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RESONANCE MEASUREMENTS USING FOURIER SPECTRUM ANALYSIS OF BEAM OSCILLATIONS

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ABSTRACT

This notes describes the possibility to measure strength and phase of linear and non linear resonances by Fourier transform analysis of the beam response to a kick.

CONTENTS

I _ INTRODUCTION

Since four years we use at lear a Fourier Transform of the beam response to a kick to measure the tune of the machine with a high quality hardware[1] and software[2][3]. The idea of this note is to show that we can measure with the same device the strength and the phase of some linear and non-linear resonances acting on the beam behaviour. The perturbation method was used to understand the Fourier spectrum and direct integration was performed to confirm the result in some cases.

II Perturbation method

Many papers[4][5][6]... describe the perturbation method to calculate the betatron motion in a storage ring and we need only to recall some results here. The perturbed Hamiltonien near an isolated resonance can be written as:

$$H(\phi_x, \phi_y, J_x, J_y; \theta) = \nu_x J_x + \nu_y J_y + 2k J_x^{k_1/2} J_y^{k_2/2} \cos\psi$$

with

$\psi = (j-k)\phi_x + (l-m)\phi_y + \gamma - p\theta$

$N = j+k+l+m$: order of resonance

$j, k, l, m \geq 0$

p : resonance harmonic

k, γ : amplitude, phase of resonance

$$k e^{i\gamma} = h_{jklm-p} = \frac{1}{2\pi}\frac{1}{2^{N/2}}\frac{1}{j!k!l!m!}\int_0^{2\pi R} \beta_x^{k_1/2} \beta_y^{k_2/2} V_{k_1,k_2} e^{i\{(j-k)W_x + (l-m)W_y + p\theta\}} ds$$

$$\theta = \frac{s}{R}, \quad k_1 = j+k, \quad k_2 = l+m, \quad W_{x,y} = \int_0^s \frac{R\,d\theta}{\beta_{x,y}} - \nu_{x,y}\theta$$

β_x, β_y : betatron function in x, y plane

The field in the perturbed magnet is

$$B = \sum_{\substack{k_1, k_2 \\ k_1+k_2=N}} V_{k_1,k_2}\, x^{k_1} y^{k_2}$$

with $V_{k_1,k_2} = (-1)^{k_2/2} \frac{1}{B\rho} \left(\frac{\partial^{N-1} B_y}{\partial x^{N-1}}\right)_{x=y=0}$ for k_2 even, Normal magnet

$W_{k_1,k_2} = (-1)^{(k_2+1)/2} \left(\frac{\partial^{N-1} B_x}{\partial x^{N-1}}\right)_{x=y=0}$ for k_2 odd, for skew magnet

With this formalism

$$x = \sqrt{2\beta_x J_x}\,\cos(\phi_x + W_x)$$
$$y = \sqrt{2\beta_y J_y}\,\cos(\phi_y + W_y)$$

For the unperturbed Hamiltonian $(k=0)$ J_α $(\alpha = x, y)$ is constant $2\beta_\alpha J_\alpha = \alpha^2 + (\beta_\alpha p_\alpha - \frac{\beta'}{2}\alpha)^2$ - with $p_\alpha = \alpha'$ - The phase advance is $\nu_\alpha \theta + \phi_\alpha$

The equation of motion obtained from the perturbed Hamiltonian can be written as:

$$\text{①}\quad \begin{cases} J'_x = -\frac{\partial H}{\partial \phi_x} = +2m_0 k\, J_x^{k_1/2} J_y^{k_2/2} \sin\psi \\ J'_y = -\frac{\partial H}{\partial \phi_y} = +2n_0 k\, J_x^{k_1/2} J_y^{k_2/2} \sin\psi \\ \phi'_x = \frac{\partial H}{\partial J_x} = \nu_x + k_1 k\, J_x^{(k_1-2)/2} J_y^{k_2} \cos\psi \\ \phi'_y = \frac{\partial H}{\partial J_y} = \nu_y + k_2 k\, J_x^{k_1/2} J_y^{(k_2-2)/2} \cos\psi \end{cases}$$

$m_0 = j - k$
$n_0 = l - m$

from equation ① we can find easily that

$$\frac{J_x}{m_0} - \frac{J_y}{n_0} = \text{const.}$$

and with $\frac{\partial H}{\partial \theta} = \frac{dH}{d\theta} = \frac{p}{m_0} J'_x \Rightarrow H = \frac{p}{m_0} J_x + \text{const}$

$$e\, J_x + 2m_0 k\, J_x^{k_1/2} J_y^{k_2/2} \cos\psi = \text{const}_x$$
$$e\, J_y + 2n_0 k\, J_x^{k_1/2} J_y^{k_2/2} \cos\psi = \text{const}_y$$

with $e = m_0 \nu_x + n_0 \nu_y - p$

To solve equations (1) we have used perturbation theory specially the Ruth's method[5].

The next canonical transformation is

$$G(\phi_x, \phi_y, J_{1x}, J_{1y}; \theta) = -2\frac{k}{e} J_{1x}^{k_1/2} J_{1y}^{k_2/2} \sin\psi + \phi_x J_{1x} + \phi_y J_{1y}$$

That gives (2)

$$J_x = J_{1x} - 2m_0 \frac{k}{e} J_{1x}^{k_1/2} J_{1y}^{k_2/2} \cos\psi$$

$$J_y = J_{1y} - 2n_0 \frac{k}{e} J_{1x}^{k_1/2} J_{1y}^{k_2/2} \cos\psi$$

$$\phi_{1x} = \phi_x - \frac{k_1 k}{e} J_{1x}^{(k_1-2)/2} J_{1y}^{k_2/2} \sin\psi$$

$$\phi_{1y} = \phi_y - \frac{k_2 k}{e} J_{1x}^{k_1/2} J_{1y}^{(k_2-2)/2} \sin\psi$$

and $H_1(\phi_{1x}, \phi_{1y}, J_{1x}, J_{1y}, \theta) =$

$$\nu_x J_{1x} + \nu_y J_{1y} - \frac{k^2}{e} J_{1x}^{k_1-1} J_{1y}^{k_2-1} (m_0 k_1 J_{1y} + n_0 k_2 J_{1x})(1 + \cos 2\psi)$$

The term in k^2 is of second order. We can neglect the faster variing term $\cos 2\psi$ and J_{1x}, J_{1y} become constant up to 2^{nd} order in J. By using ϕ'_{1x}, ϕ'_{1y} comming from H_1 or (2)

We write

$$\psi'\left(1 - \frac{\Delta e}{e}\cos\psi\right) = e\left(1 - \frac{2k^2}{e^2}\beta\right)$$

with $\Delta e = k\, J_{1x}^{(k_1-2)/2} J_{1y}^{(k_2-2)/2} (m_0 k_1 J_{1y} + n_0 k_2 J_{1x})$

and $\beta = J_{1x}^{(k_1-2)} J_{1y}^{(k_2-2)} \left[m_0^2 k_1 (k_1 - 1) J_{1y}^2 + (2 m_0 n_0) k_1 k_2 J_{1x} J_{1y} + n_0^2 k_2 (k_2 - 1) J_{1x}^2 \right]$

$\frac{2k^2}{e}\beta$ represents the second order variation of ψ'

Δe is related to the width of the resonance

if $\frac{\Delta e}{e}$ is small $\quad \psi = e\left(1 - \frac{2k^2}{e^2}\right)\theta + \psi_0$

if not $\quad \psi = e\left(1 - \frac{2k^2}{e^2}\right)\theta + \frac{\Delta e}{e}\cos\psi + \psi_0$

III Analysis of oscillations

III.1 The oscillation

The method involves kicking the beam, preferably in the two planes (x, y) simultaneously and looking at the oscillations turn by turn at the same place (H and V pick-up). If the beam has sufficiently small emittances (obtained by cooling) and with the machine tuned close to a resonance - not too close to avoid losses - it is possible to obtain the characteristics of that resonance

I

Knowing the variation of J_x and ϕ_x to first order one would look at the beam behaviour using the relationship.

