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SOME PRELIMINARY RESULTS ON COHERENT LONGITUDINAL INSTABILITIES IN THE CPS

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SUMMARY

At Brookhaven the A.G.S. had recently to modify new RF cavities in order to fight a longitudinal instability of the position of the bunches [1].

In the C.P.S. a similar instability blows up the longitudinal emittance of the beam. Even though this instability is weaker here than at the A.G.S., we have to compensate it because of the high I.S.R. sensitivity to longitudinal emittance.

In this paper we present some preliminary theoretical and experimental results of the work undertaken a few months ago. In the first part the major influence of the parasitic resonance of the CPS cavities is recognized in a straightforward way. Then a more detailed analysis is presented, the results of which are compared with experimental data.

1. A DIRECT APPROACH

1.1 Bunch induced transients in a cavity

These instabilities are created by a coupling force in between bunches. This coupling is introduced through the oscillations induced in a resonator by a bunch going through it; the following bunches will then be influenced by the voltage at this resonator.

One can characterize this resonator by three quantities:

parallel resistance R
radian frequency ω_R
quality factor Q

or parallel resistance R
inductivity L
capacity C

with the wellknown relations

$$Q = \frac{R}{L\omega_R}$$

$$LC\omega_R^2 = 1$$

If one excites this resonator by a bunch which is a δ function containing a charge q , one can calculate (appendix I) that the induced voltage will be

$$v(t) = v(0)(\cos \omega_1 t - \frac{\alpha}{\omega_1} \sin \omega_1 t) \qquad (1)$$

here

$$v(0) = \frac{R\omega_R}{Q} q\, e^{-\alpha t}$$

$$\omega_1^2 = \omega_R^2 \left[1-(\frac{1}{2Q})^2\right] \simeq \omega_R^2 \qquad (2)$$

$$\alpha = \frac{\omega_R}{2Q}$$

H.-H. Umstätter [2)] gives a description of one cavity from which we extract the following parameters for the first parasitic resonance in the cavity.

$$f_o = \frac{\omega_R}{2\pi} = 46 \text{ MHz}$$

$$R = 800\ \Omega \qquad (3)$$

$$Q \simeq 20$$

One can see then that

$$\frac{1}{\alpha} = 0.14\ \mu s$$

Since at high energy the bunches are separated by 0.11 μs, the induced voltage amplitude will be attenuated by a factor 2.14 from one bunch to the next. This partly justifies the following approximation: we only take into account the influence on the next bunch, and neglect the effect on the other bunches.

1.2 The mechanism of the instability

Through this induced voltage bunch no. 1 will affect bunch no. 2 and so on up to bunch no. 20 which will affect bunch no. 1, closing the loop.

If we assume that the bunches are rigid, the only thing which can happen to them is that they move in position. Since the focusing forces are far stronger than these perturbations, the motion will be an oscillation in phase and energy with frequency Ω, the synchrotron frequency.

As the perturbation propagates from one bunch to the other, the energy of the oscillation should be the same for all bunches. Therefore only the phase should change and by symmetry of the process the phase shift in between bunches should be constant.

The motion is then of the form

$$\Phi_m = \Phi_o \cos\left[\Omega t + (m-1)\,\Psi\right]$$

where m is the number of the bunch and Φ_m is the phase error of bunch m.

As the phenomenon is closed on h = 20 bunches,

$$\Psi = \frac{2k\pi}{20} = \frac{k\pi}{10}$$

one can easily show that there are only 20 different values for Ψ, the others being reducible to one of the 20.

$$\Psi = \pm \frac{k\pi}{10} \qquad \text{k from 1 to 10}$$

Note: One can justify this "mode" definition in a more general way: Any "closed pattern" can be Fourier analyzed and each of its components treated separately. One finds again the same definition of modes. The mode number is the harmonic number of that Fourier analysis.

One can then treat all these 20 cases. We will only treat two of these :

$$\Psi = \pm \frac{\pi}{2}$$

which turn out to be the most dangerous mode and the most stable one as it will be shown later (chapter 2). Using our previous definition, the two modes have number

$$\pm 20 \frac{\pi}{2} \Big/ 2\pi = \pm 5$$

because there are 20 bunches, mode -5 and mode 15 are the same.

We do not have to treat all 20 equations of motion for the 20 bunches for the following reason:

Suppose we start with 20 bunches oscillating with amplitude Φ : if the amplitude of bunch m goes from Φ to $\Phi + \Delta\Phi$ it will be under the only influence of bunch m-1 since bunch m does not feel the other bunches. The effect of the other bunches will be to ensure that bunch m-1 gets an amplitude increase with the right phase later on.

One can say that the pattern of oscillation grows coherently and, therefore, that if one knows the pattern, it is sufficient to calculate what happens to one bunch.

1.3 The equation of motion

We suppose we are only considering bunch m and m-1. Then we write the unperturbated equations of motion under the usual linearized form :

$$\Delta\dot{p}_m = a\ V \cos\Phi_s\ \Phi_m$$

$$\dot{\Phi}_m = b\ \Delta p_m$$

where $-\ a\ b\ V \cos\Phi_s = \Omega^2$, the synchrotron frequency.

We then develop in Taylor series of the phase difference $\Phi_m - \Phi_{m-1}$ the voltage induced by bunch m-1 on bunch m

$$v_m = v_{mo} + \frac{\partial v_m}{\partial(\Phi_m - \Phi_{m-1})} (\Phi_m - \Phi_{m-1})$$

and add this as a perturbation to our previous equations :

$$\Delta\dot{p}_m = a \left(V \cos \Phi_s \Phi_m + v_{mo} + \frac{\partial v_m}{\partial \Phi} \Phi_m - \frac{\partial v_m}{\partial \Phi} \Phi_{m-1} \right)$$

$$\dot{\Phi}_m = b \, \Delta p_m$$

Then we reduce this to a second order equation of motion

$$\ddot{\Phi}_m - ab \left(V \cos \Phi_s + \frac{\partial v_m}{\partial \Phi} \right) \Phi_m = ab \left(v_{mo} - \frac{\partial v_m}{\partial \Phi} \Phi_{m-1} \right)$$

In a first approximation we can neglect $\frac{\partial v_m}{\partial \Phi}$ in front of the main RF term $V \cos \Phi_s$.

The constant term v_{mo} will only modify the stable phase by an amount which is negligible. This effect will anyway be controlled by the radial control loop.

We are left with

$$\ddot{\Phi}_m + \Omega^2 \Phi_m = \Omega^2 \frac{\frac{\partial v_m}{\partial \Phi}}{V \cos \Phi_s} \Phi_{m-1} \qquad (4)$$

We can use here the technique [3, p.12] of first assuming a pattern. We already selected

$$\Phi_m = \pm \Phi_o \cos \Omega t$$
$$\Phi_{m-1} = \Phi_o \sin \Omega t \qquad (5)$$

and by letting Φ_o vary calculate the quantity

$$\frac{\dot{\Phi}_o}{\Phi_o} = \mp \frac{\Omega}{2} \frac{\partial v_m/\partial \Phi}{V \cos \Phi_s} \qquad (6)$$

obtained by neglecting $\ddot{\Phi}$.

Depending on the sign of $(\partial v_m/\partial \Phi) / V \cos \Phi_s$ one mode or the other will be unstable, the other being damped. It is interesting to note that at transition the sign of $(\partial v_m/\partial \Phi)/V \cos \Phi_s$ changes.

