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Minutes of the Forum of

Symbolic Computing for Accelerator Physics

held on Thursday 7 July 1994

Present: Y. Alexahin, B. Autin (Chairman), G. Dôme, M. Giovannozzi, A. Hilaire, J. Jowett (Deputy), E. Keil, M. Martini (Secretary), D. Manglunki, J. Paul, T. Pettersson, G. Sabbi, B. Sagnell, J.P. Thibonnier, A. Verdier, B. Zotter.

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1 Longitudinal coupling impedance for disc loaded wave guides: G. Dôme, J. Paul

The electromagnetic field produced by a single particle travelling on the axis of a structure made of parallel plates with circular holes, has been calculated together with the longitudinal coupling impedance.

The coupling impedance is defined in the frequency domain as the ratio of the line integral of the electric field (i.e. the wake potential across the structure) to the beam current at a given frequency. This integral involves a sum of products of two Bessel functions of fractional order. The asymptotic expansion of this function has a limited number of terms directly related to the arguments of the functions. The symbolic computing is of basic interest to compute the coefficients of the product as soon as the arguments exceed a few units.

Furthermore, the treatment of complex arguments is treated by *Mathematica* without having to resort to the real and imaginary part independently. The coupling impedance is produced in the form of graphs for which the computing time of one point may take up to one hour, an improvement of an order of magnitude over numerical techniques applied in the past. Nevertheless, efforts will be pursued to use the fastest existing computers (DEC ALPHA) to further reduce the computing time.

Details on the theory and the program are given in the attached slides.

2 Hamiltonian second order perturbation theory formulae for a two family sextupole arrangement: Y. Alexahin

The oscillatory motion of a particle travelling in nonlinear fields is studied using a second-order perturbation theory based on the Hori-Deprit algorithm. The second-order tune shift of the particles resulting from chromaticity sextupoles may not be acceptable and thus controlled by dedicated sextupolar fields.

A compensation scheme for the second-order tune shift is proposed using two intertwined sextupole families per half lattice superperiod.

See the attached slides.

The next meeting will be held on:

Thursday 29 September at 16.00 hr in the PS Auditorium - Meyrin, Bldg 6, 2-024

The date will be confirmed later.

M. Martini

Distribution list

AT, MT, PS and SL Division Leaders and Deputies.
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EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH
CERN-SL DIVISION

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(Revised)

A method for computing the longitudinal coupling impedance of circular apertures in a periodic array of infinite planes

G. Dôme [1] , L. Palumbo [2 3] , V.G. Vaccaro [4 5] , L. Verolino [6]

Abstract

The diffraction of the electromagnetic field created by a charge travelling on the axis of circular apertures in a set of perfectly conducting infinite planes is described by the field travelling with the charge itself and by the radiation from the plates, which has also a travelling wave character. Accordingly we represent all the fields as a superposition of two parts: a part generated by the charge in free space and a part created by the presence of the screens, which together must satisfy the boundary conditions. These are generally of mixed type (on the plate and in the hole) and lead to two integral equations. A general procedure is shown to transform this system into only one Fredholm integral equation of the second kind.

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[1]CERN, SL Division, CH-1211 Geneva 23, Switzerland.
[2]Dip. di Energetica, Università *La Sapienza*, Via A. Scarpa 14, I-00161, Rome, Italy.
[3]INFN-LNF, Via E. Fermi 40, I-00044, Frascati.
[4]Dip. di Scienze Fisiche, Università *Federico II*, Mostra d'Oltremare, Pad. 20, I-80125, Naples, Italy.
[5]INFN Sezione di Napoli, Mostra d'Oltremare, Pad. 20, I-80125, Naples, Italy.
[6]Fellow at CERN, SL Division, CH-1211 Geneva 23, Switzerland.

1 Introduction

Let us consider a particle of charge Q travelling at velocity $\vec{v}$ on the axis of circular holes in a set of parallel planes. The geometry is shown in Figure 1.

We shall use cylindrical coordinates whose $\hat{z}$ axis passes through the center of the apertures and is perpendicular to the plane of the screens. We shall assume that the charge moves in the positive $\hat{z}$ direction.

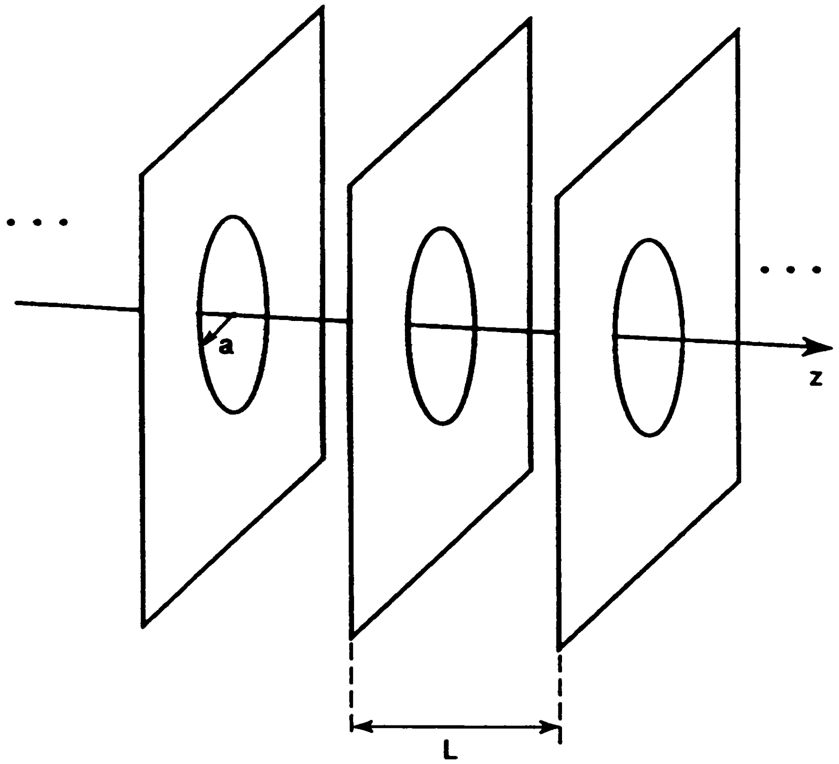


Figure 1: *Periodic array of circular apertures in infinite plane screens.*

The theory of the diffraction of a plane wave by a circular aperture in an infinite screen can easily be found in the literature [1, 2]. The problem is usually analysed by modal expansion methods which give the solution as an infinite sum of eigenfunctions of the wave equation in a particular coordinate system. However, this solution has the shortcoming of being badly convergent, especially for the case of short wavelengths. It is well known that a point charge crossing the hole will excite a continuous spectrum of frequencies, which extends to very high frequencies for ultrarelativistic charges; this general feature of the diffraction-radiation problem makes modal expansion really impracticable for our problem, even for an approximate solution.

The charge moving with uniform velocity in vacuum radiates only because of the optical inhomogeneities present near its path. The radiation is due to the diffraction of the field at the edges of the holes [3].