$$x = \sqrt{2\beta J_x} \cos(\phi_x + W_x)$$

(3) i.e. $x = \sqrt{2\beta_x J_{1x}} \sqrt{1 - 2\frac{m_0 \alpha}{J_{1x}} \cos\psi} \cos\left(\phi_{1x} - k_1 \frac{\alpha}{2J_{1x}} \sin\psi + W_x\right)$

with $\alpha = \frac{k}{e} J_{1x}^{k_1/2} J_{1y}^{k_2/2}$

if $\frac{\alpha}{J_{1x}} \ll 1$

the Fourier spectrum is $\frac{d}{d\theta}\left(\phi_{1x} - \frac{k_1\alpha}{2J_{1x}} \sin\psi + W_x\right) = f_x$

looking at one place the oscillations of the beam

$\langle f_x \rangle = \nu_x$ + phase independent perturbating term

the amplitude of the oscillation at $\langle f_x \rangle$ is

$a_x = \sqrt{2\beta J_{1x}}$ and the original phase is $\phi_{1x0} - k_1 \frac{\alpha}{2J_{1x}} \sin\psi_0 = \varphi_{xc}$

The same kind of result is obtained for vertical oscillation

$$\langle f_y \rangle = \nu_x + \text{perturbation}$$

$$a_y = \sqrt{2\beta_y J_y}$$

$$\varphi_{yo} = \phi_{1yo} - k_2 \frac{\alpha}{J_{1x}} \sin \varphi_o$$

As the x and y oscillations can be viewed as amplitude and phase modulated one can try to demodulate to obtain the amplitude and phase. Two methods could be used to find the perturbations:

- the first one uses two pick-ups in one plane and two in the other with a phase difference not too far from $\pi/2$ in each plane - for computing J_x, J_y.
- the second is just demodulation by computing the Fourier spectrum of x^2 (amplitude demodulation

III - 2 Finding J

with two signals x_1 and x_2 in the horizontal plane it is possible to find the phase difference $\Delta\varphi$ between the pick-up and the amplitude ratio $\frac{a_{1x}}{a_{2x}}$ of the signals. Using the normalized phase space we could write

$$X_1' = \frac{1}{\sin \Delta\varphi} \left(X_1 \cos \Delta\varphi - X_2' \right)$$

Introducing the electronic gain and the β_x value

$$X_1' = \frac{1}{\sin \Delta\varphi} \left(\frac{x_1}{a_{1x}} \cos \Delta\varphi - \frac{x_2}{a_{2x}} \right)$$

As we have seen in II $2J_x = X_1^2 + X_1'^2$

finally we find $2J_x a_{1x}^2 = x_2^2 + \frac{1}{\sin^2\Delta\varphi}\left(x_1 \cos\Delta\varphi - x_2 \frac{a_{1x}}{a_{2x}}\right)^2$

using the appropriate relation ② the only frequencies we could see on the Fourier spectrum of $2J_x a_{1x}^2$ are :

0 : amplitude $(\langle 2J_{1x}\rangle = \langle 2J_x\rangle) * a_{1x}^2$

$\pm\langle\psi'\rangle$: amplitude 4 mod $* a_{1x}^2$
phase ψ_0

We should be careful of the value of ψ_0

if $e>0$ i.e above the resonance
$\psi_0 = \psi_{0\,measured} + 180°$ (− sign of $\cos\psi$)
if $e<0$ i.e. under the resonance
$\psi_0 = -\psi_{0\,measured}$ (rigth frequency i.e)

By comparing $\psi_0 = m_0\phi_{x0} + n_0\phi_{y0} + \delta$ with the equation of ② we could extract $\underline{\delta = \psi_0 - m_0\varphi_{x0} - n_0\varphi_{y0}}$ which is the phase of the resonance term compared to the point of measurement

To find the strength K of the resonance we should calibrate the electronic gain

from J_x spectrum we measure

$u_1 = a_{1x}^2 \; J_x$

$u_2 = a_{1x}^2 \, \frac{2m_0 K}{e} \, J_{1x}^{k_1/2} \, J_{1y}^{k_2/2}$

from J_y spectrum we measure

$u_3 = a_{1y}^2 \; J_{1y}$

$u_4 = a_{1y}^2 \, \frac{2n_0 K}{e} \, J_{1x}^{k_1/2} \, J_{1y}^{k_2/2}$

we suppose n_0, n_0, k_1, k_2 are known as well as e & $\langle\psi'\rangle$ we get 4 equations with 5 unknowns (J_{1x}, J_{1y}, k, a_{1x}^2, a_{2x}^2)

that can be solved if $k_1 = k_2 = 1$
but for other cases we have to make another measurements just by inverting the electronic channel of H and V.

The above method could be used with the results of tracking programs to find what are the perturbating resonances.

III-3 Démodulation by squaring

From equation (3)

$$x^2 = \sqrt{\beta_x}\left\{J_{1x} - m_0\alpha\cos\psi\right\}\left\{1 + \cos\left(2\phi_{1x} - \frac{k_1}{J_{1x}}\alpha\sin\psi + 2\mu_x\right)\right\}$$

the frequencies that can be found from this signal are

"0" amplitude J_{1x}

$\langle\psi'\rangle$ ne amplitude $m_0\alpha$

phase ψ_0

$2q_H = \langle(2\phi_{1x0} - \frac{k_1\alpha}{J_{1x}}\alpha\sin\psi)'\rangle$ amplitude J_{1x}

phase $2\varphi_{x0}$

and some others $2q_H \pm \langle\psi'\rangle$. amplitude $m_0\alpha$

It's obvious that the démodulation is usefull if $2q_H$ or $(1-2q_H)$ is different from $\langle\psi'\rangle$

as in III-2 we could find the phase and amplitude of the resonant term.

III-4 Special case of $k_1 = k_2 = 1$

when $k_1 = k_2 = 1$ (resonances of type $Q_H \pm Q_V = p$) one can see from equation 1 that some problems occur when $\sqrt{\frac{J_x}{J_y}}$ or $\sqrt{\frac{J_y}{J_x}}$ is big corresponding to a non negligible value of $\frac{k_1 d}{J_{1x}}$. To be sure that the method still remain correct we propose to integrate the equations using the Guignard's formalism.

IV. The resonance $Q_H + Q_V = 5$

In this case (linear coupling) it is possible to solve the eqnations of motion analytically.
The perturbing part of the Hamiltonian is in the case of skew quadrupoles [7]

$$\mathcal{U} = h^{(2)}_{1010-5}\, a_1 a_2\, e^{i\varepsilon\theta} + h^{(2)}_{0101-5}\, a_1^* a_2^*\, e^{-i\varepsilon\theta} =$$

$$= k a_1 a_2\, e^{i\varepsilon\theta} + k^* a_1^* a_2^*\, e^{-i\varepsilon\theta}$$

where

$\varepsilon = Q_H + Q_V - 5$, distance from resonance

$|k|$, amplitude of the resonance

Hamiltons equation gives

$$\begin{cases} \dfrac{da_1}{d\theta} = i\dfrac{\partial \mathcal{U}}{\partial a_1^*} = i k^* a_2^* e^{-i\varepsilon\theta} \\[2ex] \dfrac{da_2}{d\theta} = i\dfrac{\partial \mathcal{U}}{\partial a_2^*} = i k^* a_1^* e^{-i\varepsilon\theta} \end{cases} \qquad (1)$$

a_i $(i=1,2)$ is related to the invariant of unperturbed motion by

$$\begin{cases} |a_1|^2 = R J_x \\ |a_2|^2 = R J_y \end{cases}$$

where

R average machine radius

Eq (1) can be solved by differentiating the second equation

$$\frac{d}{d\theta}\left(\frac{da_2}{d\theta}e^{i\varepsilon\theta}\right) = ik^*\frac{da_1^*}{d\theta} \Rightarrow$$

$$\left(\frac{d^2a_2}{d\theta^2} + i\varepsilon\frac{da_2}{d\theta}\right)e^{i\varepsilon\theta} = ik^*\frac{da_1^*}{d\theta}$$

Using the first equation gives

$$\left(\frac{d^2a_2}{d\theta^2} + i\varepsilon\frac{da_2}{d\theta}\right)e^{i\varepsilon\theta} = ik^*(-k)a_2e^{i\varepsilon\theta} \Rightarrow$$

$$\boxed{\frac{d^2a_2}{d\theta^2} + i\varepsilon\frac{da_2}{d\theta} - |k|^2a_2 = 0}$$