1.4 Orders of magnitude

Using (1) one can estimate the maximum slope available on the fading wave

$$\left(\frac{\partial v(t)}{\partial t}\right)_{max} = -\omega_1 \; v(t)$$

Then one has

$$\frac{\partial v}{\partial \Phi} = \frac{1}{\omega_{RF}} \frac{\partial v}{\partial t}$$

$$\left(\frac{\partial v}{\partial \Phi}\right)_{max} = - \frac{\omega_1}{\omega_{RF}} v(t)_{max}$$

Using (2) with $t = \frac{2\pi}{\omega_{RF}}$ one obtains :

$$\left(\frac{\partial v}{\partial \Phi}\right)_{max} = -q \frac{R\omega_R}{Q} \exp\left(- \frac{\omega_R}{\omega_{RF}} \frac{\pi}{Q}\right) \frac{\omega_1}{\omega_{RF}}$$

One can then put in the numbers of (3) and obtain :

$$\left(\frac{\partial v}{\partial \Phi}\right)_{max} = - 417 \text{ volts per radians}$$

for 2.10^{12} protons.

Since $V \cos \Phi_s = 8.06 \times 10^3$ volts per radians

and $\Omega = 2 \pi \cdot 300 \; s^{-1}$

the maximum e-folding time would be

$$\tau = \frac{\Phi_0}{\dot{\Phi}_0} = 20.5 \text{ ms}$$

In order to find a growth rate we have assumed in the last paragraph :

- the bunch is infinitely short
- the slope at bunch m is the maximum of the slope of the fading wave.

Both these assumptions are not valid in our case and this numerical result gives only the order of magnitude of the effect. It is possible to introduce the influence of these two effects in the previous calculations under the form of a coefficient which will depend on the shape of the bunch and the parameters of the cavity. However, the more powerful method presented in chapter 2 will be preferred.

2. THE GENERAL SOLUTION

2.1 The matrix

We start from the set of equations of the unperturbed synchrotron motion :

$$\Delta\dot{p}_m = a\ V \cos\Phi_s\ \Phi_m$$

$$\dot{\Phi}_m = b\ \Delta p_m \tag{7}$$

The first equation is perturbed by the extra voltage induced in the cavity by the other bunches. This voltage, at a time corresponding to the phase Φ_m can be written as the superposition of the effect of every bunch :

$$v_m = v_{om} + \sum_{n=1}^{h} \frac{\partial v_{mn}}{\partial \Phi} (\Phi_n - \Phi_m) \tag{8}$$

The constant term has no practical importance as it changes simply the energy gain per turn by a very small amount. The synchronous phase, around which we measure the synchrotron oscillations, is then slightly different from the theoretical value.

Obviously, the extra voltage experienced by bunch m which is due to bunch n depends only on the phase difference $\Phi_n - \Phi_m$.

We keep only the second term of (8) and transform the set of equations (7) into a single second order differential equation. We obtain :

$$\ddot{\Phi}_m + \Omega^2 \Phi_m + \sum_{n=1}^{h} \beta_{mn} (\Phi_n - \Phi_m) = 0 \tag{9}$$

Ω is the synchrotron frequency and :

$$\beta_{mn} = \frac{h\ \eta\ \omega_o}{p_o} \frac{e}{2\pi R} \frac{\partial v_{mn}}{\partial \Phi} = \frac{\Omega^2}{V \cos \Phi_s} \frac{\partial v_{mn}}{\partial \Phi}$$

For the sake of simplicity and usefulness, let us assume for the moment h equal bunches, so that the β_{mn} depend only on the distance between bunch number m and bunch number n. One can then write:

$$\beta_{mn} = \beta_\ell \quad \text{with} \quad \ell = m-n \quad \text{for } m > n$$
$$\text{and} \quad \ell = h+m-n \quad \text{for } m < n$$

For m=n the coupling term in equation (9) vanishes and the corresponding β does not appear. With this notation β_1 is the coupling coefficient between one bunch and the next. Then we have only h-1 coefficients β_ℓ and equations (9) take the form :

$$\ddot{\Phi}_m + \left(\Omega^2 - \sum_1^{h-1} \beta_\ell\right) \Phi_m + \sum_1^{h-1} \beta_\ell \Phi_n = 0 \qquad (10)$$

ℓ = m-n or h+m-n

Looking at harmonic solutions of the form exp $\pm$ jωt, equations (10) become algebraic :

$$(-\omega^2 + \Omega^2 - \Sigma \beta_\ell) \Phi_m + \Sigma \beta_\ell \Phi_n = 0 \qquad (11)$$

This set of h such homongeneous equations may be solved by a matrix method. Rejecting the trivial solutions $\Phi = 0$, the solutions in ω^2 must satisfy equations (12), written for h = 20 :

$$\begin{pmatrix} \Omega^2-B-\omega^2 & \beta_{19} & \beta_{18} & \cdots & \beta_1 \\ \beta_1 & \Omega^2-B-\omega^2 & \beta_{19} & \cdots & \beta_2 \\ \vdots & \vdots & \vdots & & \vdots \\ \beta_{19} & \beta_{18} & & & \Omega^2-B-\omega^2 \end{pmatrix} = 0 \qquad (12)$$

with $B = \sum_1^{h-1} \beta_\ell$.

It is equivalent to say that the ω^2 solutions are the eigenvalues of the following matrix :

$$\begin{pmatrix} \Omega^2-B & \beta_{19} & \beta_{18} & \cdots & \beta_1 \\ \beta_1 & \Omega^2-B & \cdots & \cdots & \cdots \\ \vdots & \vdots & & & \\ \beta_{19} & \beta_{18} & \cdots & \cdots & \Omega^2-B \end{pmatrix}$$

This matrix being cyclic, the eigenvalues can be explicitely calculated [4]. Let $\xi_k = \exp k \frac{2\pi j}{h}$ be a h^{th} root of unity.

One can verify that the set of values :

$$\Phi_1 = 1 \qquad \Phi_2 = \xi_k \qquad \Phi_3 = \xi_k^2 \ldots \quad \Phi_h = \xi_k^{h-1}$$

and
$$\omega^2 = \Omega^2 - B + \sum_{\ell=1}^{h-1} \beta_\ell \, \xi_k^{h-\ell} \tag{13}$$

is a solution of the system for every k. Then the h solutions in ω^2 are found by letting k to vary from 1 to h giving the h modes of oscillation.

The interesting feature of this result is that ω may have an imaginary part as ξ_k is complex, leading to damping or anti-damping of the modes.

A simple case:

If the coupling element is the 46 MHz parasitic resonance of the cavity, one can neglect as in chapter 1., in a first approximation, all the coupling terms except that one which couples one bunch to the next. Only β_1 remains in the matrix, and then :

$$\omega^2 = \Omega^2 - \beta_1 + \beta_1 \xi_k^{h-1}$$

The coupling terms are small compared to Ω^2 so that one can write :

$$\omega \simeq (\Omega^2 - \beta_1)^{1/2} \left(1 + \frac{\beta_1 \xi_k^{h-1}}{2(\Omega^2 - \beta_1)}\right)$$

The imaginary part of ω is maximum if $\xi_k^{h-1} = \pm j$. This solution will give the largest growth rate, which is given by :

$$\frac{1}{\tau} = \operatorname{Im} \omega = \frac{\beta_1}{2(\Omega^2 - \beta_1)^{1/2}} \simeq \frac{\beta_1}{2\Omega} \qquad (14)$$

This result agrees well with equation (6) of chapter 1. For this mode of oscillation the system (12) shows that two successive bunches oscillate in quadrature.

<u>A remark</u> : If we suppose that some bunches are missing in the machine and only short range coupling terms, the first bunch is not coupled to the others and the matrix can be written :

$$\begin{pmatrix} \Omega^2-\omega^2 & 0 & 0 & 0 & 0 \\ \beta & \Omega^2-\beta-\omega^2 & 0 & 0 & 0 \\ 0 & \beta & \Omega^2-\beta-\omega^2 & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \beta & \Omega^2-\beta-\omega^2 \end{pmatrix} = 0 \qquad (15)$$

The determinant is then simply :

$$(\Omega^2-\omega^2)\ (\Omega^2-\beta-\omega^2)^{N-1} = 0$$

N is the number of remaining bunches

and the solutions are obviously real. No instability can occur in this case.

