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STRANGE MULTIPOLES IN MAGNET ENDS

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1. Description of End Fields

The three-dimension solution of Laplace's Equation in the end field of a magnet is to be found in any standard text on electron optics[1]. Figure 1 shows a magnet's scalar potential expressed to fourth order in the transverse coordinates x and y. The dash signifies a derivative in the z direction, i.e., along the axis of the magnet.

Each of the coefficients, H, Q, etc., refers to a multipole component. We can rearrange the terms, derive fields and gradients and form Table 1, which relates the terms to the well known multipole magnets of accelerator theory.

There are two lines at the end of this table labelled "End Quadratic and Cubic" which do not correspond to a normal two dimensional multipole component. They are somewhat like sextupole and octupole fields in the x plane, but not in the y plane. In this paper we shall discuss their form and their effect on the beam.

2. Shape of the Strange Multipole

The coefficients H" and Q" are the second derivatives describing the axial variation of the horizontal bending strength and the quadrupole gradient. These second derivatives change sign in the bell-shaped end field of a magnet so that their integral along a paraxial trajectory is zero. They nevertheless have an effect for a particle which is either bent or focused in the end field.

Their transverse distribution is best described by the variation of their focusing strength, $k = \partial By/\partial x$ or $\partial Bx/\partial y$, as a function of x and y. Ideally a quadrupole's focusing strength would be constant over the whole x,y plane. An octupole:

1) Glaser - Handbuch der Physik 1956 p. 162.

$$k = 12Q_1(y^2 - x^2)$$

produces a perturbation of k as shown in Fig. 2. Imagine the magnet is standing on end like an umbrella stand. The x and y coordinates in the Figure are the normal transverse coordinates. The vertical coordinate is k as a function of x and y. The surface is a saddle. In one direction focusing is depressed as one moves off axis, in the other, it is enhanced.

The "end cubic" focusing strength:

$$k = \frac{Q''}{4}(x^2 + y^2)$$

is shown in Fig. 3. Clearly it is not an octupole, and resembles a hammock and enhances focusing in a way which is always positive.

By overlaying Figs 2 and 3 one can immediately see that an octupole could correct for the end field in one plane but would make things worse in the other. This may still be a useful thing to do where the dispersed beam is very wide and where end fields have their maximum effect.

3. The Effect on the Beam

The effect on the beam of such a cubic end field has been calculated by Wüsterfeld[2].

He takes into account that beta changes in the fringe field:

$$\beta = \beta_0 - 2d_0\Delta z + \frac{(1 + \alpha^2)}{\beta_0}\Delta z^2$$

to find the integral of the effect on focusing:

$$\Delta Q = \frac{1}{2\pi}\int \frac{k}{B\rho} B ds = \frac{\varepsilon k}{32\pi B\rho}[\alpha_0\beta_0] \quad .$$

His treatment does not include dispersion and the effect appears small:

$$\Delta Q = 1.46 \times 10^{-4} \text{ for AA,}$$
$$\Delta Q = 1.12 \times 10^{-3} \text{ for AC.}$$

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2) G, Wüsterfeld - Argonne/Jülich AAD-N-26.

We have extended the theory to include dispersion and find that the effect of a pseudo-octupole, where there is dispersion, will be much larger. The normal expressions for the effect of an octupole give an increased effect in dispersive regions of a factor:

$$1 + 4(d^2/a^2)$$

where $d = \alpha p \; \Delta p/p$ and $a = \sqrt{\beta\varepsilon}$.

This leads to larger values for ΔQ:

$$\Delta Q = 0.0127 \text{ for the AA,}$$
$$\Delta Q = 0.008 \text{ for the AC.}$$

This would appear as a curvature to Q v. $\Delta p/p$. Such a curvature would make itself apparent - and have to be corrected by octupole shims - even after all precautions had been taken to eliminate field errors in the paraxial distributions of magnet fields. This is exactly what happened in the case of the AA. See Fig. 4[3].