The solution to this equation is

$$a_2 = A_+^*e^{i(-\frac{\varepsilon}{2}-\omega')\theta} + A_-^*e^{i(-\frac{\varepsilon}{2}+\omega')\theta}$$

where

A_+, A_- complex constants given by initial conditions

$\omega' = \sqrt{(\frac{\varepsilon}{2})^2 - |k|^2}$

This can be written

$$\boxed{a_2 = A_+^*e^{i\omega_-\theta} + A_-^*e^{i\omega_+\theta}} \qquad \boxed{\omega_\pm = -\frac{\varepsilon}{2} \pm \sqrt{(\frac{\varepsilon}{2})^2 - |k|^2}}$$

This gives

$$\frac{da_1}{d\theta} = ik^*(A_+e^{-i\omega_-\theta} + A_-e^{-i\omega_+\theta})e^{-i\varepsilon\theta} =$$

$$= ik^*(A_+e^{-i(\omega_-+\varepsilon)\theta} + A_-e^{-i(\omega_+\varepsilon)\theta} =$$

$$= ik^*(A_+e^{i\omega_+\theta} + A_-e^{i\omega_-\theta})$$

This can be integrated, assuming that the integration constant is already in the unperturbed term.

$$\boxed{a_1 = k^* \left(\frac{A_+}{\omega_+} e^{i\omega_+\theta} + \frac{A_-}{\omega_-} e^{i\omega_-\theta} \right)}$$

We now get

$$|a_1|^2 = a_1 a_1^* = |k|^2 \left(\frac{A_+}{\omega_+} e^{i\omega_+\theta} + \frac{A_-}{\omega_-} e^{i\omega_-\theta} \right)\left(\frac{A_+^*}{\omega_+} e^{-i\omega_+\theta} + \frac{A_-^*}{\omega_-} e^{-i\omega_-\theta} \right) =$$

$$= |k|^2 \left[\frac{|A_+|^2}{\omega_+} + \frac{|A_-|^2}{\omega_-} + \frac{A_+ A_-^*}{\omega_+\omega_-} e^{i(\omega_+-\omega_-)\theta} + \frac{A_+^* A_-}{\omega_+\omega_-} e^{-i(\omega_+-\omega_-)} \right] =$$

We use

$$\omega_+ - \omega_- = 2\sqrt{\left(\frac{\varepsilon}{2}\right)^2 - |k|^2} = \sqrt{\varepsilon^2 - 4|k|^2} \equiv \delta$$

$$\omega_+ \cdot \omega_- = |k|^2$$

and obtain

$$|a_1|^2 = \frac{|k|^2}{\omega_+^2} |A_+|^2 + \frac{|k|^2}{\omega_-^2} |A_-|^2 + A_+ A_-^* e^{i\delta\theta} + A_+^* A_- e^{-i\delta\theta}$$

$$\boxed{|a_1|^2 = \frac{|k|^2}{\omega_+^2} |A_+|^2 + \frac{|k|^2}{\omega_-^2} |A_-|^2 + 2 Re (A_+ A_-^*) \cos\delta\theta + 2 Im (A_+ A_-^*) \sin\delta\theta}$$

$$|a_2|^2 = a_2 a_2^* = (A_+^* e^{i\omega_-\theta} + A_-^* e^{i\omega_+\theta})(A_+ e^{-i\omega_-\theta} + A_- e^{-i\omega_+\theta}) =$$

$$= |A_+|^2 + |A_-|^2 + A_+ A_-^* e^{i\delta\theta} + A_+^* A_- e^{-i\delta\theta}$$

$$\boxed{|a_2|^2 = |A_+|^2 + |A_-|^2 + 2 Re (A_+ A_-^*) \cos\delta\theta + 2 Im (A_+ A_-^*) \sin\delta\theta}$$

J's are given by

$$\begin{cases} J_x = \frac{1}{R} |a_1|^2 \\ J_y = \frac{1}{R} |a_2|^2 \end{cases}$$

From this we can see that the coherent oscillation vary due to the resonance with the angular frequency δ given by

$$\boxed{\delta = \sqrt{\varepsilon^2 - 4|\kappa|^2}}$$

The unperturbed motion is

$$\begin{cases} x = a_1 u_1 e^{iQ_H\theta} + a_1^* u_1^* e^{-iQ_x\theta} \\ z = a_2 u_2 e^{iQ_V\theta} + a_2^* u_2^* e^{-iQ_z\theta} \end{cases}$$

where the Floquet-functions u_i $(i=1,2)$ are

$$u_i = \sqrt{\frac{\beta_i}{2R}}\, e^{i(\mu_i - Q_i\theta)}$$

where

$$\mu_i = \int_0^\theta \frac{R}{\beta_i(\theta')}\, d\theta' \qquad \Rightarrow \frac{d\mu_i}{d\theta} = \frac{R}{\beta_i}$$

$$Q_i = \frac{1}{2\pi}\int_0^{2\pi} \frac{R}{\beta_i(\theta')}\, d\theta'$$

This gives

$$\begin{cases} x = \sqrt{\frac{\beta_x}{2R}}\Big[\kappa^*\Big(\frac{A_+}{\omega_+} e^{i(\mu_x + \omega_+\theta)} + \frac{A_-}{\omega_-} e^{i(\mu_x + \omega_-\theta)}\Big) + \\ \quad + \kappa\Big(\frac{A_+^*}{\omega_+} e^{-i(\mu_x + \omega_+\theta)} + \frac{A_-^*}{\omega_-} e^{-i(\mu_x + \omega\,\theta)}\Big)\Big] \\ z = \sqrt{\frac{\beta_z}{2R}}\Big[A_+^* e^{i(\mu_z + \omega_-\theta)} + A_-^* e^{i(\mu_z + \omega_+\theta)} + A_+ e^{-i(\mu_z + \omega_-\theta)} + A_- e^{-i(\mu_z + \omega_+\theta)}\Big] \end{cases}$$

By using Eulers equations this can be written in terms of sine and cosine.

$$x = \sqrt{\frac{\beta_x}{2R}}\left[\frac{1}{\omega_+}(k^*A_+ + kA_+^*)\cos(\mu_x+\omega_+\theta) + \frac{i}{\omega_+}(k^*A_+ - kA_+^*)\sin(\mu_x+\omega_+\theta)\right.$$

$$\left. + \frac{1}{\omega_-}(k^*A_- + kA_-^*)\cos(\mu_x+\omega_-\theta) + \frac{i}{\omega_-}(k^*A_- - kA_-^*)\sin(\mu_x+\omega_-\theta)\right] =$$

$$= \sqrt{\frac{2\beta_x}{R}}\left[\frac{1}{\omega_+}Re(k^*A_+)\cos(\mu_x+\omega_+\theta) - \frac{1}{\omega_+}Im(kA_+)\sin(\mu_x+\omega_+\theta) + \right.$$

$$\left. + \frac{1}{\omega_+}Re(k^*A_-)\cos(\mu_x+\omega_-\theta) - \frac{1}{\omega_-}Im(kA_-)\sin(\mu_x+\omega_-\theta)\right]$$

$$z = \sqrt{\frac{\beta_z}{2R}}\left[(A_+ + A_+^*)\cos(\mu_z+\omega_-\theta) - i(A_+ + A_+^*)\sin(\mu_z+\omega_-\theta) + \right.$$

$$\left. + (A_- + A_-^*)\cos(\mu_z+\omega_+\theta) - i(A_- - A_-^*)\sin(\mu_z+\omega_+\theta)\right] =$$

$$= \sqrt{\frac{2\beta_z}{R}}\left[Re(A_+)\cos(\mu_z+\omega_-\theta) + Im(A_+)\sin(\mu_z+\omega_-\theta) + \right.$$

$$\left. + Re(A_-)\cos(\mu_z+\omega_+\theta) + Im(A_-)\sin(\mu_z+\omega_+\theta)\right]$$

