Eur. Phys. J. C (2024) 84:751
https://doi.org/10.1140/epjc/s10052-024-13108-7

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

# Revisiting the averaged annihilation rate of thermal relics at low temperature

**A. Arbey**[1,2,a], **F. Mahmoudi**[1,2,3,b], **M. Palmiotto**[1,c]

[1] Université Claude Bernard Lyon 1, CNRS/IN2P3, Institut de Physique des 2 Infinis de Lyon, UMR 5822, 69622 Villeurbanne, France
[2] Theoretical Physics Department, CERN, 1211 Geneva 23, Switzerland
[3] Institut Universitaire de France (IUF), 75005 Paris, France

Received: 24 April 2024 / Accepted: 7 July 2024


**Abstract** We derive a low-temperature expansion of the formula to compute the average annihilation rate $\langle \sigma v \rangle$ for dark matter in $\mathbb{Z}_2$-symmetric models, both in the absence and the presence of mass degeneracy in the spectrum near the dark matter candidate. We show that the result obtained in the absence of mass degeneracy is compatible with the analytic formulae in the literature, and that it has a better numerical behaviour for low temperatures. We also provide as ancillary files two `Wolfram Mathematica` notebooks, which perform the two expansions at any order.

## 1 Introduction

One of the main areas of research in astroparticle physics today is the search for dark matter (DM). For decades, a number of astrophysical observations have been impossible to explain in the context of general relativity (GR), if we also assume that the Standard Model (SM) of fundamental interactions describes the entire particle content of the universe. Hence, a hypothesis that can explain most—or in some contexts all—observations is the existence of a kind of stable and non-relativistic matter that couples very weakly with SM fields, which is referred to as dark matter. Aside from the astrophysical observations, the existence of DM is also a necessity in cosmology in order to obtain a coherent description of the growth of perturbations.

Moreover, albeit it describes very well the phenomena observed up to the TeV scale,[1] the SM is expected to fail at an energy scale lower than the Planck energy. Thus, if DM is partly composed of stable particles, it could be explained in some extensions of the SM. Alternatively, it is possible to extend the SM with the aim of having a model that describes the nature of a fraction—or the totality—of the DM abundance, with detection limits compatible with the state of the art of the observations.

The abundance of DM has been measured by the Planck collaboration [1] as the relative energy density:

$$H_0 = 67.66 \pm 0.42\ \text{km/s/Mpc}, \tag{1}$$
$$\Omega_{\rm m} = 0.3111 \pm 0.0056, \tag{2}$$
$$\Omega_{\rm b} h^2 = 0.02237 \pm 0.00015, \tag{3}$$
$$\Omega_{\rm c} h^2 = 0.1200 \pm 0.0012, \tag{4}$$
$$\Omega_\Lambda = 0.6847 \pm 0.0073, \tag{5}$$

where $H_0$ is the Hubble parameter, $h$ is the reduced Hubble parameter, defined as:

$$h = \frac{H_0}{100\ \text{km/s/Mpc}}, \tag{6}$$

and where $\Omega_{\rm b}$ is the relative density of *baryonic* matter, i.e. of the fraction of cold matter visible via electromagnetic signals, and $\Omega_{\rm c}$ is the relative density of *cold dark matter*, i.e. of the fraction of cold matter electromagnetically invisible. The discrepancy between $\Omega_{\rm m}$ and $\Omega_{\rm b}$ is an evidence of the existence of cold dark matter, and that this component dominates the non-relativistic matter content of the Universe.

The contribution to the relative density of a particle species can be computed by solving the Boltzmann equation for its

[a] e-mail: alexandre.arbey@ens-lyon.fr
[b] e-mail: nazila@cern.ch
[c] e-mail: marco.palmiotto@univ-lyon1.fr (corresponding author)

[1] Except for neutrinos' flavour oscillations.

number density, then obtaining from it the energy density, and finally dividing this result by the critical density of the Universe today. The form of the Boltzmann equation and its resolution for the non-relativistic case can be found e.g. in Ref. [2]. In particular, it has the form

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -3Hn + \langle \sigma v \rangle (n - n_{\mathrm{eq}}), \tag{7}$$

where $n$ is the number density of the dark matter candidate, as a function of the time $t$,[2] $n_{\mathrm{eq}}$ is the value of equilibrium for $n$, at the temperature $T$ corresponding to the time $t$, and $\langle \sigma v \rangle$ is the thermal average of the product of the total cross-section of annihilation of dark matter candidates into SM particles with the relative velocity of the particles in the initial state. In order to compute the thermal average, it has been assumed that the particles in the initial state have non-relativistic velocities, allowing the replacement $s \to 4m^2 + m^2 v^2$, where $s$ is the Mandelstam variable, $m$ is the mass of the DM candidate, and $v$ its velocity in the centre-of-mass frame. The expanded $\sigma v$ is then averaged, yielding an expansion in $x = m/T$ for $\langle \sigma v \rangle$. The work by Srednicki et al. [3] aimed then to have a more reliable expansion of $\langle \sigma v \rangle$, by finding the general non-relativistic formula expressed directly in powers of $1/x$, and starting from the squared matrix elements of the annihilation reactions.

In the context of DM produced via annihilation and co-annihilation, the work of Gondolo and Gelmini [4], and of Edsjö and Gondolo [5] made a step forward, in generalising the equation in the relativistic case. Firstly, it is pointed out that $v$ should not be the relative velocity, but the Møller velocity, thus making $\langle \sigma v \rangle$ a scalar, from now on denoted as $\langle \sigma v_{\mathrm{Møl}} \rangle_{\mathrm{eff}}$. Then, the scenario of annihilation to SM particles and of co-annihilation is considered in the models with a $\mathbb{Z}_2$ symmetry that prevents the DM candidate from decaying into SM particles. The result is a Boltzmann equation for the total number density of the species with the same $\mathbb{Z}_2$-parity as the DM candidate, which has the same form as Eq. (7). In this context, the expression of $\langle \sigma v_{\mathrm{Møl}} \rangle_{\mathrm{eff}}$ is derived, yielding to formula (13) in the present article. This formula has been explicitly expanded around $T = 0$ in the work of Cannoni [6], under the assumption of a model with only one DM candidate, and with no particle degenerate (or nearly degenerate) in mass with it. The result is the same as previously obtained, under the same hypotheses, in the aforementioned work [3] by Srednicki et al. .

Thus, in the cited literature, there are two formulae for the calculation of $\langle \sigma v_{\mathrm{Møl}} \rangle_{\mathrm{eff}}$: a general one for any temperature, and an expansion for low temperatures. The former is always valid, but it presents an integral, therefore it can hardly be treated to obtain a symbolic expression. The latter, while valid for low temperatures and with a constraint on the model, can be used to have an analytic expression for $\langle \sigma v_{\mathrm{Møl}} \rangle_{\mathrm{eff}}$. Indeed, many works which present new models—such as [7]—often use the expanded formula, for providing a first order expression of $\langle \sigma v_{\mathrm{Møl}} \rangle_{\mathrm{eff}}$ around $T = 0$. In fact, usually, the accuracy of the first order expansion is good enough to have a reliable estimate of the freeze-out temperature, that may then be used for setting the initial conditions for the Boltzmann equation, and for solving it.

In this work, we present a generalisation of the work of Cannoni in [6]. In particular, we consider models in which there is small mass splitting, or mass degeneracy, in the spectrum near the mass of the DM candidate. Therefore, thanks to the formula we derive here, it is possible to apply the same method to new models with the aforementioned mass splitting or with resonances at low velocity, allowing for a reliable estimate of $\langle \sigma v_{\mathrm{Møl}} \rangle_{\mathrm{eff}}$ without needing numerical tools.

Such models are an interesting case of study, since the small mass splitting with the DM candidate is a simple condition to have in models with one or more long-lived particles that decay to the DM candidate. Other than the interest from the phenomenological perspective (see e.g. [8]), nowadays the detectors' upgrade to be able to detect long-lived particles is a topic of active discussion among many experiments at colliders (see e.g. [9]).

Thus, scenarios in which there are (one or more) particles, especially if charged, whose masses are close to the one of the DM candidate become viable cases of study. In this context, simplified effective models with mass splitting are good examples in which having a symbolic expression for $\langle \sigma v_{\mathrm{Møl}} \rangle_{\mathrm{eff}}$ at low temperature can be useful for phenomenological studies.

The rest of the present article is organised as follows. In Sect. 2 we re-consider the formula derived in [5] for $\langle \sigma v_{\mathrm{Møl}} \rangle_{\mathrm{eff}}$, also showing why the implementation of such a formula can lead to numerically unreliable results at low temperatures. In Sect. 3, we point out that a numerical evaluation of such a formula at small values of $T$ presents some numerical issues, and we derive its expansion in $1/x$, by following the procedure outlined in [6] by Cannoni. Finally, in Sect. 4 we generalise the expansion in the case of small mass splitting in the spectrum near the mass of the DM candidate. We conclude in Sect. 5 by discussing the results and their areas of application. In the appendices, there are some mathematical formulae and an example of application of the formula obtained in Sect. 4.

## 2 Freeze-out scenario for thermal relic density

The standard scenarios for dark matter particles are the so-called *thermal relic scenarios*, in which a single relic particle can explain the nature of dark matter. In the freeze-out scenarios, the new physics particles are considered in thermal

[2] In the Friedmann–Lemaître–Robertson–Walker metric.

equilibrium at a common temperature $T$. The expansion rate $H$ of the Universe is given by the Friedmann equation:

$$H^2 = \frac{8\pi G}{3} g_{\rm eff}(T) \frac{\pi^2}{30} T^4, \tag{8}$$

where $g_{\rm eff}$ is the effective number of degrees of freedom of radiation.