Even when compensated, they might still drive fourth order stopbands due to the lumped nature of the corrections. Such stopbands would be systematic and appear at all even multiples of 0.25 in the Q diagram. The AA working point is remote from such lines **but the AC is within 0.004 of 4Q = 22.**

Acknowledgement

I would like to thank V. Chohan and H. Renshall for help with the three dimensional plotting.

3) B. Autin et al., CERN/PS/AA/81-20 and 1981 PAC.

NAME	φ_m	B_y	B_x	dB_y/dx	dB_y/dy	dB_x/dx	dB_x/dy
SOLENOID	$\Phi_m - \frac{1}{4}\Phi'' r^2 + \frac{1}{64}\Phi^{(iv)} r^4$	$-\frac{1}{2}\Phi'' y + \frac{\Phi^{(iv)}}{16}(x^2y+y^3)$	$-\frac{1}{2}\Phi'' x + \frac{\Phi^{(iv)}}{16}(x^3+xy^2)$	$\frac{\Phi^{(iv)}}{8} \cdot xy$	$-\frac{1}{2}\Phi'' + \frac{\Phi^{(iv)}}{16}(x^2+3y^2)$	$-\frac{1}{2}\Phi'' + \frac{\Phi^{(iv)}}{16}(3x^2+y^2)$	$\frac{\Phi^{(iv)}}{8} xy$
HORIZONTAL BENDING	$-Hy$	H	0	0	0	0	0
VERTICAL BENDING	$-Gx$	0	G	0	0	0	0
QUADRUPOLE	Qxy	$-Qx$	$-Qy$	$-Q$	0	$-Q$	0
SKEW-QUADRUPOLE	$\frac{1}{4}\Delta(x^2-y^2)$	$-\frac{\Delta}{2}y$	$\frac{\Delta}{2}x$	0	$-\Delta/2$	$\Delta/2$	0
SEXTUPOLE	$\frac{1}{3}H_1(y^3-2x^2y)$	$H_1(y^2-x^2)$	$-2H_1xy$	$-2H_1x$	$2H_1y$	$-2H_1y$	$-2H_1x$
SKEW SEXTUPOLE	$\frac{1}{3}G_1(x^3-3xy)$	$\frac{2}{3}G_1xy$	$G_1(x^2-y^2)$	$\frac{2}{3}G_1y$	$\frac{2}{3}G_1x$	$2G_1x$	$-2G_1y$
OCTUPOLE	$4Q_1(xy^3-x^3y)$	$4Q_1(3xy^2-x^3)$	$4Q_1(y^3-3x^2y)$	$12Q_1(y^2-x^2)$	$24Q_1xy$	$-24Q_1xy$	$12Q_1(y^2-x^2)$
SKEW OCTUPOLE	$\Delta_1(x^4-6x^2y^2+y^4)$	$\Delta_1(4y^3-12x^2y)$	$\Delta_1(4x^3-12xy^2)$	$-24\Delta_1xy$	$\Delta_1 \times 12(y^2-x^2)$	$12\Delta(x^2-y^2)$	$-24\Delta_1xy$
END QUADRATIC	$\frac{1}{6}H''y^3$	$\frac{H''}{2}y^2$	0	0	$H''y$	0	0
END CUBIC	$-\frac{Q''}{12}(x^2+y^2)xy$	$-\frac{Q''}{12}(x^3+3x^2y)$	$-\frac{Q''}{12}(3x^2y+y^3)$	$-\frac{Q''}{4}(x^2+y^2)$	$-\frac{Q''}{2}xy$	$-\frac{Q''}{2}xy$	$-\frac{Q''}{4}(x^2+y^2)$

Table 1 Expansion of Magnet Ends up to Order 4

EJNW 13.2.84.