By using

$$a\cos\varphi + b\sin\varphi = \sqrt{a^2+b^2}\cos(\varphi-\varphi_0) \qquad \frac{b}{a} = \tan\varphi_0$$

$$\boxed{x = \sqrt{\frac{2\beta_x}{R}}\left[\frac{|k^*A_+|}{\omega_+}\cos(\mu_x+\omega_+\theta+\varphi_x^{(+)}) + \frac{|k^*A_-|}{\omega_-}\cos(\mu_x+\omega_-\theta+\varphi_x^{(-)})\right]}$$

where

$$\tan(-\varphi_x^{(+)}) = \frac{-Im(k^*A_+)}{Re(k^*A_-)} \Rightarrow \tan\varphi_x^{(+)} = \frac{Im(k^*A_+)}{Re(k^*A_+)}$$

$$\boxed{\varphi_x^{(+)} = \arg A_+ - \arg k}$$

$$\tan(-\varphi_x^{(-)}) = \frac{-Im(k^*A_-)}{Re(k^*A_-)} \Rightarrow \tan\varphi_x^{(-)} = \frac{Im(k^*A_-)}{Re(k^*A_-)}$$

$$\boxed{\varphi_x^{(-)} = \arg A_- - \arg k}$$

$$\boxed{z = \sqrt{\frac{2\beta_z}{R}} \left[|A_+| \cos(\mu_z + \omega_- \theta + \varphi_z^{(-)}) + |A_-| \cos(\mu_z + \omega_+ \theta + \varphi_z^{(+)}) \right]}$$

where

$$\tan(-\varphi_z^{(+)}) = \frac{Im A_-}{Re A_-} \quad \rightarrow \quad \tan \varphi_z^{(+)} = -\frac{Im A_-}{Re A_-}$$

$$\tan(-\varphi_z^{(-)}) = \frac{Im A_+}{Re A_+} \quad \rightarrow \quad \tan \varphi_z^{(-)} = -\frac{Im A_+}{Re A_+}$$

$$\boxed{\varphi_z^{(+)} = -\arg A_-} \qquad \boxed{\varphi_z^{(-)} = -\arg A_+}$$

Let us now take the square of x and z

$$\begin{cases} x^2 = \frac{2\beta_x}{R} \left[\frac{|k^* A_+|^2}{\omega_+^2} \cos^2(\mu_x + \omega_+ \theta + \varphi_x^{(+)}) + \frac{|k^* A_-|^2}{\omega_-^2} \cos^2(\mu_x + \omega_- \theta + \varphi_x^{(-)}) + \right. \\ \left. \qquad 2 \frac{|k^* A_+||k^* A_-|}{\omega_+ \omega_-} \cos(\mu_x + \omega_+ \theta + \varphi_x^{(+)}) \cos(\mu_x + \omega_- \theta + \varphi_x^{(-)}) \right] \\ z^2 = \frac{2\beta_z}{R} \left[|A_+|^2 \cos^2(\mu_z + \omega_- \theta + \varphi_z^{(-)}) + |A_-|^2 \cos^2(\mu_z + \omega_+ \theta + \varphi_z^{(+)}) + \right. \\ \left. \qquad 2 |A_+||A_-| \cos(\mu_z + \omega_- \theta + \varphi_z^{(-)}) \cos(\mu_z + \omega_+ \theta + \varphi_z^{(+)}) \right] \end{cases}$$

This can be developed to

$$\boxed{\begin{aligned} x^2 = \frac{\beta_x}{R} \left[\frac{|k^* A_+|^2}{\omega_+^2} + \frac{|k^* A_-|^2}{\omega_-^2} + \frac{|k^* A_+|^2}{\omega_+^2} \cos 2(\mu_x + \omega_+ \theta + \varphi_x^{(+)}) + \right. \\ + \frac{|k^* A_-|^2}{\omega_-^2} \cos 2(\mu_x + \omega_- \theta + \varphi_x^{(-)}) + \\ + 2 \frac{|k^* A_+||k^* A_-|}{\omega_+ \omega_-} \cos(2\mu_x + (\omega_+ + \omega_-)\theta + \varphi_x^{(+)} + \varphi_x^{(-)}) + \\ + 2 \frac{|k^* A_+||k^* A_-|}{\omega_+ \omega_-} \cos(\delta \theta + \varphi_x^{(+)} - \varphi_x^{(-)}) \end{aligned}}$$

$\delta = \omega_+ - \omega$

$$\boxed{\begin{aligned} z^2 = \frac{\beta_z}{R}\Big[& |A_+|^2 + |A_-|^2 + |A_+|^2\cos 2(\mu_z + \omega_-\theta + \varphi_z^{(-)}) + \\ & + |A_-|^2 \cos 2(\mu_z + \omega_+\theta + \varphi_z^{(+)}) + \\ & + 2|A_+||A_-|\cos(2\mu_z + (\omega_+ + \omega_-)\theta + \varphi_z^{(+)} + \varphi_z^{(-)}) + \\ & + 2|A_+||A_-|\cos(\delta\theta + \varphi_z^{(+)} - \varphi_z^{(-)}) \end{aligned}}$$

$\delta \equiv \omega_+ - \omega_-$

This can be summarized as follow:

Variable	Frequency component	Phase
x	$\mu_x' + \omega_+$	$\varphi_x^{(+)} = \arg A_+ - \arg k$
	$\mu_x' + \omega_-$	$\varphi_x^{(-)} = \arg A_- - \arg k$
z	$\mu_z' + \omega_+$	$\varphi_z^{(+)} = -\arg A_-$
	$\mu_z' + \omega_-$	$\varphi_z^{(-)} = -\arg A_+$
x^2	δ	$\varphi_x^{(+)} - \varphi_x^{(-)} = \arg A_+ - \arg A_-$
	$2(\mu_x' + \omega_+)$	$2\varphi_x^{(+)} = 2(\arg A_+ - \arg k)$
	$2(\mu_x' + \omega_-)$	$2\varphi_x^{(-)} = 2(\arg A_- - \arg k)$
	$2\mu_x' + \omega_+ + \omega_-$	$\varphi_x^{(+)} + \varphi_x^{(-)} = \arg A_+ + \arg A_- + -2\arg k$
z^2	δ	$\varphi_z^{(+)} - \varphi_z^{(-)} = -\arg A_- + \arg A_+$
	$2(\mu_z' + \omega_+)$	$2\varphi_z^{(+)} = -2\arg A_-$
	$2(\mu_z' + \omega_-)$	$2\varphi_z^{(-)} = -2\arg A_+$
	$2\mu_z + \omega_+ + \omega_-$	$\varphi_z^{(+)} + \varphi_z^{(-)} = -\arg A_- - \arg A_+$

Note that ω_+ and ω_- appear as factors in the horizontal motion. Due to this

$$\begin{cases} \omega_+, \omega_- < 0 & \text{, above the resonance} \\ \omega_+, \omega_- > 0 & \text{, below the resonance} \end{cases}$$

and π should be added to the phases of the horizontal frequency components if one are above the resonance ($Q_H + Q_V > 5$).

From the table above one can find some different relations for the phase of the resonance. For example

$$\begin{cases} \arg k = -\arg(\mu_z' + \omega_-) - \arg(\mu_x' + \omega_+) \\ \arg k = -\arg(\mu_z' + \omega_+) - \arg(\mu_x' + \omega_-) \\ \arg k = \arg \delta - \arg(\mu_z' + \omega_+) - \arg(\mu_x' + \omega_+) \\ \arg k = -\arg \delta - \arg(\mu_z' + \omega_-) - \arg(\mu_x' + \omega_-) \end{cases}$$

Which one is to be used for real measurements depends on which peaks are best seen in the measured spectra.

V. Simulation

It is possible to simulate the motion in phase-space for any given resonance by numerical integration of the equations of motion for the perturbation [7]

$$\begin{cases} \dfrac{da_1}{d\theta} = iKk a_1^j a_1^{*(k-1)} a_2^l a_2^{*m} e^{i\varepsilon\theta} + iK^* j a_1^{*(j-1)} a_1^k a_2^{*l} a_2^m e^{-i\varepsilon\theta} \\ \dfrac{da_2}{d\theta} = iKm a_1^j a_1^{*k} a_2^l a_2^{*(m-1)} e^{i\varepsilon\theta} + iK^* l a_1^{*j} a_1^k a_2^{*(l-1)} a_2^m e^{-i\varepsilon\theta} \end{cases}$$

where

$$\varepsilon = n_x Q_x + n_z Q_z - p$$

and a_1, a_2 are related to the invariant of the unperturbed motion by

$$\begin{cases} |a_1|^2 = R \cdot J_x \\ |a_2|^2 = R \cdot J_y \end{cases}$$

the unperturbed motion was given by

$$\begin{cases} x = a_1 u_1 e^{iQ_x\theta} + a_1^* u_1^* e^{-iQ_x\theta} \\ z = a_2 u_2 e^{iQ_z\theta} + a_2^* u_2^* e^{-iQ_z\theta} \end{cases}$$

and

$$u_i = \sqrt{\frac{\beta_i}{2R}}\, e^{i(\mu_i - Q_i\theta)}$$

$$\mu_i = \int_0^\theta \frac{R}{\beta_i(\theta')}\, d\theta'$$

$$Q_i = \frac{1}{2\pi}\int_0^{2\pi} \frac{R}{\beta_i(\theta')}\, d\theta'$$

where a_1 and a_2 now are functions of θ due to the perturbation.

