JB/tk

PS/LEA/Note 87-3
February 1987

SPECTRAL ANALYSIS OF THE MOTION AT A SINGLE NONLINEAR RESONANCE BY CANONICAL PERTURBATION THEORY

Johan Bengtsson

I) INTRODUCTION

In an earlier paper[1] we described a method to measure the amplitude and phase of a single nonlinear resonance by doing spectral analysis of the signals from a horizontal- and vertical pick-up. We are here going to develop this theory further, in particular to find a way to get the amplitude and phase of a resonance by only doing spectral analysis of the x and y coordinates, whereas before it was also necessary to do spectral analysis of x^2 and y^2. We also obtain the different frequencies related to a resonance. This knowledge may be used for particle tracking followed by spectral analysis of the coordinates to deduce which resonances the particle feels in it's motion.

We want to point out that in the calculations we use the definition of Ando[2] for the amplitude of a resonance. This one differs from the one used by Guignard[3] by a factor $R^{(N-2)/2}$ where R is the mean radius of the machine and N the order of the resonance.

1. CANONICAL PERTURBATION THEORY (Time independent)

The Hamiltonian is divided in two parts :

$$H = H_0 + \varepsilon H_1 \tag{1}$$

where H_1 is a perturbation to the unperturbed Hamiltonian and ε is a parameter assumed to be small. The idea is to find a canonical transformation where the new action-variable is a constant of motion. This is normally possible only for some order in ε so that the new action-variable may depend on the angle and time in higher order.

We will follow the procedure given by Ruth[4] to study a single isolated resonance. The Hamiltonian is in the case of a single resonance $n_x Q_x + n_y Q_y = p$:

$$H(\phi_x, \phi_y, J_x, J_y; \theta) = H_0 + H_1 = \nu_x J_x + \nu_y J_y + 2|\kappa| J_x^{k_x/2} J_y^{k_y/2} \cos\psi \tag{2}$$

where :

$$H_0 = \nu_x J_x + \nu_y J_y \quad , \quad k_x = j + k \quad , \quad k_y = l + m$$

$$\psi = n_x\phi_x + n_y\phi_y + \gamma - p\theta \quad , \quad n_x = j - k \quad , \quad n_y = l - m \tag{3}$$

$$N = j + k + l + m \text{ , order of the resonance}$$

and

- $|\kappa|$ amplitude of the resonance
- γ phase of the resonance

or :

$$\kappa = |\kappa| e^{i\gamma} = 2^{-N/2} \binom{j+k}{k} \binom{l+m}{m} \frac{1}{2\pi} \int_0^{2\pi R} \beta_x^{(j+k)/2} \beta_y^{(l+m)/2} V_{j+k,l+m} \, e^{i[(j-k)W_x + (l-m)W_y + p\frac{s}{R}]} ds$$

$$R_e \sum_{n\geq 1} \frac{1}{n!} (B^{(n)} + iC^{(n)}) (x + iy)^n = \sum_{m,n\geq 0} V_{mn} x^m y^n$$

$$B^{(n)} = \frac{1}{(B\varrho)_0} \left[\frac{\partial^{n-1}\beta_y}{\partial x^{n-1}}\right]_{x=y=0} \quad , \quad C^{(n)} = \frac{1}{(B\varrho)_0} \left[\frac{\partial^{n-1}\beta_x}{\partial x^{n-1}}\right]_{x=y=0}$$

The action-angle-variables are related to coordinates and momenta by :

$$z = \sqrt{2J_z \beta_z} \; \cos(\phi_z + W_z) \tag{4}$$

$$p_z = -\sqrt{\frac{2J_z}{\beta_z}} \left[\sin(\phi_z + W_z) - \frac{\beta'_z}{2} \cos(\phi_z + W_z)\right] \qquad , \; z = x \text{ or } y \tag{5}$$

where :

$$W_z = \int_0^\theta \frac{R}{\beta(\theta')} \, d\theta' - \nu_z \theta = \mu_z(\theta) - \nu_z \theta \tag{6}$$

The generating function for the canonical transformation is :

$$G = \phi_x J_{1x} + \phi_y J_{1y} - 2|\kappa| \frac{J_{1x}^{k_x/2} \; J_{1y}^{k_y/2}}{e} \sin \psi \tag{7}$$

and the new variables are given by :

$$\begin{aligned}
\phi_{1x} &= \frac{\partial G}{\partial J_{1x}} = \phi_x - \frac{k_x|\kappa|}{e} \; J_{1x}^{(k_x-2)/2} \; J_{1y}^{k_y/2} \; \sin \psi \\
\phi_{1y} &= \frac{\partial G}{\partial J_{1y}} = \phi_y - \frac{k_y|\kappa|}{e} \; J_{1x}^{k_x/2} \; J_{1y}^{(k_y-2)/2} \; \sin \psi \\
J_x &= \frac{\partial G}{\partial \phi_x} = J_{1x} - 2 \; \frac{n_x|\kappa|}{e} \; J_{1x}^{k_x/2} \; J_{1y}^{k_y/2} \; \cos \psi \\
J_y &= \frac{\partial G}{\partial \phi_x} = J_{1y} - 2 \; \frac{n_y|\kappa|}{e} \; J_{1x}^{k_x/2} \; J_{1y}^{k_y/2} \; \cos \psi
\end{aligned} \tag{8}$$

The new Hamiltonian is :

$$\begin{aligned}
K &= H_0(J_{1x}, J_{1y}) + \frac{\partial H_1}{\partial J_{1x}} \frac{\partial G}{\partial \phi_x} + \frac{\partial H_1}{\partial J_{1y}} \frac{\partial G}{\partial \phi_y} + \frac{1}{2} \frac{\partial H_0}{\partial J_{1x}} \left[\frac{\partial G}{\partial \phi_x}\right]^2 + \frac{1}{2} \frac{\partial H_0}{\partial J_{1y}} \left[\frac{\partial G}{\partial \phi_y}\right]^2 \\
&= \nu_x J_{1x} + \nu_y J_{1y} - \frac{|\kappa|^2}{e} J_{1x}^{k_x-1} \; J_{1y}^{k_y-1} \; (n_x k_x J_{1y} + n_y k_y J_{1x})(1 + \cos 2\psi)
\end{aligned} \tag{9}$$

The oscillating term cos 2ψ has zero average and may therefore by neglected. The Hamiltonian is then only a function of the new action-variables. Observe that the perturbation in the new Hamiltonian is of second order in the amplitude of the resonance so that the new action-variable is now a constant of motion to second order.

The new Hamiltonian is (neglecting the oscillating part) :

$$K = \nu_x J_{1x} + n_y J_{1y} - \frac{|\kappa|^2}{e} J_{1x}^{k_x - 1} J_{1y}^{k_y - 1} (n_x k_x J_{1y} + n_y k_y J_{1x}) \tag{10}$$

new frequencies are now given by :

$$\nu_{1x} = \phi'_{1x} = \frac{\partial K}{\partial J_{1x}} = \nu_x - \frac{|\kappa|^2}{e} J_{1x}^{k_x - 2} J_{1y}^{k_y - 2} \left[n_x k_x (k_x - 1) J_{1y}^{\,2} + n_y k_x k_y J_{1x} J_{1y} \right]$$

$$\nu_{1y} = \phi'_{1y} = \frac{\partial K}{\partial J_{1y}} = \nu_y - \frac{|\kappa|^2}{e} J_{1x}^{k_x - 2} J_{1y}^{k_y - 2} \left[n_x k_x k_y J_{1x} J_{1y} + n_y k_y (k_y - 1) J_{1x}^{\,2} \right] \tag{11}$$

or :

$$\nu_{1x} = \nu_x + \Delta\nu_x$$

$$\nu_{1y} = \nu_y + \Delta\nu_y \tag{12}$$

where :

$$\boxed{\begin{aligned} \Delta\nu_x &= - \frac{|\kappa|^2}{e} J_{1x}^{k_x - 2} J_{1y}^{k_y - 2} \left[n_x k_x (k_x - 1) J_{1y}^{\,2} + n_y k_x k_y J_{1x} J_{1y} \right] \\ \Delta\nu_y &= - \frac{|\kappa|^2}{e} J_{1x}^{k_x - 2} J_{1y}^{k_y - 2} \left[n_x k_x k_y J_{1x} J_{1y} + n_y k_y (k_y - 1) J_{1x}^{\,2} \right] \end{aligned}} \tag{13}$$

The equation (11) may be integrated :

$$\phi_{1x} = (\nu_x + \Delta\nu_x)\, \theta + \phi_{1x}(0)$$

$$\phi_{1y} = (\nu_y + \Delta\nu_y)\, \theta + \phi_{1y}(0) \tag{14}$$

where the constants $\phi_{1x}(0)$, $\phi_{1y}(0)$ are given by (8) :

$$\phi_{1x}(0) = \phi_x(0) - \frac{k_x |\kappa|}{e} J_{1x}^{(k_x - 2)/2} J_{1y}^{k_y/2} \sin \psi(0)$$

$$\phi_{1y}(0) = \phi_x(0) - \frac{k_y |\kappa|}{e} J_{1x}^{k_x/2} J_{1y}^{(k_y - 2)/2} \sin \psi(0) \tag{15}$$

where :

$$\psi\ (0) = n_x\ \phi_x\ (0) + n_y\ \phi_y\ (0) + \gamma \tag{16}$$

2. THE MOTION OF A PARTICLE

The motion is given by (4), (5). If we use the new action-variable which is a constant of motion to second order of $|\kappa|$ then by (8) we get :

$$\begin{aligned} x &= \sqrt{2\beta_x\ J_{1x}}\ \sqrt{1 - 2\ \frac{n_x|\kappa|}{e}\ J_{1x}^{\ (k_x-2)/2}\ J_{1x}^{\ k_y/2}\ \cos\psi}\ .\ \cos\ (\phi_x + W_x) \\ y &= \sqrt{2\beta_y\ J_{1y}}\ \sqrt{1 - 2\ \frac{n_y|\kappa|}{e}\ J_{1x}^{\ k_x/2}\ J_{1y}^{\ (k_y-2)/2}\ \cos\psi}\ .\ \cos\ (\phi_y + W_y) \end{aligned} \tag{17}$$