At thermal equilibrium, under the assumption of the Maxwell–Boltzmann statistics, the total number density of new physics particles is given by

$$n_{\rm eq} = \frac{T}{2\pi^2} \sum_i g_i m_i^2 K_2\left(\frac{m_i}{T}\right), \tag{9}$$

where $g_i$ and $m_i$ are the number of degrees of freedom and the mass of the $i$-th new physics particle, respectively, and $K_2$ the modified Bessel function of the second kind of order 2.

To compute the present relic density of dark matter particles, one needs to solve the Boltzmann evolution equation [4,10,11]:

$$\frac{{\rm d}n}{{\rm d}t} = -3Hn - \langle\sigma_{\rm eff} v\rangle (n^2 - n_{\rm eq}^2), \tag{10}$$

where $n$ is the total number density of new physics particles and $\langle\sigma_{\rm eff} v\rangle$ is the thermal average of the annihilation rate of the new physics particles to the Standard Model particles.

The case with non-thermalised species—as non-relativistic ones—can be treated by appropriately modifying the expression of $H$ (8), and the Boltzmann equation (10) as follows (see e. g. [12,13]):

$$H^2 = \frac{8\pi G}{3} \left( g_{\rm eff}(T) \frac{\pi^2}{30} T^4 + \rho_{\rm nt} \right), \tag{11}$$

$$\frac{{\rm d}n}{{\rm d}t} = -3Hn - \langle\sigma_{\rm eff} v\rangle (n^2 - n_{\rm eq}^2) + N_{\rm nt}, \tag{12}$$

where the parameters $\rho_{\rm nt}$ and $N_{\rm nt}$ are respectively the effective energy density, and the variation of the number density due to non-thermalised particles. Therefore, considering such a case in the freeze-out scenario does not affect the expression of $\langle\sigma_{\rm eff} v\rangle$.

The thermal average of the effective cross-section at temperature $T$ is obtained, under the assumptions of kinetic equilibrium and Maxwell–Boltzmann statistics:

$$\langle\sigma_{\rm eff} v\rangle(T) = \frac{\displaystyle\int_0^\infty {\rm d}p_{\rm eff}\, p_{\rm eff}^2 W_{\rm eff}(\sqrt{s}) K_1\left(\frac{\sqrt{s}}{T}\right)}{m_{\rm DM}^4 T \left[\displaystyle\sum_i \frac{g_i}{g_{\rm DM}} \frac{m_i^2}{m_{\rm DM}^2} K_2\left(\frac{m_i}{T}\right)\right]^2}, \tag{13}$$

where $K_1$ is the modified Bessel function of the second kind of order 1, $g_{\rm DM}$ and $m_{\rm DM}$ are the number of degrees of freedom and the mass of the dark matter particle, and

$$p_{\rm eff}(\sqrt{s}) = \frac{1}{2}\sqrt{s - 4m_{\rm DM}^2}, \tag{14}$$

where $\sqrt{s}$ is the centre-of-mass energy. We can obtain $W_{\rm eff}$ by integrating over the outgoing directions of the final particles [11]:

$$\frac{{\rm d}W_{\rm eff}}{{\rm d}\cos(\theta)} = \sum_{ijkl} \frac{p_{ij}\, p_{kl}}{8\pi g_{\rm DM}^2 p_{\rm eff} S_{kl} \sqrt{s}} \sum_{\rm helicities} \left| \sum_{\rm diagrams} \mathcal{M}(ij \to kl) \right|^2, \tag{15}$$

where $\mathcal{M}(ij \to kl)$ is the amplitude of two new physics particles $(i, j)$ giving two Standard Model particles $(k, l)$, and $\theta$ is the angle between particles $i$ and $k$, $S_{kl}$ is a symmetry factor equal to 2 for identical final particles and to 1 otherwise, and $p_{kl}$ is the final centre-of-mass momentum such that

$$p_{kl} = \frac{\left[s - (m_k + m_l)^2\right]^{1/2} \left[s - (m_k - m_l)^2\right]^{1/2}}{2\sqrt{s}}. \tag{16}$$

The current density of dark matter particles can be obtained by integrating the Boltzmann equation (10) between a high temperature where all particles are in thermal equilibrium, and the current Universe temperature $T_0 = 2.726$ K. The freeze-out temperature $T_f$ is defined as the temperature at which the dark matter particles leave thermal equilibrium.

There exist several codes for the calculation of dark matter relic density, such as `SuperIso Relic` [14–16], `MicrOMEGAs` [17,18], `DarkSUSY` [19,20], `MadDM` [21,22] and `DarkPACK` [23], which use different methods of integration of the Boltzmann equation and calculation of the thermal average of the effective cross-section.

In particular, one can observe that the formula (13) can present some numerical instabilities for small values of $T$. In fact, both the Bessel functions have an asymptotic behaviour, as their expansion is given in (A1), so for large $1/T$ both the integrand function in the numerator and the sum in the denominator tend to 0, leading to an undefined value. Thus, when the evaluation of the numerator or the denominator returns a number close to the minimum value of the adopted floating number precision, the value of $\langle\sigma v\rangle$ cannot be reliable.

We see in the following that in our case of study, the threshold for the reliability of the implementation of the formula (13) with double floating precision is $T/m_1 \approx 10^{-6}$, which is related to the ratio of infinitesimally small Bessel functions in this regime.

Before doing the expansion, let us define $\widetilde{W}_{\rm eff} = g_{\rm DM}^2 W_{\rm eff}$, such that:

$$\frac{{\rm d}\widetilde{W}_{\rm eff}}{{\rm d}\cos(\theta)} = \sum_{ijkl} \frac{p_{ij}\, p_{kl}}{8\pi p_{\rm eff} S_{kl} \sqrt{s}} \sum_{\rm helicities} \left| \sum_{\rm diagrams} \mathcal{M}(ij \to kl) \right|^2, \tag{17}$$

and

$$\langle \sigma_{\text{eff}} v \rangle(T) = \frac{\displaystyle\int_0^\infty \mathrm{d}p_{\text{eff}}\, p_{\text{eff}}^2 \widetilde{W}_{\text{eff}}(\sqrt{s}) K_1\left(\frac{\sqrt{s}}{T}\right)}{m_{\text{DM}}^4 T \left[\displaystyle\sum_i g_i \frac{m_i^2}{m_{\text{DM}}^2} K_2\left(\frac{m_i}{T}\right)\right]^2}, \tag{18}$$

where we removed the parameter $g_{\text{DM}}$—originally introduced *ad hoc* in the definition of $W_{\text{eff}}$ [5] - as its square is a common term in the numerator and in the denominator.

## 3 Averaged annihilation rate at low temperature

The definition of the averaged annihilation rate given in Eq. (18), is the central part of the Boltzmann equation. In fact, for small values of $T$, the arguments of the Bessel functions tend to infinity, and both $K_1$ and $K_2$ become infinitesimally small since their arguments tend to infinity. This generates some computational issues, if $T$ is very small, which is the case in the recent Universe.

From a phenomenological perspective, often the freeze-out temperature will not be small enough to require a specific expansion for $\langle \sigma v_{\text{Møl}} \rangle_{\text{eff}}$. In fact, it is typically equal to the mass of the DM candidate times a factor ranging from 1/30 to 1/20, and therefore there is no need to evaluate $\langle \sigma v_{\text{Møl}} \rangle_{\text{eff}}$ at temperatures as low as $10^{-14}$GeV. However, we found this expansion useful, since in some cases it is possible to calculate, or to find in the literature, some formulae for $\langle \sigma v_{\text{Møl}} \rangle_{\text{eff}}$ in the non-relativistic case. Thus, providing a correct numerical expansion at low temperature, independent of a full formula prone to numerical instabilities, allows us to detect possible errors in the numerical implementation of the model, or in the algorithms which derive an analytical expression of the non-relativistic $\langle \sigma v_{\text{Møl}} \rangle_{\text{eff}}$ in a specific model.

It is therefore useful to study the expansion of the averaged annihilation rate at low temperatures, in order for example to verify that the relativistic result is consistent with the non-relativistic one. The derivation of the latter can be found in Ref. [3].

In this subsection, we will outline the steps of the expansion of (18), showing that it can be performed up to any given order. We also show that the lowest order is the order zero, hence proving that the formula (18) does not present singularities at $T = 0$. The original procedure has been suggested in Ref. [6], and in the following we describe the final computational steps in a way that they can be reproduced by hand or even with symbolic manipulation algorithms. We also provide as an ancillary file a `Mathematica notebook` [24] which performs such an expansion.

To begin, we make a change of variable for the integral (18):

$$p_{\text{eff}} \to y = \frac{p_{\text{eff}}^2}{m_1^2} + 1, \tag{19}$$

where $m_1$ is the mass of the lightest new physics particle, i.e. the dark matter particle, which we will denote in the following $\chi_1$. In the denominator, we keep in the sum only the contribution of $\chi_1$, since it is the lightest particle leading to the dominant contribution to the sum. Using the asymptotic form of $K_n$ provided in the Appendix in Eq. (A1), we therefore obtain:

$$\langle \sigma v_{\text{Møl}} \rangle_{\text{eff}} = \frac{x}{2 g_1^2 m_1^2 K_2^2(x)} \int_1^{+\infty} \mathrm{d}y \sqrt{y-1}\, \widetilde{W}_{\text{eff}}(y) K_1\left(2x\sqrt{y}\right), \tag{20}$$

where $x = m_1/T$.