The field created by the charge in the presence of the screen will interact with the charge itself so that, together with the phenomenon of radiation, we should find a decrease of the particle velocity [4, 5]. Such a radiation problem is very difficult to solve and therefore a

We can now integrate Green's function multiplied by the current $\vec{J}(r_0, z_0; \omega)$ over the whole space V_0, even if the actual current flows on the plates only for $r > a$ (hole radius). Eventually we shall impose the condition that the current vanishes in the hole, which is a condition for the Hankel transform of the current. We obtain

$$\begin{aligned} \Pi_r(r, z; \omega) &= \frac{j}{4\pi^2 \omega \varepsilon} \sum_{n=-\infty}^{+\infty} \int_{V_0} J_r^n(r_0, z_0; \omega) \left\{ \int_0^\infty k_r J_1(r k_r) J_1(r_0 k_r) \cdot \right. \\ &\cdot \left. \left[\int_{-\infty}^{+\infty} \frac{e^{j k_z (z_0 - z)}}{k^2 - k_z^2 - k_r^2} dk_z \right] dk_r \right\} dV_0 , \end{aligned}$$

where $J_r^n(r_0, z_0; \omega)$ is given by equation (4) and $dV_0 = r_0 dr_0 dz_0 d\varphi_0$. Performing the integration over z_0 and φ_0, we get:

$$\begin{aligned} \Pi_r(r, z; \omega) &= \frac{j}{2\pi \omega \varepsilon} \sum_{n=-\infty}^{+\infty} e^{-j\omega nL/v} \int_0^\infty k_r J_1(r k_r) \left\{ \int_{-\infty}^{+\infty} \frac{e^{-j(z-nL)k_z}}{k^2 - k_z^2 - k_r^2} \cdot \right. \\ &\cdot \left. \left[\int_0^\infty r_0 J_1(r_0 k_r) J_r^0(r_0; \omega) dr_0 \right] dk_z \right\} dk_r . \end{aligned} \qquad (6)$$

The integral over r_0 is simply the Hankel transform of the current $J_r^0(r_0; \omega)$:

$$F(u) = \int_0^\infty r_0 J_1(u r_0) J_r^0(r_0; \omega) dr_0$$

where u is the radial wavenumber k_r. The inverse transform reads

$$\boxed{J_r^0(r_0; \omega) = \int_0^\infty u F(u) J_1(u r_0) du .} \qquad (7)$$

From now on we choose the transform $F(u)$ as the unknown of the problem. Equation (6) becomes

$$\Pi_r(r, z; \omega) = \frac{j}{2\pi \omega \varepsilon} \sum_{n=-\infty}^{+\infty} e^{-j\omega nL/v} \int_0^\infty u F(u) J_1(ur) \left[\int_{-\infty}^{+\infty} \frac{e^{-j(z-nL)k_z}}{k^2 - u^2 - k_z^2} dk_z \right] du . \qquad (8)$$

The integration in square brackets over k_z may be performed [2] by means of the residue theorem; in fact, putting $U = \sqrt{k^2 - u^2}$, the integrand function exhibits two simple poles at $k_z = \pm U$. It is found that

$$\int_{-\infty}^{+\infty} \frac{e^{-j(z-nL)k_z}}{U^2 - k_z^2} dk_z = j\pi \frac{e^{-jU|z-nL|}}{U} .$$

Finally expression (8) can be rewritten as

$$\Pi_r(r, z; \omega) = -\frac{1}{2\omega\varepsilon} \int_0^\infty \frac{u F(u) J_1(ur)}{U} \left[\sum_{n=-\infty}^{+\infty} e^{-j(U|z-nL| + \omega nL/v)} \right] du . \qquad (9)$$

[2] In the evaluation of the residues it is necessary to take into account a small imaginary part for $k = \omega\sqrt{\varepsilon\mu}$, where $\varepsilon\mu$ is considered to be complex [23], so that $\mathrm{Im}(k) < 0$. For the two poles at $k_z = \pm U$, we then have $\mathrm{Im}(U) < 0$ or $U = \sqrt{k^2 - u^2} = -j\sqrt{u^2 - k^2}$ when $u > \mathrm{Re}(k)$. By taking $\mathrm{Im}(U) < 0$, we implicitly satisfy the radiation condition at infinity.

Because the fields $\vec{E}$ and $\vec{H}$ are related to the Hertz potential $\vec{\Pi}$ by [21]

$$\begin{aligned}\vec{E} &= k^2\vec{\Pi} + \nabla(\nabla\cdot\vec{\Pi})\\ \vec{H} &= j\omega\varepsilon\nabla\times\vec{\Pi}\end{aligned}$$

introducing the notation

$$S(u,z) = \sum_{n=-\infty}^{+\infty} e^{-j(U|z-nL|+\omega nL/v)} \tag{10}$$

we have the following expressions for the fields [3]:

$$E_r(r,z) = -\frac{\zeta_0}{2k}\int_0^\infty uF(u)\sqrt{k^2-u^2}S(u,z)J_1(ur)du \tag{11}$$

$$E_z(r,z) = -\frac{\zeta_0}{2k}\int_0^\infty \frac{u^2F(u)}{\sqrt{k^2-u^2}}\frac{\partial S(u,z)}{\partial z}J_0(ur)du \tag{12}$$

$$H_\varphi(r,z) = -\frac{j}{2}\int_0^\infty \frac{uF(u)}{\sqrt{k^2-u^2}}\frac{\partial S(u,z)}{\partial z}J_1(ur)du \tag{13}$$

($\zeta_0 = 120\pi\,\Omega$ is the free space impedance). Let us remember that $\sqrt{k^2-u^2} = -j\sqrt{u^2-k^2}$. The series that defines the function $S(u,z)$ can be summed in closed form [24]; we obtain

$$S(u,z) = j\frac{\sin[(L-|z|)\sqrt{k^2-u^2}] + e^{-j\omega\mathrm{sgn}(z)L/v}\sin(|z|\sqrt{k^2-u^2})}{\cos(L\sqrt{k^2-u^2}) - \cos(\omega L/v)} \qquad \text{for } |z|\le L\,.$$

If z is not in the range $|z|\le L$, then the function $S(u,z)$ can be computed by means of the relation

$$S(u,z+L) = S(u,z)e^{-j\omega L/v}\,.$$

It should be noted for future use that

$$S(u,0) = \frac{j\sin\left(L\sqrt{k^2-u^2}\right)}{\cos\left(L\sqrt{k^2-u^2}\right)-\cos\left(\omega L/v\right)} = \frac{\sinh\left(L\sqrt{u^2-k^2}\right)}{\cosh\left(L\sqrt{u^2-k^2}\right)-\cos\left(\omega L/v\right)}\,. \tag{14}$$

It is not difficult to compute the derivative along z of the function $S(u,z)$ for $|z|<L$:

$$\frac{1}{\sqrt{k^2-u^2}}\frac{\partial S(u,z)}{\partial z} = -j\mathrm{sgn}(z)\frac{\cos\left[(L-|z|)\sqrt{k^2-u^2}\right] - e^{-j\omega\mathrm{sgn}(z)L/v}\cos\left(z\sqrt{k^2-u^2}\right)}{\cos\left(L\sqrt{k^2-u^2}\right)-\cos(\omega L/v)}\,,$$

[3]Remembering that $\vec{\Pi} = \hat{r}\Pi_r$, we can write

$$\begin{aligned}E_r &= \frac{\partial^2\Pi_r}{\partial r^2} + \frac{1}{r}\frac{\partial\Pi_r}{\partial r} + \left(k^2 - \frac{1}{r^2}\right)\Pi_r\,,\\ E_z &= \frac{1}{r}\frac{\partial^2(r\Pi_r)}{\partial z\partial r}\,,\\ H_\varphi &= j\omega\varepsilon\frac{\partial\Pi_r}{\partial z}\,.\end{aligned}$$

and its value for $z = 0$:

$$\left[\frac{1}{\sqrt{k^2-u^2}}\frac{\partial S(u,z)}{\partial z}\right]_{z=0} = -j\mathrm{sgn}(z) + \frac{\sin(\omega L/v)}{\cos\left(L\sqrt{k^2-u^2}\right) - \cos(\omega L/v)}. \qquad (15)$$

From equation (14) we see that

$$\lim_{u\to\infty} S(u,0) = 1,$$

and, because $\mathrm{Im}\sqrt{k^2-u^2} < 0$, we also have

$$\lim_{L\to\infty} S(u,0) = 1 \qquad \text{for any u}.$$

Similarly, the last term of equation (15) vanishes when $L \to \infty$. Therefore, as expected when $L \to \infty$, equations (11) to (13) reduce to the formulae given in the Appendix for the case of a single screen.
As already mentioned, the solution can be found as a superposition of the solution of the inhomogeneous equations in free space and a solution of the homegeneous equations, chosen in such a way as to fulfill the boundary conditions on the plates. Accordingly the two conditions which are to be satisfied are

$$J_r^0(r;\omega) = 0, \qquad 0 \le r \le a, \qquad (16)$$

$$E_{0r}(r, z=0;\omega) + E_r(r, z=0;\omega) = 0, \qquad r > a. \qquad (17)$$

Therefore the two conditions that allow us to find the expression of the current transform $F(u)$ can easily be obtained as a system of *dual integral equations* (extensively studied in Ref. [22]) from equations (2), (16), and (17) :

$$\int_0^\infty uF(u)J_1(ur)du = 0, \qquad 0 \le r \le a, \qquad (18)$$

$$\int_0^\infty uF(u)\sqrt{u^2-k^2}S(u,0)J_1(ur)du = Q\frac{jk^2}{\pi\beta^2\gamma}K_1\left(\frac{\omega r}{v\gamma}\right), \qquad r > a, \qquad (19)$$

where $\omega > 0$. In the Appendix are given the formulae for the case of a single screen.

