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THE LIFETIME AGAINST GAS SCATTERING OF A CIRCULATING ELECTRON BEAM OF KINETIC ENERGY BETWEEN 1.5 AND 100 MeV.

Introduction.

The lifetime of a circulating electron beam against residual gas scattering depends on the gas density, the electron energy and the design parameters of the machine.

The electrons can be scattered on the nuclei and the electrons of the residual molecules. In the case of nuclear scattering, the electron can be lost from the beam in two different ways. The first is single scattering where an electron is deflected by a single collision through such a big angle that it hits the wall and is lost. The second is multiple scattering, where numerous collisions through small angles lead to the build-up of betatron oscillations, until the electrons hit the chamber wall as well. In the following the lifetimes against elastic scattering of the single and multiple type are calculated separately.

An electron colliding with an electron of a residual molecule looses a considerable portion of its energy even at relatively small scattering angles and is lost. It will be shown, that only in unfavourable cases this process comes in the order of magnitude of nuclear scattering.

The scattering on nuclei is inelastic if bremsstrahlung quanta are emitted. If the energy loss is big enough, the electron is lost at once. With small energy losses, synchrotron oscillations are built up by multiple scattering.

For energies of 500 MeV the lifetime is mostly limited by bremsstrahlung (1) while at 2 MeV it can certainly be neglected. At about 100 MeV the inelastic scattering may well be as serious as the elastic scattering. But no calculations are done on this subject in these notes.

Multiple Scattering on Nuclei.

The following theory is based on a paper by Blachman and Courant [2] on "The Scattering of Protons by the Gas in a Synchrotron".

We consider electrons of constant energy confined on a circular orbit by a constant gradient or alternating gradient magnetic field.

An electron, striking a nucleus, is scattered through an angle ϑ, and consequently its radial and axial betatron amplitudes are changed. We assume ϑ small, in the sense that changes in amplitude from a single encounter are small compared with the aperture of the vacuum chamber. The changes in amplitude are statistical, so that the root mean square amplitude of both components of betatron oscillations, averaged over all electrons, increase equally.

The axial aperture of the chamber of any storage ring would always be considerably smaller than the radial one, for the reason of injection difficulties and magnet economics. It is therefore sufficient to consider the build-up of the axial betatron oscillations only.

The scattering angle has a component in the axial direction

$$\vartheta_y \simeq \vartheta \sin \varphi \qquad (1)$$

where φ is random. Taking the mean square of eq. (1) we obtain

$$\overline{\vartheta_y^2} = 1/2 \, \overline{\vartheta^2} \qquad (2)$$

An electron on the equilibrium orbit - with no betatron oscillations - has after a collision, neglecting higher harmonics, an axial betatron amplitude a.

It can be seen from Fig. 1, that the amplitude a is given by

$$a = \frac{R \vartheta_y}{q_y} \qquad (3)$$

where R is the orbit radius and q_y the number of axial betatron oscillations per revolution.

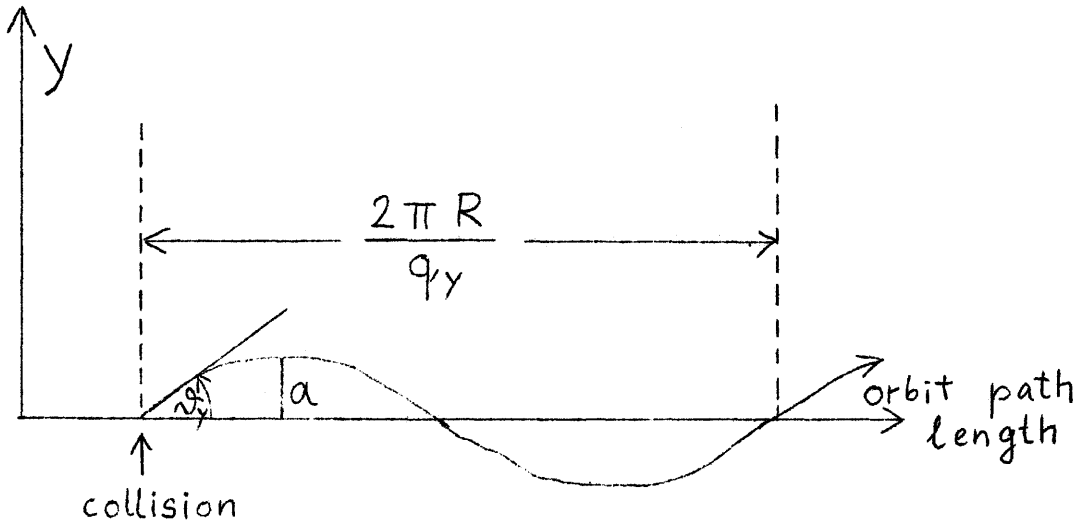

Fig. 1

If an electron had already a betatron amplitude A_o before the encounter, the result is a superposition of the former oscillation and that one induced by the scattering. The two oscillations add up as vectors. See Fig. 2.