NAME	φ_m	By	Bx	dBy/dx	dBy/dy	dBx/dx	dBx/dy
SOLENOID	$\Phi_m - \frac{1}{4}\Phi'' r^2 + \frac{1}{64}\Phi^{(iv)} r^4$	$-\frac{1}{2}\Phi'' y + \Phi^{(iv)}/16\,(x^2y+y^3)$	$-\frac{1}{2}\Phi'' x + \frac{\Phi^{(iv)}}{16}(x^3+xy^2)$	$\frac{\Phi^{(iv)}}{8}\cdot xy$	$-\frac{1}{2}\Phi'' + \frac{\Phi^{(iv)}}{16}(x^2+3y^2)$	$-\frac{1}{2}\Phi'' + \frac{\Phi^{(iv)}}{16}(3x^2+y^2)$	$\frac{\Phi^{(iv)}}{8}xy.$
HORIZONTAL BENDING	$-Hy$	H	0	0	0	0	0
VERTICAL BENDING	$-Gx$	0	G	0	0	0	0
QUADRUPOLE	Qxy	$-Qx$	$-Qy$	$-Q$	0	$-Q$	0
SKEW-QUADRUPOLE	$\frac{1}{4}\Delta(x^2-y^2)$	$-\frac{\Delta}{2}y$	$\frac{\Delta}{2}x$	0	$-\Delta/2$	$\Delta/2$	0
SEXTUPOLE	$\frac{1}{3}H_1(y^3-2x^2y)$	$H_1(y^2-x^2)$	$-2H_1xy$	$-2H_1x$	$2H_1y$	$-2H_1y,$	$-2H_1x$
SKEW SEXTUPOLE	$\frac{1}{3}G_1(x^3-3xy)$	$\frac{2}{3}G_1xy$	$G_1(x^2-y^2)$	$\frac{2}{3}G_1y$	$\frac{2}{3}G_1x$	$2G_1x$	$-2G_1y$
OCTUPOLE	$4Q_1(xy^3-x^3y)$	$4Q_1(3xy^2-x^3)$	$4Q_1(y^3-3x^2y)$	$12Q_1(y^2-x^2)$	$24Q_1xy$	$-24Q_1xy$	$12Q_1(y^2-x^2)$
SKEW OCTUPOLE	$\Delta_1(x^4-6x^2y^2+y^4)$	$\Delta_1(4y^3-12x^2y)$	$\Delta_1(4x^3-12xy^2)$	$-24\Delta_1xy$	$\Delta_1\times 12(y^2-x^2)$	$12\Delta(x^2-y^2)$	$-24\Delta_1xy$
END QUADRATIC	$\frac{1}{6}H''y^3$	$\frac{H''}{2}y^2$	0	0	$H''y$	0	0
END CUBIC	$-\frac{\varphi''}{12}(x^2+y^2)xy$	$-\frac{\varphi''}{12}(x^3+3x^2y)$	$-\frac{\varphi''}{12}(3x^2y+y^3)$	$-\frac{\varphi''}{4}(x^2+y^2)$	$-\frac{\varphi''}{2}xy$	$-\frac{\varphi''}{2}xy$	$-\frac{\varphi''}{4}(x^2+y^2)$

Table 1 Expansion of Magnet Ends up to Order 4

EJNW. 13.2.84.

$$
\begin{aligned}
\varphi_m(x,y,z) = {} & \Phi_m - Gx - Hy - \tfrac{1}{4}(\Phi''_m - \Delta)\,x^2 + Q\,x\,y - \tfrac{1}{4}(\Phi''_m + \Delta)\,y^2 + \\
& + \tfrac{1}{3}\left(\tfrac{1}{2}G'' + G_1\right)x^3 - H_1\,x^2 y - G_1\,x\,y^2 + \tfrac{1}{3}\left(\tfrac{1}{2}H'' + H_1\right)y^3 + \\
& + \left(\tfrac{1}{64}\Phi_m^{(4)} - \tfrac{1}{48}\Delta'' + \Delta_1\right)x^4 - \left(\tfrac{1}{12}Q'' - 4Q_1\right)x^3 y + \\
& + \left(\tfrac{1}{32}\Phi_m^{(4)} - 6\Delta_1\right)x^2 y^2 - \left(\tfrac{1}{12}Q'' + 4Q_1\right)x\,y^3 + \\
& + \left(\tfrac{1}{64}\Phi^{(4)} + \tfrac{1}{48}\Delta'' + \Delta_1\right)y^4 .
\end{aligned}
$$