This system of integral equations can be transformed into one equation with a singular kernel. In fact if we introduce the unitary step function $\mathcal{U}(x)$, the previous system becomes formally

$$\int_0^\infty uF(u)J_1(ur)du = \frac{1}{C}\left\{Q\frac{jk^2}{\pi\beta^2\gamma}K_1\left(\frac{\omega r}{v\gamma}\right) + \int_0^\infty uF(u)\left[C - \sqrt{u^2-k^2}S(u,0)\right]J_1(ur)du\right\}\mathcal{U}(r-a), \qquad (20)$$

where C is any constant independent of u, different from 0 and ∞. Equation (20) can be interpreted as a Hankel transform of $F(u)$ [22]; so, making use of the inversion formula (7) and of the fundamental result

$$\int_0^\infty rJ_1(ur)J_1(vr)dr = \frac{\delta(u-v)}{u} = \frac{\delta(u-v)}{v},$$

$$F(u)\sqrt{u^2-k^2}S(u,0) = jk\frac{Q}{\pi\beta}\frac{u}{u^2+\kappa^2}\overbrace{\sum_{n=0}^{\infty} A_n \frac{J_{2n+\nu}(au)}{(au)^{\nu}}}^{G(au)} \qquad \nu \geq -\frac{1}{2}$$

$$\int_0^{\infty} \frac{u^2}{u^2+\kappa^2} \sum_{n=0}^{\infty} A_n \frac{J_{2n+\nu}(au)}{(au)^{\nu}} J_1(ur)\,du = \kappa K_1(\kappa r) \qquad r > a \qquad \kappa = \frac{k}{\beta\gamma}$$

But

$$\int_0^{\infty} \frac{u^2}{u^2+\kappa^2} \frac{J_{2n+\nu}(au)}{(au)^{\nu}} J_1(ur)\,du = (-1)^n \frac{I_{2n+\nu}(\kappa a)}{(\kappa a)^{\nu}} \kappa K_1(\kappa r) \qquad r > a \qquad \nu > -1$$

therefore

$$\sum_{n=0}^{\infty} A_n (-1)^n \frac{I_{2n+\nu}(\kappa a)}{(\kappa a)^{\nu}} = 1 \qquad \nu > -1$$

$$\int_0^{\infty} uF(u)J_1(ur)\,du = 0 \qquad 0 \leq r \leq a$$

$$\int_0^{\infty} \frac{u^2}{u^2+k^2} \frac{1}{\sqrt{u^2-\kappa^2}S(u,0)} \sum_{n=0}^{\infty} A_n \frac{J_{2n+\nu}(au)}{(au)^{\nu}} J_1(ur)\,du = 0 \qquad 0 \leq r \leq a$$

For some suitable functions $f_m(r)$,

$$\int_0^a dr\, f_m(r) J_1(ur) = \frac{J_{2m+\mu}(au)}{(au)^{1+\mu}} \qquad m = 1, 2, \ldots \qquad \mu > -1$$

hence

$$\int_0^{\infty} \frac{u}{u^2+\kappa^2} \frac{1}{\sqrt{u^2-k^2}S(u,0)} \sum_{n=0}^{\infty} A_n \frac{J_{2n+\nu}(au)}{(au)^{\nu}} \frac{J_{2m+\nu}(au)}{(au)^{\nu}}\,du = 0 \qquad m = 1, 2, \ldots \qquad \nu > -1$$

In the hole,

$$E_r(r) = -\frac{\zeta_0}{2\pi}\frac{Q}{\beta}\int_0^\infty \frac{u^2}{u^2+\kappa^2}\sum_{n=0}^\infty A_n \frac{J_{2n+\nu}(au)}{(au)^\nu} J_1(ur)\,du \qquad 0 \le r < a$$

When $r \to a$,

$$\boxed{E_r \sim (a^2 - r^2)^{-\frac{1}{2}}}$$

By choosing $\nu = -\frac{1}{2}$, one guarantees that each term in $\sum_{n=0}^\infty$ has the correct singularity at $r = a$.

Indeed,

$$\frac{J_{2n+\nu}(au)}{(au)^\nu} = (-1)^n\sqrt{\frac{2}{\pi}}\cos(au) + O\Big(\frac{1}{au}\Big)$$

when $\nu = -\frac{1}{2}$ and $au \to \infty$.

The singularity comes from the behaviour of the integrand as $u \to \infty$.

$$\int_0^\infty \sum_{n=0}^\infty A_n \frac{J_{2n+\nu}(au)}{(au)^\nu} J_1(ur)\,du = \sum_{n=0}^\infty A_n(-1)^n\sqrt{\frac{2}{\pi}}\underbrace{\int_0^\infty \cos(au)J_1(ur)\,du}_{-\frac{r}{\sqrt{a^2-r^2}(a+\sqrt{a^2-r^2})}} + (\text{regular part at } r = a)$$

Therefore $\nu = -\frac{1}{2}$ yields the fastest convergence.

Normalised Coupling Impedance Calculations

The Program

The Program consists of two packages written in Mathematica code to calculate the normalised coupling impedance as a function of frequency. Its principal aim is to produce output files containing plot points, so that a graph of impedance against a function of frequecy can be plotted.

The function of frequency was chosen to be kL/Pi where k=w/c is the wave number and L is the distance between the screens. This results in an oscillation of the coupling impedance occuring every integer value of kL/Pi.

The program is structured in the form of two packages which are found in the files,

pkge1
pkge2.

Description of Packages

pkge2 (MatrixElements)

Exportable Function :
Matel[n,m,kL/Pi,L/a,Beta*Gamma,limit]

Aim : To calculate the matrix elements a[m,n] and d[m,n]

Input variables : n, column index
m, row index
L/Pi, normalised frequency
L/a, distance between discs to aperture ratio
Beta*Gamma, relativistic factor
imit, the limit of the finite sum

There are two routines in the package, one is for the case Beta*Gamma finite and the other for the special case Beta*Gamma infinite (which is in fact a simplified case). Since the matrices a[m,n] and d[m,n] are symmetric matrices, the elements a[[m,n]], d[[m,n]] for n>=0 and m>=n are calculated, where the case n,m=0 is treated seperately from the the general case.

Internal Functions :

Bess : Bess[x_] computes the trigonometric series expansion for the Bessel function of the first kind Jn(x) of order,n = 2x -1/2

arg : arg[q_] defines the variable a*alpha_q

ChSh : ChSh[x_] is an improvised hyperbolic function of which the output is in vector form.
ChSh[x_][[1]] is the improvised hyperbolic cosine function,

Ch(x) = [exp(x)*E i(-x) + exp(-x)* Ei(x)]/2

ChSh[x_][[2]] is the improvised hyperbolic sine function,

Sh(x) = [exp(x)*E i(-x) - exp(-x)* Ei(x)]/2

int : int[x_] is the solution of the integral equation I, using exponential integrals, ie. for larger arguements of x

hypergeo : hypergeo[x_] is the solution to the integral equation, I, using hyperbgeometric functions, ie. for smaller arguments of x

funct0(d) : funct0(d)[Q_] is a function to compute the approximation of the latter terms of the matrix element a[0,0] (d[0,0]), ie. to compute the part of a[0,0]d[0,0]) containing the sum from Q to infinity.

funct(d) : funct(d)[Q_] is a function to compute the approximation of the latter terms of the general matrix element a[m,n] (d[m,n]) for n>=0, m>0, ie. to compute the part of a[m,n] (d[m,n]) containing the sum from Q to infinity.