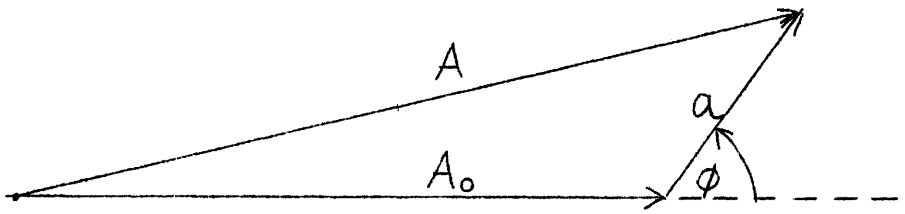

Fig. 2

The phase angle ϕ is random. The square of the new amplitude is

$$A^2 = A_o^2 + a^2 + 2 A_o a \cos \phi \qquad (4)$$

By averaging this equation over ϕ, the term with $\cos \phi$ vanishes, and subsequent averaging over A_o and a yields

$$\overline{A^2} = \overline{A_o^2} + \overline{a^2} \qquad (5)$$

Using eqs. (2), (3) and (5) we obtain

$$\overline{A^2} = \overline{A_o^2} + \frac{R^2}{2 q_y^2} \overline{\vartheta^2} \tag{6}$$

Making the approximation, that all electrons have undergone the same number n of collisions, the mean square amplitude is

$$\overline{A^2} = \overline{A_o^2} + n \frac{R^2}{2 q_y^2} \overline{\vartheta^2} \tag{7}$$

The mean number n of collisions is

$$n = N \sigma c t \tag{8}$$

where N is the number of atoms per volume unit. σ the total cross section, c the velocity of the electrons, which is here taken as the velocity of light and t the time. Hence

$$\overline{A^2}(t) = \overline{A_o^2} + \frac{c N R^2 \sigma \overline{\vartheta^2}}{2 q_y^2} t \tag{9}$$

The r.m.s. amplitude increases with the root only of the time.

Now we can define as lifetime τ_m against multiple scattering the time in which the r.m.s. ampltidue becomes equal to the axial half aperture of the vacuum chamber, and obtain

$$\boxed{\tau_m = \frac{2}{cN} \left(\frac{q_y Y}{R}\right)^2 \left(1 - \frac{\overline{A_o^2}}{Y^2}\right) \frac{1}{\sigma \overline{\vartheta^2}}} \tag{10}$$

The first factor shows the dependence on the gas density, the second one on the machine parameters, the third one on the relative mean square amplitude of the undisturbed beam and the last one on the energy of the electrons.

For the last factor we need the total cross section σ and the mean square scattering angle $\overline{\vartheta^2}$. They are derived from the scattering formula $\frac{d\sigma}{d\omega}(\vartheta)$

$$\sigma = 2\pi \int \frac{d\sigma}{d\omega}(\vartheta) \sin\vartheta \, d\vartheta \tag{11}$$

$$\overline{\vartheta^2} = \frac{2\pi}{\sigma} \int \frac{d\sigma}{d\omega}(\vartheta)\, \vartheta^2 \sin\vartheta \, d\vartheta \tag{12}$$

Hence

$$\sigma\overline{\vartheta^2} = 2\pi \int_{\vartheta_1}^{\vartheta_2} \frac{d\sigma}{d\omega}(\vartheta)\, \vartheta^2 \sin\vartheta \, d\vartheta \tag{13}$$

The upper integration limit is defined by the equation

$$Y^2 = \overline{A_o^2} + \frac{R^2 \vartheta_2^2}{2 q_y^2} \tag{14}$$

which is obtained from eq. (6) by putting $\vartheta = \vartheta_2$ and $A = Y$. It means that for $\vartheta > \vartheta_2$, the electron is lost by single scattering.

$$\vartheta_2 = \frac{q_y}{R} \sqrt{2 (Y^2 - \overline{A_o^2})} \tag{15}$$

The lower integration limit ϑ_1 will be discussed together with the scattering formula.