To make life simple we assume β to be constant around the machine for the simulation. The numerical integration is uses the Runge-Khutta method and we enter the unperturbed Q-values and oscillation amplitude, strength and phase for the resonance ($|k|, \varphi_k$) as parameters.

We show two examples of the simulation. The first case) is for the resonance) $Q_H + Q_V = 5$ where we also give the calculated pertubed Q-values which can be compared with the ones measured by Fast Fourier Transform. We also calculate the phase of the resonance from one of the relations in the previous chapter, for comparison with the given phase.

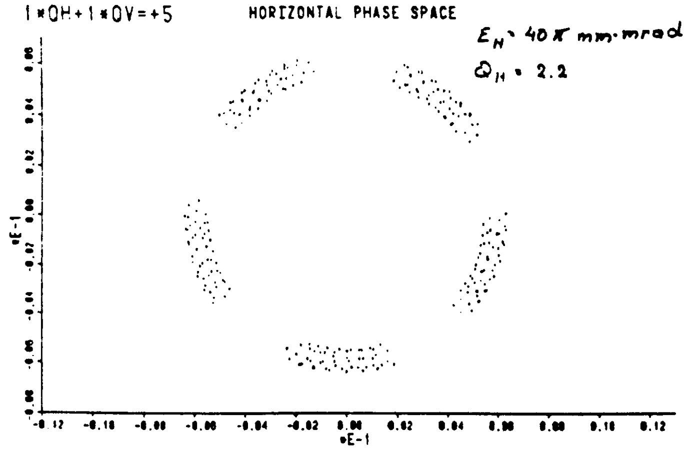
In the second case we simulate the 3rd order resonance $Q_H + 2Q_V = 8$, and we calculate the phase of the resonance given in chapter III. We also identify the different peaks from the measured perturbed Q-values.

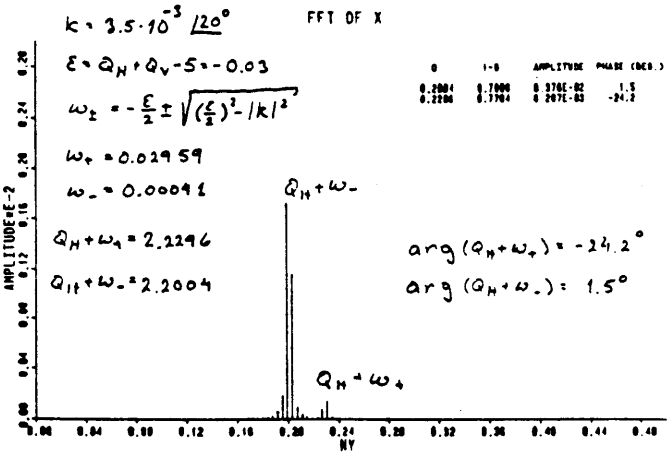
VI Conclusions

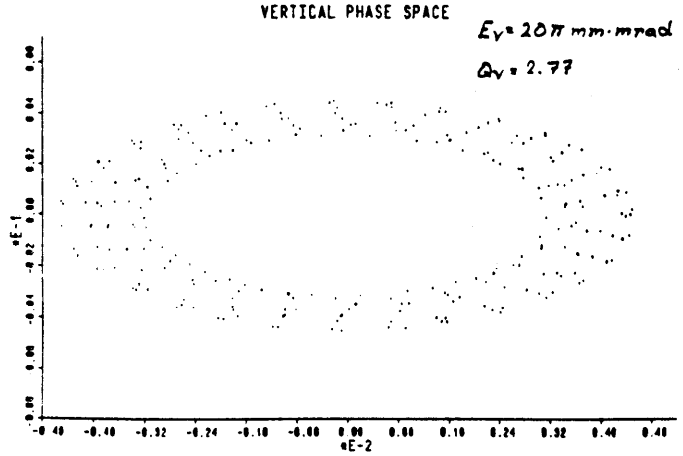
With the clean signals that we expect to obtain from the measuring device we hope to be able to measure the strength and phase of perturbing resonances and hence decrease their effects

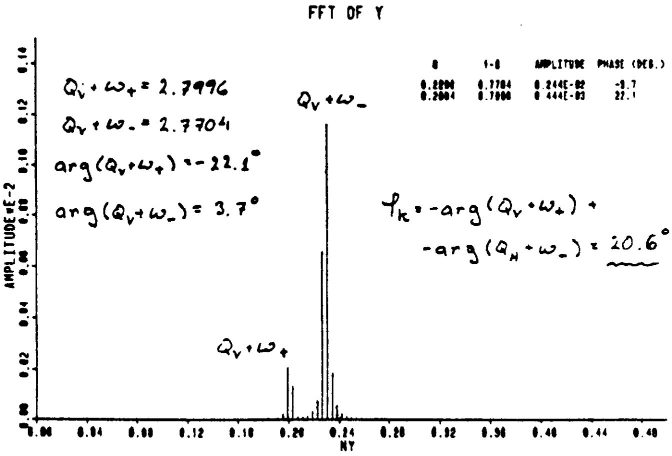
We would like to thank P. Lefevre and D. Möhl for their continuous support and discussions. We also thank E Asseo for introducing us to the subtleties of Fourier Transform and for building high quality hardware