array : array[[m,n+1]] gives the value of the real argument x at which the evaluation of the integral equation, I, switches. If x>=array[[m,n+1]] the function int is used, otherwise the function hypergeo is employed.

arraycomp : array[[m,n+1]] gives the value of the imaginary argument x at which the evaluation of the integral equation, I, switches. If Im[x]>=arraycomp[[m,n+1]] the function int is used, otherwise the function hypergeo is employed.

first : first[x_] calculates one of the components of the function hypergeo.

case1 : case1[x_,y_,z_] is used for the computation of a[m,n]. It effectively puts the qth term of the finite sum equal to zero if x or y = 0 (ie. a*alpha_q = 0 or kL/(Pi Beta) = q). case[x_,y_] is for the case when Beta*Gamma = infinity, it effectively puts the qth term of the finite sum equal to zero if x = 0, (ie. a*alpha_q = 0).

case2 : case2[x_,y_,z_] is used for the computation of d[m,n]. It effectively puts the qth term of the finite sum equal to zero if x =0, (ie. a*alpha_q =0) and correctly evaluates the term when y = 0 (ie. kL/(Pi Beta) = q).

enum : enum[x_] gives Euler Constant.

pkgel (Imp)

Exportable Function :
ImpPackage[L/a,Beta*Gamma,l,u,s]

Aim : To calculate and store the points for the plot of the normalised coupling impedance against the function of frequency, kL/Pi. To calculate and store the points for the plot of p(a) against kL/Pi. To store the results of the impedance against kL/Pi when the dimension of the matrices a[m,n] and d[m,n] are reduced. To store all calculated matrix elements a[m,n] and d[m,n].

Input Variables :

L/a, distance between discs to aperture ratio
Beta*Gamma, relativistic factor
l,u and s are concerned with the values of kL/Pi at which the normalised coupling impedance will be calculated and hence plotted.
l = lowest value of kL/Pi (>=0)
u = highest value of kL/Pi
s = step length

As for the previous package, the cases Beta*Gamma finite and Beta*Gamma infinite have to be treated seperately. This package calls the exportable function Matel in pkge1 (MatrixElements) to obtain the matrix elements a[m,n] and d[m,n]. Using these results it calculates the normalised coupling impedance and p(a).

Internal Functions :

Impedance : Impedance[x_,y_,z_] performs the calculations of the impedance and p(a) for the particular function of frequency, kL/Pi, given by the first arguement, x. The latter two arguements are L/a and Beta*Gamma respectively.

To Use the program

Load the two packages into Mathematica as follows :

<<pkge1
<<pkge2

Then call the function ImpPackage with the desired numerical arguments,

ImpPackage[L/a,Beta*Gamma,l,u,s]

(NB. If Beta*Gamma = infinity is desired, type Infinity as the second arguement.)

Output files produced

Impedance_Pts_Real : Stores a list of vectors {x,y}, x being the function of frequency, kL/Pi and y the real component of the normalised coupling impedance.

Impedance_Pts_Imag : Stores a list of vectors {x,y}, x being the function of frequency, kL/Pi and y the imaginary component of the normalised coupling impedance.

p_a_real : Stores a list of vectors {x,y}, x being the function of frequency, kL/Pi and y the real component of the function p(a).

p_a_imag : Stores a list of vectors {x,y}, x being the function of frequency, kL/Pi and y the imaginary component of the function p(a).

redmat : Stores a list of elements of the following form, {x,{y,z}}, where x represents the function of frequency, kL/Pi, and y, z represent the normalised coupling impedance, calculated using matrices (namely, a[m,n] and d[m,n]) of dimension 2 and 3 respectively.

mata_elements_save and matd_elements_save : Produced for reference only. These files contain all calculated matrix elements, a[m,n] and d[m,n] respectively.

How to view the results

Load the third package pkge3, as follows :

<<pkge3

To plot the graphs of the results use the following command,

plot[?]

where ? is one of the following options :

impedancereal - plots only the real part of the normalised coupling impedance against the function of frequency, kL/Pi.

impedanceimag - plots only the imaginary part of the normalised coupling impedance against the function of frequency, kL/Pi.

impedance - plots both the real and imaginary components of the normalised coupling impedance against the function of frequency, kL/Pi.

p(a)real - plots only the real part of p(a) against the function of frequency, kL/Pi.

p(a)imag - plots only the imaginary part of p(a) against the function of frequency, kL/Pi.

p(a) - plots both the real and imaginary components of p(a) against the function of frequency, kL/Pi.

redmat(2) - plots both the real and imaginary components of the normalised coupling impedance calculated with only 2*2 matrices, a[m,n] and d[m,n] (where 0<=m,n<=1), against the function of frequency, kL/Pi.

redmat(3) - plots both the real and imaginary components of the normalised coupling impedance calculated with only 3*3 matrices, a[m,n] and d[m,n] (where 0<=m,n<=2), against the function of frequency, kL/Pi.

To change the scale of the plots, you can use the following command,

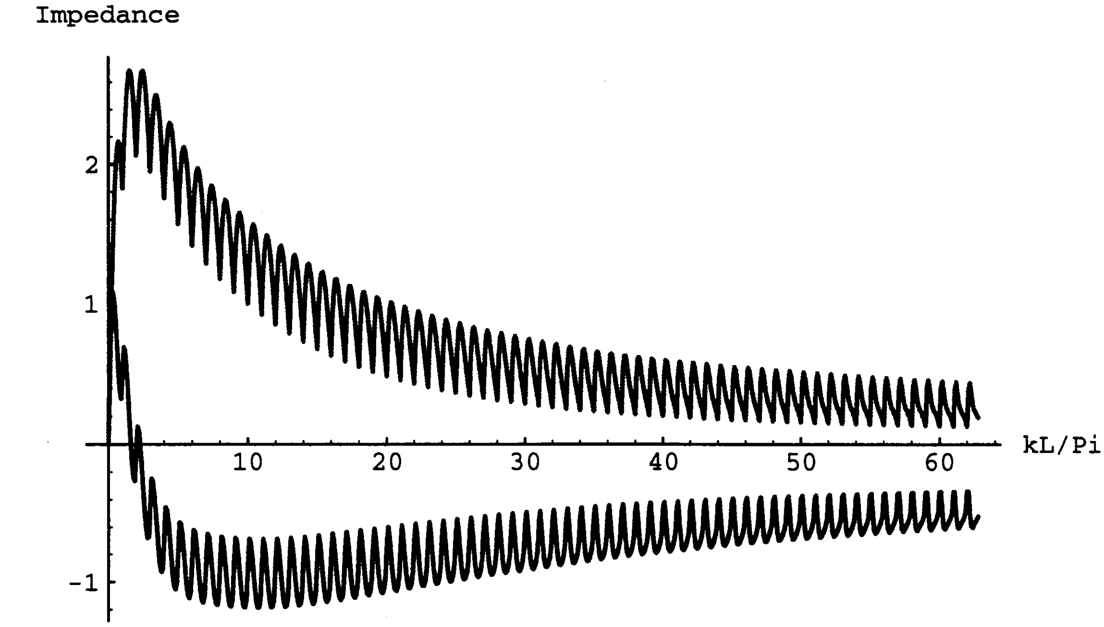
plot[?,x1,x2,y1,y2]

where ? is again one of the options above.
{x1,x2} is the range of the horizontal axis, ie. the function of frequency, kL/Pi and {y1,y2} is the range of the vertical axis, ie. the normalised coupling impedance or p(a).