2.2 The dispersion relation

Until now, we have considered the presence of one frequency of oscillation for the particles. In fact, in each bunch there exists a spread in the synchrotron frequencies of the individual particles. This spread can arise from the R.F. non linearities or from non linearities in the transverse focalisation ($\gamma_{transition}$ spread).

We have to take this into account if we want to calculate the Landau damping of the instability.

An intuitive and simple way to do that is used in [5] and [6]. We can follow it in our case.

Let $\varphi_{\ell m}$ be the phase of particle ℓ of bunch m, and

$$\Phi_m = \frac{1}{N}\sum_{\ell=1}^{N} \varphi_{\ell m}$$

the phase of the center of mass of bunch m, containing N particles. If Ω_ℓ is the frequency of the free oscillation of particle ℓ, we have in the presence of the coupling :

$$\ddot{\varphi}_{\ell m} + \Omega^2_{\ell m}\,\varphi_{\ell m} + \sum_{n=1}^{h} \beta_{\ell mn}(\Phi_n - \varphi_{\ell m}) = 0 \qquad (16)$$

(see equ. (9))

We can write $\varphi_{\ell m} = \zeta_{\ell m}(t)\, e^{-j\omega t}$, ω being the collective frequency.

The coupling terms are small, so that we can suppose $\omega - \Omega_{\ell m} << \omega$: then $\zeta_{\ell m}(t)$ oscillates at very low frequency, and we can neglect its time derivatives. Doing this we get :

$$\ddot{\varphi}_{\ell m} \simeq - \omega^2 \varphi_{\ell m}$$

and (16) becomes :

$$\left(\Omega^2_{\ell m} - \omega^2\right) \varphi_{\ell m} + \sum_{n=1}^{h} \beta_{\ell mn} (\Phi_n - \varphi_{\ell m}) = 0 \qquad (17)$$

dividing by $\Omega^2_{\ell m} - \omega^2$ and summing over the ℓ gives :

$$N\Phi_m + \sum_{\ell} \frac{\sum_{n=1}^{h} \beta_{\ell mn} (\Phi_n - \varphi_{\ell m})}{\Omega^2_{\ell m} - \omega^2} = 0 \qquad (18)$$

Note that the coupling term β, as defined in 2.1, is proportional to the first derivative of the voltage induced by bunch n at the rest position of the center of mass of bunch m. Then, in this approximation, one can write: $\beta_{\ell mn} = \beta_{mn}$ independent of ℓ.

Now we can describe the bunch m by a two dimensional, normalized distribution function $f(\Omega,\varphi)$ (there are $f\, d\Omega\, d\varphi$ particles having their Ω between Ω and $\Omega+d\Omega$, and their φ between φ and $\varphi+d\varphi$). We replace the sum by an integral and get :

$$\Phi_m + \sum_{n=1}^{h} \beta_{mn} \int \frac{f(\Omega,\varphi)\ (\Phi_n - \varphi)\ d\Omega\ d\varphi}{\Omega^2 - \omega^2} = 0$$

If we assume that $f(\Omega,\varphi)$ can be put into the form :

$$f(\Omega,\varphi) = f_1(\Omega).\ f_2(\varphi)$$

the above integral writes :

$$\int \frac{f_1(\Omega)\ d\Omega}{\Omega^2 - \omega^2} \cdot \int f_2(\varphi)\ d\varphi\ (\Phi_n - \varphi)$$

The second integral is just $\Phi_n - \Phi_m$.

Putting

$$\frac{1}{\lambda} = \int \frac{f_1(\Omega)\ d\Omega}{\Omega^2 - \omega^2}$$

(18) can be written, in the case of equal bunches :

$$(-B-\lambda)\ \Phi_m + \sum_n \beta_n\ \Phi_n = 0 \qquad (19)$$

with $B = \sum_1^{h-1} \beta_n$

Then the λ's are the eigenvalues of the matrix written for h = 20 (see (12)) :

$$\begin{pmatrix} -B & \beta_{19} & \beta_{18} \cdots\cdots & \beta_1 \\ \beta_1 & -B & \beta_{19} \cdots\cdots & \beta_2 \\ \vdots & & & \\ \beta_{19} & \beta_{18} & \cdots\cdots\cdots & -B \end{pmatrix}$$

For each mode k, we have a λ_k which can be written in the form :

$$\lambda = U + jV$$

The ω's will be obtained by solving the dispersion relation

$$1 = (U + jV) \int \frac{f_1(\Omega)\, d\Omega}{\Omega^2 - \omega^2} \tag{20}$$

We can put (20) into a form more appropriate to calculations by remarking that $\Omega - \Omega_0 << \Omega$ (Ω_0 = frequency of the synchronous particle). Then $\Omega^2 - \omega^2 \simeq 2\Omega_0(\Omega - \omega)$.
If we put

$$\frac{\Omega_0 - \Omega}{\Omega_0} = X \; ; \qquad \frac{\Omega_0 - \omega}{\Omega_0} = X_1$$

(20) becomes

$$1 = \frac{U + jV}{2\Omega_0} \int \frac{h(X)\, dX}{X - X_1} \tag{21}$$

It can then be solved numerically by a method similar to the one used in [7], These calculations have not yet been done and will be presented in a future paper.

<u>Particular case</u> : if $f(\Omega) = \delta(\Omega_0)$ δ = Dirac function

$\lambda = \Omega_0^2 - \omega^2$ and we find the result of chapter 2.1.

<u>Remark</u>: As pointed out by Hereward [8], (20) might not be the true dispersion relation: in particular, a rectangular distribution function in phase space must not give rise to Landau damping (at least as far as RF non linearities are concerned) as it would do here. In fact, the above assumption that $f(\Omega, \varphi) = f_1(\Omega) \cdot f_2(\varphi)$ is valid only if the spread in Ω arises only from non linearities that have nothing to do with φ (for example: γ_t spread arising from the non zero betatronic amplitudes of the particles). In this case (20) is correct. Otherwise, to solve the problem rigorously, we have to attack it by means of the Vlasov equation. This calculation is under way.

Nevertheless, only the form of the dispersion integral will change. The general form of (20) will still be right.

2.3 Calculation of the coupling terms

In all this theory we assume the synchrotron frequency much smaller than the revolution frequency, so that the current corresponding to one bunch is periodic and can be expressed by a Fourier series :

$$i(t) = I_o \sum_{p=-\infty}^{+\infty} A_p \exp\,(j\ p\ \omega_0\, t) \qquad (22)$$

$$A(-p) = A^*(p) \text{ as } i(t) \text{ is real}$$

I_o is the DC current of one bunch, the A_p's are dimensionless Fourier coefficients, and $\omega_0/2\pi$ is the revolution frequency ($T = 2\pi/\omega_0$ is the period).

If the bunch is shifted in time by some amount t_o, equation (22) becomes :

$$i(t) = I_o \sum_{p=-\infty}^{+\infty} A_p \exp\,(j\ p\ \omega_o\ (t-t_o))$$

The current for h bunches can be written :

$$i(t) = \sum_{m=1}^{h} I_{om} \sum_{p} A_p \exp\,(j\ p\ \omega_o\ (t-t_{om}))$$

t_{om} is related to the RF phase error (Φ_m) of the m^{th} bunch by :

$$t_{om} = \frac{mT}{h} + \frac{\Phi_m}{h\omega_o} = \frac{2m\pi}{h\,\omega_o} + \frac{\Phi_m}{h\,\omega_o} \qquad (23)$$

Now we are able to calculate the total voltage induced in the cavity by 20 bunches having some phase shifts Φ_m.

$$v(t) = \sum_{m=1}^{h} I_{om} \sum_{p=-\infty}^{+\infty} A_p\ Z(p)\ \exp(jp\omega_o t)\ \exp\left[-jp\left(\frac{2m\pi}{h} + \frac{\Phi_m}{h}\right)\right] \qquad (24)$$

$Z(p)$ is the cavity impedance at the frequency $p\,\omega_o/2\pi$. (Note that $Z(-p) = Z^*(p)$).