Single Scattering on Nuclei.

The lifetime due to single scattering is

$$\tau_1 = \frac{1}{N c \sigma_1} \tag{16}$$

where

$$\sigma_1 = 2\pi \int_{\vartheta_2}^{\pi} \frac{d\sigma}{d\omega}(\vartheta) \sin\vartheta \, d\vartheta \tag{17}$$

The lower integration limit ϑ_2 is the upper integration limit from eq. (13) for the multiple scattering.

The Scattering Formula for Nuclear Scattering.

For the calculation of $\sigma \vartheta^2$ we use the Mott formula in a form presented by Feshbach and McKinley [3], which is valid for atomic numbers $Z \lesssim 27$.

$$\frac{d\sigma}{d\omega} = \frac{Z^2 r_o^2}{4} \frac{1-\beta^2}{\beta^4} \frac{1}{\sin^4 \vartheta/2} \left\{ 1 + \pi\alpha Z\beta \sin \vartheta/2 - (\beta^2 + \pi\alpha Z\beta) \sin^2 \vartheta/2 \right\} \tag{18}$$

where r_o is the "electron radius", $r_o = e^2/mc^2 = 2.8 \times 10^{-13}$ cm, $\beta = v/c$, the velocity of the electrons and $\alpha = 1/137$ the fine structure constant.

Equation (18) can be written in the form

$$\frac{d\sigma}{d\omega} = \frac{\alpha_o}{\sin^4 \vartheta/2} (1 + \alpha_1 \sin \vartheta/2 - \alpha_2 \sin^2 \vartheta/2) \tag{19}$$

where the α's are defined by comparison of eqs. (18) and (19). For $\vartheta/2 \ll 1$

$$\frac{d\sigma}{d\omega} = \frac{16\,\alpha_o}{\vartheta^4} \left(1 + \frac{\alpha_1}{2}\vartheta - \frac{\alpha_2}{4}\vartheta^2\right) \tag{20}$$

For the integral

$$\sigma \overline{\vartheta^2} = 2\pi \int_{\vartheta_1}^{\vartheta_2} \frac{d\sigma}{d\omega}(\vartheta)\, d\vartheta \tag{14}$$

we obtain, with $\vartheta_1 \ll \vartheta_2$,

$$\sigma \overline{\vartheta^2} = 32\pi\chi_o \left(\log \vartheta_2/\vartheta_1 + \frac{\alpha_1}{2}\vartheta_2 - \frac{\alpha_2}{2}\vartheta_2^2\right) \tag{21}$$

where ϑ_2 is defined by eq. (15).

It is impossible to choose zero for the lower integration limit ϑ_1. In fact the scattering is limited by the angle for which the impact parameter (the distance of the undisturbed path from the nucleus) about equals the "radius" of the atom

$$r_a = 4.7 \times 10^{-9} Z^{-1/3} \text{ cm}, \tag{22}$$

because for longer distances the nuclear Coulomb field is screened by the electron shells.

The impact parameter is $\lambda\!\!\!^{-}/\vartheta$, where $\lambda\!\!\!^{-}$ is the electron wave length. Hence

$$\vartheta_1 = \frac{\lambda\!\!\!^{-}}{r_a} = \frac{\lambda\!\!\!^{-}_o}{r_a \sqrt{\gamma^2 - 1}} \tag{23}$$

where $\lambda\!\!\!^{-}_o = 3.82 \times 10^{-11}$ cm, the Compton wave length of the electron and $\gamma = \frac{E}{mc^2}$, with E_o the energy.

For the single scattering lifetime we need the integral

$$\sigma_1 = 2\pi \int_{\vartheta_2}^{\pi} \frac{d\sigma}{d\omega}(\vartheta) \sin\vartheta \, d\vartheta \tag{17}$$

for which we obtain

$$\sigma_1 = 4\pi\alpha_o \left\{ \cot^2 \vartheta_2/2 + 2\alpha_1 \left(\frac{1}{\sin \vartheta_2/2} - 1 \right) + \right.$$
$$\left. + 2\alpha_2 \log \sin \vartheta_2/2 \right\} \tag{24}$$

Now we have all formulas we need for a numerical estimate.

Numerical Results for Multiple and Single Scattering on Nuclei.