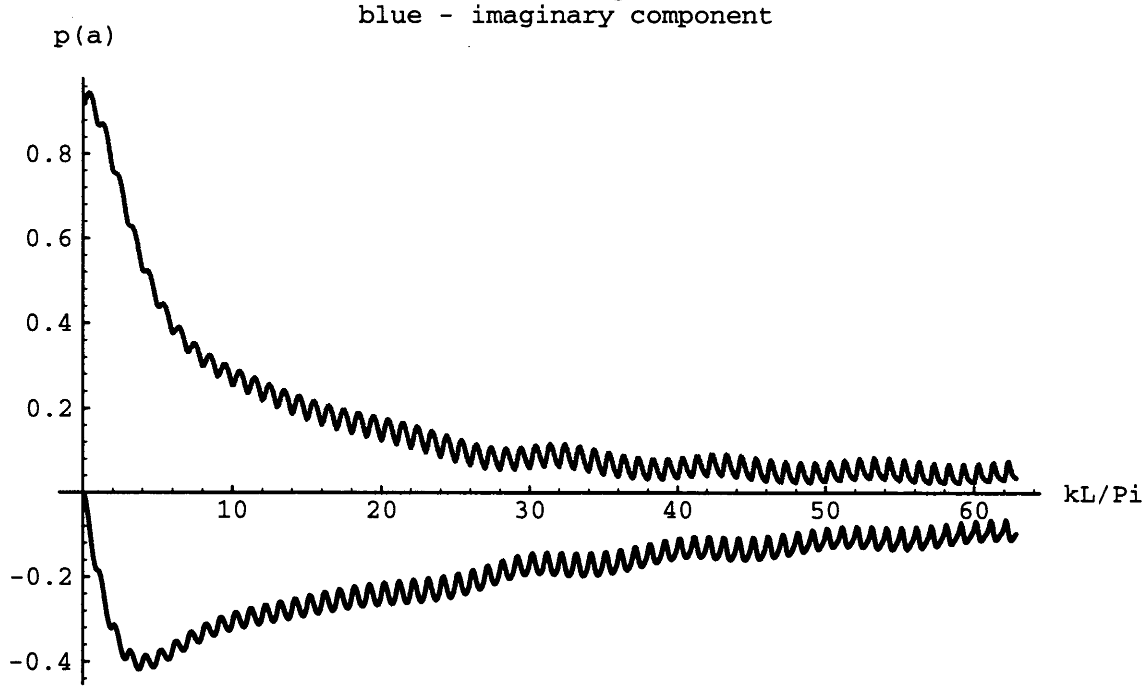
Examples of plots

For the case L/a =10, Beta*Gamma = Infinity

Normalised Coupling Impedance
against a function of frequency
red - real component
blue - imaginary component

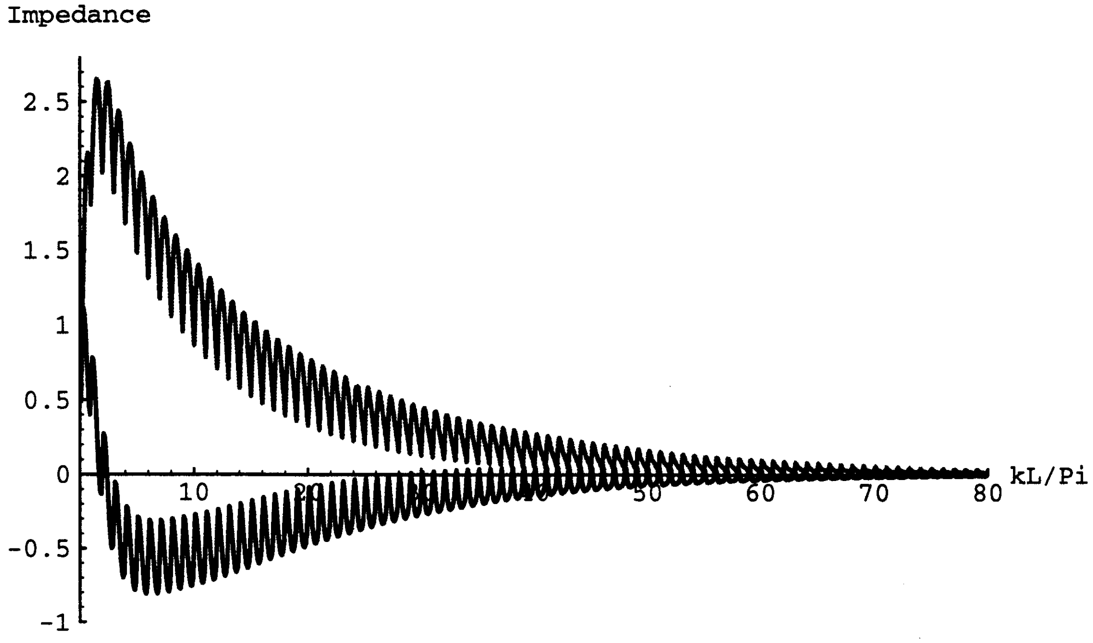


p(a) against a function of frequency
red - real component
blue - imaginary component