We now have to calculate the influence of this extra voltage on bunch number n. As we are interested in the coherent motion of that bunch, we assume it rigid and calculate the mean voltage seen by it :

$$\mathbf{v} = \frac{1}{q} \int_0^T v(t)\,\rho\,(t)\ dt \qquad (25)$$

q is the total charge within the bunch, and $\rho(t)$ is the longitudinal density of the bunch.

Obviously $\rho(t)$ has the same shape as i(t) and can be expressed as :

$$\rho(t) = \frac{q}{T} \sum_{r=-\infty}^{+\infty} A_r \exp j\, r\, \omega_0 \,(t - t_{on}) \qquad (26)$$

for bunch number n.

The contribution of the mth bunch on bunch number n is therefore, from (23), (24), (25) and (26) :

$$v_{mn} = \frac{I_{om}}{T} \sum_{r=-\infty}^{+\infty} \sum_{p=-\infty}^{+\infty} A_p\, A_r\, Z(p) \exp\left[-jp\left(\frac{2m\pi}{h} + \frac{\Phi_m}{h}\right)\right] \exp\left[-jr\left(\frac{2n\pi}{h} + \frac{\Phi_n}{h}\right)\right] \times$$

$$\times \int_0^T \exp j\left((p + r)\,\omega_0\, t\right) dt. \qquad (27)$$

The integral vanishes except for r = -p, where it equals T; therefore:

$$v_{mn} = I_{om} \sum_{p=-\infty}^{+\infty} A_p\, A_{-p}\, Z(p) \exp\left[2jp\,(n-m)\,\frac{\pi}{h}\right] \exp\left[j\,\frac{p}{h}\,(\Phi_n - \Phi_m)\right] \qquad (28)$$

The coupling term we are interested in: $\frac{\partial v_{mn}}{\partial \Phi}$ is then :

$$\frac{\partial v_{mn}}{\partial \Phi} = I_{om} \sum_{-\infty}^{+\infty} A_p\, A_{-p}\, Z(p) \exp\left[2j\,p(n-m)\,\frac{\pi}{h}\right] \frac{jp}{h}$$

As $A_{-p} = A^*_p$ and $Z(-p) = Z^*(p)$, one can write:

$$\frac{\partial v_{mn}}{\partial \Phi} = 2 \, I_{om} \, R_e \sum_{o}^{\infty} A_p^{\,2} \, Z(p) \, \exp\left[2jp \, (n-m) \, \frac{\pi}{h} \right] \frac{jp}{h} \tag{29}$$

2.4 Numerical results

The general solution of the problem implies the eigenvalues and the eigenvectors of a 20 x 20 matrix. As pointed out in 2.1., in the case of 20 equal bunches, the matrix is cyclic and the problem simplifies considerably. One has just to calculate the β_m and carry over the summation (13). We made this using a Fortran program kindly supplied by M. Q. Barton (who made similar calculations for the A.G.S.).

As the impedance determining the instability we used either a resonant LCR parallel circuit simulating the 46 MHz parasitic resonance of our accelerating cavities, or the true impedance measured by H.H. Umstätter [2)] on the PS spare cavity (cutting it down under 20 MHz to eliminate the effect of the main resonance). We had to modify the Barton program in this later case.

At present, 5 of our 14 accelerating cavities have been modified, and possess a new coupling "figure-of-eight" loop. A number of cavities (old and new loop) possesses a tuning capacity on the gap, ranging from zero to 15 pF.

We have calculated the growth rate for different cases:

1) 14 cavities, all with old loop and same tuning capacity. We varied the frequency of the parasitic resonance from 49.4 MHz to 43 MHz by letting the tuning capacity vary from

0 to 30 picofarads.

Within this range of frequency, the ratio N = frequency of parasitic resonance /revolution frequency varies from 104 to 94.

The results are displayed in Fig. 1 : we have plotted only the modes with $10 < k < 20$. The others are obtainable by symmetry about the horizontal axis. We see that there is a gap around N = 100 where no modes have an e folding time τ of less than, say, 100 ms, that is five times the minimum τ. But this gap is rather narrow, approximately .35 MHz.

2) 14 cavities, all with the new loop and same tuning capacity. The results are not much different from the previous case, except for a global horizontal shift of the curves corresponding to those of Fig. 1. (The cavities with the new loop have a slightly different parasitic resonance frequency.)

3) 14 cavities simulated by a resonant circuit, the characteristics of which are : $Q = 24.4$
$f_R = 49.4$ MHz with tuning capacity = 0
$R = 830\ \Omega$

The results are not much different from case 1. The dotted line on Fig. 1 indicates the variation of the growth rate for the mode k = 16.

4) 14 cavities with the old loop :
7 tuned to $\omega_R = 104\ \omega_0$ (capacity = 0 pF)
7 tuned to $\omega_R = 94\ \omega_0$ (capacity = 22 pF)
(ω_0 = revolution frequency)

The e-folding times τ corresponding to the most dangerous mode numbers k are :

k	1	2	3
τ_{ms}	110	104	193

So we can see that this kind of "stagger tuning" increases the e-folding time by a factor ~ 6 (the minimum τ for case 1) was for mode 16 with capacity = 0, and reached 18 ms).

5) 14 cavities with the old loop, each tuned to different frequencies equally spread from 94 ω_0 to 104 ω_0. The more dangerous mode numbers are :

k	4	5	6	7	8
τ_{ms}	89	86	92	92	98

So this scheme seems less efficient than the previous one.

6) The present situation in the PS :

	old loop			new loop	
Capacity (pF)	5	10	15	0	5
Number of cavities	5	3	1	4	1

The more dangerous modes are :

k	4	5	6
τ_{ms}	216	178	203

We have also varied the width of the bunches from 0 (δ-function) to 20 ns, in the cases 1) and 3). The results are displayed in Fig. 2. We assumed for the calculations a bunch of sinusoidal form.

The program giving also the real part of the complex frequency shift $\Delta\omega$, we could make an estimate of the minimum bunch width for which Landau damping might become efficient. For that, we calculated for each bunch width the frequency shift $\Delta\Omega$ arising from RF non linearities at the edges of the bunch. We stated that there is Landau damping if $\Delta\Omega \geq |\Delta\omega|$. A vertical line in Fig. 2 indicates the limit.

2.5 Comparison with experiments

In the CPS, oscillations of the first moment (position) of the bunches had been observed from time to time at medium intensities ($.8 \times 10^{12}$ protons/cycle). They disappeared when the intensity reached 1.5×10^{12} and above. The explanation is that we had then a large blow-up of the longitudinal emittance at transition. When we suppressed this blow-up by the Q-jump method [9,10] the oscillations reappeared with stronger amplitudes.

We used two methods of observation of this instability:

- Firstly, we took photos of one or several bunches, by displaying the signal from the wide band electrostatic puck-up station on a scope with the "Cappi type" trigger*). This was suitable for observing the mode number, the maximum amplitude, the appearance of higher order oscillations.

- Secondly, we gated the signal of one bunch from a radial pick-up station (radial position of the bunch), amplified and detected it. This gave us the growth rate of the instability.

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*) The scope is triggered in synchronism with the revolution frequency. A linear ramp of tension is added on the signal so as to displace the traces vertically at each revolution.

The observed characteristics of the instability are the following:

- It starts as an oscillation of the first moment, without distortions of the form of the bunches. Distortions appear later on, when amplitudes are such that non linearities become important.

- The maximum peak to peak amplitudes are of the order of 7 ns (24 RF degrees, which is equivalent to 2 mm radial displacement).

- The mode numbers currently observed are k = 3, 4, 5. From time to time we saw evidence of mode numbers 1 and 2 (remember that mode k looks like mode 20 -k).