Fig.1 Scalar Magnetic Potential

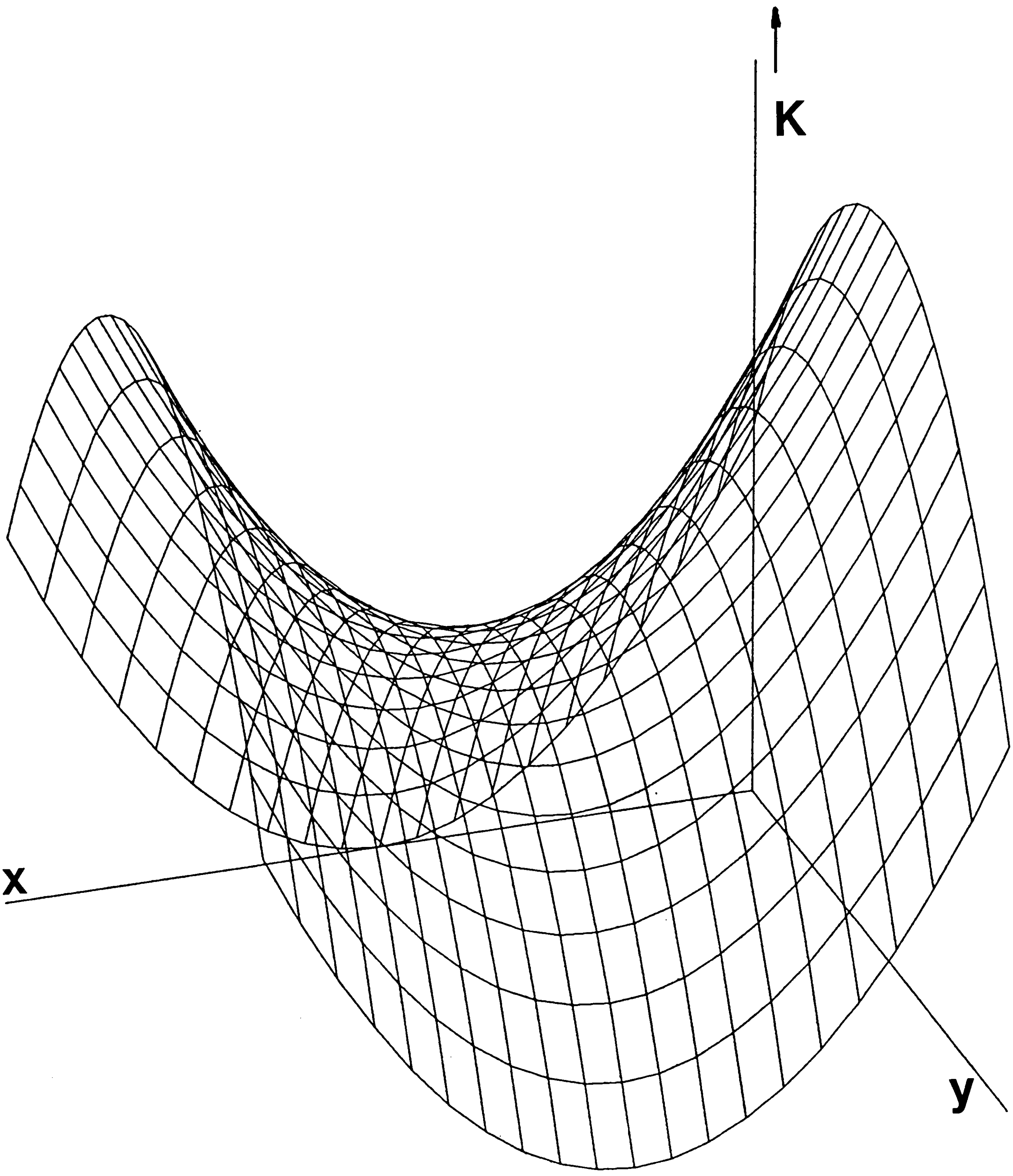