If we expand the square root :

$$\begin{aligned} x &= 2\beta_x J_{1x}\ (1 - n_x\Delta_x\ \cos\psi - \frac{1}{2}\, n_x^{\ 2}\ \Delta_x^{\ 2}\ \cos^2\psi)\ \cos\ (\phi_x + W_x) \\ y &= 2\beta_y J_{1y}\ (1 - n_y\Delta_y\ \cos\psi - \frac{1}{2}\, n_y^{\ 2}\ \Delta_y^{\ 2}\ \cos^2\psi)\ \cos\ (\phi_y + W_y) \end{aligned} \tag{18}$$

where :

$$\boxed{\begin{aligned} \Delta_x &= \frac{|\kappa|}{e}\ J_{1x}^{\ (k_x-2)/2}\ J_{1y}^{\ k_y/2} \\ \Delta_y &= \frac{|\kappa|}{e}\ J_{1x}^{\ k_x/2}\ J_{1y}^{\ (k_y-2)/2} \end{aligned}} \tag{19}$$

by using trigonometric relations this may be written :

$$\begin{aligned} x = \sqrt{2\beta_x J_{1x}}\ \Big[&(1 - \frac{1}{4}\, n_x^{\ 2}\Delta_x^{\ 2})\ \cos\ (\phi_x + W_x) - \frac{1}{2}\, n_x\Delta_x\ \cos\ (\psi + \phi_x + W_x) + \\ &- \frac{1}{2}\, n_x\Delta_x \cos(\psi - \phi_x - W_x) - \frac{1}{8}\, n_x^{\ 2}\Delta_x^{\ 2} \cos(2\psi + \phi_x + W_x) - \frac{1}{8}\, n_x^{\ 2}\Delta_x^{\ 2} \cos(2\psi - \phi_x - W_x) \Big] \\ y = \sqrt{2\beta_y J_{1y}}\ \Big[&(1 - \frac{1}{4}\, n_y^{\ 2}\Delta_y^{\ 2})\ \cos\ (\phi_y + W_y) - \frac{1}{2}\, n_y\Delta_y\ \cos\ (\psi + \phi_y + W_y) + \\ &- \frac{1}{2}\, n_y\Delta_y \cos(\psi - \phi_y - W_y) - \frac{1}{8}\, n_y^{\ 2}\Delta_y^{\ 2} \cos(2\psi + \phi_y + W_y) - \frac{1}{8}\, n_y^{\ 2}\Delta_y^{\ 2} \cos(2\psi - \phi_y - W_y) \Big] \end{aligned} \tag{20}$$

The phases are given by (8)

$$\phi_{1x} = \phi_x - \frac{k_x|\kappa|}{e} J_{1x}^{(k_x-2)/2} J_{1y}^{k_y/2} \sin\psi$$

$$\phi_{1y} = \phi_y - \frac{k_y|\kappa|}{e} J_{1x}^{k_x/2} J_{1y}^{(k_y-2)/2} \sin\psi$$

The equations have to be inverted. Since this can not be done analytically, we do it by succesive approximations :

$$\phi_x^{(1)} = \phi_{1x}, \qquad \phi_y^{(1)} = \phi_{1y} \tag{21}$$

$$\phi_x^{(2)} = \phi_{1x} + \frac{k_x|\kappa|}{e} J_{1x}^{(k_x-2)/2} J_{1y}^{k_y/2} \sin\psi_1$$

$$\phi_y^{(2)} = \phi_{1y} + \frac{k_y|\kappa|}{e} J_{1x}^{k_x/2} J_{1y}^{(k_y-2)/2} \sin\psi_1 \tag{22}$$

where :

$$\psi_1 = n_x\,\phi_{1x} + n_y\,\phi_{1y} + \gamma - p\theta \tag{23}$$

or by using (19) :

$$\phi_x^{(2)} = \phi_{1x} + k_x\,\Delta_x\, \sin\psi_1$$

$$\phi_y^{(2)} = \phi_{1y} + k_y\,\Delta_y\, \sin\psi_1 \tag{24}$$

We use (24) for the phases and keep only first order terms in Δ_x and Δ_y. Then (20) gives :

$$x = \sqrt{2\beta_x J_{1x}} \left[\cos(\phi_{1x} + W_x) - \frac{1}{2}(k_x + n_x)\Delta_x \cos(\psi_1 - \phi_{1x} - W_x) + \frac{1}{2}(k_x - n_x)\Delta_x \cos(\psi_1 + \phi_{1x} + W_x)\right]$$

$$y = \sqrt{2\beta_y J_{1y}} \left[\cos(\phi_{1y} + W_y) - \frac{1}{2}(k_y + n_y)\Delta_y \cos(\psi_1 - \phi_{1y} - W_y) + \frac{1}{2}(k_y - n_y)\Delta_y \cos(\psi_1 + \phi_{1y} + W_y)\right] \tag{25}$$

Where we have used :

$$\cos\left[(k_x \pm 1)\,\Delta_x + k_y\,\Delta_y\,\sin\psi_1\right] \approx 1 + 0\,(\Delta^2)$$
$$\sin\left[(k_x \pm 1)\,\Delta_x + k_y\,\Delta_y\,\sin\psi_1\right] \approx (k_x \pm 1)\Delta_x + k_y\Delta_y\,\sin\psi_1 + 0(\Delta^3)$$

and similar for the vertical plane. If higher order terms in Δ_x and Δ_y are included we get terms like :

$\cos\,(2\psi_1 \pm \phi_{1x} \pm W_x)$ in the horizontal plane

$\cos\,(2\psi_1 \pm \phi_{1y} \pm W_y)$ in the vertical plane

The phases for the frequencies in (25) are given by (15). We find the following frequencies :

FREQUENCIES	AMPLITUDE	PHASE
$\mu_x' + \Delta\nu_x$	$\sqrt{2\beta_x J_{1x}}$	$\phi_{1x}(0)$
$(n_x-1)(\mu_x'+\Delta\nu_x) + n_y(\mu_y'+\Delta\nu_y)-p$	$\sqrt{2\beta_x J_{1x}}\cdot\frac{1}{2}\,(k_x+n_x)\Delta_x$	$(n_x-1)\phi_{1x}(0)+n_y\phi_{1y}(0)+\gamma$
$(n_x+1)(\mu_x'+\Delta\nu_x) + n_y(\mu_y'+\Delta\nu_y)-p$	$\sqrt{2\beta_x J_{1x}}\cdot\frac{1}{2}\,(k_x-n_x)\Delta_x$	$(n_x+1)\phi_{1x}(0)+n_y\phi_{1y}(0)+\gamma$
$\mu_y' + \Delta\nu_y$	$\sqrt{2\beta_y J_{1y}}$	$\phi_{1y}(0)$
$n_x(\mu_x'+\Delta\nu_x)+(n_y-1)\,(\mu_y'+\Delta\nu_y)-p$	$\sqrt{2\beta_y J_{1y}}\cdot\frac{1}{2}\,(k_y+n_y)\Delta_y$	$n_x\phi_{1x}(0)+(n_y-1)\phi_{1y}+(0)\gamma$
$n_x(\mu_x'+\Delta\nu_x)+(n_y+1)\,(\mu_y'+\Delta\nu_y)-p$	$\sqrt{2\beta_y J_{1y}}\cdot\frac{1}{2}\,(k_y-n_y)\Delta_y$	$n_x\phi_{1x}(0)+(n_y+1)\phi_{1y}+(0)\gamma$

We conclude that in first order of Δ_x and Δ_y we will find new frequencies in each plane due to the resonance which is :

FREQUENCIES	AMPLITUDE	PHASE
$(n_x \pm 1)(\mu_x'+\Delta\nu_x) + n_y(\mu_y'+\Delta\nu_y)-p$	$\sqrt{2\beta_x J_{1x}}\cdot\frac{1}{2}\,(k_x \mp n_x)\Delta_x$	$(n_x \pm 1)\phi_{1x}(0)+n_y\phi_{1y}(0)+\gamma$
$n_x(\mu_x'+\Delta\nu_x) + (n_y \pm 1)(\mu_y'+\Delta\nu_y)-p$	$\sqrt{2\beta_y J_{1y}}\cdot\frac{1}{2}\,(k_y \mp n_y)\Delta_y$	$n_x\phi_{1x}(0)+(n_y \pm 1)\phi_{1y}(0)+\gamma$

The amplitude $|\kappa|$ of the resonance may then be calculated by the following, where δ is the amplitude of the perturbed betatron-frequency divided with the amplitude of the new frequency :

$$\delta_z = \frac{1}{\frac{1}{2}(k_z \pm n_z)\Delta_z} , \quad z = x \text{ or } y$$

From (19) we get :

$$\boxed{|\kappa| = \frac{e}{\frac{1}{2}(k_x \pm n_x)\delta_x J_{1x}^{(k_x-2)/2} J_{1y}^{k_y/2}} = \frac{e}{\frac{1}{2}(k_y \pm n_y)\,\delta_y J_{1x}^{k_x/2} J_{1y}^{(k_y-2)/2}}} \qquad (26)$$

Where the first equation is valid in the horizontal plane, and the second in the vertical plane.

The phase may be calculated from :

$$\boxed{\delta = \arg \delta_x - (n_x \pm 1)\phi_{1x}(0) - n_y \phi_{1y}(0) = \arg \delta_y - n_x \phi_{1x}(0) - (n_y \pm 1)\phi_{1y}(0)} \qquad (27)$$

Where the sign to be used depends on which frequency component one is looking at.

$\phi_{1x}(0)$ and $\phi_{1y}(0)$ are the phases of the perturbed betatron frequencies.

Observe that Δ_x and Δ_y in (25) are positive under the following conditions from (19).

$$e > 0 \longrightarrow \Delta_x > 0$$
$$e > 0 \longrightarrow \Delta_y > 0 \qquad (28)$$

So that π should be substracted from the phase if the term in (25) is negative.

3. SIMULATION

We simulate the motion of a particle close to a single resonance by numerical integration of the equations of motion for the perturbed motion[3]. The unperturbed motion is added with the assumption that β is constant. We then use FFT[5] for the frequency analysis. Note that if one is observing the motion at one point in the machine the phase advance μ should be replaced by the averaged frequency.

In appendix A we give examples of the following resonances where the simulation has been done with measured values for the amplitude in LEAR by the method described in (3) and $k_x = |n_x|$, $k_y = |n_y|$.

$$Q_H + Q_V = 5$$

$$3Q_H = 7$$

$$Q_H + 2Q_V = 8$$

$$Q_H - 2Q_V = -3$$

We calculate the amplitude $|\kappa|$ and the phase γ of the resonance. We also calculate the phase $\phi_{1x}(0)$, $\phi_{1y}(0)$ and amplitude of the perturbed betatron motion from (15) to compare with the measured.

In the calculation for $|\kappa|$ we calculate an approximation to e by the measured perturbed frequencies. We also use measured values for J_{1x} and J_{1y} which may be calculated from the amplitude of the perturbed betatron frequency which is :

$$\sqrt{2\beta_z J_{1z}} \quad (z = x \text{ or } y)$$

In the simulation we have used :

$$\beta_z = 1 \quad (z = x \text{ or } y)$$

Note that if the frequency is bigger than 0.5 the sign of the phase should be changed.