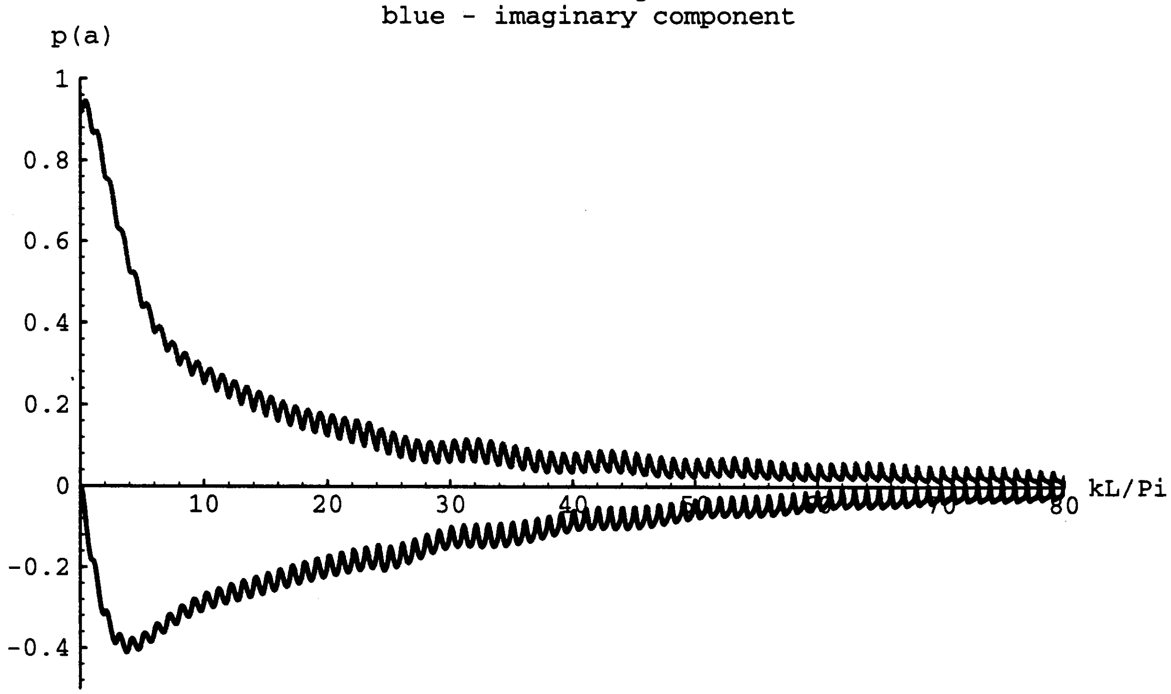


For the case L/a =10, Beta*Gamma = 10

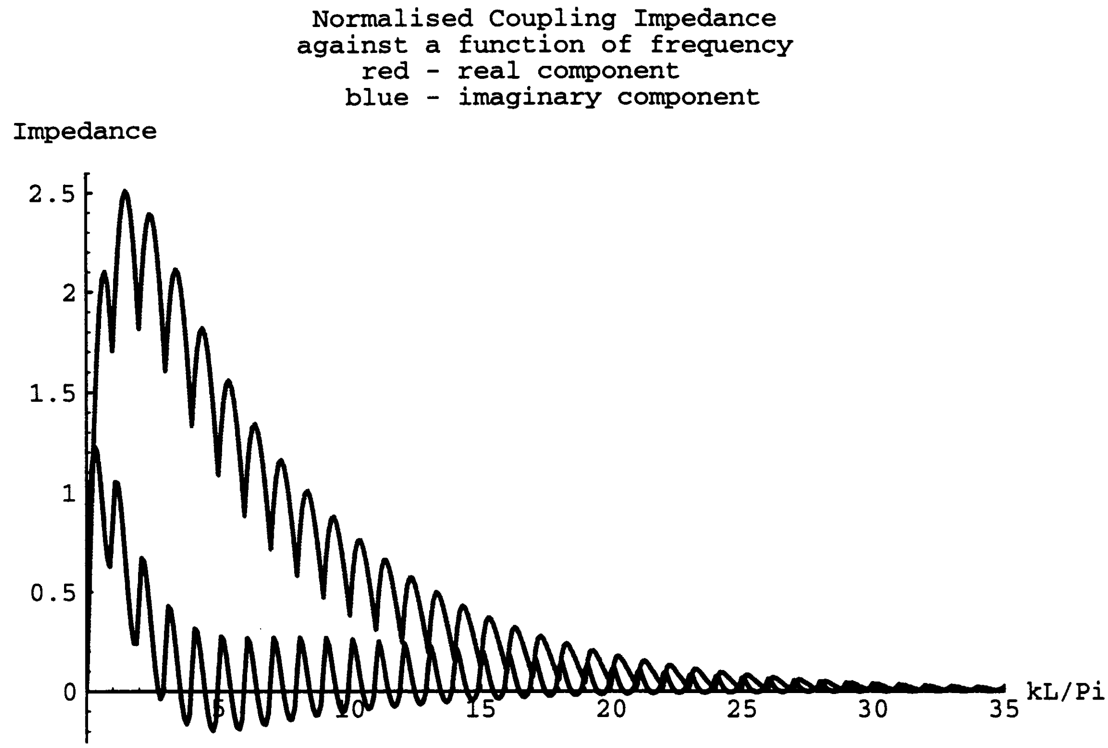
Normalised Coupling Impedance
against a function of frequency
red - real component
blue - imaginary component



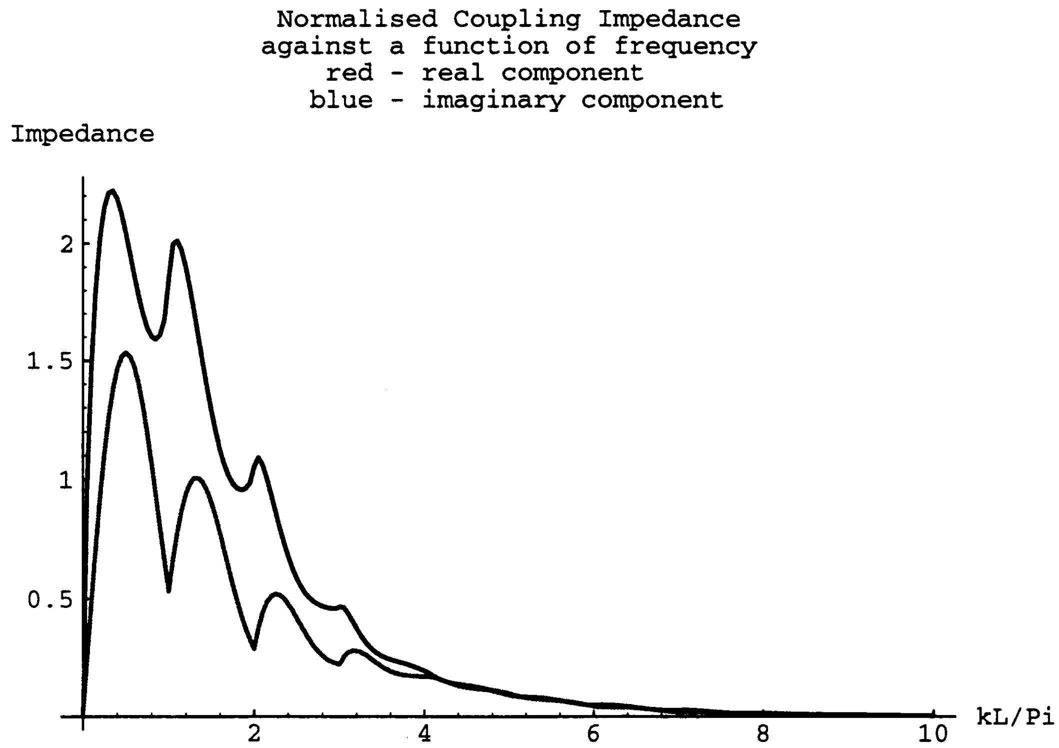
p(a) against a function of frequency
red - real component
blue - imaginary component



For the case L/a = 10, Beta*Gamma = 4



For the case L/a =10, Beta*Gamma = 1



Normalised Longitudinal Coupling Impedance Calculations

Background

The normalised longitudinal coupling impedance is given by the following formulea,

$$\begin{aligned}
\frac{2\pi}{\zeta_0} Z_{\|}(k) &= \frac{2jk}{\beta^2}\int_0^\infty \frac{u^3}{(u^2+\kappa^2)^2\sqrt{u^2-k^2}S(u,0)} \sum_{n=0}^{\infty} A_n \frac{J_{2n+\nu}(au)}{(au)^\nu}\, du \\
&= \frac{2jk}{\beta^2}\int_0^\infty \frac{u^3}{(u^2+\kappa^2)^2\sqrt{u^2-k^2}S(u,0)} \left|\sum_{n=0}^{\infty} A_n \frac{J_{2n+\nu}(au)}{(au)^\nu}\right|^2 du \\
&= \frac{2jk}{\beta^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_m^* A_n b_{m,n}
\end{aligned}$$

where

$$b_{m,n} = \int_0^\infty \frac{u^3}{(u^2+\kappa^2)^2\sqrt{u^2-k^2}S(u,0)} \frac{J_{2m+\nu}(au)}{(au)^\nu}\frac{J_{2n+\nu}(au)}{(au)^\nu}\, du$$

and

$$S(u,0) = \frac{\sinh(L\sqrt{u^2-k^2})}{\cosh(L\sqrt{u^2-k^2}) - \cos\frac{wL}{v}}$$

$\zeta_0 = 120\pi\Omega$ is the characteristic impedance of free space

L is the distance between the screens

$\beta = v/c$, where v is the velocity of the charge and c is the velocity of light

$k = w/c$ is the wave number, where w is the angular frequecy

$\kappa = |k|/(\beta\gamma)$, where γ is the energy of the charge expressed in rest mass units

The A_i's are calculated using the following equalities,

$$\sum_{n=0}^{\infty} A_n a_{m,n} = 0\,, \qquad m = 1, 2, \cdots$$

where

$$a_{m,n} = \int_0^\infty \frac{u}{u^2+\kappa^2} \frac{1}{\sqrt{u^2-k^2}S(u,0)} \frac{J_{2n+\nu}(au)}{(au)^\nu} \frac{J_{2m+\nu}(au)}{(au)^\nu} du$$

and

$$\sum_{n=0}^{\infty} A_n (-1)^n \frac{I_{2n+\nu}(a\kappa)}{(a\kappa)^\nu} = 1 , \qquad \nu > -1$$

To obtain a good convergence of this solution, $\nu = -\frac{1}{2}$ is chosen. To evaluate this solution the following approximation is used,

$$\frac{1}{\sqrt{u^2-k^2}S(u,0)} = \frac{1}{L} \sum_{q=0}^{\infty} \varepsilon_q \left[\frac{1-(-1)^q \cos\left(\frac{wL}{v}\right)}{u^2+\alpha_q^2} \right]$$

where

$$\alpha_q^2 = \left(\frac{q\pi}{L}\right)^2 - k^2$$

Finally we obtain for the normalised coupling impedance

$$\frac{2\pi}{\zeta_0} Z_{\|}(k) = \frac{2jk}{\beta^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_m^* A_n [a_{m,n} + d_{m,n}]$$

where

$$a_{m,n} = -\frac{L}{\pi^2} \sum_{q=0}^{\infty} \varepsilon_q \left[\frac{1-(-1)^q \cos\left(\frac{kL}{\beta}\right)}{q^2 - \left(\frac{kL}{\pi\beta}\right)^2} \right] \left[\int_0^\infty u \left[\frac{1}{u^2+a\alpha_q^2} - \frac{1}{u^2} \right] \frac{J_{2n+\nu}(u)}{u^\nu} \frac{J_{2m+\nu}(u)}{u^\nu} du \right]$$

and

$$d_{m,n} = \frac{L(L\kappa)^2}{\pi^4} \sum_{q=0}^{\infty} \varepsilon_q \left[\frac{1-(-1)^q \cos\left(\frac{kL}{\beta}\right)}{\left[q^2 - \left(\frac{kL}{\pi\beta}\right)^2\right]^2} \right] \left\{ \int_0^\infty u \left[\frac{1}{u^2+(a\kappa)^2} - \frac{1}{u^2} \right] \frac{J_{2n+\nu}(u)}{u^\nu} \frac{J_{2m+\nu}(u)}{u^\nu} du \right.$$
$$\left. - \int_0^\infty u \left[\frac{1}{u^2+(a\alpha_q)^2} - \frac{1}{u^2} \right] \frac{J_{2n+\nu}(u)}{u^\nu} \frac{J_{2m+\nu}(u)}{u^\nu} du \right\}$$

The difficulty of this solution is evaluating the matrix elements $a_{m,n}$ and $d_{m,n}$.