Fig. 2 - Octupole Gradient

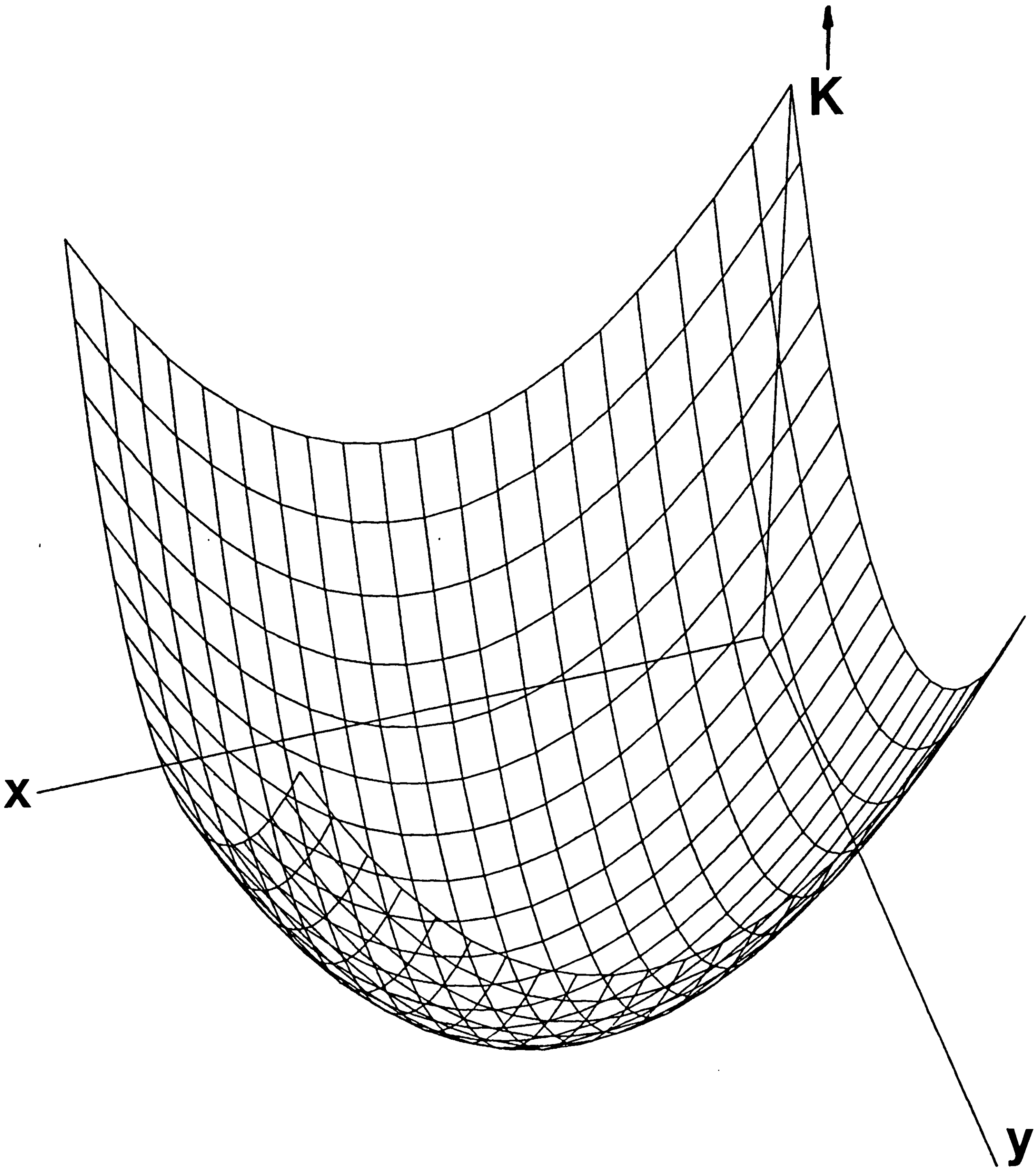

Fig. 3 - Gradient of End