π should be substracted depending on the condition in (28). The sign of the phase should also be changed if we have a "negative frequency" like in the case $Q_H - 2Q_V = -3$ where we get a peak for $-2Q_V$ in the horizontal plane.

The simulation also shows that in the case of linear coupling one will still excite the resonance in the case when the action-variable has the value zero at the beginning (this is not true in the general case).

In the plots we use the following parameters :

$$FIK = \gamma$$

$$EH = 2J_x$$

$$EV = 2J_y$$

$$Q_z = \frac{1}{2\pi} \int_0^{2\pi} \frac{R}{\beta(\theta)'} \, d\theta' \quad (z = x \text{ or } y)$$

Observe that $|\kappa|$ in the plots are given by Guignard's definition[3] which differs from Ando's[2] by a factor $R^{(N-2)/2}$

We have put measured or given values in brackets to compare with.

The simulation shows that one obtains a quite good value of the amplitude and the phase of the resonance if only the amplitude of the new frequencies are big enough.

4. PARTICLE TRACKING

Appendix B gives some examples of particle tracking in LEAR for the working point at extraction. There is one case with a small value for the action-variable in the horizontal plane and big value in the vertical plane and another case with the contrary. We find a lot of frequencies in the spectra due to resonances excited by the sextupoles.

The following frequencies are found :

Horizontal plane :

$$Q_H,\ 2Q_H,\ 2Q_V,\ 4Q_V,\ Q_H + 4Q_V,\ Q_H - 2Q_V,\ Q_H - 4Q_V,\ 6Q_V,\ 4Q_H,\ 3Q_H - 2Q_V$$

Vertical plane :

$$Q_V,\ Q_H + Q_V,\ Q_H - Q_V,\ 3Q_V,\ 3Q_H + Q_V,\ 2Q_H - Q_V,\ 4Q_H + Q_V,\ Q_H - 3Q_V,\ 3Q_H - Q_V,\ 5Q_V$$

From the chapter 2. we know that a single resonance $n_H Q_H + n_V Q_V = p$ gives the following frequencies.

$$(n_H \pm 1)\ (\mu_H' + \Delta v_H) + n_V(\mu_V' + \Delta v_V)$$

$$n_x(\mu_H' + \Delta v_H) + (n_V \pm 1)\ (\mu_V' + \Delta v_V)$$

The observed frequencies could then be caused by the following resonances :

$Q_H + 2Q_V = 8$ N = 3 sextupole

$Q_H - 2Q_V = -3$

$3Q_H = 7$

$4Q_V = 11$ N = 4 Sextupole, 2nd order

$2Q_H - 2Q_V = -1$

The other frequencies are probably from the neglected terms in chapter 2, caused by the same resonances.

5. A WAY TO MEASURE k_x AND k_y FROM TUNE SHIFTS DEPENDENCE ON AMPLITUDE

The tune shifts are given by [13] :

$$\Delta\nu_x = -\frac{|\kappa|^2}{e} J_{1x}^{k_x-2} J_{1y}^{k_y-2} \left[n_x k_x(k_x-1) J_{1y}^{2} + n_y k_x k_y J_{1x} J_{1y}\right]$$

$$\Delta\nu_y = -\frac{|\kappa|^2}{e} J_{1x}^{k_x-2} J_{1y}^{k_y-2} \left[n_x k_x k_y J_{1x} J_{1y} + n_y k_y(k_y-1) J_{1x}^{2}\right]$$

or :

$$\Delta\nu_x = -\frac{|\kappa|^2}{e} \left[n_x k_x(k_x-1) J_{1x}^{k_x-2} J_{1y}^{k_y} + n_y k_x k_y J_{1x}^{k_x-1} J_{1y}^{k_y-1}\right]$$

$$\Delta\nu_y = -\frac{|\kappa|^2}{e} \left[n_y k_y(k_y-1) J_{1x}^{k_x} J_{1y}^{k_y-2} + n_x k_x k_y J_{1x}^{k_x-1} J_{1y}^{k_y-1}\right] \tag{30}$$

If we interpret $\Delta\nu_x$ and $\Delta\nu_y$ as functions of J_{1x} and J_{1y}, then we have the following propertionalities :

$$\Delta\nu_x(J_{1x}) \sim \alpha_1\; n_x k_x(k_x-1) J_{1x}^{k_x-2} + \alpha_2\; n_y k_x k_y J_{1x}^{k_x-1}$$

$$\Delta\nu_y(J_{1x}) \sim \alpha_3\; n_y k_y(k_y-1) J_{1x}^{k_x} + \alpha_4\; n_x k_x k_y J_{1x}^{k_x-1}$$

$$\Delta\nu_x(J_{1y}) \sim \alpha_5\; n_x k_x(k_x-1) J_{1y}^{k_y} + \alpha_4\; n_y k_x k_y J_{1x}^{k_y-1} \tag{31}$$

$$\Delta\nu_y(J_{1y}) \sim \alpha_7\; n_y k_y(k_y-1) J_{1y}^{k_y-2} + \alpha_8\; n_x k_x k_y J_{1y}^{k_y-1}$$

If one plots the logaritm of $\Delta\nu_x$ or $\Delta\nu_y$ as function of J_{1x} or J_{1y} then the slope will be given by the potens in the dominating term in [31].

We therefore expect that the curve will have two limits with different slopes where only one term contributes. We give an example in appendix C where the data has been obtained by simulating the resonance $Q_x + 2Q_y = 8$ when $k_x = 1$ and $k_y = 2$ for different values of J_{1x} and J_{1y}. Note that $e < 0$ in the simulation.

Only one type of driving element has been assumed for the above analysis.

6. CONCLUSIONS

We conclude from the simulation that the second order perturbation theory seems to give the correct frequency shifts, frequency spectra and phases near a single isolated resonance. It also gives expressions for the amplitude and the phase of the resonance that can be used to calculate their values from the frequency spectra.

By knowing the frequency spectra associated with a single resonance one can from tracking obtain information of which resonances the particle feels by doing spectral analysis of the coordinates.

From the simulations done we get quite good estimates of the amplitude and the phase of a resonance when the new frequencies caused by the resonance are strong enough.

We have also shown how one can get information about the type of driving element, by study how the tune shift varies with the amplitude of the oscillation.

ACKNOWLEDGMENT

I would like to thank E. Asseo for the development of the algoritms to obtain accurate values from the FFT. I would also like to thank M. Chanel for many helpful discussions.

REFERENCES

1. J. Bengtsson and M. Chanel, Resonance Measurements using Fourier Sepctrum Analysis of Beam Oscillations, PS/LEA/Note 86-15.

2. A. Ando, Distortion of Beam Emittance with Nonlinear Magnetic Fields, Particle Accelerators 1985, Vol.15 pp. 177-207.

3. G. Guignard, A General Treatment of Resonances in Accelerators, CERN 78-11.

4. R.D. Ruth, Single Particle Dynamics and Nonlinear Resonances in Circular Accelerators.

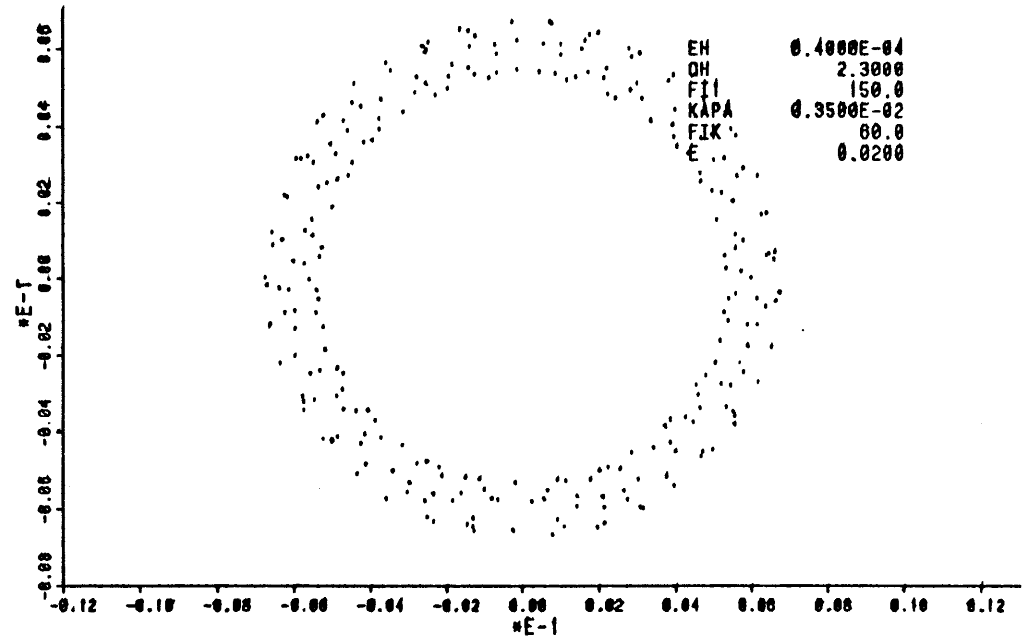
5. E. Asseo, Moyens de calcul pour la mesure des force et phase des effets perturbateurs des resonances sur le faisceau, PS/LEA/Note 87-01.