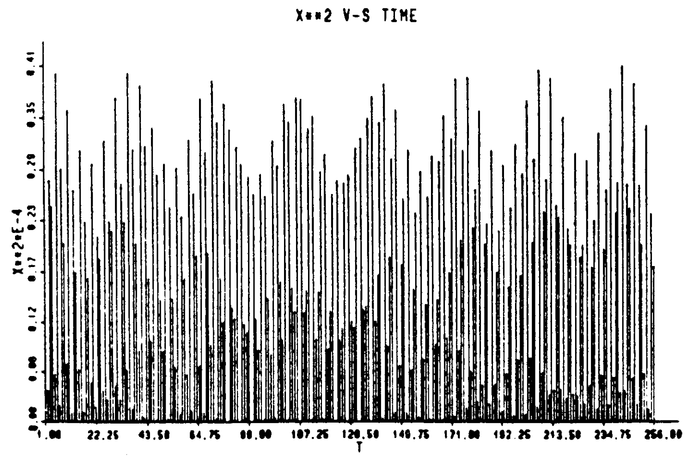
Appendix

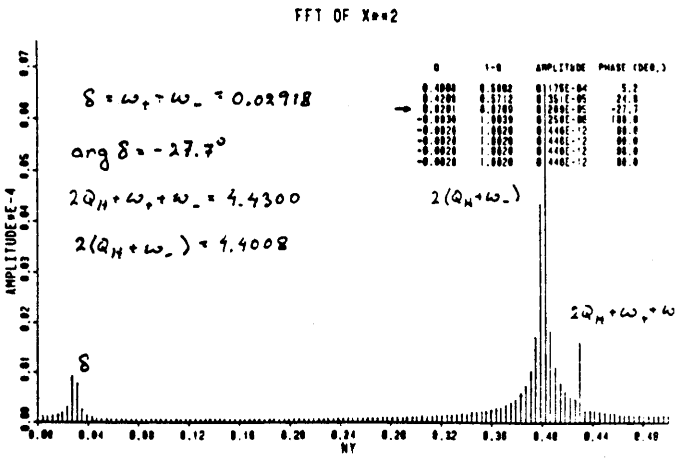

1*QH+1*QV=+5
HORIZONTAL PHASE SPACE
$E_H = 40\pi$ mm·mrad
$Q_H = 2.2$
*E-1
*E-1

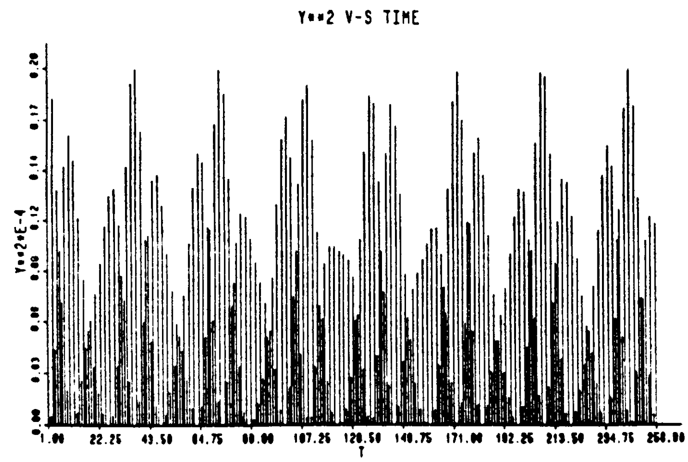

$k = 3.5\cdot10^{-3}$ /20°
FFT OF X
$\varepsilon = Q_H + Q_V - 5 = -0.03$
$\omega_\pm = -\frac{\varepsilon}{2} \pm \sqrt{(\frac{\varepsilon}{2})^2 - |k|^2}$
$\omega_+ = 0.02959$
$\omega_- = 0.00041$
$Q_H + \omega_-$
$Q_H + \omega_+ = 2.2296$
$Q_H + \omega_- = 2.2004$
$\arg(Q_H + \omega_+) = -24.2°$
$\arg(Q_H + \omega_-) = 1.5°$
$Q_H + \omega_+$
AMPLITUDE*E-2
NY

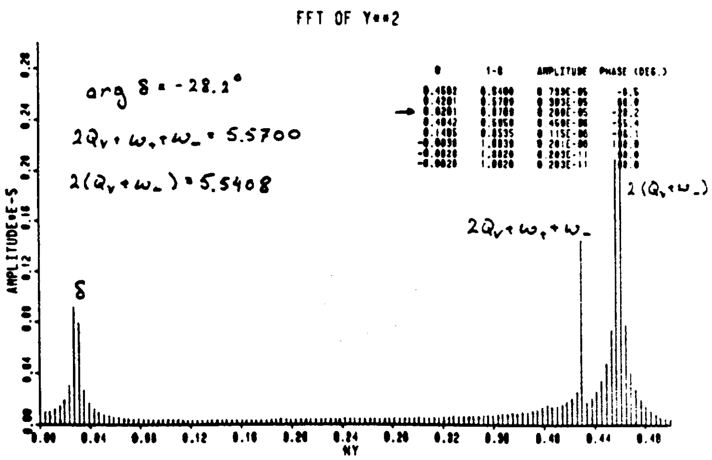

VERTICAL PHASE SPACE
$E_V = 20\pi$ mm·mrad
$Q_V = 2.77$
*E-1
*E-2


FFT OF Y
$Q_V + \omega_+ = 2.7996$
$Q_V + \omega_- = 2.7704$
$Q_V + \omega_-$
$\arg(Q_V + \omega_+) = -22.1°$
$\arg(Q_V + \omega_-) = 3.7°$
$\varphi_k = -\arg(Q_V + \omega_+) +$
$-\arg(Q_H + \omega_-) = 20.6°$
$Q_V + \omega_+$
AMPLITUDE*E-2
NY

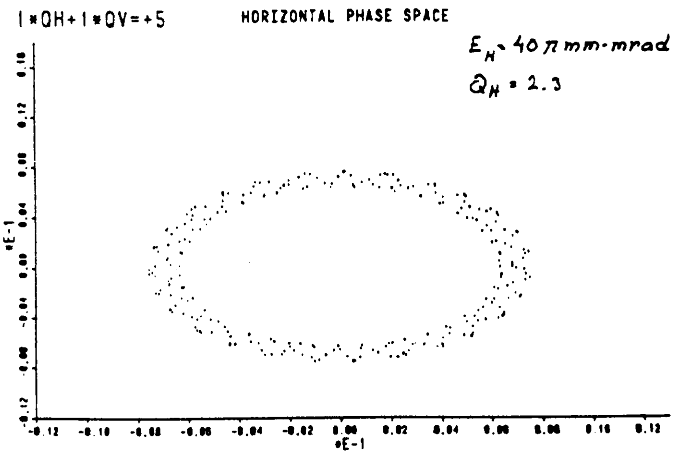

X**2 V-S TIME
X**2*E-4
T

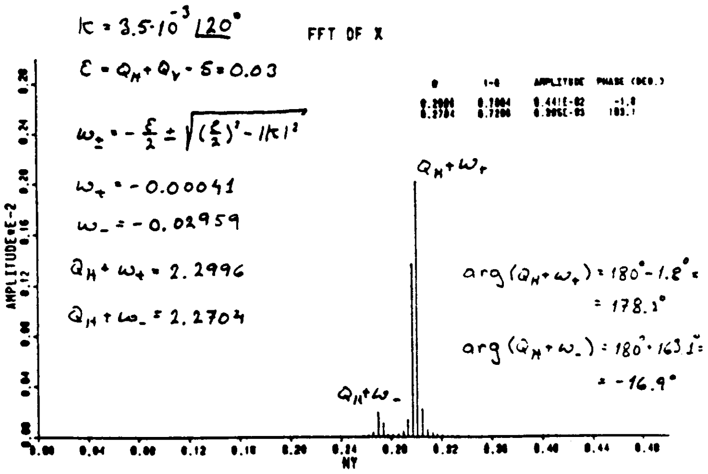

FFT OF X**2
$\delta = \omega_+ - \omega_- = 0.02918$
$\arg\delta = -27.7°$
$2Q_H + \omega_+ + \omega_- = 4.4300$
$2(Q_H + \omega_-) = 4.4008$
$2(Q_H + \omega_-)$
$2Q_H + \omega_+ + \omega_-$
δ
AMPLITUDE*E-4
NY

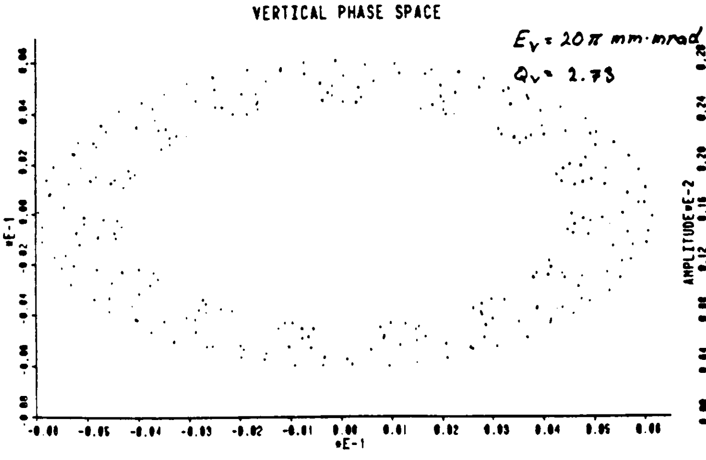

Y**2 V-S TIME
Y**2*E-4
T

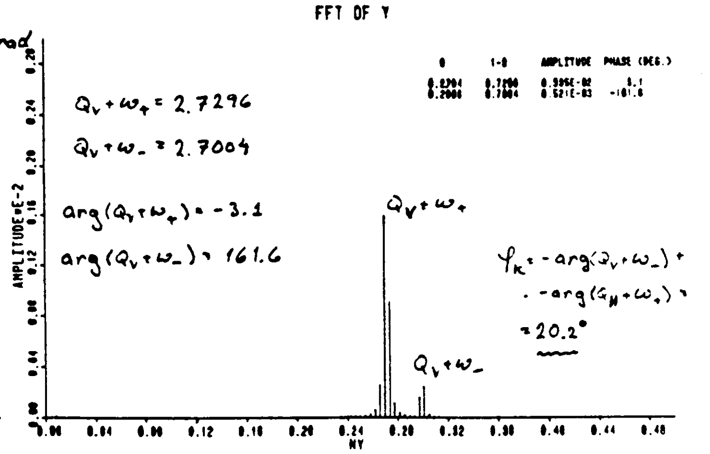

FFT OF Y**2
$\arg\delta = -28.2°$
$2Q_V + \omega_+ + \omega_- = 5.5700$
$2(Q_V + \omega_-) = 5.5408$
$2(Q_V + \omega_-)$
$2Q_V + \omega_+ + \omega_-$
δ
AMPLITUDE*E-5
NY