- The amplitudes of oscillation vary from bunch to bunch, some bunches appearing at rest. This can be explained by a mixing of modes (say mode 4+5, or 3+5). The observation time being not longer than a few synchrotron periods, the phase shift between modes is negligible.

- The measured e-folding time is 40 - 50 ms for accelerated intensities $I_p = 1.5 \times 10^{12}$ protons and bunch width $\Delta t = 13$ ns.

- We did not measure systematically the growth rate as a function of bunch width. Nevertheless, we noticed that the instability disappears quite rapidly when Δt reaches 16 - 17 ns, confirming the curve of Fig. 2. (Without Q-jump at transition, the Δt for $I_p = 1.5 \times 10^{12}$ was larger than 22 ns, so that we could no longer observe the instability. At $I_p = .8 \times 10^{12}$, Δt falls in the range 14 - 15 ns even without Q-jump, so that we could observe the instability.)

All this compares very well with the case 1) of our theoretical calculations, i.e. all cavities with the same coupling loop, and tuned to the same frequency. The curves of Figs. 1 and 2 are drawn for $I_p = 1 \times 10^{12}$, the curves of Fig. 1 for $\Delta t = 0$. So, for the mode 4, $I_p = 1.5 \times 10^{12}$ and $\Delta t = 13$ ns, we find from Figs. 1 and 2, $\underline{\tau_{min} = 45 \text{ ms}}$.

On the other hand, the case 6) of 2.4, which pretends to represent the actual situation for the PS cavities, gives in the same conditions $\tau_{min} = 250$ ms. This would give rise to no observable effects.

But let us point out that impedance measurements were made on the spare cavity, not on the ones in the ring. So we cannot be sure that the effective tuning frequencies of the parasitic resonances of the accelerating cavities are those we have assumed.

Another fact does not agree with our previous calculations and asks for further explanations: The Q of our parasitic resonance being around 20, the wake field in the cavities fades rapidly (see 1.1). So, if we take three or four bunches out of the machine, we would expect the remaining bunches to be entirely decoupled, and the instability suppressed. We did an experiment which contradicts all this: we accelerated only eight adjacent bunches, and saw oscillations still present, with comparable amplitudes. The outphasing between two adjacent bunches was around $\frac{2\pi}{20}$.

We cannot apply our calculations to the case of missing bunches (as we assumed exactly equal bunches). But we tried to find an explanation under the form of an effect which might be efficient in coupling bunches separated by a distance of the order of the machine circumference. We consider in the next chapters the effect of the beam control system, and the case of a high Q-resonator (main resonance of the cavities).

3. THE EFFECT OF THE MAIN RESONANCE OF ACCELERATING CAVITIES

Around the main resonance, the cavity impedance can be represented by an equivalent parallel RLC circuit, whose impedance equals :

$$Z(\omega) = \frac{R}{1+2jQ\,\dfrac{\Delta\omega}{\omega_R}} \qquad (30)$$

We have to calculate the coupling coefficients with equation (29). Let us assume the cavity tuned near $h\omega_0$, and having a quality factor Q high enough, so that we can in a first approximation consider only the three terms : $p = h-1$, $p = h$ and $p = h+1$.

For $p = h$ $\qquad 2jQ\,\dfrac{\Delta\omega}{\omega_R} = jX$

X represents the mistuning of the cavity (due for instance to some errors in the tuning loop, or to the beam loading effect).

For $p = h-1$ $\qquad 2jQ\,\dfrac{\Delta\omega}{\omega_R} = jX - 2j\,\dfrac{Q}{h}$

$p = h+1$ $\qquad 2jQ\,\dfrac{\Delta\omega}{\omega_R} = jX + 2j\,\dfrac{Q}{h}$

Assuming h equal bunches, the p=h term in equation (29) will give a contribution which is independent of n-m :

$$2\, I_o\, A_h^{\,2}\, \frac{RX}{1+X^2} \tag{31}$$

whereas the p = h+1 and h-1 terms can be combined :

$$I_m 2 I_o \Big[\left(1-\frac{1}{h}\right) A_{h-1}^{\,2} \frac{R}{1+jX-\frac{2jQ}{h}} \left(\cos\frac{2\pi}{h}(n-m) - j\sin\frac{2\pi}{h}(n-m)\right)$$

$$+ \left(1+\frac{1}{h}\right) A_{h+1}^{\,2} \frac{R}{1+jX+\frac{2jQ}{h}} \left(\cos\frac{2\pi}{h}(n-m) + j\sin\frac{2\pi}{h}(n-m)\right) \Big] \tag{32}$$

Let us assume a linear variation of A_p^2 around p = h, so that one can write :

$$A_{h+1}^{\,2} - A_h^{\,2} = A_h^{\,2} - A_{h-1}^{\,2}$$

and define :

$$u = \frac{1}{h} + \frac{A_{h+1}^{\,2} - A_h^{\,2}}{A_h^{\,2}} = \frac{1}{h} + \frac{A_h^{\,2} - A_{h-1}^{\,2}}{A_h^{\,2}} \tag{33}$$

After some algebraic manipulations, expression (32) takes the form :

$$\frac{2\,X\,R}{\left(1+X^2+\left(\frac{2Q}{h}\right)^2\right)^2-4X^2\left(\frac{2Q}{h}\right)^2}\left[\left(\left(\frac{2Q}{h}\right)^2-1-X^2\right)\cos\frac{2\pi}{h}(n-m)-\frac{4Q}{h}\sin\frac{2\pi}{h}(n-m)\right]$$

$$+\frac{2\,u\,R}{\left(1+X^2+\left(\frac{2Q}{h}\right)^2\right)^2-4X^2\left(\frac{2Q}{h}\right)^2}\left[\left(X^2-1-\left(\frac{2Q}{h}\right)^2\right)\frac{2Q}{h}\cos\frac{2\pi}{h}(n-m)+\left(1+X^2+\left(\frac{2Q}{h}\right)^2\right)\sin\frac{2\pi}{20}(n-m)\right] \quad (34)$$

In the cases we are interested in, X is of the order of unity, or less (phase shift of 45^o) and $\left(\frac{2Q}{h}\right)^2 \gg 1$. Then, the following approximation holds :

$$\frac{2XR}{\left(\frac{2Q}{h}\right)^3}\left(\frac{2Q}{h}\cos\frac{2\pi}{h}(n-m)-2\sin\frac{2\pi}{h}(n-m)\right)$$

$$+\frac{2uR}{\left(\frac{2Q}{h}\right)^2}\left(-\frac{2Q}{h}\cos\frac{2\pi}{h}(n-m)+\sin\frac{2\pi}{h}(n-m)\right) \quad (35)$$

$$= A\cos\frac{2\pi}{h}(n-m)+B\sin\frac{2\pi}{h}(n-m)$$

Let us now look at possible instabilities by putting expressions (31) and (35) into equation (13).

Obviously, the p=h contribution, which is independent of ℓ (ℓ = n - m or h + m - n) has no imaginary part because

$$\sum_{\ell=1}^{h-1} \xi_k^{h-\ell} \quad \text{is purely real.}$$

Then we have to find the imaginary part of expressions of the form:

$$\sum_{\ell=1}^{h-1} \left(A \cos 2\pi \frac{\ell}{h} + B \sin 2\pi \frac{\ell}{h}\right)\left(\cos k\, 2\pi \frac{\ell}{h} - j \sin k \frac{2\pi\ell}{h}\right) \qquad (36)$$

We shall continue the calculations for h = 20 (PS case). Then, if k is even, the imaginary part :

$$- \sum_{1}^{19} \sin k \frac{2\pi\ell}{h} \left(A \cos 2\pi \frac{\ell}{h} + B \sin 2\pi \frac{\ell}{h}\right) \qquad (37)$$

vanishes because the ℓ and $\ell+10$ components cancel each other. If k is odd, it remains :

$$- 2 \sum_{1}^{9} \sin k \frac{2\pi\ell}{h} \left(A \cos 2\pi \frac{\ell}{h} + B \sin 2\pi \frac{\ell}{h}.\right)$$

By combining ℓ and $10-\ell$ terms we get :

$$- 4\, B \sum_{1}^{4} \sin k \frac{2\pi\ell}{h} \sin 2\pi \frac{\ell}{h} - 2\, B \sin k \frac{\pi}{2} \qquad (38)$$

Now, if k changes into 20-k, only the sign of this expression changes. Therefore, we only have to look at k = 1, 3, 5, 7 and 9 in order to find he growth rates of the instability.