Cubic Potential

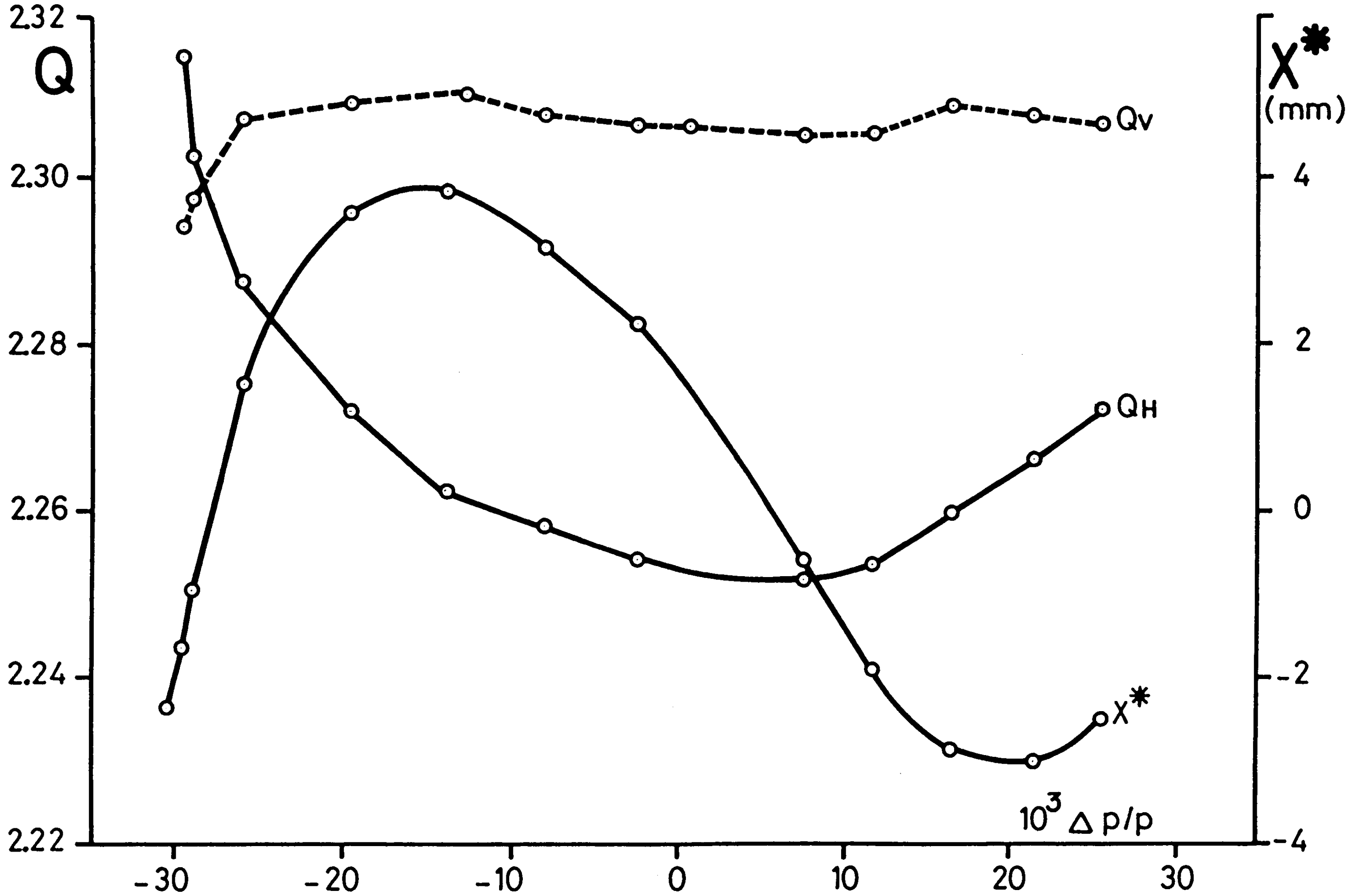

Fig.4 _ Q Variation at Start