Similarly to $K_2$, $K_1$ has its maximum value when its argument has its smallest value in the integral. This means that the largest contributions to the integral come from the region with $y \gtrsim 1$. Let us then expand $\widetilde{W}_{\text{eff}}$ around $y = 1$:

$$\widetilde{W}_{\text{eff}}(y) = \sum_{n=0}^{\infty} \frac{1}{n!} (y-1)^n \widetilde{W}_n, \tag{21}$$

where $\widetilde{W}_n = \left.\dfrac{\mathrm{d}^n \widetilde{W}_{\text{eff}}}{\mathrm{d}y}\right|_{y=1}$. Then

$$\langle \sigma v_{\text{Møl}} \rangle_{\text{eff}} = \sum_{n=0}^{\infty} \frac{1}{n!} \widetilde{W}_n \mathcal{K}_n(x), \tag{22}$$

where we have defined:

$$\mathcal{K}_n(x) = \frac{x}{2 g_1^2 m_1^2 K_2^2(x)} I_n(x),$$
$$I_n(x) = \int_1^{+\infty} \mathrm{d}y (y-1)^{\frac{1}{2}+n} K_1(2x\sqrt{y}). \tag{23}$$

The integral in $I_n$ corresponds to Eq. (B2) with $\lambda = 0$, $\mu = 3/2 + n$ and $\nu = 1$. Hence,

$$I_n(x) = \frac{1}{2}\Gamma\left(n + \frac{3}{2}\right) G_{1,3}^{3,0}\left(x^2 \middle| \begin{matrix} 0 \\ -n - \frac{3}{2}, \frac{1}{2}, -\frac{1}{2} \end{matrix}\right). \tag{24}$$

The coefficients $\mathcal{K}_n(x)$ of expansion (22) are therefore

$$\mathcal{K}_n(x) = \Gamma\left(n + \frac{3}{2}\right) \frac{x}{4 g_1^2 m_1^2 K_2^2(x)} \times G_{1,3}^{3,0}\left(x^2 \middle| \begin{matrix} 0 \\ -\frac{1}{2}, \frac{1}{2}, -n - \frac{3}{2} \end{matrix}\right). \tag{25}$$

Using the results in Appendix B, we can write the asymptotic form of $\mathcal{K}_n(x)$ as:

$$\mathcal{K}_n(x) = \Gamma\left(n+\frac{3}{2}\right)\frac{\sqrt{\pi}}{4g_1^2m_1^2}\mathrm{e}^{-2x}\frac{x}{K_2^2(x)}\sum_{p=n+2}^{\infty} g_{n,p}x^{-p}. \tag{26}$$

We consider now the asymptotic form of $K_2(x)$. By using Eq. (A4), and keeping the same notation as in Appendix A, we can write:

$$\mathcal{K}_n(x) = \Gamma\left(n+\frac{3}{2}\right)\frac{1}{2\sqrt{\pi}g_1^2m_1^2}\frac{x^2}{1+B(x)}\sum_{p=n+2}^{\infty} g_{n,p}x^{-p}. \tag{27}$$

Since $B(x)$ is a small quantity, we can use the properties of the geometric sum and write:

$$\mathcal{K}_n(x) = \Gamma\left(n+\frac{3}{2}\right)\frac{1}{2\sqrt{\pi}g_1^2m_1^2}x^2 \times\sum_{r=0}^{\infty}(-1)^r B^r(x)\sum_{p=n+2}^{\infty} g_{n,p}x^{-p}. \tag{28}$$

At this point, by knowing the $g_{n,p}$, the procedure becomes straightforward.

First, we use the expansions (22) and (26) to factorise the terms independent of $n$ in the expression of $\langle\sigma v_{\text{Møl}}\rangle_{\text{eff}}$:

$$\langle\sigma v_{\text{Møl}}\rangle_{\text{eff}} = \frac{1}{2\sqrt{\pi}g_1^2m_1^2}x^2\sum_{r=0}^{\infty}(-1)^r B^r(x) \times\sum_{n=0}^{\infty}\left[\frac{1}{n!}\widetilde{W}_n\Gamma\left(n+\frac{3}{2}\right)\sum_{p=n+2}^{\infty} g_{n,p}x^{-p}\right]. \tag{29}$$

The term in square brackets is a Laurent series whose maximum order is 2. Thsus, we can define a set of $g_p$ such that[3]:

$$\langle\sigma v_{\text{Møl}}\rangle_{\text{eff}} = \frac{1}{2\sqrt{\pi}g_1^2m_1^2}x^2\sum_{r=0}^{\infty}(-1)^r B^r(x)\sum_{p=2}^{\infty}\frac{g_p}{x^p}. \tag{30}$$

Moreover, using Eq. (A7) we can define the coefficients $\beta_r$ such that

$$\langle\sigma v_{\text{Møl}}\rangle_{\text{eff}} = \frac{1}{2\sqrt{\pi}g_1^2m_1^2}\sum_{r=0}^{\infty}\frac{\beta_r}{x^r}\sum_{p=0}^{\infty}\frac{\tilde{g}_p}{x^p}, \tag{31}$$

where $\tilde{g}_p = g_{p+2}$. Written in this form, it is clear that the lowest order is zero, as it should be.[4]

The next step is to determine the coefficients of the powers of $1/x$ up to a given order $N_{\text{max}}$. This can be done once we know the coefficients $\beta_r$ and $g_p$ up to $r = N_{\text{max}}$ and $p = N_{\text{max}} + 2$. We can also show that there is a maximum contribution from $n$, which can be obtained from the range of the sum in $p$:

$$n+2 \le p \le N_{\text{max}} + 2 \tag{32}$$

from which we obtain the condition $n \le N_{\text{max}}$. To summarise, in order to truncate the expansion at the order $N_{\text{max}}$, the indices have the following ranges:

$$0 \le r \le N_{\text{max}}, \quad 0 \le n \le N_{\text{max}}, \quad n+2 \le p \le N_{\text{max}} + 2\,. \tag{33}$$

Let us now discuss how to perform the expansion, considering for instance $N_{\text{max}} = 4$ to illustrate the intermediate steps and $N_{\text{max}} = 10$ for the final result. The `Mathematica notebook` provided as an ancillary file provides the algorithm valid for any values of $N_{\text{max}}$.

For a given $N_{\text{max}}$ the maximum order of the derivative of $W_{\text{eff}}$ that contributes to $\langle\sigma v_{\text{Møl}}\rangle_{\text{eff}}$ is exactly $N_{\text{max}}$:

$$\langle\sigma v_{\text{Møl}}\rangle_{\text{eff}} = \sum_{n=0}^{N_{\text{max}}}\frac{1}{n!}\widetilde{W}_n\mathcal{K}_n(x)\,. \tag{34}$$

$\langle\sigma v_{\text{Møl}}\rangle_{\text{eff}}$ is defined as a finite sum, and each $\mathcal{K}_n$ is the product of two series that we know where to truncate. Let us define the quantity:

$$D(x) = \frac{1}{1+B(x)} = \sum_{r=0}^{N_{\text{max}}}(-1)^r B^r(x)\,. \tag{35}$$

Then, we can write $\mathcal{K}_n$ in the form:

$$\mathcal{K}_n(x) = \Gamma\left(n+\frac{3}{2}\right)\frac{1}{2\sqrt{\pi}g_1^2m_1^2}x^2 D(x)\sum_{p=n+2}^{N_{\text{max}}+2} g_{n,p}x^{-p}\,. \tag{36}$$

The coefficients $g_{n,p}$ are tabulated in Eq. (B5). Therefore, we are left with determining the coefficients of the expansion of $D$ and calculating the product of the two truncated series. Firstly, we write each power of $B$ in $D$ by expanding the

[3] Note that for the sum over $p$ we kept the same name for the index for clarity. In fact, the powers of $x$ are expressed as functions of $p$ in the original sum. This means that if truncated at a certain order, the upper limit of the sums is the same.

[4] Note that for the sum over $r$ we kept the same name for the index for clarity. $B^r$ gives the highest contribution to the term $\eta_r/x^r$. Therefore, if we truncate at a certain order, the range of the two sums is the same.

**Table 1** Symbolical expressions and values for the coefficients $\beta_r$

| | Symbolic expression | Value |
|---|---|---|
| $\beta_0$ | $1$ | $1$ |
| $\beta_1$ | $-b_1$ | $-\frac{15}{4}$ |
| $\beta_2$ | $b_1^2 - b_2$ | $\frac{285}{32}$ |
| $\beta_3$ | $-b_1^3 + 2b_2b_1 - b_3$ | $-\frac{2115}{128}$ |
| $\beta_4$ | $b_1^4 - 3b_2b_1^2 + 2b_3b_1 + b_2^2 - b_4$ | $\frac{51435}{2048}$ |

series of $B$ in $B^r$ up to the order $N_{\max} - r + 1$. Then, up to the 4th order, the non-trivial powers of $B$ are:

$$B^2 = \frac{b_2^2 + 2b_1b_3}{x^4} + \frac{2b_2b_1}{x^3} + \frac{b_1^2}{x^2}, \tag{37}$$

$$-B^3 = -\frac{3b_2b_1^2}{x^4} - \frac{b_1^3}{x^3}, \tag{38}$$

$$B^4 = \frac{b_1^4}{x^4}. \tag{39}$$

The coefficients $b_i$ are given in Table 3, and the coefficients $\beta_r$ of the expansion of $D(x)$ are given in Table 1.