The integral, I, below,

$$\mathbf{I} = \int_0^\infty u\left[\frac{1}{u^2+x^2} - \frac{1}{u^2}\right]\frac{J_{2n+\nu}(u)}{u^\nu}\frac{J_{2m+\nu}(u)}{u^\nu}\,du$$

is solved in two ways depending on the size of the argument x.

For small arguments, x

We obtain a solution for I using hypergeometric functions.

$$\mathbf{I} = \left(\frac{a\alpha_q}{2}\right)^2 \Gamma(m+n)\frac{2}{\sqrt{\Pi}}\times$$

$$\sum_{s=0}^{m+n-2} \frac{\Gamma(\frac{3}{2}+\nu+s)\Gamma(2+\nu+s)(a\alpha_q)^{2s}}{(-m-n+1)_{1+s}\Gamma(m-n+2+\nu+s)\Gamma(n-m+2+\nu+s)\Gamma(m+n+2+2\nu+s)}$$

$$+(-1)^{m+n}\Bigg\{$$

$$\frac{1}{2^{1+2\nu}}\left(\frac{a\alpha_q}{2}\right)^{2m+2n}\sum_{s=0}^{\infty}\Bigg[\frac{(m+n+\nu+\frac{1}{2})_s(m+n+\nu+1)_s(a\alpha_q)^{2s}}{s!\Gamma(2m+\nu+1+s)\Gamma(2n+\nu+1+s)(2m+2n+2\nu+1)_s}\times$$

$$[\psi(1+s)+\psi(2m+\nu+1+s)+\psi(2n+\nu+1+s)+\psi(2m+2n+2\nu+1+s)-2\psi(2m+2n+2\nu+1+2s)]\Bigg]$$

$$-\frac{1}{(a\alpha_q)^{2\nu}}\log\left(\frac{a\alpha_q}{2}\right)I_{2m+\nu}(a\alpha_q)I_{2n+\nu}(a\alpha_q)\Bigg\}$$

For large arguments, x

we obtain a solution for I using exponential integrals.

$$\mathbf{I} = \frac{(-1)^{m+n}}{\pi}\Bigg\{-\left(1+\frac{A1}{(a\alpha_q)^2}+\frac{A2}{(a\alpha_q)^4}+\frac{A3}{(a\alpha_q)^6}+\cdots\right)\left(\gamma+\log(2a\alpha_q)\right)$$
$$+\left(B1+\frac{B2}{(a\alpha_q)^2}+\frac{B3}{(a\alpha_q)^4}+\cdots\right)$$
$$-\left(1+\frac{C1}{(a\alpha_q)^2}+\frac{C2}{(a\alpha_q)^4}+\frac{C3}{(a\alpha_q)^6}+\cdots\right)Ch(2a\alpha_q)$$
$$+\left(\frac{D1}{a\alpha_q}+\frac{D2}{(a\alpha_q)^3}+\frac{D3}{(a\alpha_q)^5}+\cdots\right)Sh(2a\alpha_q)\Bigg\}$$

where,

$$Ch(x) = \frac{1}{2}\left[e^x E_i(-x) + e^{-x}\overline{E}_i(x)\right]$$

$$Sh(x) = \frac{1}{2}\left[e^x E_i(-x) - e^{-x}\overline{E}_i(x)\right]$$

The A, B, C and D coefficients are obtained from the trigonometric series expansion of the Bessel function product, below

$$\frac{J_{2n+\nu}(au)}{(au)^\nu}\frac{J_{2m+\nu}(au)}{(au)^\nu}$$

The Electric Field

In the hole, when $r \to a$,

$$E_r \sim (a^2 - r^2)^{-\frac{1}{2}}$$

p(a) is the constant of proportionality, where

$$p(a) = \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} (-1)^n A_n$$

Hamiltonian Second Order Perturbation Theory Formulae for a Two Families Sextupole Arrangement

The purpose of this notebook is to provide an analytical tool for study of nonlinear betatron oscillations. It is based on the Hori-Deprit normalization procedure in the form suggested by L.Michelotti [1]. At present the consideration is limted to the
second order in the sextupole strength.
The problem geometry is shown in Fig.A2. The arrangement consists of two interleaved sextupole families (S1 and S2) of ns1 and ns2=ns1+1 sextupoles per half the superperiod reflected w.r.t. the symmetry point theta=Pi. It is assumed that the phase advances between two adjacent sextupoles belonging to different families are half those between sextupoles of the same family.

Variables and units

Number of superperiods:

```
nsuper=4;
```

Betatron tunes per a superperiod :

```
q:={qx,75.18}/nsuper;
```

Phase advances per cell [deg]&[rad] :

```
mudeg:={mux,muy}; mu:=mudeg*Pi/180;
```

Phase advances from IP to the first S1-sextupole [rad] :

```
phi1={2.276*2*Pi, 1.945*2*Pi;}; phi2:=phi1-.5 mu;
```

Betatron functions [m] at the S1 and S2 sextupoles:

```
bx1=169; by1=38; bx2=11; by2=167;
bx12=Sqrt[bx1*bx2]; by12=Sqrt[by1*by2];
```

Integrated sextupole strength [m^-2]

```
k211=.2765*.4; k212=-.2585*.76;
```

Number of sextupoles:

```
ns1=31; ns2=ns1+1;
```

Mode numbers:

```
m[1]={1,0}; m[2]={1,2}; m[3]={1,-2}; m[4]={3,0};
pmq[i_]:=Pi m[i].q; mm[i_]:=m[i].mu;
```

Detuning with amplitude

Trigonometric sums for correlation between sextupoles of the same family:

```
tr[i_,phi_,ns_]:=
(ns Sin[mm[i]]-Sin[ns mm[i]]+(1-Cos[ns mm[i]])*
(Cos[pmq[i]]+Cos[pmq[i]-2m[i].phi-(ns-1) mm[i]])/
        Sin[pmq[i]])/(1-Cos[mm[i]]);
```

and between sextupoles of the different families:

```
trs[i_]:=
(-Cos[pmq[i]-(ns1+.5) mm[i]]+Cos[pmq[i]-.5 mm[i]]+
  Cos[pmq[i]-2m[i].phi1-(ns1-1.5) mm[i]]*
(1-Cos[ns1 mm[i]]))/
Sin[pmq[i]]/(1-Cos[mm[i]])+ns1/Sin[.5 mm[i]];
```