APPENDIX A

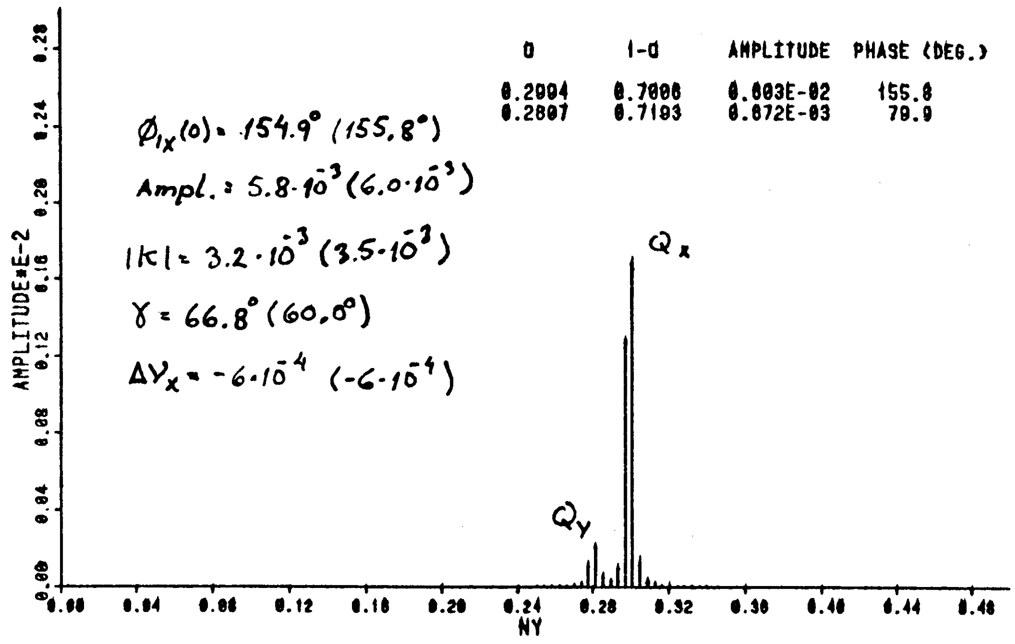
Simulation of single resonances
by numerical integration (chapter 3)