We can plug those expressions into $D$, $\mathcal{K}_n$ and $\langle \sigma v_{\text{Møl}} \rangle_{\text{eff}}$, obtaining the result (to the 10th order):

$$\begin{aligned}
\langle \sigma v_{\text{Møl}} \rangle_{\text{eff}} = {} & \frac{1}{4g_1^2m_1^2}\Bigg\{\widetilde{W}_0 + \frac{1}{x}\left(-3\widetilde{W}_0 + \frac{3\widetilde{W}_1}{2}\right) \\
& + \frac{1}{x^2}\left(6\widetilde{W}_0 - 3\widetilde{W}_1 + \frac{15\widetilde{W}_2}{8}\right) \\
& + \frac{1}{x^3}\left(-\frac{75\widetilde{W}_0}{8} + \frac{75\widetilde{W}_1}{16} - \frac{15\widetilde{W}_2}{16} + \frac{35\widetilde{W}_3}{16}\right) \\
& + \frac{1}{x^4}\left(\frac{23445\widetilde{W}_0}{2048} - \frac{1485\widetilde{W}_1}{256} - \frac{1575\widetilde{W}_2}{256} - \frac{525\widetilde{W}_3}{64} + \frac{315\widetilde{W}_4}{128}\right) \\
& + \frac{1}{x^5}\left(-\frac{17505\widetilde{W}_0}{2048} + \frac{19395\widetilde{W}_1}{4096} + \frac{11925\widetilde{W}_2}{512} + \frac{9975\widetilde{W}_3}{512} - \frac{4725\widetilde{W}_4}{512} + \frac{693\widetilde{W}_5}{256}\right) \\
& + \frac{1}{x^6}\left(-\frac{222885\widetilde{W}_0}{32768} + \frac{13095\widetilde{W}_1}{8192} - \frac{878175\widetilde{W}_2}{16384} - \frac{74025\widetilde{W}_3}{2048} + \frac{89775\widetilde{W}_4}{4096} - \frac{10395\widetilde{W}_5}{1024} + \frac{3003\widetilde{W}_6}{1024}\right) \\
& + \frac{1}{x^7}\left(\frac{1661715\widetilde{W}_0}{32768} - \frac{1264815\widetilde{W}_1}{65536} + \frac{3173175\widetilde{W}_2}{32768} + \frac{1800225\widetilde{W}_3}{32768} + -\frac{666225\widetilde{W}_4}{16384} + \frac{197505\widetilde{W}_5}{8192} + \frac{6435\widetilde{W}_7}{2048} - \frac{45045\widetilde{W}_6}{4096}\right) \\
& + \frac{1}{x^8}\left(-\frac{1379496825\widetilde{W}_0}{8388608} + \frac{32645025\widetilde{W}_1}{524288} - \frac{76137975\widetilde{W}_2}{524288} - \frac{8594775\widetilde{W}_3}{131072} + {} + \frac{16202025\widetilde{W}_4}{262144} - \frac{1465695\widetilde{W}_5}{32768} + \frac{855855\widetilde{W}_6}{32768} + \frac{109395\widetilde{W}_8}{32768} - \frac{96525\widetilde{W}_7}{8192}\right) \\
& + \frac{1}{x^9}\left(-\frac{13671950879025\widetilde{W}_0}{4294967296} - \frac{2855855475\widetilde{W}_1}{16777216} + \frac{186553125\widetilde{W}_2}{1048576} + \frac{45808875\widetilde{W}_3}{1048576} - \frac{77352975\widetilde{W}_4}{1048576} + \frac{35644455\widetilde{W}_5}{524288} - \frac{6351345\widetilde{W}_6}{131072} + \frac{1833975\widetilde{W}_7}{65536} - \frac{1640925\widetilde{W}_8}{131072} + \frac{230945\widetilde{W}_9}{65536}\right) \\
& + \frac{1}{x^{10}}\left(-\frac{38822473644075\widetilde{W}_0}{8589934592} - \frac{43047242435475\widetilde{W}_1}{8589934592} - \frac{10584016875\widetilde{W}_2}{67108864} + \frac{275065875\widetilde{W}_3}{4194304} + \frac{412279875\widetilde{W}_4}{8388608} - \frac{170176545\widetilde{W}_5}{2097152} \right. \\
& \left. + \frac{154459305\widetilde{W}_6}{2097152} - \frac{13610025\widetilde{W}_7}{262144} + \frac{31177575\widetilde{W}_8}{1048576} - \frac{3464175\widetilde{W}_9}{262144} + \frac{969969\widetilde{W}_{10}}{262144}\right)\Bigg\},
\end{aligned} \tag{40}$$

which correctly reproduces the results in Ref. [3] and Ref. [6]. From a numerical perspective, the error on $\widetilde{W}_n$ for $n \geq 2$ will be large. Therefore, it is recommended to stop at the order 1 or 2. We show the results obtained for the pMSSM[5] in DarkPACK in a scenario where the dark matter candidate has a mass $m_1 \approx 200$GeV in Fig. 1. We see that truncating at the first order can give a very satisfactory result, since the obtained curves for $\langle \sigma v_{\text{Møl}} \rangle_{\text{eff}}(T)$ computed with the numerical evaluation of Eq. (18) and with the asymptotic behaviour (40) are compatible. Moreover, from the figure, one can notice that the numerical implementation of the full formula for $\langle \sigma v \rangle$ fails to deliver reliable results for $T \lesssim 10^{-6} m_1$. In the calculation, DarkPACK uses floating numbers with double precision.

[5] The phenomenological Minimal Supersymmetric extension of the Standard Model [25].

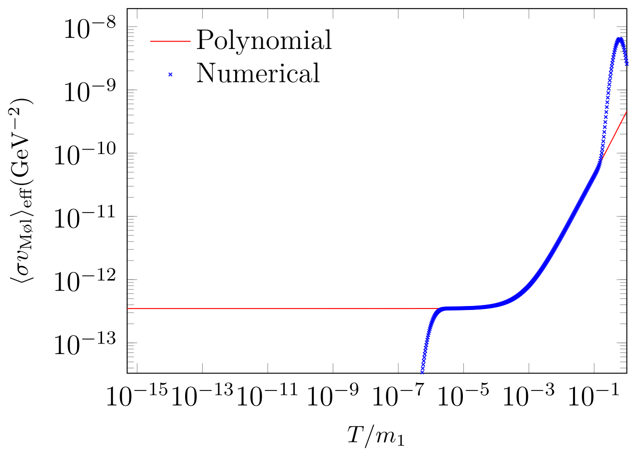


**Fig. 1** Comparison between the results obtained for $\langle\sigma v_{\rm Møl}\rangle_{\rm eff}$ by using the full expression in Eq. (18) and the polynomial expansion at the first order by using Eq. (40)

## 4 Case of particles with small mass splittings

The result shown in Sect. 3 is correct up to a defined order, under the hypothesis that there are no new physics species with a mass close to the one of the dark matter candidate.[6] In models with a $\mathbb{Z}_2$ symmetry, such a particle is the lightest of a set. Let us suppose that there are $M \leq N$ particles nearly degenerate in mass. In such a case, we need to retain their contributions to the denominator in Eq. (18). Let us define:

$$x=\frac{m_1}{T}, \quad x_i=\frac{m_i}{T}, \quad c_i=g_i\frac{m_i^2}{m_1^2}, \quad \Delta x_i=x_i-x, \tag{41}$$

for $i=1,\ldots,M$. We can perform the same change of variable as in Sect. 3 which leads to:

$$\langle\sigma v_{\rm Møl}\rangle_{\rm eff}=\frac{x}{2m_1^2}\frac{1}{\left[\sum_{i=1}^M c_iK_2\left(\frac{m_i}{T}\right)\right]^2}\times\int_1^{+\infty}\mathrm{d}y\sqrt{y-1}\,\widetilde{W}_{\rm eff}(y)K_1\left(2x\sqrt{y}\right). \tag{42}$$

After expanding $W_{\rm eff}$ around $y=1$ as in the previous case we obtain:

$$\langle\sigma v_{\rm Møl}\rangle_{\rm eff}=\sum_{n=0}^{\infty}\frac{1}{n!}\widetilde{W}_n\mathcal{H}_n(x), \tag{43}$$

where we have defined:

$$\mathcal{H}_n(x)=\frac{x}{2m_1^2}\frac{1}{\left[\sum_{i=1}^M c_iK_2\left(\frac{m_i}{T}\right)\right]^2}I_n(x), \tag{44}$$

[6] Except, of course, the candidate itself.

and where $I_n$ is the same as the one defined in Eq. (23). Note that $\mathcal{H}_n$ and $\mathcal{K}_n$ differ only for the Bessel functions in the denominator. With some manipulations, we can treat $\mathcal{H}_n$ in the same way as done with $\mathcal{K}_n$. In fact, we already know how to write $I_n$ as Laurent series.

Let us define the quantity:

$$D(x)=\left[\sum_{i=1}^M c_iK_2\left(x_i\right)\right]^2. \tag{45}$$

Our goal is to expand $D(x)$ at its first order in $\Delta x_i$ and at an arbitrary order in $x$. Let us expand the square:

$$\begin{aligned}D(x)&=\left[\sum_{i=1}^M c_iK_2\left(x+\Delta x_i\right)\right]^2\\&=2\sum_{i=1}^M c_iK_2(x+\Delta x_i)\sum_{j=i}^M\left(1-\frac{\delta_{ij}}{2}\right)c_jK_2(x+\Delta x_j).\end{aligned} \tag{46}$$

At the first order in $\Delta x_i$ we have also:

$$K_2\left(x+\Delta x_i\right)=K_2(x)+K_2'(x)\Delta x_i, \tag{47}$$

where $K_2'(x)=\frac{\mathrm{d}K_2}{\mathrm{d}x}(x)$. Therefore:

$$\begin{aligned}D(x)&=2\sum_{i=1}^M\sum_{j=i}^M c_i\left(1-\frac{\delta_{ij}}{2}\right)c_j\\&\quad\times\left(K_2(x)+K_2'(x)\Delta x_i\right)\left(K_2(x)+K_2'(x)\Delta x_j\right)\\&=2\sum_{i=1}^M\sum_{j=i}^M c_i\left(1-\frac{\delta_{ij}}{2}\right)c_j\\&\quad\times\left(K_2^2(x)+K_2(x)K_2'(x)\left(\Delta x_i+\Delta x_j\right)+o\left(\Delta x^2\right)\right).\end{aligned} \tag{48}$$