We will assume a residual gas density of 1×10^{-9} Torr nitrogen, so that

$$N = 7.2 \times 10^{7} \text{ cm}^{-3} \tag{25}$$

As reasonable machine parameters we will take

$$\left.\begin{aligned} R &= 3 \times 10^{2} \text{ cm} \\ Y &= 5 \text{ cm} \\ q_y &= 6 \end{aligned}\right\} \tag{26}$$

Hence

$$\frac{q_y Y}{R} = 10^{-1} \tag{27}$$

and

$$\frac{2}{cN}\left(\frac{q_y Y}{R}\right) = 0.93 \times 10^{-20} \text{ cm}^2\text{sec} \tag{28}$$

For the critical angle ϑ_2 we obtain

$$\vartheta_2 = \sqrt{2}\,\frac{q_y Y}{R} = 0.14 \simeq 8^\circ \tag{29}$$

by neglecting $\overline{A_o^2}$ against Y^2.

The Thomas-Fermi radius of the nitrogen atom, as defined in eq. (22) is

$$r_a = 2.46 \times 10^{-9} \text{ cm} \tag{30}$$

From r_a we obtain the minimum angles ϑ_1 as a function of energy, as given in Table I.

Table I.

γ	$\lambda = \frac{\lambda_o}{\sqrt{\gamma^2 - 1}}$ (cm)	$\vartheta_1 = \lambda / r_a$
4	9.88×10^{-12}	4.00×10^{-3}
10	3.84×10^{-12}	1.56×10^{-3}
20	1.92×10^{-12}	7.78×10^{-4}
40	9.55×10^{-13}	3.87×10^{-4}
100	3.82×10^{-13}	1.55×10^{-4}
200	1.92×10^{-13}	7.78×10^{-4}

The parameters α_o, α_1 and α_2 are listed in Table II.

Table II.

γ	α_o (cm^2)	α_1	α_2
4	6.85×10^{-26}	1.56	2.50
10	1.00×10^{-26}	1.60	2.59
20	2.40×10^{-27}	1.61	2.61
40	6.00×10^{-28}	1.61	2.61
100	9.6×10^{-29}	1.61	2.61
200	2.40×10^{-29}	1.61	2.61

For the lifetime against multiple scattering we obtain finally the values given in Table III.

Table III

γ	σ_{θ^2} (cm^2)	τ_m
4	2.51×10^{-23}	3.7×10^2 sec = 6 min
10	4.61×10^{-24}	2.0×10^3 sec = 33 min
20	1.28×10^{-24}	7.3×10^3 sec = 2 h.
40	3.60×10^{-25}	2.6×10^4 sec = 7 h.
100	6.65×10^{-26}	1.4×10^5 sec = 39 h.
200	1.83×10^{-26}	5.1×10^5 sec = 147 h.

The lifetimes given in Table III must be multiplied with a factor

$$F = \left(1 - \frac{\overline{A_o^2}}{Y^2}\right) \qquad (31)$$

in the case where the mean square amplitude of the undisturbed beam cannot be neglected against the square of the half aperture Y^2.

For the pessimistic assumption of an initial r.m.s. amplitude of 80 o/o of the half-aperture, we obtain a factor $F = 0.36$.

The mean square amplitude $\overline{A_o^2}$ has, beside the factor F, an influence on the critical angle ϑ_2 (eq. (15)) and hence on the integral $\sigma\overline{\vartheta^2}$. But this influence is less serious than that described in eq. (31)

The lifetime against single scattering is given in Table IV.

Table IV

γ	σ_1 (cm^2)	τ_1
4	2.40×10^{-22}	1.93×10^3 sec = 32 min.
10	3.54×10^{-23}	1.31×10^4 sec = 218 min.
20	8.50×10^{-24}	5.4×10^4 sec = 15 h.
40	2.12×10^{-24}	2.2×10^5 sec = 69 h.
100	3.40×10^{-25}	1.36×10^6 sec = 378 h.
200	8.50×10^{-26}	5.4×10^6 sec = 1500 h.

In practice single scattering can be neglected compared to multiple scattering.

Comparison with the MURA Minutes No. 16 [4].

The MURA Minutes No. 16 present estimates of electron beam lifetime assuming that loss of beam is due to multiple scattering at a pressure of 1.4×10^{-6} Torr.

Their results are compared with those from our formula eq. (11), for the machine parameter $q_y Y/R = 0.1$ in Table V.

Table V.

γ	Our formula	MURA
10	1.45 sec	1.72 sec
100	100 sec	120 sec
200	360 sec	440 sec

The agreement is good enough.

Scattering of Electrons by the Electrons of the Residual Gas.

In the case of scattering of electrons by the electrons of the residual gas, the energy loss is more serious for the lifetime than the change in betatron amplitude.