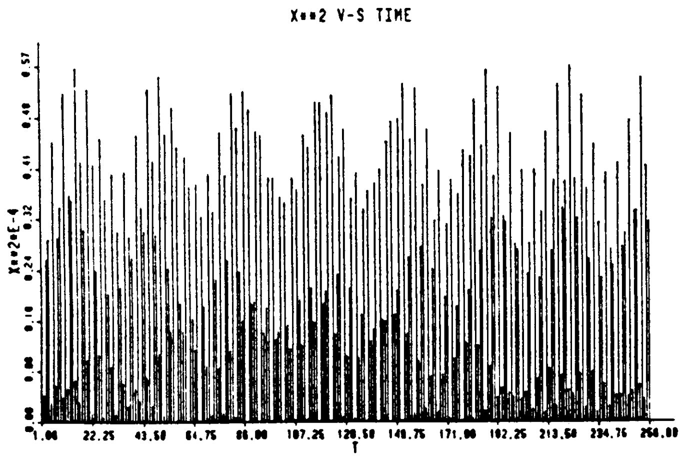
1*QH+1*QV=+5

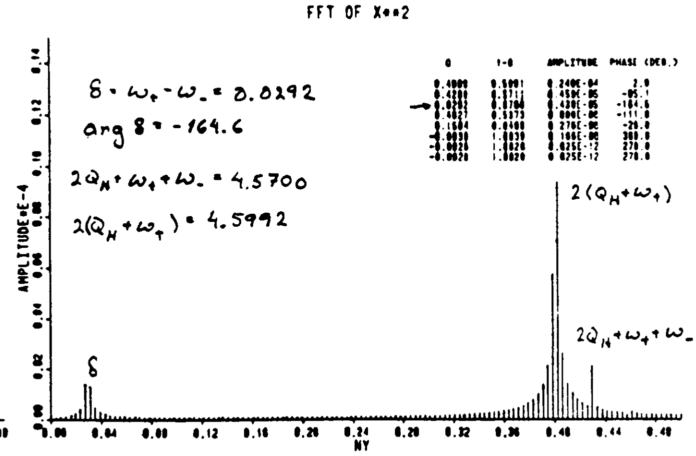

HORIZONTAL PHASE SPACE
$E_H = 40\pi$ mm·mrad
$Q_H = 2.3$
*E-1

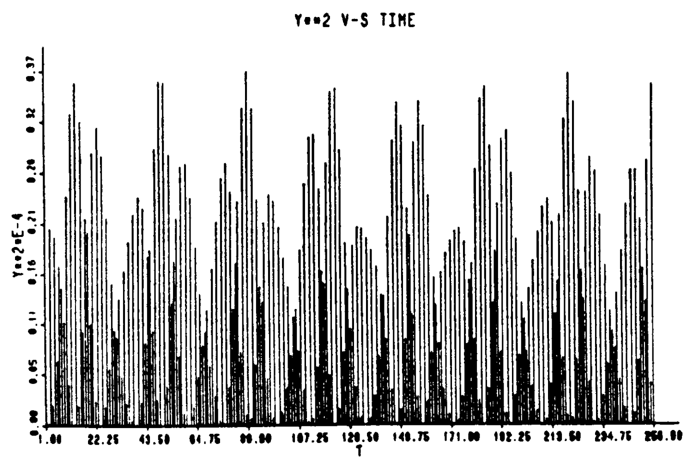

FFT OF X
$\kappa = 3.5 \cdot 10^{-3} \angle 20^\circ$
$\varepsilon = Q_H + Q_V - 5 = 0.03$
$\omega_\pm = -\frac{\varepsilon}{2} \pm \sqrt{(\frac{\varepsilon}{2})^2 - |\kappa|^2}$
$\omega_+ = -0.00041$
$\omega_- = -0.02959$
$Q_H + \omega_+ = 2.2996$
$Q_H + \omega_- = 2.2704$
$Q_H + \omega_+$
$Q_H + \omega_-$
$\arg(Q_H + \omega_+) = 180^\circ - 1.8^\circ =$
$= 178.2^\circ$
$\arg(Q_H + \omega_-) = 180^\circ - 163.1^\circ =$
$= -16.9^\circ$
AMPLITUDE*E-2
NY

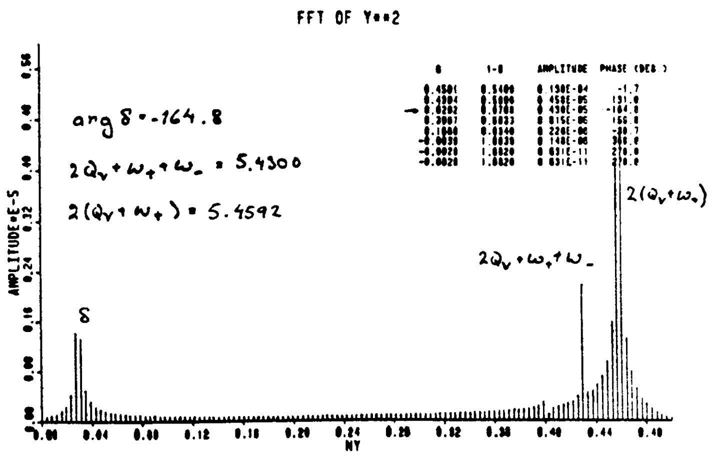

VERTICAL PHASE SPACE
$E_V = 20\pi$ mm·mrad
$Q_V = 2.73$
*E-1


FFT OF Y
$Q_V + \omega_+ = 2.7296$
$Q_V + \omega_- = 2.7004$
$\arg(Q_V + \omega_+) = -3.1$
$\arg(Q_V + \omega_-) = 161.6$
$Q_V + \omega_+$
$Q_V + \omega_-$
$\varphi_K = -\arg(Q_V + \omega_-) +$
$-\arg(Q_H + \omega_+) =$
$= 20.2^\circ$
AMPLITUDE*E-2
NY

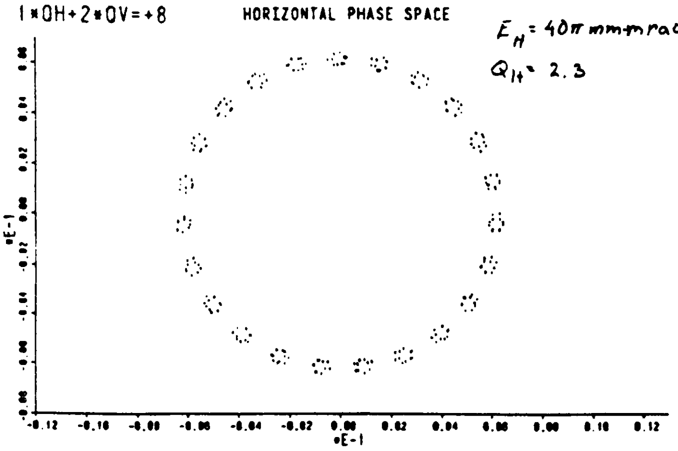

X**2 V-S TIME
X**2*E-4
T

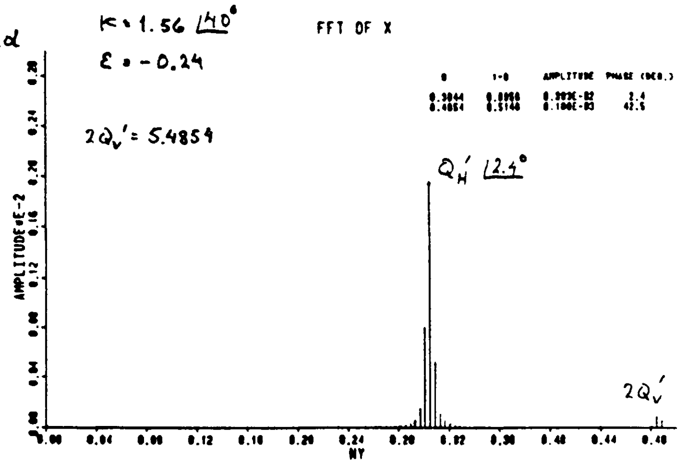

FFT OF X**2
$\delta = \omega_+ - \omega_- = 0.0292$
$\arg \delta = -164.6$
$2Q_H + \omega_+ + \omega_- = 4.5700$
$2(Q_H + \omega_+) = 4.5992$
δ
$2(Q_H + \omega_+)$
$2Q_H + \omega_+ + \omega_-$
AMPLITUDE*E-4
NY