1*QH+1*QV= +5

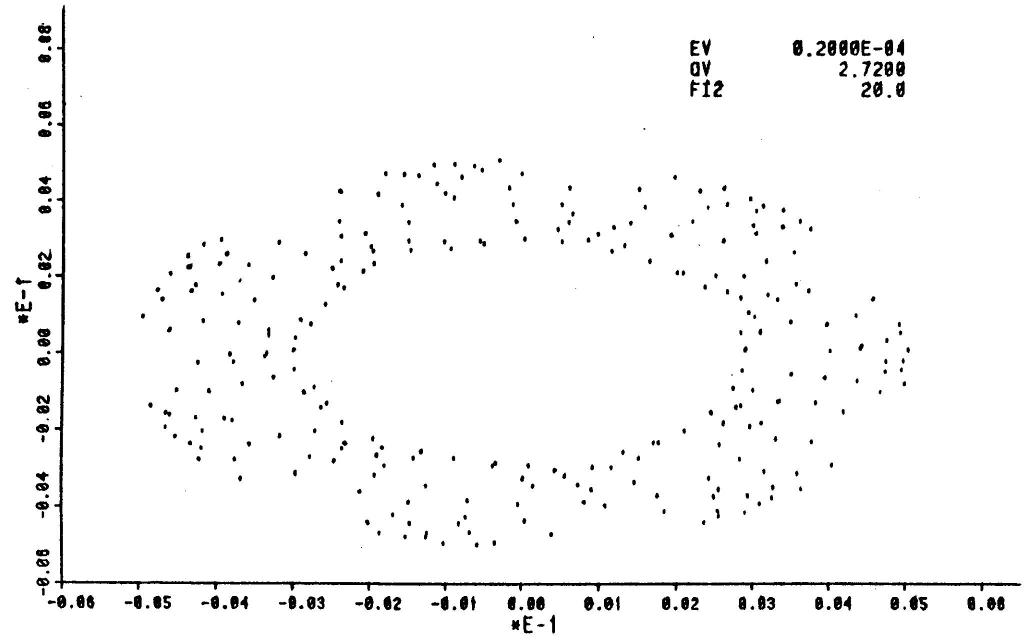
HORIZONTAL PHASE SPACE



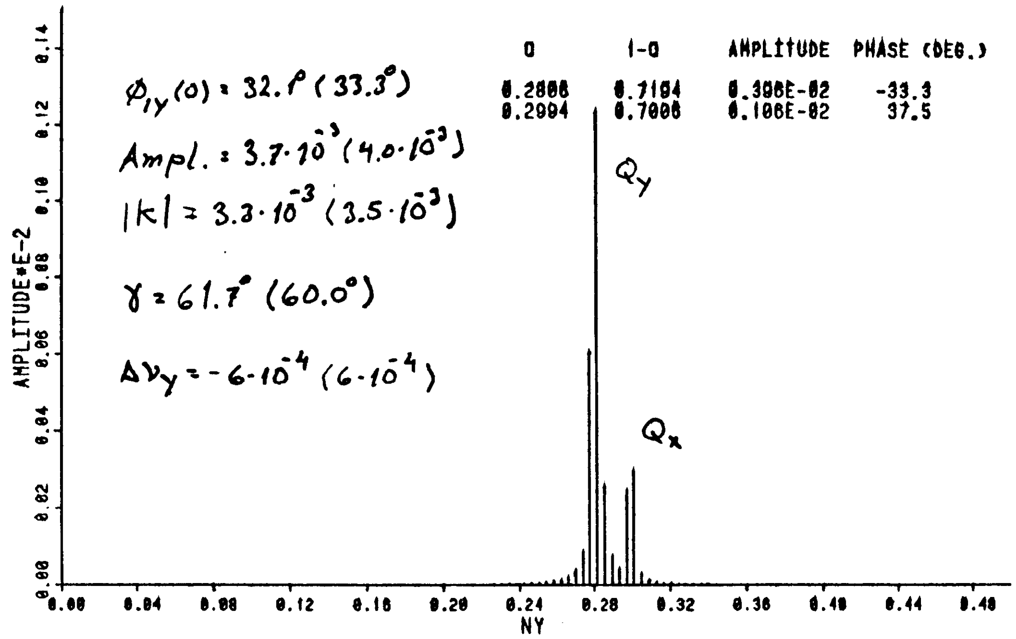
FFT OF X



VERTICAL PHASE SPACE

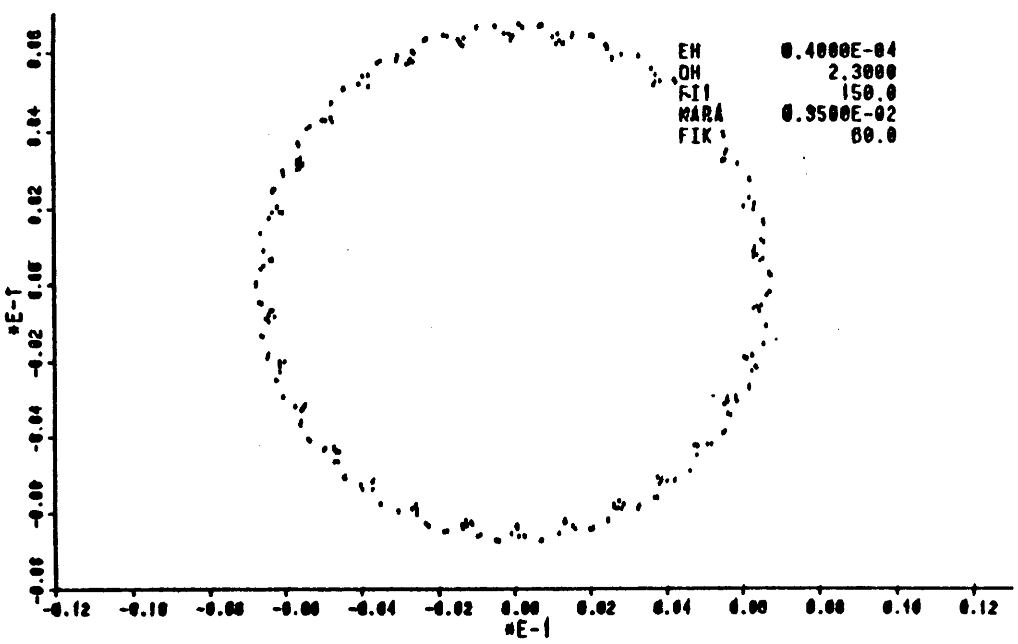


FFT OF Y

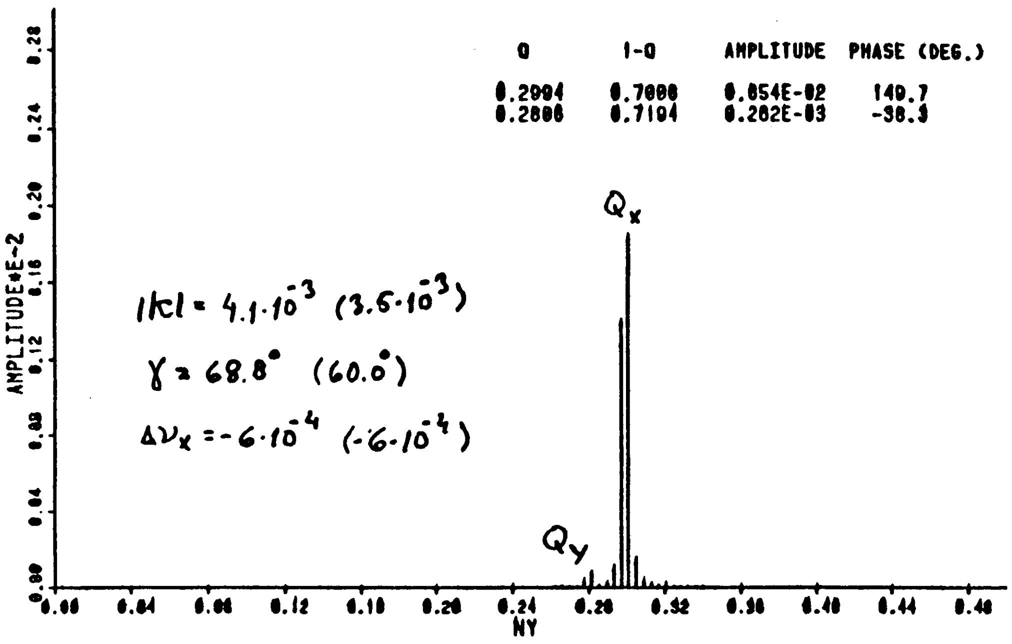


1*QH+1*QV=+5

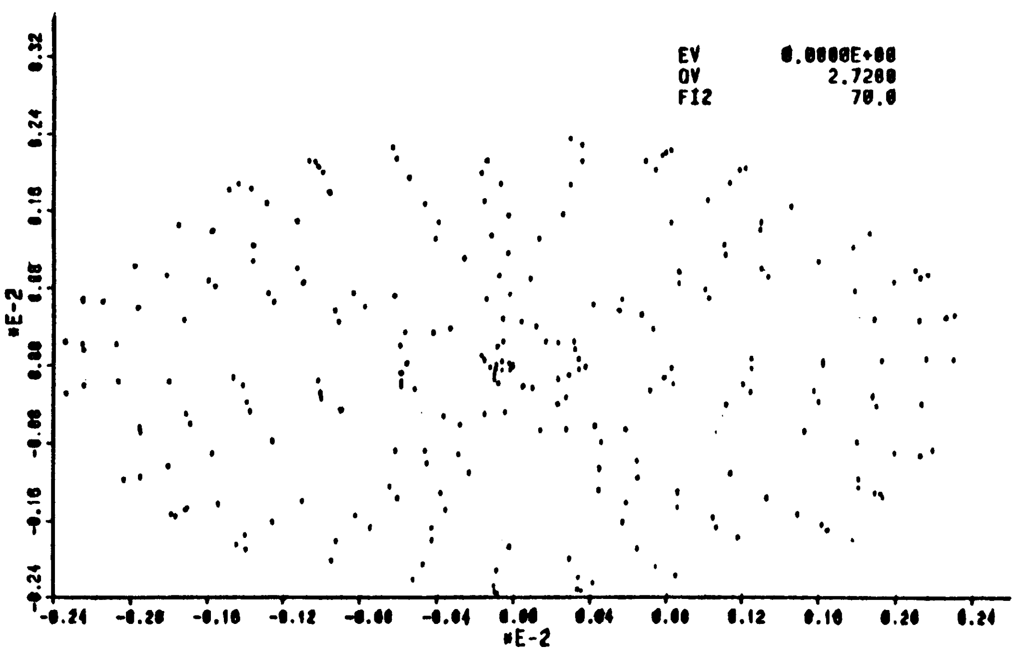
HORIZONTAL PHASE SPACE



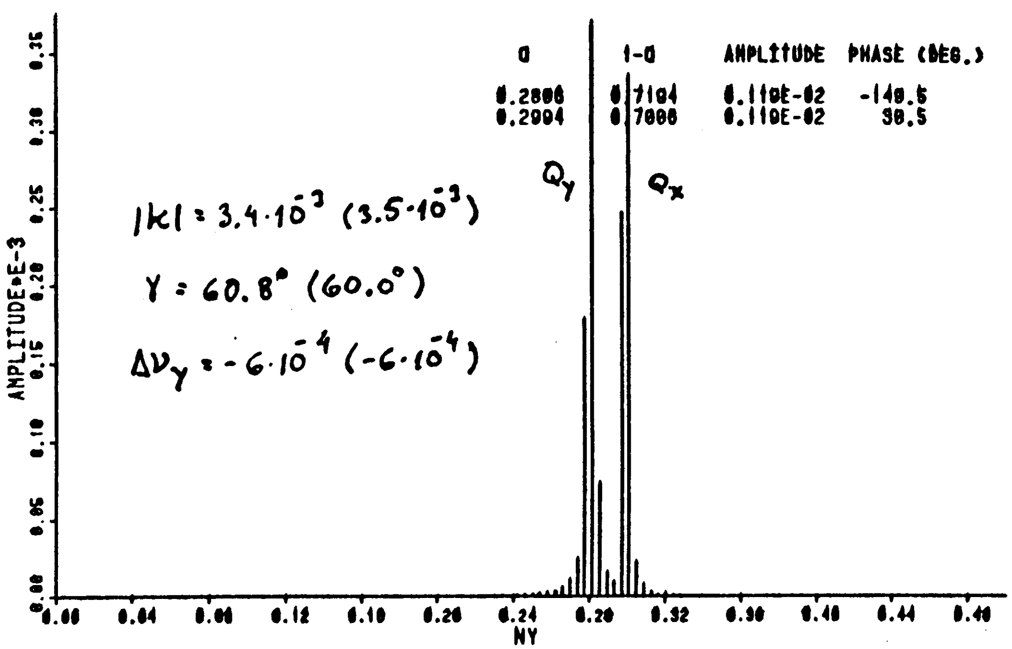
FFT OF X



VERTICAL PHASE SPACE

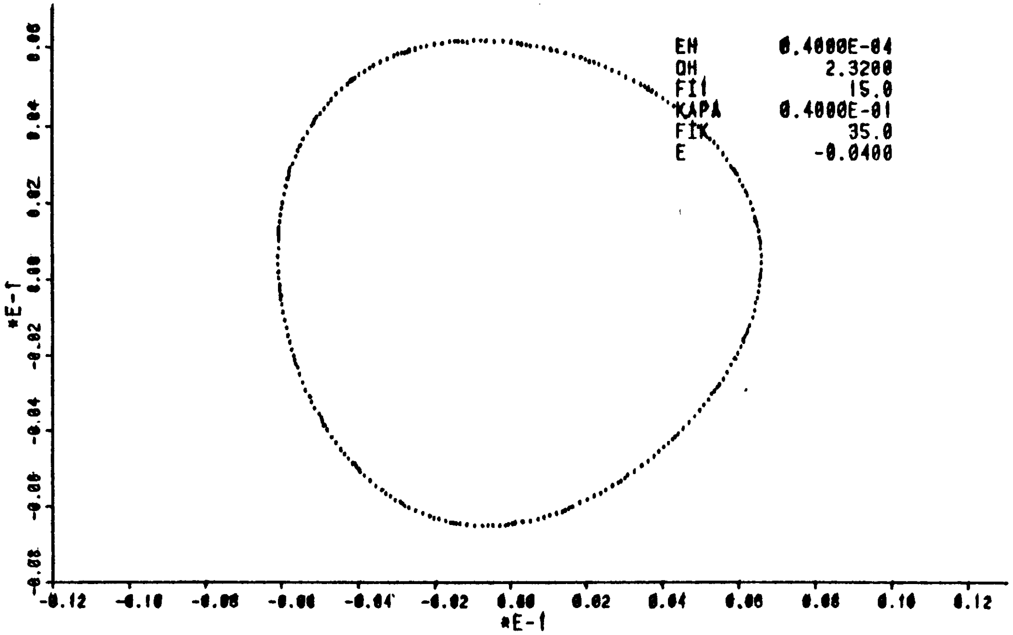


FFT OF Y

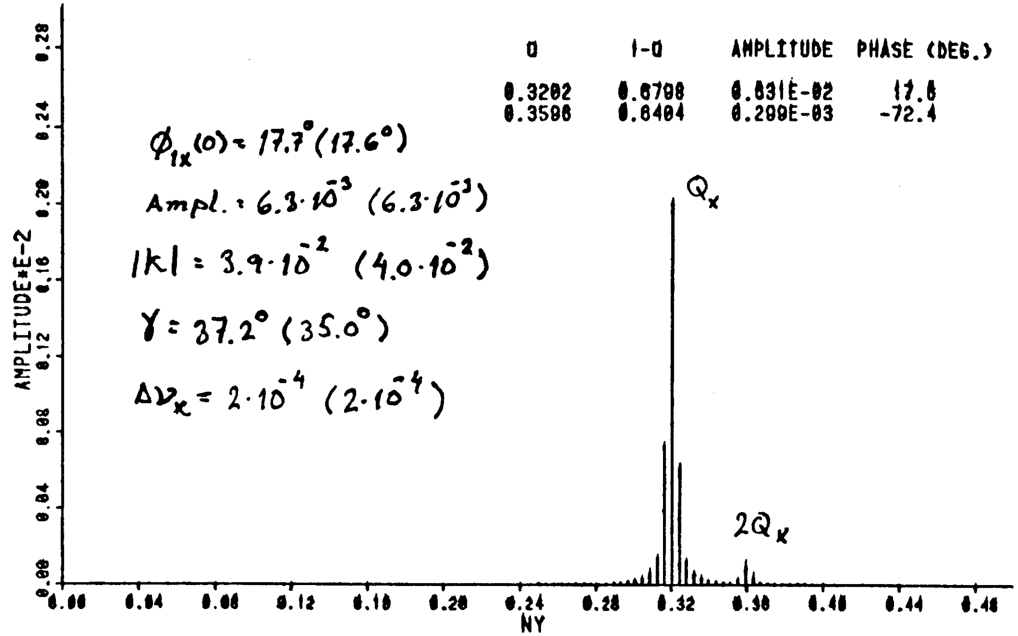


3*QH+0*QV= +7

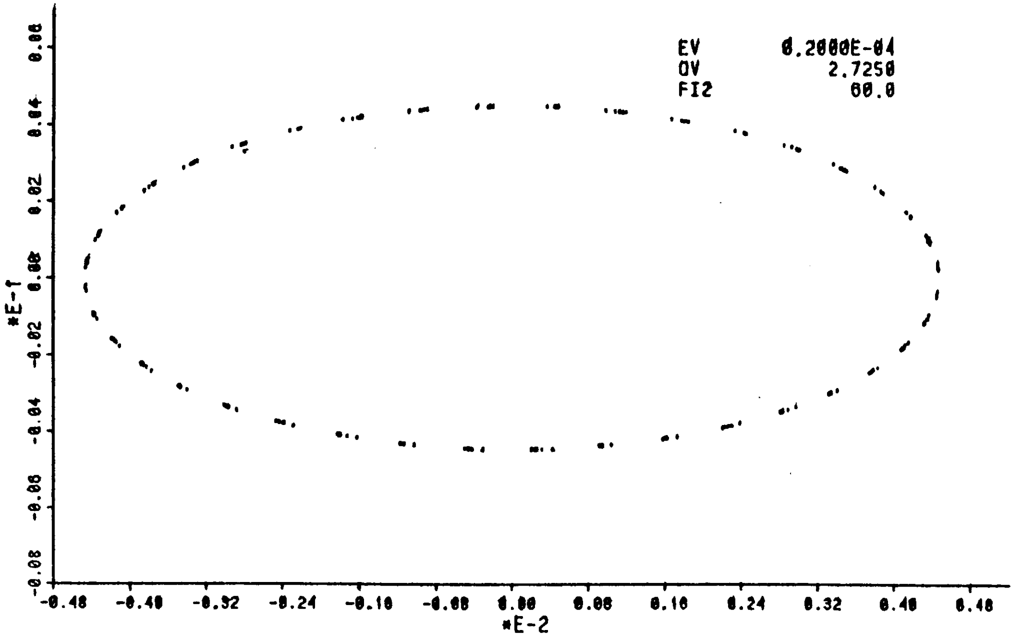
HORIZONTAL PHASE SPACE



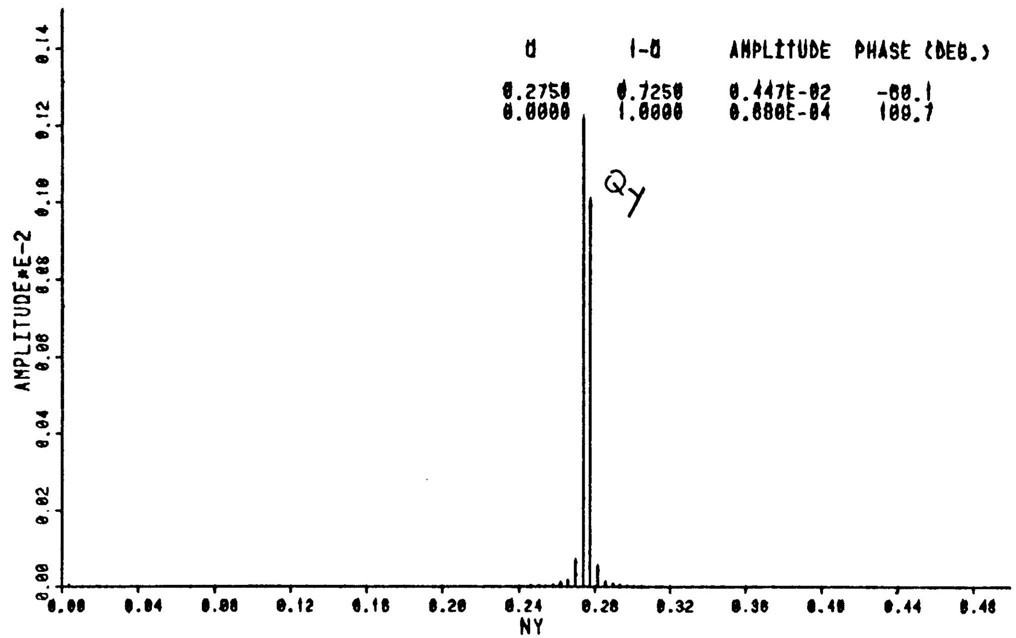
FFT OF X

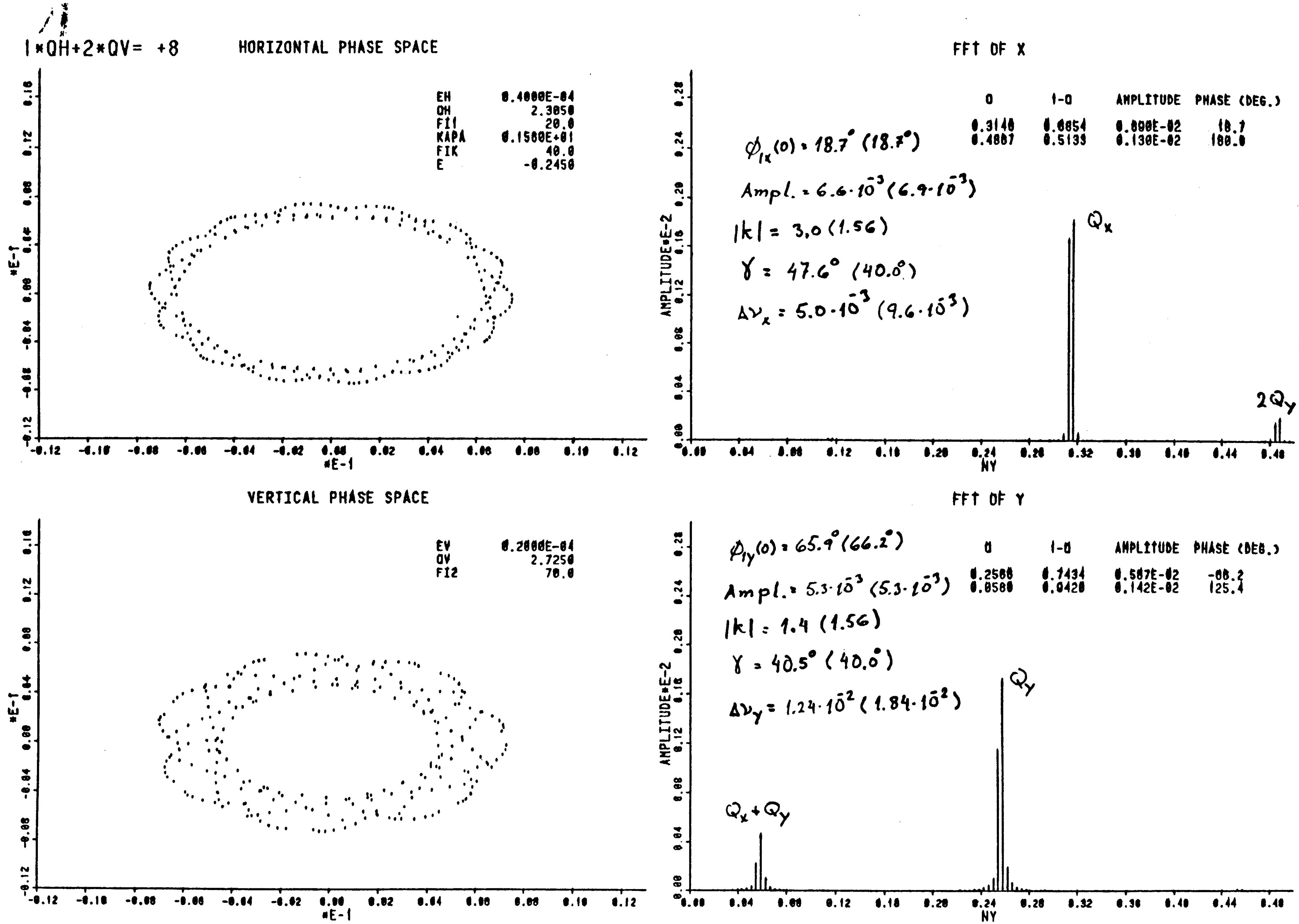


VERTICAL PHASE SPACE

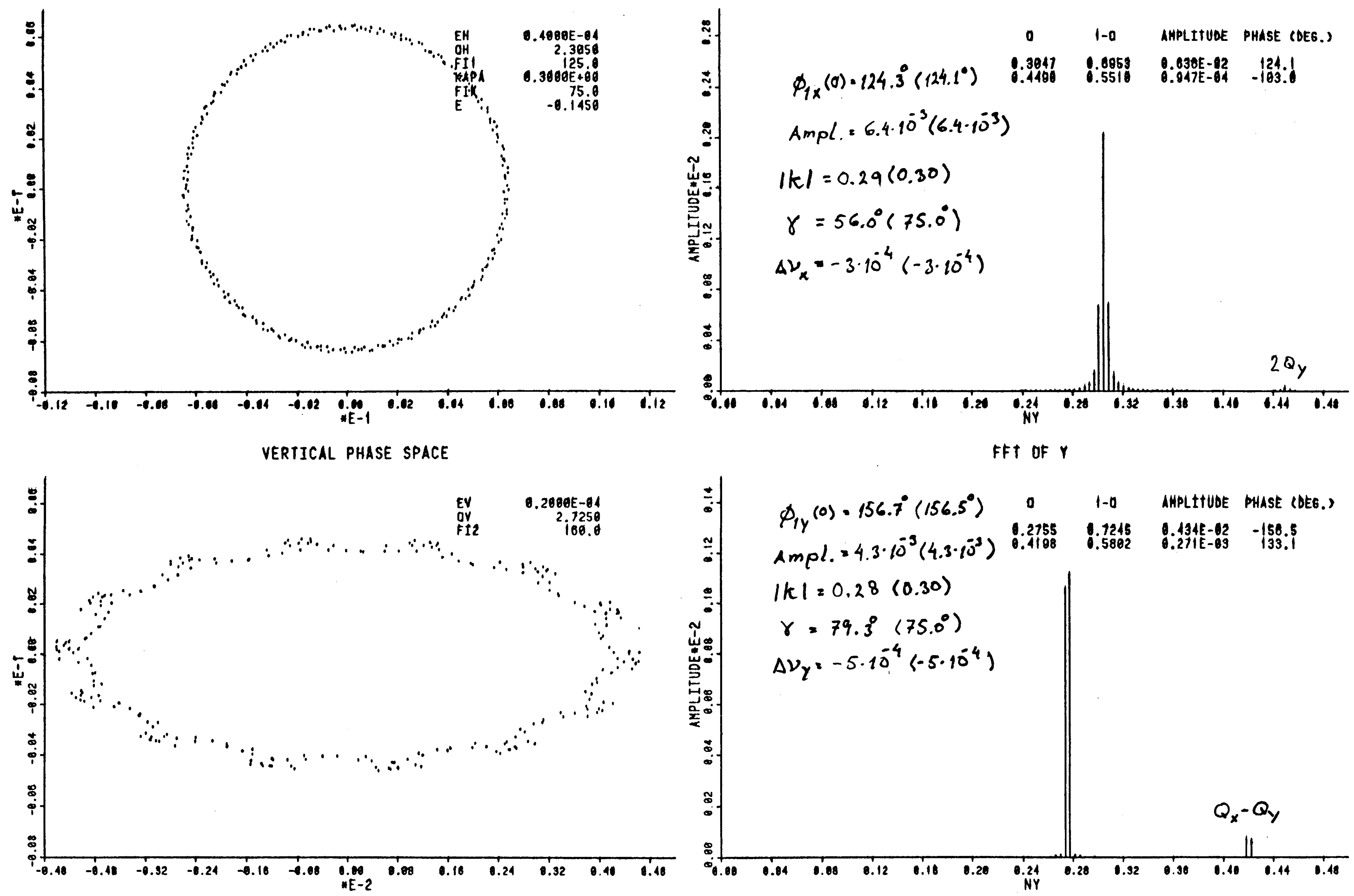


FFT OF Y




1*QH+2*QV= +8