The lifetime can be calculated by assuming that an electron is lost, if its energy loss exceeds a certain critical value $\Delta\gamma_{crit.}$

The energy loss is directly correlated with the scattering angle. The correlation for the elastic scattering of two identical particles is given by the formula (see F. Sauter [5])

$$\Delta\gamma = (\gamma - 1) \sin^2 \vartheta'/2 \tag{32}$$

where ϑ' is the scattering angle in the c.m.s. and γ the energy in the rest system of one of the particles.

For $\gamma \gg 1$, $\vartheta' \ll 1$ and after transformation of the angle into the laboratory system by

$$\vartheta' \simeq \sqrt{2\gamma}\,\vartheta \tag{33}$$

we obtain

$$\Delta\gamma = \frac{\gamma^2 \vartheta^2}{2} \tag{34}$$

Hence

$$\vartheta^2_{crit} = \frac{2}{\gamma^2}\,\Delta\gamma_{crit.} \tag{35}$$

The lifetime can be calculated with the formula

$$\tau_e = \frac{1}{c\,N_e\,\sigma} \tag{36}$$

where N_e is the number of electrons per unit volume and the integration in the total cross section σ runs from ϑ_{crit} to π.

$$\sigma = 2\pi \int_{\vartheta_{crit}}^{\pi} \frac{d\sigma}{d\omega}(\vartheta) \sin\vartheta \, d\vartheta \tag{37}$$

For the differential cross section we use for simplicity the Rutherford formula

$$\frac{d\sigma}{d\omega'} = \frac{r_o^2}{4\gamma'^2} \sin^{-4} \vartheta'/2 \tag{38}$$

instead of the more correct Møller formula, where the primes denote the c.m.s. Equation (38) can be further simplified, for $\vartheta' \ll 1$, to

$$\frac{d\sigma}{d\omega'} = \frac{4 r_o^2}{\gamma'^2 \vartheta'^4} \tag{39}$$

After transformation to the laboratory system, using eq. (33) and

$$\gamma' \approx \sqrt{\frac{\gamma}{2}} \tag{40}$$

$$\frac{d}{d\omega'} \simeq \frac{1}{2\gamma} \frac{d}{d\omega} \tag{41}$$

we obtain from eq. (39) the scattering formula in the laboratory system

$$\frac{d\sigma}{d\omega} = \frac{4 r_o^2}{\gamma^2 \vartheta^4} \tag{42}$$

which incidentally looks identical to eq. (39).

By using eq. (42) for the integral in eq. (37) we obtain

$$\sigma = \frac{4\pi r_o^2}{\gamma^2 \vartheta_{crit}^2} \tag{43}$$

and combining eqs. (35), (36) and (42)

$$\boxed{\tau_e = \frac{\Delta\gamma_{crit}}{2\pi c N_e r_o^2}} \tag{44}$$

The electron density for nitrogen at a gas density of 10^{-9} Torr is

$$N_e = 5.0 \times 10^8 \text{ cm}^{-3} \tag{45}$$

Hence

$$\tau_e = (1.35 \times 10^5 \text{ sec}) \, \Delta\gamma_{crit} \tag{46}$$

or

$$\tau_e = (0.27 \frac{\text{sec}}{\text{eVolt}}) \quad \Delta E_{crit} \tag{47}$$

Two examples are given in Table VI

Table VI

ΔE_{crit}	τ_e
1 keV	270 sec = 4 1/2 min
10 keV	2700 sec = 45 min

The maximum energy loss which can be tolerated depends, of course, on the design of a special machine. But it seems that in an unfavourable case, the life-time against electron scattering can be the same order of magnitude as that against multiple nuclear scattering.

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References.

1) G.K. O'Neill: International Conference etc., CERN 1959, p. 125: Storage Rings for Electrons and Protons.

2) N.M. Blachman and E.D. Courant: Phys. Rev. 74, 140 (1948), The Scattering of Protons by the Gas in a Synchrotron.

3) H. Feshbach, W.A. McKinley: Phys. Rev. 74, 1759 (1948), The Coulomb Scattering of Relativistic Electrons by Nuclei.

4) Minutes of the MURA Technical Group Staff Meeting, Madison, Wisc., January 28, 1957, No. 16.

5) F. Sauter: Die theoretischen Grundlagen für Streuung und Bremsung geladener Teilchen. In "Vorträge über kosmische Strahlung", edited by W. Heisenberg. Springer Verlag, Berlin-Göttingen-Heidelberg, 1953.

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