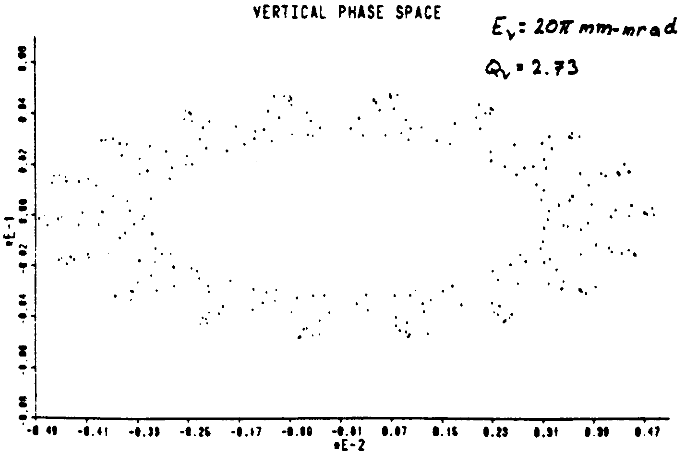

Y**2 V-S TIME
Y**2*E-4
T

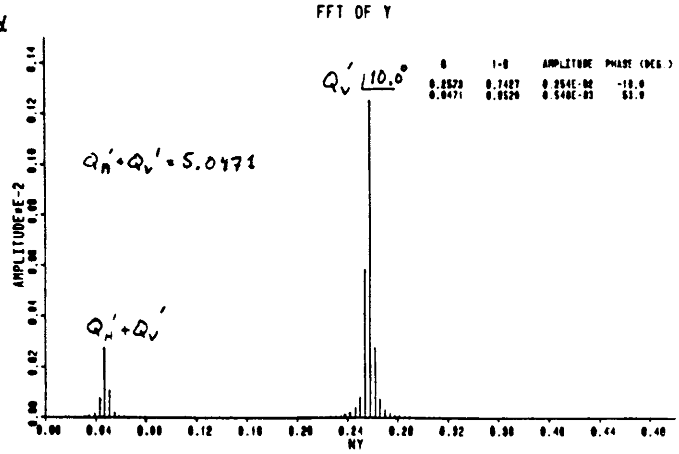

FFT OF Y**2
$\arg \delta = -164.8$
$2Q_V + \omega_+ + \omega_- = 5.4300$
$2(Q_V + \omega_+) = 5.4592$
δ
$2Q_V + \omega_+ + \omega_-$
$2(Q_V + \omega_+)$
AMPLITUDE*E-5
NY

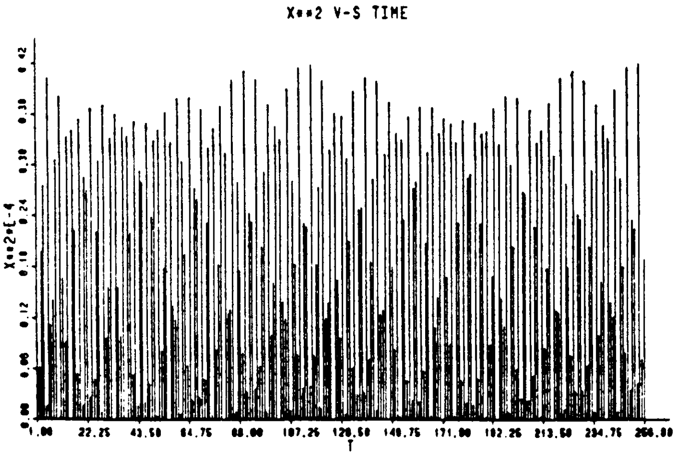
1*QH+2*QV=+8

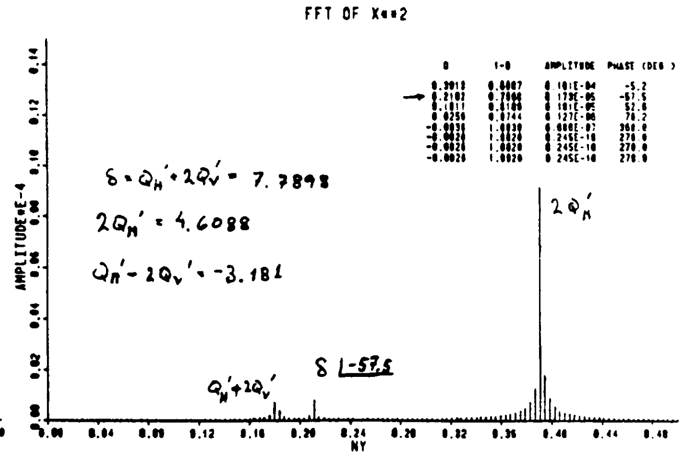

HORIZONTAL PHASE SPACE
$E_H = 40\pi$ mm-mrad
$Q_H = 2.3$
*E-1

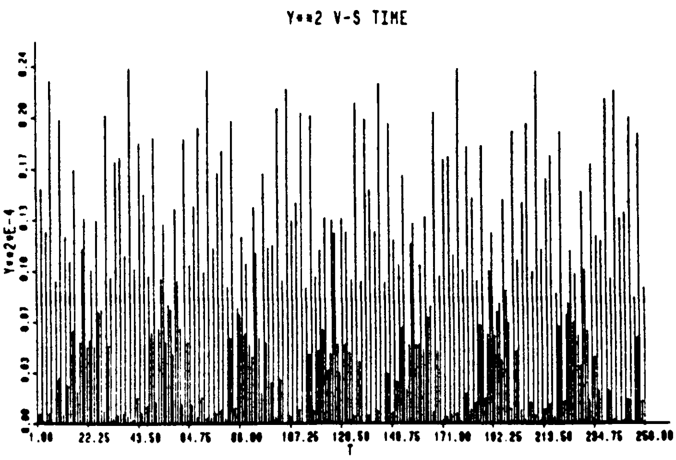

FFT OF X
$\kappa = 1.56 \angle 40^\circ$
$\varepsilon = -0.24$
$2Q_V' = 5.4854$
$Q_H' \angle 2.4^\circ$
$2Q_V'$
AMPLITUDE*E-2
NY

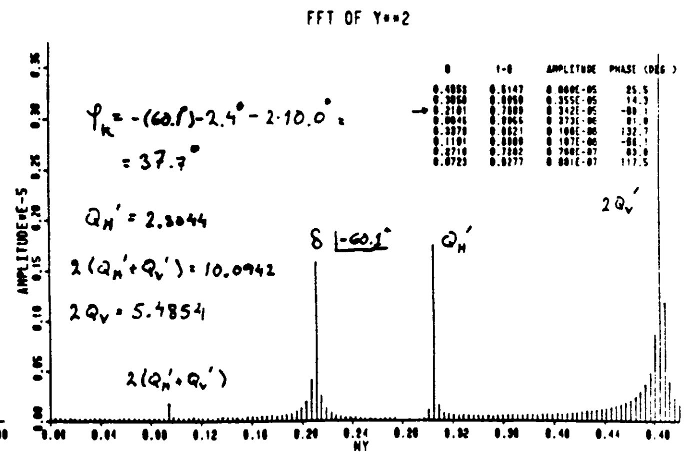

VERTICAL PHASE SPACE
$E_V = 20\pi$ mm-mrad
$Q_V = 2.73$
*E-1
*E-2


FFT OF Y
$Q_V' \angle 10.0^\circ$
$Q_H' + Q_V' = 5.0471$
$Q_H' + Q_V'$
AMPLITUDE*E-2
NY


X**2 V-S TIME
X**2*E-4
T


FFT OF X**2
$\delta = Q_H' + 2Q_V' = 7.7898$
$2Q_H' = 4.6088$
$Q_H' - 2Q_V' = -3.181$
$2Q_H'$
$\delta \angle -57.5$
$Q_H' + 2Q_V'$
AMPLITUDE*E-4
NY


Y**2 V-S TIME
Y**2*E-4
T


FFT OF Y**2
$\varphi_k = -(60.1) - 2.4^\circ - 2 \cdot 10.0^\circ =$
$= 37.7^\circ$
$Q_H' = 2.3044$
$2(Q_H' + Q_V') = 10.0942$
$2Q_V = 5.4854$
$\delta \angle -60.1^\circ$
Q_H'
$2Q_V'$
$2(Q_H' + Q_V')$
AMPLITUDE*E-5
NY

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⑦ G. Guignard CERN 78-11
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