HORIZONTAL PHASE SPACE
EH 0.4000E-04
QH 2.3050
FI1 20.0
KAPA 0.1500E+01
FIK 40.0
E -0.2450
*E-1
FFT OF X
0 1-0 AMPLITUDE PHASE (DEG.)
0.3140 0.6854 0.800E-02 18.7
0.4807 0.5139 0.130E-02 180.0
$\phi_{1x}(0) = 18.7^\circ$ (18.7°)
Ampl. = $6.6\cdot10^{-3}$ ($6.9\cdot10^{-3}$)
$|k| = 3,0$ (1.56)
$\gamma = 47.6^\circ$ (40.0°)
$\Delta\nu_x = 5.0\cdot10^{-3}$ ($9.6\cdot10^{-3}$)
Q_x
$2Q_y$
AMPLITUDE*E-2
NY
VERTICAL PHASE SPACE
EV 0.2000E-04
QV 2.7250
FI2 70.0
*E-1
FFT OF Y
0 1-0 AMPLITUDE PHASE (DEG.)
0.2566 0.7434 0.567E-02 -66.2
0.0580 0.9420 0.142E-02 125.4
$\phi_{1y}(0) = 65.9^\circ$ (66.2°)
Ampl. = $5.3\cdot10^{-3}$ ($5.3\cdot10^{-3}$)
$|k| = 1.4$ (1.56)
$\gamma = 40.5^\circ$ (40.0°)
$\Delta\nu_y = 1.24\cdot10^{-2}$ ($1.84\cdot10^{-2}$)
Q_y
Q_x+Q_y
AMPLITUDE*E-2
NY

