (2018).
- [88] Y. Aoki, P.A. Boyle, N.H. Christ, C. Dawson, M.A. Donnellan, *et al.*, “Non-perturbative renormalization of quark bilinear operators and $B(K)$ using domain wall fermions,” *Phys.Rev.* **D78**, 054510 (2008), arXiv:0712.1061 [hep-lat].
- [89] Peter Boyle, Luigi Del Debbio, and Ava Khamseh, “Massive momentum-subtraction scheme,” *Phys. Rev. D* **95**, 054505 (2017), arXiv:1611.06908 [hep-lat].
- [90] Luigi Del Debbio, Felix Erben, Jonathan Flynn, Rajnandini Mukherjee, and J. Tobias Tsang, “Charm quark mass using a massive nonperturbative renormalisation scheme,” (2023), arXiv:2312.16537 [hep-lat].
- [91] Nicolas Garron, Caroline Cahill, Martin Gorbahn, J. A. Gracey, and P. E. L. Rakow, “Exploring interpolating momentum schemes,” *PoS LATTICE2021*, 438 (2022), arXiv:2202.04394 [hep-lat].

- [92] “[Mathematica, Version 14.0](#),” (2024), Wolfram Research Inc., Champaign, IL.
- [93] Marco Ciuchini, E. Franco, V. Lubicz, G. Martinelli, I. Scimemi, and L. Silvestrini, “Next-to-leading order QCD corrections to $\Delta F = 2$ effective Hamiltonians,” *Nucl. Phys. B* **523**, 501–525 (1998), [arXiv:hep-ph/9711402](#).
- [94] Andrzej J. Buras, Mikolaj Misiak, and Jorg Urban, “Two loop QCD anomalous dimensions of flavor changing four quark operators within and beyond the standard model,” *Nucl. Phys. B* **586**, 397–426 (2000), [arXiv:hep-ph/0005183](#).
- [95] Mauro Papinutto, Carlos Pena, and David Preti, “On the perturbative renormalization of four-quark operators for new physics,” *Eur. Phys. J. C* **77**, 376 (2017), [Erratum: *Eur.Phys.J.C* 78, 21 (2018)], [arXiv:1612.06461 \[hep-lat\]](#).
- [96] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, “Five-Loop Running of the QCD coupling constant,” *Phys. Rev. Lett.* **118**, 082002 (2017), [arXiv:1606.08659 \[hep-ph\]](#).
- [97] F. Herzog, B. Ruijl, T. Ueda, J. A. M. Vermaseren, and A. Vogt, “The five-loop beta function of Yang-Mills theory with fermions,” *JHEP* **02**, 090 (2017), [arXiv:1701.01404 \[hep-ph\]](#).
- [98] Thomas Luthe, Andreas Maier, Peter Marquard, and York Schroder, “Complete renormalization of QCD at five loops,” *JHEP* **03**, 020 (2017), [arXiv:1701.07068 \[hep-ph\]](#).
- [99] K. G. Chetyrkin, Johann H. Kuhn, and Christian Sturm, “QCD decoupling at four loops,” *Nucl. Phys. B* **744**, 121–135 (2006), [arXiv:hep-ph/0512060](#).
- [100] Y. Schroder and M. Steinhauser, “Four-loop decoupling relations for the strong coupling,” *JHEP* **01**, 051 (2006), [arXiv:hep-ph/0512058](#).
- [101] Tao Liu and Matthias Steinhauser, “Decoupling of heavy quarks at four loops and effective Higgs-fermion coupling,” *Phys. Lett. B* **746**, 330–334 (2015), [arXiv:1502.04719 \[hep-ph\]](#).
- [102] K. G. Chetyrkin, Johann H. Kuhn, and M. Steinhauser, “RunDec: A Mathematica package for running and decoupling of the strong coupling and quark masses,” *Comput. Phys. Commun.* **133**, 43–65 (2000), [arXiv:hep-ph/0004189](#).
- [103] Barbara Schmidt and Matthias Steinhauser, “CRUnDec: a C++ package for running and decoupling of the strong coupling and quark masses,” *Comput. Phys. Commun.* **183**, 1845–1848 (2012), [arXiv:1201.6149 \[hep-ph\]](#).
- [104] Florian Herren and Matthias Steinhauser, “Version 3 of RunDec and CRUnDec,” *Comput. Phys. Commun.* **224**, 333–345 (2018), [arXiv:1703.03751 \[hep-ph\]](#).
- [105] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, “Quark Mass and Field Anomalous Dimensions to $\mathcal{O}(\alpha_s^5)$,” *JHEP* **10**, 076 (2014), [arXiv:1402.6611 \[hep-ph\]](#).
- [106] Thomas Luthe, Andreas Maier, Peter Marquard, and York Schröder, “Five-loop quark mass and field anomalous dimensions for a general gauge group,” *JHEP* **01**, 081 (2017), [arXiv:1612.05512 \[hep-ph\]](#).
- [107] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, “Five-loop fermion anomalous dimension for a general gauge group from four-loop massless propagators,” *JHEP* **04**, 119 (2017), [arXiv:1702.01458 \[hep-ph\]](#).
- [108] S. Aoki *et al.* (Flavour Lattice Averaging Group), “FLAG review 2019: Flavour lattice averaging group (FLAG),” *Eur. Phys. J. C* **80**, 113 (2020), [arXiv:1902.08191 \[hep-lat\]](#).