Separating the constant terms from the linear terms in $\Delta x_i$, and using the identity $2K_2(x)K_2'(x)=\frac{\mathrm{d}K_2^2}{\mathrm{d}x}(x)$ we obtain:

$$\begin{aligned}D(x)&=\left[\sum_{i=1}^M\sum_{j=i}^M c_i\left(2-\delta_{ij}\right)c_j\right]K_2^2(x)\\&\quad+\left[\sum_{i=1}^M\sum_{j=i}^M c_i\left(1-\frac{\delta_{ij}}{2}\right)c_j\left(\Delta x_i+\Delta x_j\right)\right]\left(K_2^2\right)'(x),\end{aligned} \tag{49}$$

where $\left(K_2^2\right)'(x) = \frac{\mathrm{d}K_2^2}{\mathrm{d}x}(x)$. Let us define:

$$\tilde{\rho} = \left[\sum_{i=1}^{M}\sum_{j=i}^{M} c_i \left(2-\delta_{ij}\right) c_j\right],$$
$$\tilde{\eta} = -\left[\sum_{i=1}^{M}\sum_{j=i}^{M} c_i \left(1-\frac{\delta_{ij}}{2}\right) c_j \left(\Delta x_i + \Delta x_j\right)\right], \quad (50)$$

and use the expansions (A4) and (A5):

$$\begin{aligned} D(x) &= \frac{\pi}{2x}\mathrm{e}^{-2x}\left[\tilde{\rho}(1+B(x)) + \tilde{\eta}(2+\tilde{B}(x))\right] \\ &= \frac{\pi}{2x}\mathrm{e}^{-2x}\left[\tilde{\rho} + 2\tilde{\eta} + \tilde{\rho}B(x) + \tilde{\eta}\tilde{B}(x)\right] \\ &= \frac{\pi}{2x}\mathrm{e}^{-2x}\gamma\left[1 + \rho B(x) + \eta\tilde{B}(x)\right], \end{aligned} \quad (51)$$

with

$$\gamma = \tilde{\rho} + 2\tilde{\eta}, \qquad \rho = \frac{\tilde{\rho}}{\gamma}, \qquad \eta = \frac{\tilde{\eta}}{\gamma}. \quad (52)$$

We can therefore write:

$$\frac{1}{D(x)} = \frac{2x}{\pi\gamma}\mathrm{e}^{2x}\frac{1}{1+F(x)}, \quad (53)$$

with:

$$F(x) = \rho B(x) + \eta\tilde{B}(x). \quad (54)$$

The asymptotic behaviour at large $x$ for $F$ is proportional to $1/x$, since for both $B$ and $\tilde{B}$ it is proportional to $1/x$. This means that we can treat (53) with the geometric expansion:

$$\frac{1}{D(x)} = \frac{2x}{\pi\gamma}\mathrm{e}^{2x}\sum_{r=0}^{\infty}(-1)^r F^r(x). \quad (55)$$

At this point we have found the same form as in the previous case, and we can treat it similarly. Moreover, we have chosen to use the definition (54) for $F$, because it has the advantage of being straightforward to reduce to the order 0 in $\Delta x_i$, which is the case if more particles have exactly the same mass. In fact, $\tilde{B}$ is from the first order of the expansion of $K_2^2(x+\Delta x_i)$ and also for $\Delta x_i = 0$ we have $\eta = 0$.

Thus, apart from a factor $1/\gamma$ and the replacement $B \to F$, we have found the same expression as in the previous case:

$$\langle\sigma v_{\mathrm{Møl}}\rangle_{\mathrm{eff}} = \frac{1}{2\sqrt{\pi}m_1^2\gamma}x^2\sum_{r=0}^{\infty}(-1)^r F^r(x)$$
$$\times\sum_{n=0}^{\infty}\left[\frac{1}{n!}\widetilde{W}_n\Gamma\left(n+\frac{3}{2}\right)\sum_{p=n+2}^{\infty} g_{n,p}x^{-p}\right]. \quad (56)$$

This does not change the orders at which we need to truncate the series once we know that we want the result to a given $N_{\max}$.

By using the results in Eq. (33):

$$0 \le r \le N_{\max}, \quad 0 \le n \le N_{\max}, \quad n+2 \le p \le N_{\max}+2. \quad (57)$$

The difference is that here the coefficients of $F$ up to a given order depend on model-dependent quantities, i.e. $\rho$ and $\eta$. We can use the same procedure as before, since we know that the geometric sum in $F$ has to be truncated at the order $N_{\max}$. Hence, it is enough to expand each power $F^r$ separately as a function of $B$ and $\tilde{B}$, for which we know the coefficients, at the first order in $\eta$.[7] We automated this calculation in the `Wolfram Mathematica` notebook in the ancillary files.

Parametrising the geometric expansion as:

$$\frac{1}{1+F(x)} = \sum_{r=0}^{\infty}\frac{\phi_r}{x^r}, \quad (58)$$

we have, for $N_{\max} = 4$, the following expressions:

$$\phi_0 = 1, \quad (59a)$$
$$\phi_1 = -\eta\tilde{b}_1 - \rho b_1, \quad (59b)$$
$$\phi_2 = \eta\left(2\rho b_1\tilde{b}_1 - \tilde{b}_2\right) + \rho^2 b_1^2 - \rho b_2, \quad (59c)$$
$$\phi_3 = \eta\left(-3\rho^2 b_1^2\tilde{b}_1 + 2\rho b_1\tilde{b}_2 + 2\rho b_2\tilde{b}_1 - \tilde{b}_3\right) - \rho^3 b_1^3 + 2\rho^2 b_2 b_1 - \rho b_3, \quad (59d)$$
$$\phi_4 = \eta\left(4\rho^3 b_1^3\tilde{b}_1 - 3\rho^2 b_1^2\tilde{b}_2 - 6\rho^2 b_2 b_1\tilde{b}_1 + 2\rho b_1\tilde{b}_3 + 2\rho b_3\tilde{b}_1 + 2\rho b_2\tilde{b}_2 - \tilde{b}_4\right) + \rho^4 b_1^4 - 3\rho^3 b_2 b_1^2 + 2\rho^2 b_3 b_1 + \rho^2 b_2^2 - \rho b_4. \quad (59e)$$

The values of the $b_i$'s and $\tilde{b}_i$'s can be found in Table 3.

Plugging into the expression of $\langle\sigma v_{\mathrm{Møl}}\rangle_{\mathrm{eff}}$, and replacing the $g_{n,p}$ with their values,[8] we obtain:

$$\langle\sigma v_{\mathrm{Møl}}\rangle_{\mathrm{eff}} = \frac{1}{4\gamma m_1^2}\left\{\widetilde{W}_0\phi_0 + \frac{1}{x}\left[\widetilde{W}_0\left(\frac{3\phi_0}{4} + \phi_1\right) + \frac{3\widetilde{W}_1\phi_0}{2}\right]\right.$$

[7] Since we are treating the first order in $\Delta x_i$.

[8] This simplifies dramatically the expressions, since many of them vanish.

$$
\begin{aligned}
&+\frac{1}{x^2}\Bigg[\widetilde{W}_0\left(-\frac{3\phi_0}{32}+\frac{3\phi_1}{4}+\phi_2\right)\\
&+\widetilde{W}_1\left(\frac{21\phi_0}{8}+\frac{3\phi_1}{2}\right)+\frac{15\widetilde{W}_2\phi_0}{8}\Bigg]\\
&+\frac{1}{x^3}\Bigg[\widetilde{W}_0\phi_0\left(\frac{15\phi_0}{128}-\frac{3\phi_1}{32}+\frac{3\phi_2}{4}+\phi_3\right)\\
&+\widetilde{W}_1\left(\frac{75\phi_0}{64}+\frac{21\phi_1}{8}+\frac{3\phi_2}{2}\right)+\\
&+\widetilde{W}_2\left(\frac{195\phi_0}{32}+\frac{15\phi_1}{8}\right)+\frac{35\widetilde{W}_3\phi_0}{16}\Bigg]\\
&+\frac{1}{x^4}\Bigg[\widetilde{W}_0\left(\frac{15\phi_1}{128}-\frac{3\phi_2}{32}+\frac{3\phi_3}{4}+\phi_4\right)\\
&+\widetilde{W}_1\left(\frac{75\phi_1}{64}+\frac{21\phi_2}{8}+\frac{3\phi_3}{2}\right)\\
&+\widetilde{W}_2\left(\frac{195\phi_1}{32}+\frac{15\phi_2}{8}\right)+\frac{35\widetilde{W}_3\phi_1}{16}\\
&+\frac{315\widetilde{W}_4\phi_0}{128}\Bigg]\Bigg\}. \qquad (60)
\end{aligned}
$$

From this expression, one can check that in the previous hypothesis (i.e. $\gamma=\rho=1$ and $\eta=0$) we recover the same coefficients as in Eq. (40). This expression, however, is correctly expanded until the order 4 in $1/x$, but it contains some spurious terms of higher orders in $\Delta x_i$. In order to eliminate them and consistently truncate at the first order, we have to do some more manipulations. Recalling the definitions (52) and (50):

$$
\rho=\frac{1}{1+2\frac{\tilde{\eta}}{\tilde{\rho}}}, \qquad \eta=\frac{\tilde{\eta}}{\tilde{\rho}+2\tilde{\eta}}, \qquad (61)
$$

we can now identify $\tilde{\eta}$ as the expansion parameter. At the first order we have:

$$
\gamma^{-1}=\frac{1}{\tilde{\rho}}\left(1-2\frac{\tilde{\eta}}{\tilde{\rho}}\right), \quad \rho^n=1-2n\frac{\tilde{\eta}}{\tilde{\rho}}, \quad \eta=\frac{\tilde{\eta}}{\tilde{\rho}}. \qquad (62)
$$

As we know already, $\eta\sim\Delta x_i$, justifying the truncation of higher powers of $\eta$. Since $\tilde{\eta}$ is always divided by $\tilde{\rho}$, we can choose $\eta$ as the expansion parameter.