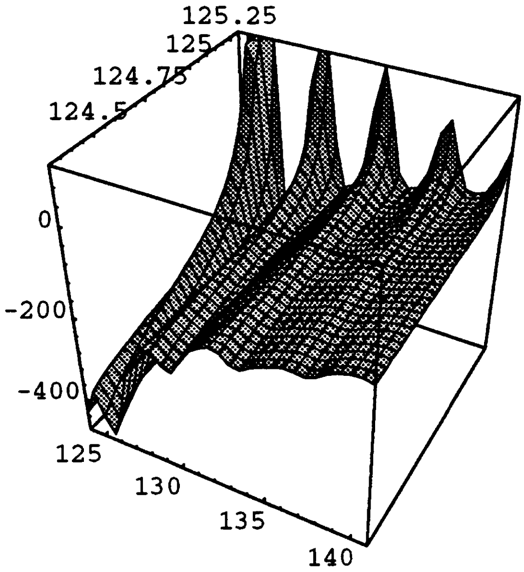
Detuning coefficients:

```
dq1de1:=
-(k211^2*bx1^3*(tr[4,phi1,ns1]+3*tr[1,phi1,ns1])+
  k212^2*bx2^3*(tr[4,phi2,ns2]+3*tr[1,phi2,ns2])+
  2*k211*k212*bx12^3*(trs[4]+3*trs[1])
 )/64/Pi*nsuper;
dq1de2:=
-(k211^2*bx1*by1*(-bx1*tr[1,phi1,ns1]+.5*by1*
 (tr[2,phi1,ns1]-tr[3,phi1,ns1]))+
  k212^2*bx2*by2*(-bx2*tr[1,phi2,ns2]+.5*by2*
 (tr[2,phi2,ns2]-tr[3,phi2,ns2]))+
  k211*k212*bx12*(-(bx1*by2+bx2*by1)*trs[1]+
  by1*by2*(trs[2]-trs[3]))
 )/16/Pi*nsuper;
dq2de2:=
-(k211^2*bx1*by1^2*(tr[1,phi1,ns1]+
 (tr[2,phi1,ns1]+tr[3,phi1,ns1])/4)+
  k212^2*bx2*by2^2*(tr[1,phi2,ns2]+
 (tr[2,phi2,ns2]+tr[3,phi2,ns2])/4)+
  2*k211*k212*bx12*by1*by2*(trs[1]+(trs[2]+trs[3])/4)
 )/16/Pi*nsuper;
```

```
N[dq1de1]
```

```
mux=.
```

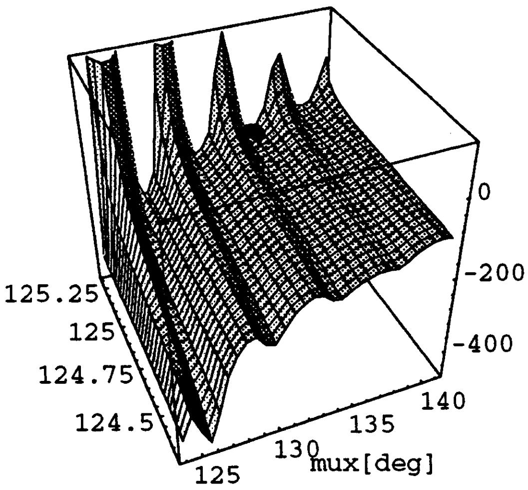
```
gr=Plot3D[ dq1de1/1000, {mux,124,141},{qx,124.3,125.3},
BoxRatios->{1,1,1}, PlotPoints->30];
```




```
mux=135; qx=125.23;
dtn=dq1de1//N
```

-136358.

```
workingPoint=Point[{135, 125.23, dtn/1000+3}];
reference=Point[{140,125.3,0}];
wp=Graphics3D[ {PointSize[.04], workingPoint}];
rf=Graphics3D[ {PointSize[.01], reference}];
Show[gr, wp,ViewPoint->{-1,-2.4,2},
LightSources->{{{-1,.0,1},
RGBColor[1,0,0]},{{-1,.0,1},RGBColor[0,1,0]},{{-1,.0,1},
RGBColor[0,0,1]}},AxesLabel->{"mux[deg]","",""}];
```




```
dq1de1//N
```

-136358.

```
dq1de2//N
-5238.64
dq2de2//N
23564.7
,AxesLabel->{"mux[deg]","Qx","dQx/dEx"}
mux=.
```

Resonance excitation

The term in the Hamiltonian, exciting the (i-j)*qx+(k-l)*qy=n0 resonance, may be represented as a product of numerical coefficient rijkl and Cos[m.delta-n0*theta-Pi/2*Apply[Plus,m]]*Ix^((i+j)/2)*Iy^((k+l)/2), where m={i-j,k-l}, delta is a list of angle variables (see Ref.[1]), Ix and Iy are the action variables.
For performing Fourier analysis some geometry parameters are needed: superperiod and cell lengths [m]:

```
lsuper=6664.72; lcell=79.;
```

```
General::spell1:
   Possible spelling error: new symbol name "lsuper"
     is similar to existing symbol "nsuper".
```

distance from the IP to the first sextupoles in the families [m]:

```
ls1=482.46; ls2=ls1-lcell/2;
```

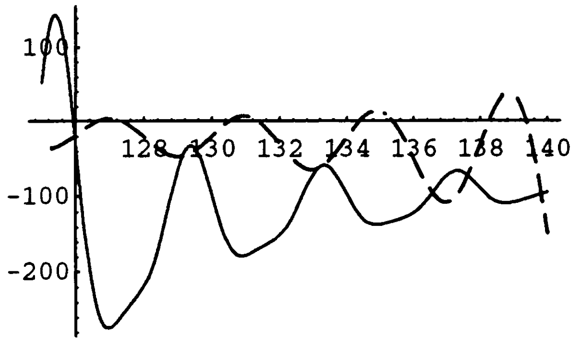
Corresponding angular values:

```
thel1=2*Pi*lcell/lsuper; thet1=2*Pi*ls1/lsuper;
thet2=2*Pi*ls2/lsuper;
```

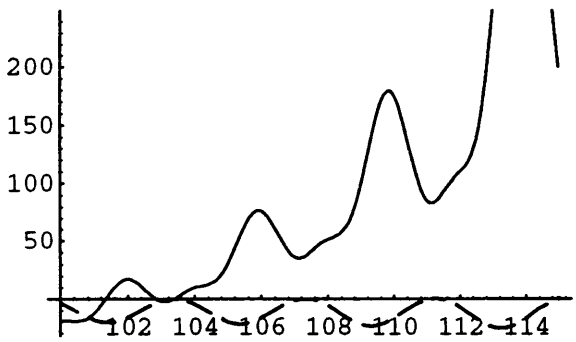
□ **First order (in the sextupole strength) resonance driving terms.**

```
alf[i_]:=mm[i]+(n0-m[i].q)*thel1;
cont[i_,phi_,thet_,ns_]:=
 Sin[alf[i]/2*ns]/Sin[alf[i]/2]*
 Cos[m[i].phi+(n0-m[i].q)*thet+(ns-1)*alf[i]/2]/4*
 Sqrt[2]/Pi*nsuper;
cs[2]=-1/2.; cs[3]=-1/2.; cs[4]=1/6.;
bx1s=Sqrt[bx1]; bx2s=Sqrt[bx2];
r3000:=cs[4]*(k2l1*bx1s^3*cont[4,phi1,thet1,ns1]+
              k2l2*bx2s^3*cont[4,phi2,thet2,ns2]);
r1020:=cs[2]*(k2l1*bx1s*by1*cont[2,phi1,thet1,ns1]+
              k2l2*bx2s*by2*cont[2,phi2,thet2,ns2]);
r1002:=cs[3]*(k2l1*bx1s*by1*cont[3,phi1,thet1,ns1]+
              k2l2*bx2s*by2*cont[3,phi2,thet2,ns2]);
r2100:=cs[4]*(k2l1*bx1s^3*cont[1,phi1,thet1,ns1]+
              k2l2*bx2s^3*cont[1,phi2,thet2,ns2])*3;
r1011:=cs[3]*(k2l1*bx1s*by1*cont[1,phi1,thet1,ns1]+
              k2l2*bx2s*by2*cont[1,phi2,thet2,ns2])*2;
```

```
Plot[{dq1de1/1000,r3000},{mux,125,140},
PlotStyle->{Thickness[.003],Dashing[{.05,.025}]}];
```




```
Plot[{dq1de1/1000,r3000},{mux,100,115},
PlotStyle->{Thickness[.003],Dashing[{.05,.025}]}];
```




```
n0=Round[m[4].q]; r3000//N
2.41624
n0=Round[m[2].q]; r1020//N
10.1342
n0=Round[m[3].q]; r1002//N
-76.7943
```