1*QH-2*QV= -3

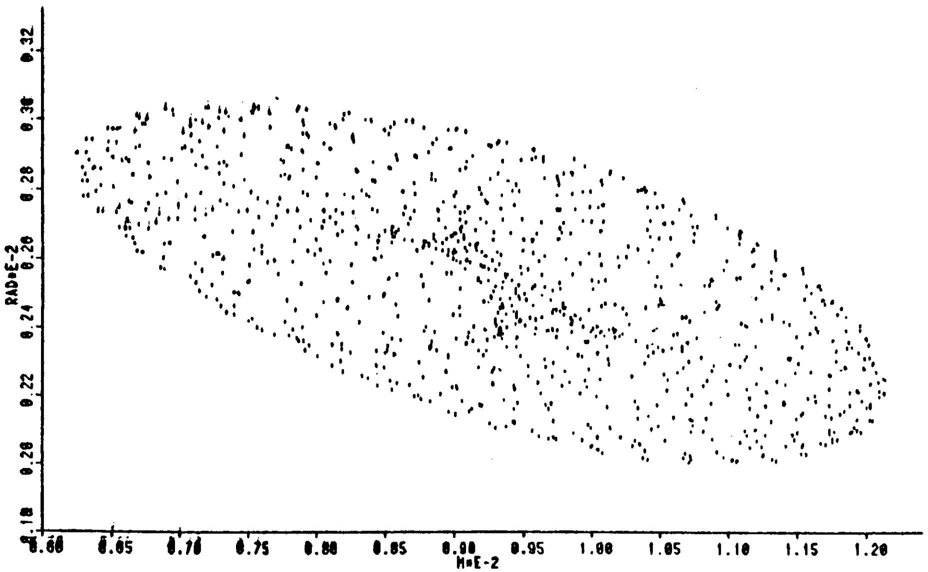

HORIZONTAL PHASE SPACE
EH 0.4000E-04
QH 2.3050
FI1 125.0
KAPA 0.3000E+00
FIK 75.0
E -0.1450
*E-1
*E-T
FFT OF X
0 1-0 AMPLITUDE PHASE (DEG.)
0.3047 0.6953 0.630E-02 124.1
0.4400 0.5510 0.947E-04 -103.0
$\phi_{1x}(0) = 124.3^\circ\ (124.1^\circ)$
$Ampl. = 6.4 \cdot 10^{-3} (6.4 \cdot 10^{-3})$
$|k| = 0.29\ (0.30)$
$\gamma = 56.0^\circ\ (75.0^\circ)$
$\Delta\nu_x = -3 \cdot 10^{-4}\ (-3 \cdot 10^{-4})$
$2Q_y$
AMPLITUDE*E-2
NY
VERTICAL PHASE SPACE
EV 0.2000E-04
QV 2.7250
FI2 160.0
*E-2
*E-T
FFT OF Y
0 1-0 AMPLITUDE PHASE (DEG.)
0.2755 0.7245 0.434E-02 -156.5
0.4198 0.5802 0.271E-03 133.1
$\phi_{1y}(0) = 156.7^\circ\ (156.5^\circ)$
$Ampl. = 4.3 \cdot 10^{-3} (4.3 \cdot 10^{-3})$
$|k| = 0.28\ (0.30)$
$\gamma = 79.3^\circ\ (75.0^\circ)$
$\Delta\nu_y = -5 \cdot 10^{-4}\ (-5 \cdot 10^{-4})$
$Q_x - Q_y$
AMPLITUDE*E-2
NY

APPENDIX B

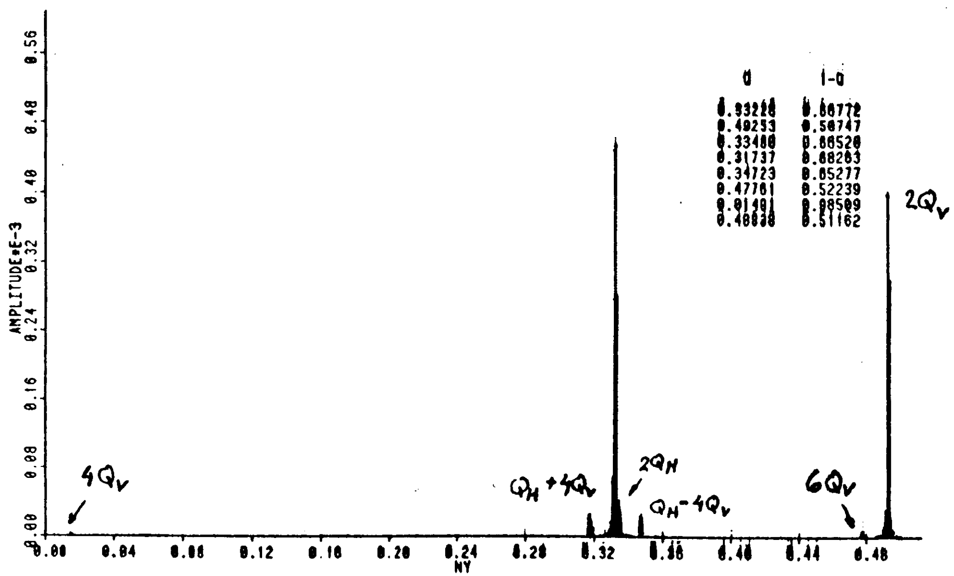
Tracking with DIMAT at extraction in LEAR (chapter 4)