In the formula (60), we can replace $\gamma$ with $\tilde{\rho}$ and $\phi_n$ with:

$$
\tilde{\phi}_n=(1-2\eta)\,\phi_n. \qquad (63)
$$

We are left with expressing $\tilde{\phi}_n$ at the first order in $\eta$:

$$
\tilde{\phi}_0=1-2\eta, \qquad (64a)
$$

$$
\tilde{\phi}_1=\eta\left(3b_1-\tilde{b}_1\right)-b_1, \qquad (64b)
$$

$$
\tilde{\phi}_2=\eta\left(2b_1\tilde{b}_1-\tilde{b}_2-4b_1^2+3b_2\right)+b_1^2-b_2, \qquad (64c)
$$

$$
\begin{aligned}
\tilde{\phi}_3=\eta\Big(&-3b_1^2\tilde{b}_1+2b_1\tilde{b}_2+2b_2\tilde{b}_1\\
&-\tilde{b}_3+5b_1^3-8b_2b_1+3b_3\Big)-b_1^3+2b_2b_1-b_3,
\end{aligned} \qquad (64d)
$$

$$
\begin{aligned}
\tilde{\phi}_4=\eta\Big(&4b_1^3\tilde{b}_1-3b_1^2\tilde{b}_2-6b_2b_1\tilde{b}_1+2b_1\tilde{b}_3\\
&+2b_3\tilde{b}_1+2b_2\tilde{b}_2-\tilde{b}_4-6b_1^4\\
&+15b_2b_1^2-8b_3b_1-4b_2^2+3b_4\Big)\\
&+b_1^4-3b_2b_1^2+2b_3b_1+b_2^2-b_4.
\end{aligned} \qquad (64e)
$$

The expression for $\langle\sigma v_{\mathrm{M\o l}}\rangle_{\mathrm{eff}}$ at the first order in $\Delta x_i$ and at the 4th order in $x^{-1}$ reads then:

$$
\begin{aligned}
\langle\sigma v_{\mathrm{M\o l}}\rangle_{\mathrm{eff}}=\frac{1}{4\tilde{\rho}m_1^2}\Bigg\{&\widetilde{W}_0\tilde{\phi}_0+\frac{1}{x}\Bigg[\widetilde{W}_0\left(\frac{3\tilde{\phi}_0}{4}+\tilde{\phi}_1\right)+\frac{3\widetilde{W}_1\tilde{\phi}_0}{2}\Bigg]\\
&+\frac{1}{x^2}\Bigg[\widetilde{W}_0\left(-\frac{3\tilde{\phi}_0}{32}+\frac{3\tilde{\phi}_1}{4}+\tilde{\phi}_2\right)\\
&+\widetilde{W}_1\left(\frac{21\tilde{\phi}_0}{8}+\frac{3\tilde{\phi}_1}{2}\right)+\frac{15\widetilde{W}_2\tilde{\phi}_0}{8}\Bigg]\\
&+\frac{1}{x^3}\Bigg[\widetilde{W}_0\left(\frac{15\tilde{\phi}_0}{128}-\frac{3\tilde{\phi}_1}{32}+\frac{3\tilde{\phi}_2}{4}+\tilde{\phi}_3\right)\\
&+\widetilde{W}_1\left(\frac{75\tilde{\phi}_0}{64}+\frac{21\tilde{\phi}_1}{8}+\frac{3\tilde{\phi}_2}{2}\right)\\
&+\widetilde{W}_2\left(\frac{195\tilde{\phi}_0}{32}+\frac{15\tilde{\phi}_1}{8}\right)+\frac{35\widetilde{W}_3\tilde{\phi}_0}{16}\Bigg]\\
&+\frac{1}{x^4}\Bigg[\widetilde{W}_0\left(\frac{15\tilde{\phi}_1}{128}-\frac{3\tilde{\phi}_2}{32}+\frac{3\tilde{\phi}_3}{4}+\tilde{\phi}_4\right)\\
&+\widetilde{W}_1\left(\frac{75\tilde{\phi}_1}{64}+\frac{21\tilde{\phi}_2}{8}+\frac{3\tilde{\phi}_3}{2}\right)\\
&+\widetilde{W}_2\left(\frac{195\tilde{\phi}_1}{32}+\frac{15\tilde{\phi}_2}{8}\right)+\frac{35\widetilde{W}_3\tilde{\phi}_1}{16}\\
&+\frac{315\widetilde{W}_4\tilde{\phi}_0}{128}\Bigg]\Bigg\},
\end{aligned} \qquad (65)
$$

with $\tilde{\rho}$ defined in Eq. (50), $\eta$ defined in Eq. (62) and $\tilde{\phi}_n$s defined in Eq. (64). Note that we showed the results at the order 4 since the expression is already very complicated, but the `Wolfram Mathematica` notebook in the ancillary files allows us to obtain the correct result at any given order. We would like to comment on the result, by noticing that, in this form, the coefficients of $x^{-n}$ depend on $T$, but they are linear in $1/T$, therefore, there is a term proportional to $\eta\sim 1/T$ in $\tilde{\phi}_0$. However, we recall that the requirement for the mass degeneracy to contribute is to have $\Delta x_i$ small enough to allow the expansion (47). At very low temperatures, only

**Table 2** Parameters of $\tilde{\phi}_n$ in the form $\tilde{\phi}_n = \beta_n + \eta\lambda_n$

| n | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\beta_n$ | 1 | $-\frac{15}{4}$ | $\frac{285}{32}$ | $-\frac{2115}{128}$ | $\frac{51435}{2048}$ |
| $\lambda_n$ | -2 | $-\frac{11}{4}$ | $\frac{165}{32}$ | $-\frac{5295}{128}$ | $\frac{297135}{2048}$ |

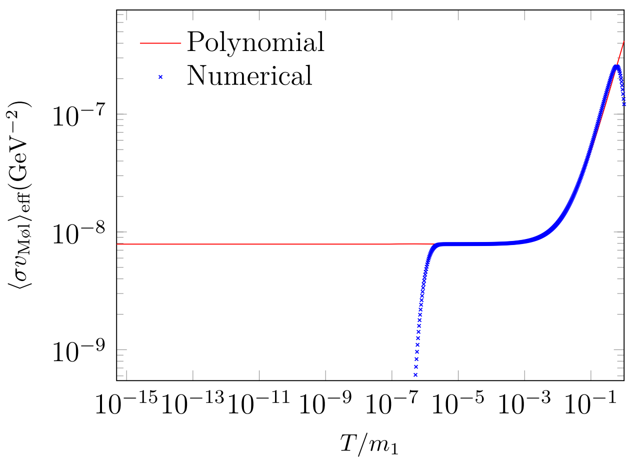


**Fig. 2** Comparison between the results obtained for $\langle\sigma v_{\text{Møl}}\rangle_{\text{eff}}$ using the full expression (18) and the polynomial expansion at the first order by using (65). Plot obtained using `DarkPACK` [23]

the fully degenerate species contribute, and for $\Delta x_i = 0$ we have $\eta = 0$, so the 0th order contribution does not have a divergence, and the only deviation from the previous formula in Eq. (40) is contained in the $\tilde{\rho}$ in the prefactor. In Fig. 2 we show the results for the pMSSM, in which we considered the three lightest neutralinos to have the same mass of 200GeV. As in the case without mass degeneracy, `DarkPACK` uses floating numbers with double precision. In such conditions we find again that the numerical expression breaks down for $T/m_1 \lesssim 10^{-6}$.

## 5 Conclusion

We showed in detail how to derive the formula for the expansion of $\langle\sigma v_{\text{Møl}}\rangle_{\text{eff}}$ at low temperatures at an arbitrary order in $x = m_1/T$ and at the first order in the mass splittings $x_i - x$, providing also two `Wolfram Mathematica` notebooks that allow us to perform both the expansions at an arbitrary order in $x$.

The implementation of the formula (40) in the software `DarkPACK` has shown the necessity of the usage of such an expansion at low temperatures, as the implementation of the full formula (18) fails to provide a numerically stable result. The result is publicly available in `DarkPACK 1.2`.

In many NP scenarios, the freeze-out temperature lies in the interval $[m_1/30, m_1/20]$, thus the calculations of the relic density made by using formula (18) are reliable. Therefore, the result presented in the present work will hardly improve the numerical computation of the relic density in many NP scenarios, apart in case of resonances at low velocities. However, it can be used to validate the numerical result obtained by using (18) for temperatures which are low, but larger than the breakdown threshold of $10^{-6}m_1$. This check can be extremely useful, especially in very recent codes (such as `DarkPACK`) or while implementing a new model with mass degeneracy near the DM candidate, since it provides the computation of an important quantity in an independent way of the full numerical one. The latter involves in fact a numerical integration, while the former involves a numerical derivative. Both calculations are prone to different kinds of numerical errors, and from the agreement of the two calculations of $\langle\sigma v_{\text{Møl}}\rangle_{\text{eff}}$ one can ensure that there are no inconsistency.

Moreover, as pointed out in the introduction, the formula derived in this work can be used to compute symbolically—by hand or with symbolic manipulation programs—the expression of $\langle\sigma v_{\text{Møl}}\rangle_{\text{eff}}$ at low temperatures in models where a small mass splitting or a resonance is present. This allows for a better profiling of the parameter space of the model, since it is possible to study its impact on $\langle\sigma v_{\text{Møl}}\rangle_{\text{eff}}$, and therefore on the corresponding relic density.

**Data Availability Statement** This manuscript has data included as electronic supplementary material. [Author's comment: The Mathematica notebooks used for the derivation of the formulae can be found in the attachments of the article.]

**Code Availability Statement** This manuscript has associated code/software in a data repository. [Author's comment: The Mathematica notebooks used for the derivation of the formulae can also be found in the repositories of DarkPACK (https://gitlab.in2p3.fr/darkpack/darkpack-public).]

Funded by SCOAP[3].