It turns out that $k = 1$ gives the largest value for expression (38) and leads to the most dangerous instability. Its growth rate is proportional to B, which, from (35), has the form :

$$B = \frac{2\ R}{(\frac{2Q}{h})^2} \left(u - X\ \frac{h}{Q}\right) \qquad (39)$$

This shows that there is still an instability for a cavity exactly tuned on $h\ \omega_0 (X = 0)$, but for a detuning :

$$X = \frac{Q\ u}{h}$$

we get the minimum growth rate.

Numerical results for the PS obtained with Barton's program which does not make the simplified assumptions of these analytical calculations, are presented in Fig. 3. They show the pre-eminence of mode $k = 19$ (or $k = 1$), and a gap at $X = \frac{Qu}{h}$ as the analytical calculations did.

During one M.D. we tried to get an experimental confirmation of this theoretical result.

Using the so-called detuning control of the cavities (designed for the debunching before slow extraction) during the accelerating cycle, it was possible to get an average phase shift: Arc tan $X \simeq 20^{\circ}$ for all the cavities.

As the instabilities given by the main resonance of the cavity are believed to be quite independent on bunch length, we tried to observe them with long bunches, therefore avoiding the usual instabilities.

At the far end of the cycle we noticed some instabilities with a mode number $k = 20 \pm 1$, which we thought might be explained by this theory.

4. INFLUENCE OF THE BEAM CONTROL SYSTEM

4.1 Equations of motion

In the case of beam-controlled acceleration, the RF frequency is no longer constant and the time equation in (7) :

$$\dot{\Phi}_m = b \, \Delta p_m$$

does not hold if Φ is the phase error between RF and bunch.

Let us define a fictitious RF having a constant frequency (we neglect the adiabatic variations of parameters), Φ_m and Φ_{RF} measuring the phases of bunch m and of the actual RF voltage referred to this clock. The linearized synchrotron equations are now :

$$\dot{\Phi}_m = b \, \Delta p_m$$
$$\Delta \dot{p}_m = a \, V \cos \Phi_s \, (\Phi_m - \Phi_{RF}) \tag{40}$$

which can be written :

$$\ddot{\Phi}_m + \Omega^2 \, \Phi_m - \Omega^2 \, \Phi_{RF} = 0 \tag{41}$$

Φ_{RF} is determined by the beam control system. The radial loop averages the radial positions of bunches, or the $\dot{\Phi}_m$'s, which is equivalent, so that the error signal of the radial loop is given by :

$$\frac{G}{N} \sum_{1}^{N} \dot{\Phi}_{\ell}$$

for N equal bunches in the machine ($N \leq h$).
G is the gain of the radial loop including some dynamical constants.

For the sake of simplicity, we shall assume the phase loop gain at the synchrotron frequency infinite so that the RF phase is simply the average of bunch phases, corrected by the error signal :

$$\Phi_{RF} = \frac{1}{N} \sum_{1}^{N} \Phi_{\ell} + \frac{G}{N} \sum_{1}^{N} \dot{\Phi}_{\ell} \qquad (42)$$

Now, if we consider coupling terms, equation (9) without beam control must be replaced, according to (41), by :

$$\ddot{\Phi}_m + (\Omega^2 - B) \Phi_m + \Sigma \beta_{\ell} \Phi_n - \Omega^2 \Phi_{RF} = r_m \qquad (43)$$

These N equations, with N+1 unknowns must be completed by equation (42).

4.2 A particular solution

The right hand side of equation (43) may not vanish, because of the v_{om} term in equation (8). But now the influence of the constant term r_m cannot be derived as simply as in chapter 2.1.

In order to get rid of the right hand side of (14) we shall search for a particular solution of the form :

$$\begin{aligned} \ddot{\Phi}_m &= 0 \\ \Phi_m &= \dot{\Phi} t + \alpha_m \end{aligned} \qquad (44)$$

$\dot{\Phi}$ being the same for all the bunches.

Putting equations (44) and (42) into (43) gives :

$$(\Omega^2 - B)\,(\dot{\Phi}t + \alpha_m) + \sum_1^{N-1} \beta_\ell (\dot{\Phi}t + \alpha_n) - \Omega^2 \left(\frac{1}{N} \sum_1^N \dot{\Phi}t + \frac{1}{N} \sum_1^N (\alpha_\ell + G\dot{\Phi}) \right) = r_m$$

The t dependent terms cancel out and it remains only :

$$\left(\Omega^2 - B - \frac{\Omega^2}{N}\right) \alpha_m + \sum \left(\beta_\ell - \frac{\Omega^2}{N}\right) \alpha_n - \Omega^2 G \dot{\Phi} = r_m \qquad (45)$$

This system of N algebraic equations has N+1 unknowns, but one can easily check that shifting all the Φ_m solutions by a constant amount does not change $\dot{\Phi}$. Therefore, one can choose arbitrarily one Φ_m, and derive one unique value of $\dot{\Phi}$, which gives one particular solution *).

*) Note: To demonstrate the existence of $\dot{\Phi}$, we have to calculate the characteristic determinant of the system and to show that it does not vanish. We can do that neglecting the β's which are, in all practical cases, very small compared to Ω^2. It can be shown that this determinant equals $G(\Omega^2)^N$; then the $\dot{\Phi}$ solution exists and is of the form :

$$\dot{\Phi} \simeq \frac{F(r_m)}{G(\Omega^2)^N} \qquad \text{F is a linear function of } r_m$$

This result simply shows that the frequency has been somewhat shifted from its fictitious value : The radial position of the beam is not exactly on the theoretical orbit, and this radial shift is inversely proportional to the gain of the radial loop.

The $\dot{\Phi}$ = const. solution plays the role of the zero solution when the right hand side of equations (43) vanishes, and the instabilities may be found by looking only at the solutions of the homogeneous system.

4.3 Solutions of the homogeneous system

The associated matrix (of the N+1[th] order) is of the form :

$$\begin{pmatrix} \Omega^2-B-\omega^2 & \beta_1 & \cdots\cdots & -\Omega^2 \\ \beta_{N-1} & \Omega^2-B-\omega^2 & \beta_1 \ \cdots\cdots & -\Omega^2 \\ \cdots & \cdots & \cdots & \cdots \\ A & A & \cdots\cdots & -1 \end{pmatrix} \tag{46}$$

with

$$A = \frac{1}{N}\,(1 + j\,G\,\omega)$$

Let us develop the determinant from the last line; we get :

$$\begin{aligned} & A\,P\,(\omega^2, \Omega^2, \beta_\ell) \quad \text{for the N first terms, and} \\ & (-1)^{N+1}\,(-1)\,D \quad \text{for the last term} \end{aligned} \tag{47}$$

P is a polynominal in ω^2, Ω^2 and β_ℓ and D is the determinant associated with the problem, not taking into account the influence of the beam control. Note that D does not contain any G term.

Now, we can easily verify that the determinant of (46) vanishes for :

$$\omega^2 = \Omega^2 \, (1 - N A) \tag{48}$$

For instance, one can add the N first columns and obtain one new column which is proportional to the last one.