VD12:H1V30

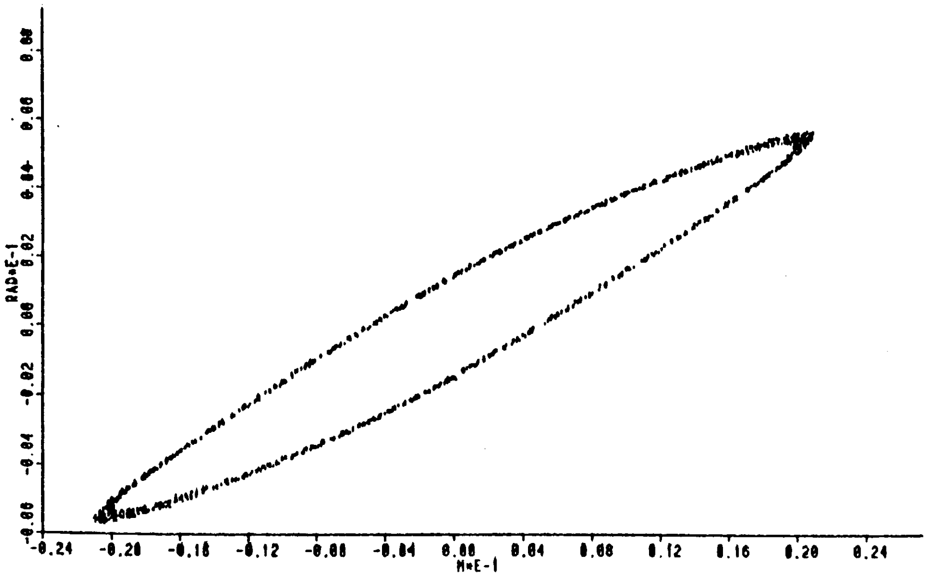
HORIZONTAL PHASE SPACE



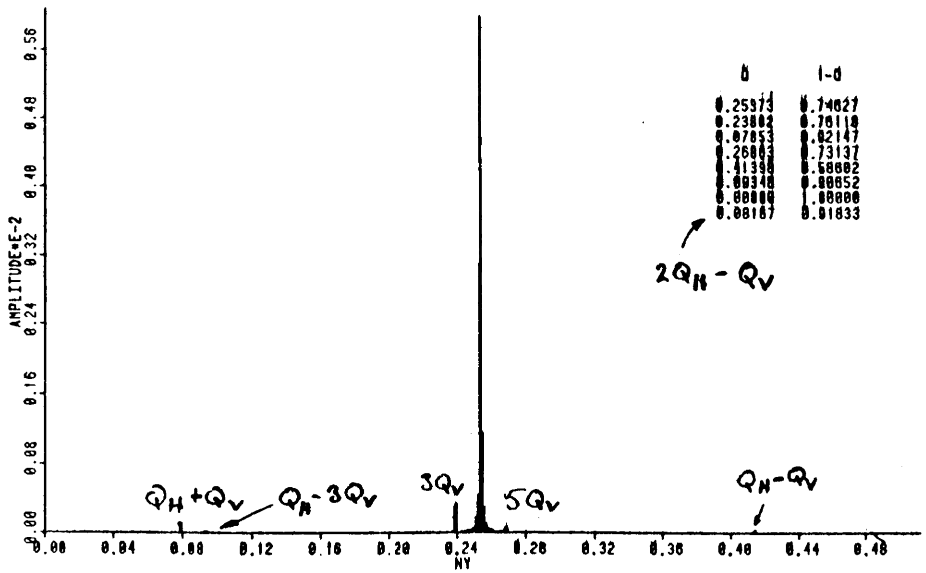
FFT OF POSITION



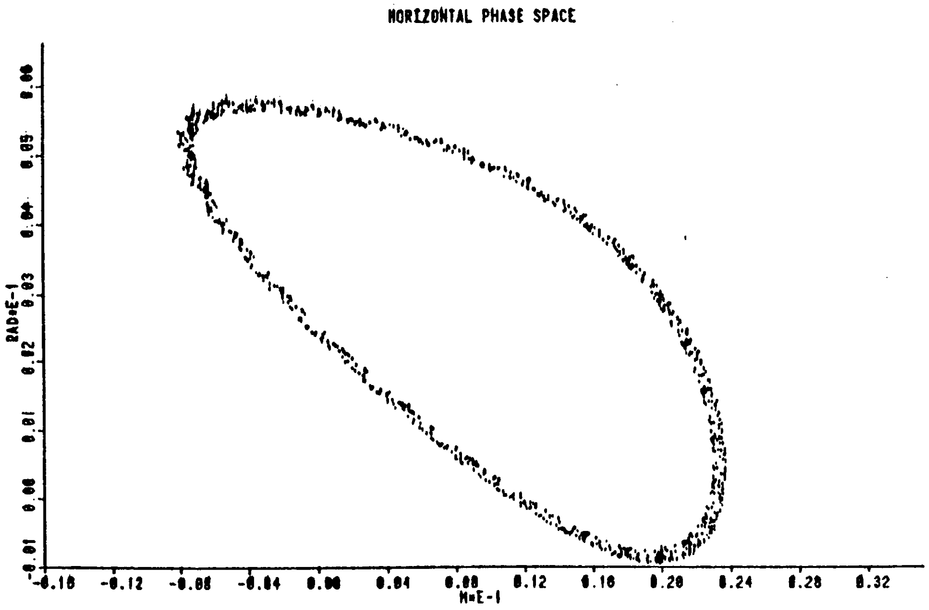
VERTICAL PHASE SPACE

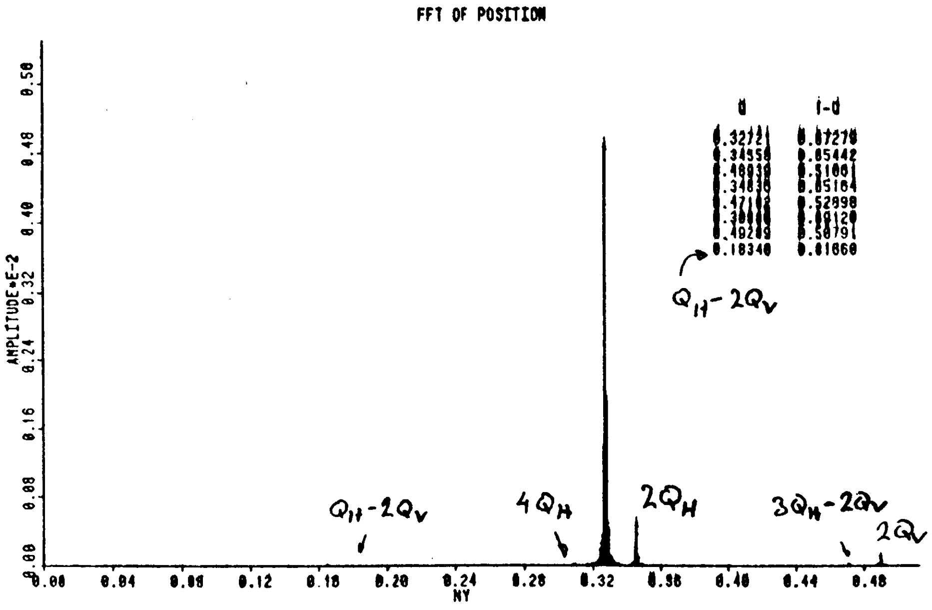


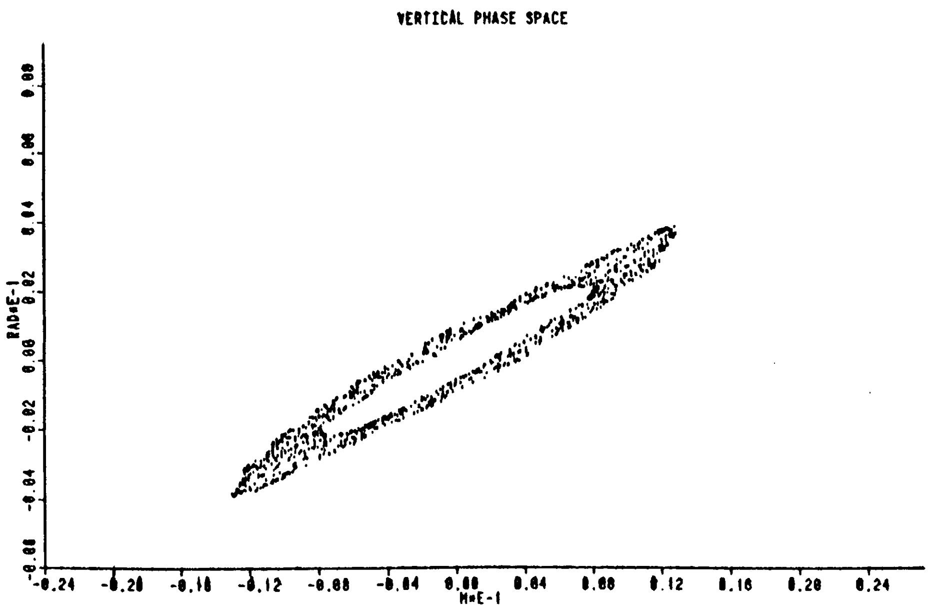
FFT OF POSITION

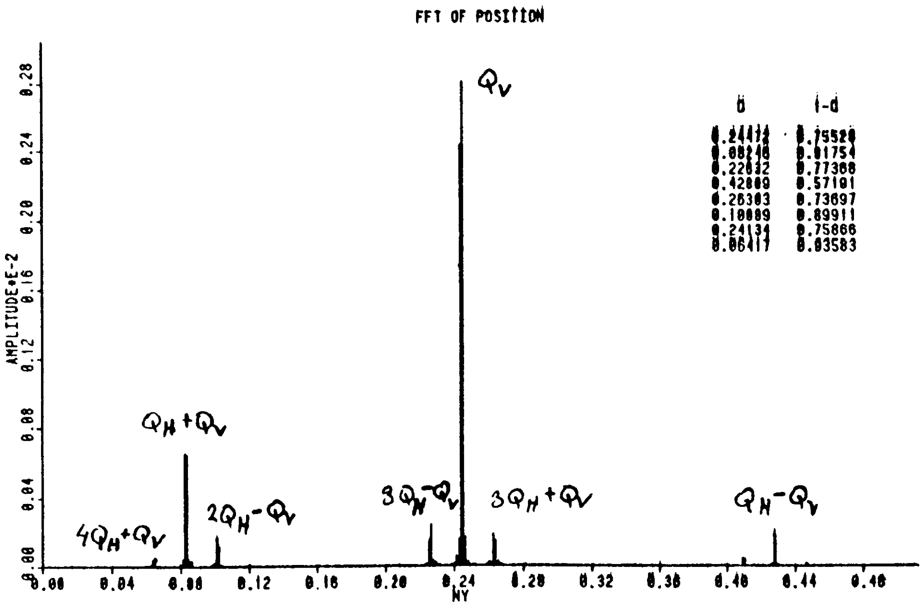


VD12:H30V10


HORIZONTAL PHASE SPACE
RAD*E-1
M*E-1


FFT OF POSITION
AMPLITUDE*E-2
NY
$Q_H - 2Q_V$
$4Q_H$
$2Q_H$
$3Q_H - 2Q_V$
$2Q_V$


VERTICAL PHASE SPACE
RAD*E-1
M*E-1

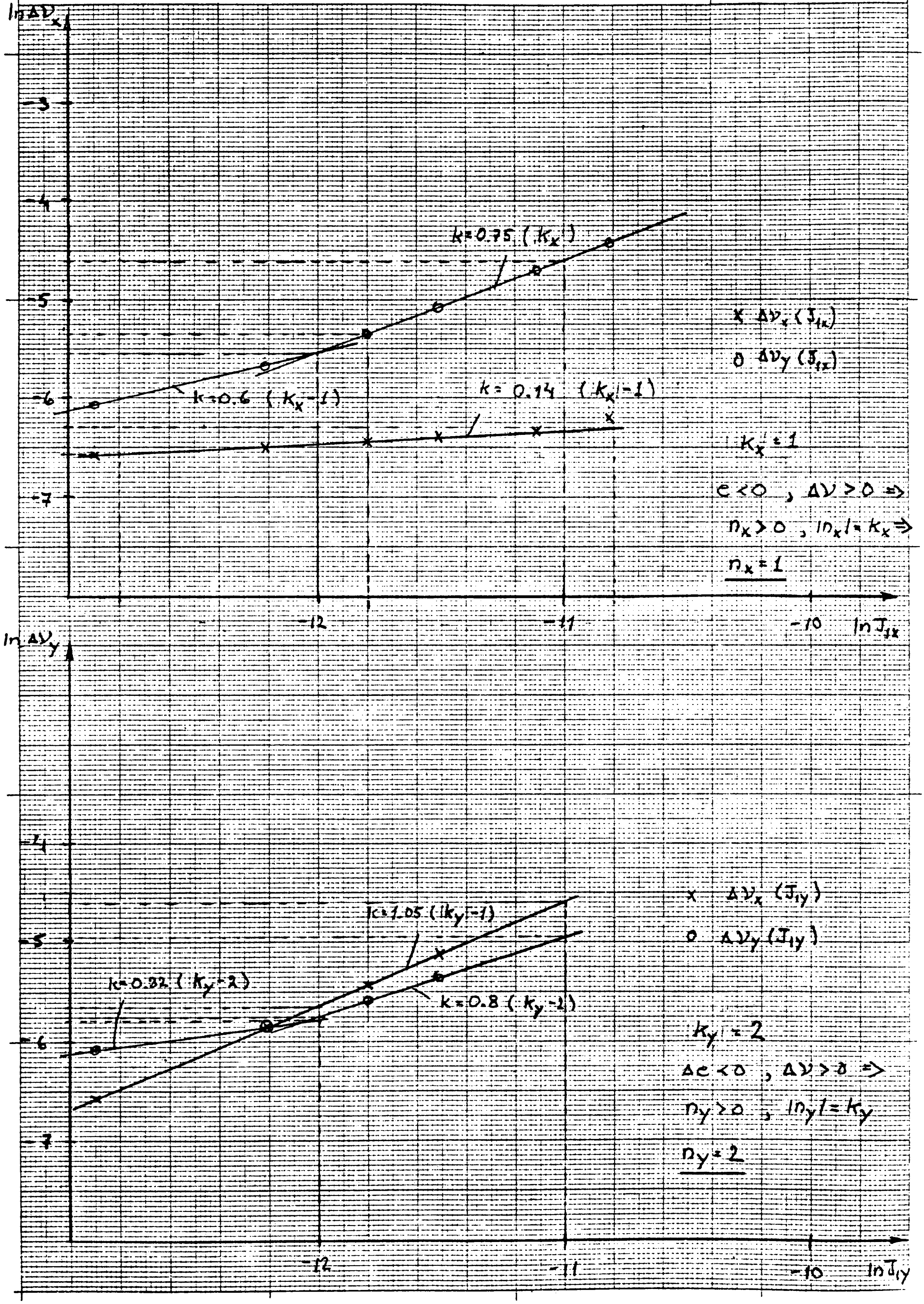

FFT OF POSITION
AMPLITUDE*E-2
NY
Q_V
$Q_H + Q_V$
$4Q_H + Q_V$
$2Q_H - Q_V$
$3Q_H - Q_V$
$3Q_H + Q_V$
$Q_H - Q_V$
0.22032 0.77368
0.42809 0.57191
0.26303 0.73697
0.10089 0.89911
0.24134 0.75866

APPENDIX C

Plots of : Δv_x (J_{1x}), Δv_y (J_{1y}) and

Δv_x (J_{1y}), Δv_y (J_{1y})

$\Delta\nu$ versus J_1



*/2

DISTRIBUTION

Groupe LEAR

D. ALLEN
E. ASSEO
S. BAIRD
J. BENGTSSON
M. CHANEL
J. CHEVALLIER
R. GALIANA
R. GIANNINI
P. LEFEVRE
F. LENARDON
R. LEY
D. MANGLUNKI
E. MARTENSSON
J.L. MARY
C. MAZELINE
D. MOEHL
G. MOLINARI
J.C. PERRIER
T. PETTERSSON
P. SMITH
N. TOKUDA
G. TRANQUILLE
H. VESTERGAARD