## Appendix A: Asymptotic expansions of the Bessel functions

The asymptotic form of $K_n(z)$ for $|z| \to +\infty$ and $|argz| < 3\pi/2$ can be written as [26]:

$$K_n(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left[ 1 + \frac{\mu - 1}{8z} + \frac{(\mu-1)(\mu-9)}{2!(8z)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + o\left(z^{-4}\right) \right] \tag{A1}$$

where $\mu = 4n^2$.

For $n = 2$ we define the quantity $A(x)$ such that:

$$K_2(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \left[1 + A(x)\right], \tag{A2}$$

and we write it in the form:

$$A(x) = \sum_{m=1}^{\infty} a_m x^{-m}. \tag{A3}$$

Analogously, we can define the quantity $B(x)$ such that:

$$K_2^2(x) \sim \frac{\pi}{2x} e^{-2x} \left[1 + B(x)\right], \tag{A4}$$

implying the relation $B = 2A + A^2$.

It is helpful to write the asymptotic form of the first derivative of $K_2^2$ as:

$$\left(K_2^2\right)'(x) \sim -\frac{\pi}{2x} e^{-2x} \left[2 + \tilde{B}(x)\right], \tag{A5}$$

with:

$$\tilde{B}(x) = 2B(x) + \frac{1}{x} + \frac{B(x)}{x} - B'(x). \tag{A6}$$

where the $'$ denotes the derivative with respect to $x$.

Note that since $A \sim 1/x$,[9] we have $B \sim 1/x$ and $\tilde{B} \sim 1/x$:

$$B(x) = \sum_{m=1}^{\infty} b_m x^{-m}, \qquad \tilde{B}(x) = \sum_{m=1}^{\infty} \tilde{b}_m x^{-m}. \tag{A7}$$

[9] By neglecting a non-null real factor.

Therefore, the coefficients can be determined using the relations

$$b_m = 2a_m + \sum_{k=1}^{m} a_k a_{m-k}, \qquad \forall m \geq 1 \tag{A8}$$

$$\tilde{b}_1 = 2b_1 + 1, \qquad \tilde{b}_m = 2b_m + m b_{m-1}. \quad \forall m \geq 2 \tag{A9}$$

The coefficients $a_m$ are known, and their values are given in Table 3, thus we can compute all the other parameters.

## Appendix B: Meijer functions

The Meijer functions are defined as[10]

$$G^{m,n}_{p,q}\left(z \middle| \begin{matrix} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1} \ldots, b_q \end{matrix}\right) = \frac{1}{2\pi i} \int ds \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} x^s, \tag{B1}$$

where $0 \leq m \leq q$, $0 \leq n \leq p$, and the poles of $\Gamma(b_j - s)$ must not coincide with the poles of $\Gamma(1 - a_j + s)$ for any pair $(j, k)$ with $1 \leq j \leq n$, $1 \leq k \leq m$.

The following property holds[11]

$$\int_1^{\infty} dx x^{\lambda} (x-1)^{\mu-1} K_\nu(a\sqrt{x}) = \Gamma(\mu) 2^{2\lambda-1} a^{-2\lambda} G^{3,0}_{1,3} \times \left(\frac{a^2}{4} \middle| \begin{matrix} 0 \\ -\mu, \frac{\nu}{2} + \lambda, -\frac{\nu}{2} + \lambda \end{matrix}\right). \tag{B2}$$

For $\lambda = 0$, $\mu = 3/2 + n$ and $\nu = 1$, the asymptotic form of the $G$-function at the right hand side is:

$$G^{3,0}_{1,3}\left(x^2 \middle| \begin{matrix} 0 \\ -\frac{1}{2}, \frac{1}{2}, -n - \frac{3}{2} \end{matrix}\right) = \sqrt{\pi} e^{-2x} G_n(x), \tag{B3}$$

where $G_n(x)$ is a generalised series:

$$G_n(x) = \sum_{p=n+2}^{\infty} g_{n,p} x^{-p}. \tag{B4}$$

The results of the expansion of $G_n$ for $0 \leq n \leq 10$ to the 12th order are:

$$G_0 = \frac{1}{x^2} + \frac{3}{4x^3} - \frac{3}{32x^4} + \frac{15}{128x^5}, \tag{B5a}$$

[10] See e.g.: Definition (9.301) in Ref. [27].

[11] See e.g.: Eq. (6.592.4) in Ref. [27].

**Table 3** Coefficients of the Laurent series in the asymptotic expansion of $K_2$, $K_2^2$ and $\left(K_2^2\right)'$

| m | $a_m$ | $b_m$ | $\tilde{b}_m$ |
|---|---|---|---|
| 1 | $\frac{15}{8}$ | $\frac{15}{4}$ | $\frac{17}{2}$ |
| 2 | $\frac{105}{128}$ | $\frac{165}{32}$ | $\frac{285}{16}$ |
| 3 | $-\frac{315}{1024}$ | $\frac{315}{128}$ | $\frac{1305}{64}$ |
| 4 | $\frac{10{,}395}{32{,}768}$ | $\frac{315}{2048}$ | $\frac{10{,}395}{1024}$ |
| 5 | $-\frac{135{,}135}{262{,}144}$ | $-\frac{2835}{8192}$ | $\frac{315}{4096}$ |
| 6 | $\frac{4{,}729{,}725}{4{,}194{,}304}$ | $\frac{61{,}425}{65{,}536}$ | $-\frac{6615}{32{,}768}$ |
| 7 | $-\frac{103{,}378{,}275}{33{,}554{,}432}$ | $-\frac{779{,}625}{262{,}144}$ | $\frac{80{,}325}{131{,}072}$ |
| 8 | $\frac{21{,}606{,}059{,}475}{2{,}147{,}483{,}648}$ | $\frac{90{,}904{,}275}{8{,}388{,}608}$ | $-\frac{8{,}887{,}725}{4{,}194{,}304}$ |
| 9 | $-\frac{655{,}383{,}804{,}075}{17{,}179{,}869{,}184}$ | $-\frac{1{,}497{,}971{,}475}{33{,}554{,}432}$ | $\frac{138{,}305{,}475}{16777216}$ |
| 10 | $\frac{45{,}221{,}482{,}481{,}175}{274{,}877{,}906{,}944}$ | $\frac{55{,}124{,}944{,}875}{268{,}435{,}456}$ | $-\frac{4{,}793{,}914{,}125}{134{,}217{,}728}$ |

$$G_1 = \frac{25}{32x^5} + \frac{7}{4x^4} + \frac{1}{x^3}, \tag{B5b}$$

$$G_2 = \frac{13}{4x^5} + \frac{1}{x^4}, \tag{B5c}$$

$$G_\nu = \frac{1}{x^{\nu+2}}, \qquad \forall\, 3 \le \nu \le 10. \tag{B5d}$$

## Appendix C: A practical example

In this appendix, we provide a practical example on how to use formula (65). Let us consider a generalisation of the model presented e.g. in [28], in which the SM is extended by adding a Dirac fermion $\chi_1$ and a parity-even real scalar $\phi$, according to a variation of lagrangian density by:

$$\Delta\mathcal{L}_1 = \frac{1}{2}m_\phi^2(\partial\phi)^2 + \bar{\chi}_1(\mathrm{i}\not{\partial} - m_1)\chi_1 - g_{1,\phi}\phi\bar{\chi}_1\chi_1 - \sum_f \frac{1}{\sqrt{2}} g_{f\phi} y_f \phi \bar{f} f, \tag{C1}$$

where the sum runs over all the SM fermions $f$, and $y_f = \sqrt{2}m_f/v$, with $v$ the Higgs boson's v.e.v. of about 246 GeV, and $m_f$ the mass of $f$. For such a model, considering $m_\phi > m_1 > m_f$, the only process contributing to $\langle\sigma v_{\text{Møl}}\rangle_{\text{eff}}$ is $\bar{\chi}_1\chi_1 \to \bar{f} f$, whose sum over the spins of the squared amplitude is:

$$\mathcal{M} = \sum_{\text{spins}} |M^2| = 2\left(g_{1,\phi}g_{f\phi}y_f\right)^2 \frac{\left(s - 4m_1^2\right)\left(s - 4m_f^2\right)}{\left(s - m_\phi^2\right)^2 + m_\phi^2\Gamma_\phi^2}, \tag{C2}$$

where $s$ is the Mandelstam variable, and $\Gamma_\phi$ is the total decay width of $\phi$. From this expression, we can use the relation (17) to compute $\widetilde{W}_{\text{eff}}$:

$$\widetilde{W}_{\text{eff}}(s) = \frac{1}{8\pi\sqrt{s}} \sum_f N_f C_f \frac{\left(s - 4m_1^2\right)\left(s - 4m_f^2\right)^{3/2}}{\left(s - m_\phi^2\right)^2 + m_\phi^2\Gamma_\phi^2}, \tag{C3}$$

where $N_f$ is the number of colors' degrees of freedom of $f$, and $C_f = 2\left(g_{1,\phi}g_{f\phi}y_f\right)^2$.

At this point, we can apply formula (40) to determine $\langle\sigma v_{\text{Møl}}\rangle_{\text{eff}}$ at the first order in $T$:

$$\langle\sigma v_{\text{Møl}}\rangle_{\text{eff}} = \frac{3}{8}\frac{T}{g_1^2 m_1^3} W_1 = \frac{3}{8}\frac{T}{g_1^2 m_1^3}\frac{\mathrm{d}s}{\mathrm{d}y}\frac{\mathrm{d}\widetilde{W}_{\text{eff}}}{\mathrm{d}s}\bigg|_{s=4m_1^2} = \frac{3}{2}\frac{T}{g_1^2 m_1}\frac{\mathrm{d}\widetilde{W}_{\text{eff}}}{\mathrm{d}s}\bigg|_{s=4m_1^2}. \tag{C4}$$

By computing the derivative, we obtain the result:

$$\langle \sigma v_{\rm M\o l} \rangle_{\rm eff} = \frac{3}{4\pi g_1^2 m_1^2} \sum_f N_f C_f \frac{\left(m_1^2 - m_f^2\right)^{3/2}}{\left(4m_1^2 - m_\phi^2\right)^2 + m_\phi^2 \Gamma_\phi^2} T, \tag{C5}$$

which, after replacing $C_f$ and $g_1 = 2$, reproduces the result in [28].