Therefore, the determinant of (46) can be written also :

$$(\Omega^2 - NA\Omega^2 - \omega^2) \; Q \; (\omega^2, \Omega^2, \beta_\ell) \tag{49}$$

From (47) and (49) we obtain :

$$(-1)^N \; D + \frac{1}{N} \, (1 + jG\omega) \; P = \left[\Omega^2 - \omega^2 - \Omega^2 (1 + jG\omega) \right] Q \tag{50}$$

As G is an independent parameter, we should have :

$$\begin{aligned} \frac{G \, \omega \, P}{N} &= - G \, \omega \, \Omega^2 \, Q \\ P &= - Q \, N \, \Omega^2 \end{aligned} \tag{51}$$

Equation (50) becomes now :

$$(-1)^N \; D = Q \; (\Omega^2 - \omega^2) \tag{52}$$

From (49) the solutions of the problem with beam control are given by :

$$\omega = 0, \quad \omega = - j\, G\, \Omega^2 \quad \text{and} \quad Q = 0.$$

The first one shows that shifting all the Φ_m and Φ_{RF} do not change the problem, and the second one is associated with the transient decay of the radial loop. Possible instabilities are given by $Q = 0$.

Now, from (52) one can say that the solutions of $Q = 0$ are those of $D = 0$ except the purely real $\omega = \pm \Omega$ solution. ($D = 0$ has always this solution which is not interesting from the instability point of view.)

Therefore, one can conclude that under the assumptions made along these calculations, namely equal bunches having equal synchrotron frequencies, the stability conditions of the problem are not affected by the beam control system.

5. CONCLUSION

Experimental evidences with 20 equal bunches seem to fit roughly the estimation but we do not have yet a satisfactory explanation for the instabilities observed in the missing bunches.

The next steps, as far as theory is concerned, will be the complete (20 x 20) matrix-treatment for unequal bunches with different synchrotron frequencies being worked out now by Mary Bell. Also we would like to understand better the various aspects of Landau damping involved.

We do not have yet enough experimental evidence to confirm a theory and we will continue to improve our instrumentation.

Later we hope to turn to the stabilizing problem either through compensating devices or using Landau damping techniques. This should prove useful especially for the design of the new PS RF cavities.

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APPENDIX I

Voltage induced by a δ function bunch in a cavity

We have to use Laplace transforms. The cavity impedance can be written :

$$Z(p) = \left(\frac{1}{R} + \frac{1}{Lp} + Cp \right)^{-1}$$

$$= \frac{R\omega_R}{Q} \frac{p}{(p-p_1)(p-p_2)}$$

$$p_1, p_2 = - \frac{\omega_R}{2Q} \pm j\,\omega_1$$

$$\omega_1^2 = \omega_R^2 \left[1 - \left(\frac{1}{2Q}\right)^2 \right]$$

The bunch is represented by

$$I(p) = q$$

where q is the charge in the bunch.

Then

$$V(p) = Z(p)\, I(p) = \frac{R\omega_R}{Q}\, q \, \frac{p}{(p-p_1)(p-p_2)}$$

which can be inverted using the residues theorem to :

$$v(t) = \frac{R\,\omega_0}{Q}\, q\, e^{-\alpha t} \left[\cos\, \omega_1 t - \frac{\alpha}{\omega_1} \sin \omega_1\, t \right]$$

with $\alpha = \frac{\omega_0}{2Q}$

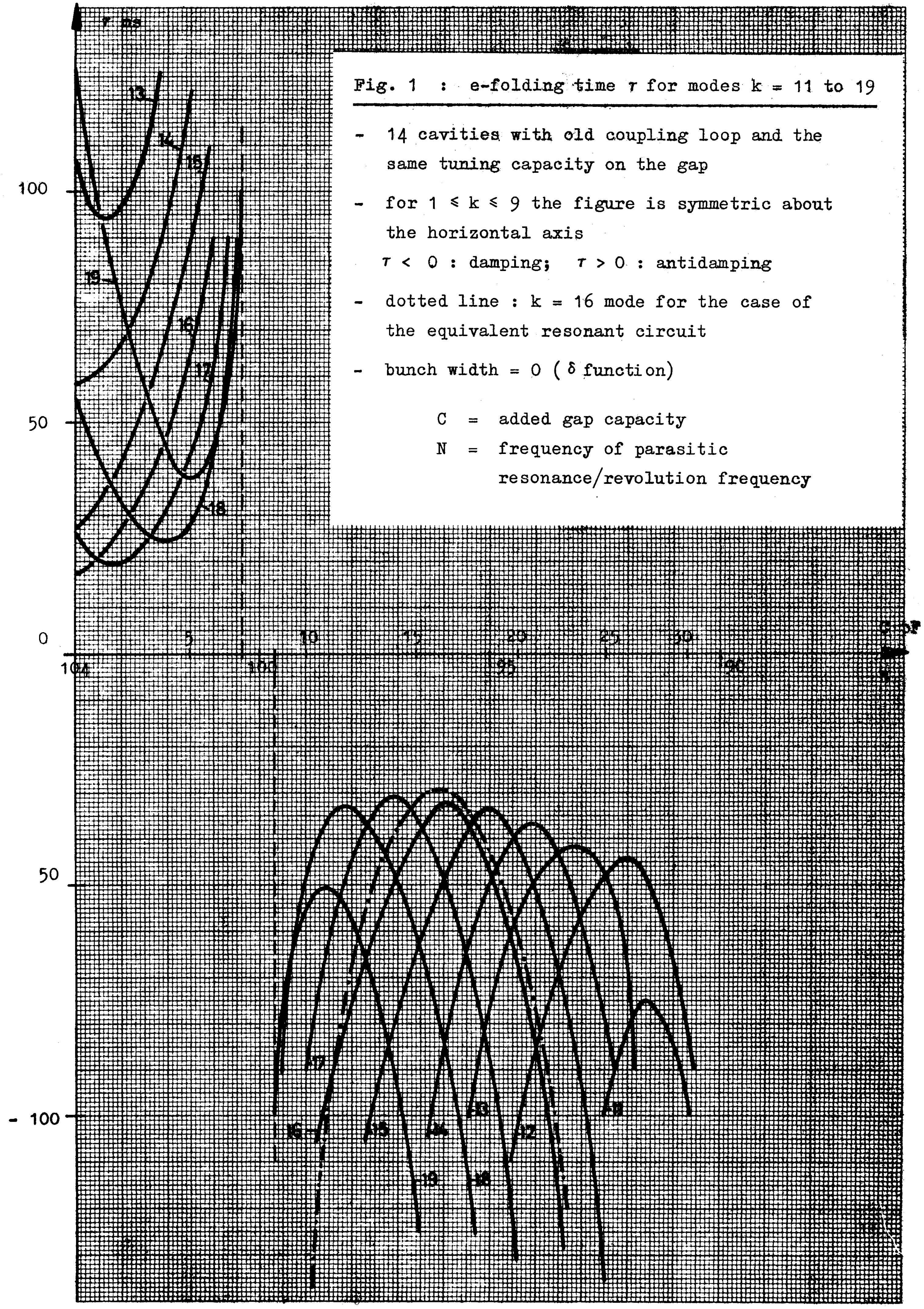

Fig. 1 : e-folding time τ for modes k = 11 to 19

- 14 cavities with old coupling loop and the same tuning capacity on the gap
- for $1 \leq k \leq 9$ the figure is symmetric about the horizontal axis
  $\tau < 0$: damping; $\tau > 0$: antidamping
- dotted line : k = 16 mode for the case of the equivalent resonant circuit
- bunch width = 0 (δ function)

C = added gap capacity
N = frequency of parasitic resonance/revolution frequency

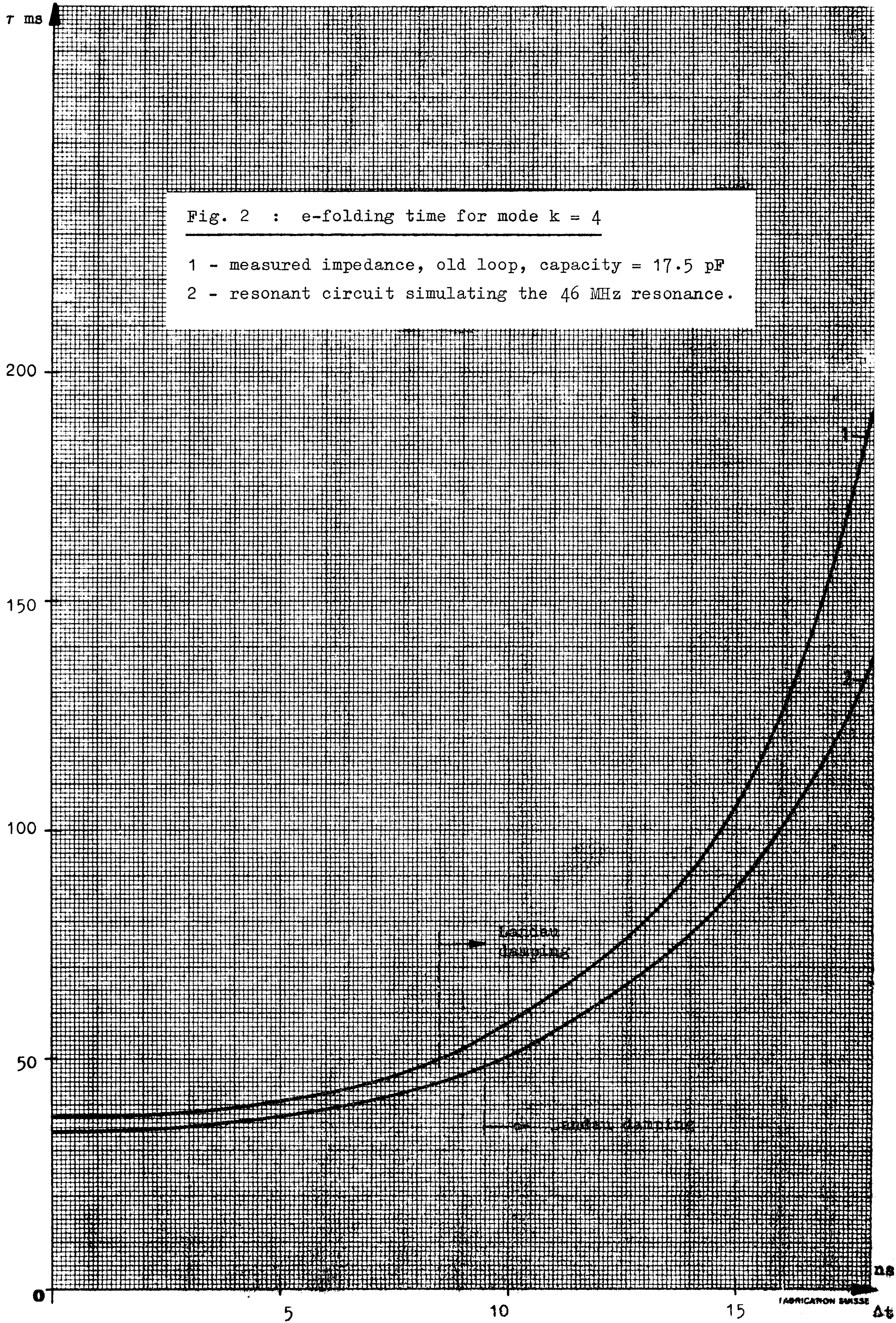

Fig. 2 : e-folding time for mode k = 4

1 - measured impedance, old loop, capacity = 17.5 pF

2 - resonant circuit simulating the 46 MHz resonance.

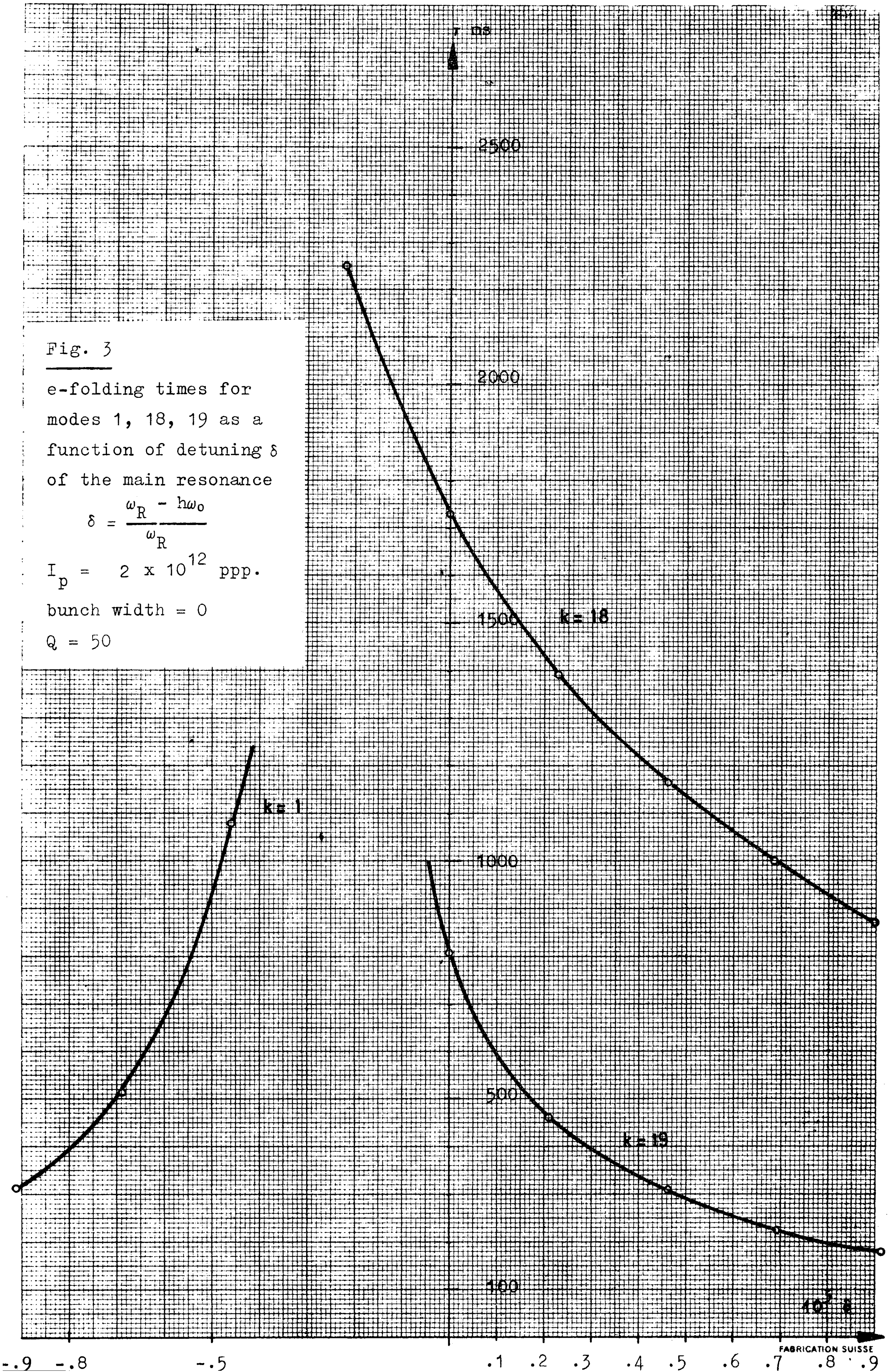

Fig. 3

e-folding times for modes 1, 18, 19 as a function of detuning δ of the main resonance

$$\delta = \frac{\omega_R - h\omega_o}{\omega_R}$$

$I_p = 2 \times 10^{12}$ ppp.

bunch width = 0

Q = 50