To show an example about the application of formula (65), let us consider a generalisation of the presented model: in particular, let us add a new Dirac fermion $\chi_2$ with a mass $m_2 \gtrsim m_1$, that couples with $\phi$ with a different coupling $g_{2,\phi}$. Then, the variation of lagrangian density with respect to $\Delta\mathcal{L}_1$ is:

$$\Delta\mathcal{L}_2 = \bar{\chi}_2(i\partial\!\!\!/ - m_2)\chi_2 - g_{2,\phi}\phi\bar{\chi}_2\chi_2. \tag{C6}$$

Next, we need to compute $\widetilde{W}_{\rm eff}(s)$ and its derivatives for $s = 4m_1^2$: the only contribution to these quantities comes from the process $\bar{\chi}_1\chi_1 \to \bar{f} f$, which is the only one kinematically allowed at the threshold. Therefore, the expression of $\widetilde{W}_{\rm eff}(s)$ takes the form (C3), and also the expression of its derivative is the same.

Let us write formula (65) at the first order, knowing that $\widetilde{W}_{\rm eff}(s = 4m_1^2) = 0$:

$$\langle \sigma v_{\rm M\o l} \rangle_{\rm eff} = \frac{3T}{8\tilde{\rho} g_1^2 m_1^3} \tilde{\phi}_0 \frac{{\rm d}s}{{\rm d}y} \frac{{\rm d}\widetilde{W}_{\rm eff}}{{\rm d}s}\bigg|_{s=4m_1^2} = \frac{\tilde{\phi}_0}{\tilde{\rho}} \left(\langle \sigma v_{\rm M\o l} \rangle_{\rm eff}\right)_1 \tag{C7}$$

where $(\langle \sigma v_{\rm M\o l} \rangle_{\rm eff})_1$ has been already computed, and it takes the form of (C5). Therefore, we need to compute the ratio $\tilde{\phi}_0/\tilde{\rho}$. By using their definitions, and with the further simplification $c_i = g_i$,[12] we obtain:

$$\frac{\tilde{\phi}_0}{\tilde{\rho}} = \frac{1}{24} + \frac{\Delta x_2}{24}, \tag{C8}$$

where $\Delta x_2 = (m_2 - m_1)/T$.

[12] In fact $m_i/m_1$ is very close to 1.

## References


1. N. Aghanim et al. (Planck), Planck 2018 results. VI. Cosmological parameters, Astron. Astrophys. **641**, A6 (2020), [Erratum: Astron.Astrophys. 652, C4 (2021)]. https://doi.org/10.1051/0004-6361/201833910. arXiv:1807.06209 [astro-ph.CO]
2. E. Kolb, M. Turner, *The early universe (Addison-Wesley)* (Reading, MA, 1990)
3. M. Srednicki, R. Watkins, K.A. Olive, Calculations of Relic Densities in the Early Universe. Nucl. Phys. B **310**, 693 (1988). https://doi.org/10.1016/0550-3213(88)90099-5
4. P. Gondolo, G. Gelmini, Cosmic abundances of stable particles: Improved analysis. Nucl. Phys. B **360**, 145 (1991). https://doi.org/10.1016/0550-3213(91)90438-4
5. P. Gondolo, J. Edsjo, Neutralino relic density including coannihilations. Phys. Atom. Nucl. **61**, 1081 (1998)
6. M. Cannoni, Relativistic <$\sigma v_{\rm rel}$> in the calculation of relics abundances: a closer look. Phys. Rev. D **89**, 103533 (2014). https://doi.org/10.1103/PhysRevD.89.103533. arXiv:1311.4494 [astro-ph.CO]
7. J. March-Russell, J. Unwin, S. M. West, Closing in on Asymmetric Dark Matter I: Model independent limits for interactions with quarks. JHEP **08**, 029. https://doi.org/10.1007/JHEP08(2012)029. arXiv:1203.4854 [hep-ph]
8. M.J. Strassler, K.M. Zurek, Echoes of a hidden valley at hadron colliders. Phys. Lett. B **651**, 374 (2007). https://doi.org/10.1016/j.physletb.2007.06.055. arXiv:hep-ph/0604261
9. J. Alimena et al., Searching for long-lived particles beyond the Standard Model at the Large Hadron Collider. J. Phys. G **47**, 090501 (2020). https://doi.org/10.1088/1361-6471/ab4574. arXiv:1903.04497 [hep-ex]
10. J. Edsjo, P. Gondolo, Neutralino relic density including coannihilations. Phys. Rev. D **56**, 1879 (1997). https://doi.org/10.1103/PhysRevD.56.1879. arXiv:hep-ph/9704361
11. A. Arbey, F. Mahmoudi, Dark matter and the early Universe: a review. Prog. Part. Nucl. Phys. **119**, 103865 (2021). https://doi.org/10.1016/j.ppnp.2021.103865. arXiv:2104.11488 [hep-ph]
12. A. Arbey, F. Mahmoudi, SUSY Constraints, Relic Density, and Very Early Universe. JHEP **05**, 051. https://doi.org/10.1007/JHEP05(2010)051. arXiv:0906.0368 [hep-ph]
13. A. Arbey, J. Ellis, F. Mahmoudi, G. Robbins, Dark Matter Casts Light on the Early Universe. JHEP **10**, 132 (2018). https://doi.org/10.1007/JHEP10(2018)132. arXiv:1807.00554 [hep-ph]
14. A. Arbey, F. Mahmoudi, SuperIso Relic: A Program for calculating relic density and flavor physics observables in Supersymmetry. Comput. Phys. Commun. **181**, 1277 (2010). https://doi.org/10.1016/j.cpc.2010.03.010. arXiv:0906.0369 [hep-ph]
15. A. Arbey, F. Mahmoudi, SuperIso Relic v3.0: A program for calculating relic density and flavour physics observables: Extension to NMSSM. Comput. Phys. Commun. **182**, 1582 (2011). https://doi.org/10.1016/j.cpc.2011.03.019
16. A. Arbey, F. Mahmoudi, G. Robbins, SuperIso Relic v4: A program for calculating dark matter and flavour physics observables in Supersymmetry. Comput. Phys. Commun. **239**, 238 (2019). https://doi.org/10.1016/j.cpc.2019.01.014. arXiv:1806.11489 [hep-ph]
17. G. Belanger, F. Boudjema, A. Pukhov, A. Semenov, MicrOMEGAs: A Program for calculating the relic density in the MSSM. Comput. Phys. Commun. **149**, 103 (2002). https://doi.org/10.1016/S0010-4655(02)00596-9. arXiv:hep-ph/0112278
18. G. Bélanger, F. Boudjema, A. Goudelis, A. Pukhov, B. Zaldivar, micrOMEGAs5.0: Freeze-in. Comput. Phys. Commun. **231**, 173 (2018). https://doi.org/10.1016/j.cpc.2018.04.027. arXiv:1801.03509 [hep-ph]
19. P. Gondolo, J. Edsjo, P. Ullio, L. Bergstrom, M. Schelke, E. A. Baltz, DarkSUSY: Computing supersymmetric dark matter properties numerically. JCAP **07**, 008 (2004). https://doi.org/10.1088/1475-7516/2004/07/008. arXiv:astro-ph/0406204
20. T. Bringmann, J. Edsjö, P. Gondolo, P. Ullio, L. Bergström, DarkSUSY 6: An Advanced Tool to Compute Dark Matter Properties Numerically. JCAP **07**, 033 (2018). https://doi.org/10.1088/1475-7516/2018/07/033. arXiv:1802.03399 [hep-ph]
21. F. Ambrogi, C. Arina, M. Backovic, J. Heisig, F. Maltoni, L. Mantani, O. Mattelaer, G. Mohlabeng, MadDM v.3.0: a Comprehensive Tool for Dark Matter Studies. Phys. Dark Univ.

**24**, 100249 (2019). https://doi.org/10.1016/j.dark.2018.11.009. arXiv:1804.00044 [hep-ph]
22. C. Arina, J. Heisig, F. Maltoni, L. Mantani, D. Massaro, O. Mattelaer, G. Mohlabeng, Studying dark matter with MadDM 3.1: a short user guide. PoS **TOOLS2020**, 009 (2021). https://doi.org/10.22323/1.392.0009. arXiv:2012.09016 [hep-ph]
23. M. Palmiotto, A. Arbey, F. Mahmoudi, DarkPACK: A modular software to compute BSM squared amplitudes for particle physics and dark matter observables. Comput. Phys. Commun. **294**, 108905 (2024). https://doi.org/10.1016/j.cpc.2023.108905. arXiv:2211.10376 [hep-ph]
24. Wolfram Research, Inc., Wolfram mathematica, Computer Software ( 2023), version 13 https://www.wolfram.com/mathematica/
25. A. Djouadi *et al.* ( MSSM Working Group), The Minimal supersymmetric standard model: Group summary report, in *GDR (Groupement De Recherche) - Supersymetrie* (1998). arXiv:hep-ph/9901246
26. M. Abramowitz, I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Vol. 55 ( US Government printing office, 1968)
27. I. Gradshteyn, I. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, 2007)
28. J. Abdallah et al., Simplified Models for Dark Matter Searches at the LHC. Phys. Dark Univ. **9–10**, 8 (2015). https://doi.org/10.1016/j.dark.2015.08.001. arXiv:1506.03116 [hep-ph]