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LONGITUDINAL SPACE-CHARGE FORCES AT TRANSITION

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INTRODUCTION

The effect of longitudinal space-charge forces has been studied previously for energies well below [1] transition energy and for energies well above [2], [3], [4] transition energy. The longitudinal space-charge forces are stronger the shorter are the bunches. As the bunch length passes through a minimum at transition, the longitudinal space-charge forces have a maximum at transition. The longitudinal space-charge forces will distort the particle orbits in different ways, and they may cause particle losses.

In the CPS today, the intensity is probably limited by transverse space-charge forces, and longitudinal space-charge forces are only moderately important. If a new injector of higher energy is built, the transverse space-charge forces will be less important than they are today, as the effect of transverse space-charge forces decreases with increasing energy. However, the effect of longitudinal space-charge forces near transition does not decrease much with increased injection energy. With a new injector, the longitudinal space-charge forces may constitute an intensity limitation of the same order of importance as the transverse space-charge forces. The same is true for the planned 300 GeV accelerator. And even moderately large orbit distortions, not causing particle losses. effectively dilute phase space, which is harmful for the Intersecting Storage Rings.

It is the purpose of this paper to calculate the effects of longitudinal space-charge forces in the range around the transition energy, and investigate two possible methods of reducing them.

This paper is divided in three main parts : In the first part we shall assume that there are no space-charge forces, but we shall allow for a timing error of the phase-jump. In the second part, we shall allow for longitudinal space-charge forces, but we shall assume that there is no timing error of the phase-jump. These space-charge forces are treated as linear, and the effect of the space-charge forces is handled in the first approximation only. In the third part we allow for longitudinal space-charge forces as well as a timing error of the phase-jump. We find the interesting result that the distortion of the bunches can be compensated, to some extent, by performing the phase-jump somewhat later than transition. We also investigate the possibility of artificial blow_up of the bunches before transition to minimize the resulting bunch length.

The reader who is only interested in the results, and not in the computational method, may drop the chapters 1 and 2.

There is planned an extension of this paper where the effect of longitudinal space-charge forces are handled more accurately than in the first approximation. One should also investigate more complicated tricks with the R.F. system [14], to see if such tricks could further reduce the effect of longitudinal space-charge forces.

1. PHASE-JUMP TIMING ERROR, NO SPACE-CHARGE FORCES

THE EQUATION OF MOTION

Let us first completely ignore space-charge forces. The effect of a timing error of the phase-jump has been handled in the literature [5], [6], [7] as far as the amplitude blow-up of the synchrotron oscillations is concerned. We shall also investigate the phase relationships, as this is important in the third part of this paper, where we handle the combined effect from a timing error and longitudinal space-charge forces. We shall here start the calculation from the very beginning to develop a notation which will be used in the second and third part of the paper as well, even if we by doing this to some extent duplicate already existing work.

The equations of motion are ([5], eqs. (5.5a) and (5.5b)):

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_s}\right) = \frac{eV}{2\pi} \left(\sin\varphi - \sin\varphi_s\right) \tag{1.1}$$

$$\frac{d\varphi}{dt} = \frac{\eta h \omega_s}{\beta^2} \frac{\Delta E}{E} \tag{1.2}$$

The quantity denoted by

$$\eta = \frac{p}{T} \frac{dT}{dp} \qquad (1.3)$$

where p is the momentum of the particle and T is its revolution time, obeys the equation ([5] , eq. (5.3))

$$\eta = \left(\frac{mc^2}{E_{tr}}\right)^2 - \left(\frac{mc^2}{E}\right)^2 \qquad (1.4)$$

where E is the energy of the particle and E_{tr} is a constant of the machine which is called the transition energy.

We shall assume that the synchronous phase angle φ_s is constant at all t except at $t = t_o$, where it jumps discontinuously. From eq. (1.4) we see that when $E = E_{tr}$, we have $\eta = 0$. We shall measure t from this instant. With this origin for the time, t_o is the timing error of the phase jump. The phase jump is performed in such a way that

$$\sin \varphi_s \qquad (1.5)$$

is constant and

$$\operatorname{sgn} (\cos \varphi_s) = -\operatorname{sgn} (t - t_o) . \qquad (1.6)$$

By the symbol $\operatorname{sgn} (x)$ we mean :

$$\operatorname{sgn} (x) = -1 \quad \text{for} \quad x < 0 \qquad (1.7.1)$$
$$\operatorname{sgn} (x) = +1 \quad \text{for} \quad x > 0 \quad . \qquad (1.7.2)$$

Eqs. (1.1) and (1.2) can be combined into one equation :

$$\frac{d}{dt} \left(\frac{\beta^2 E}{h\omega_s{}^2 \eta} \frac{d\varphi}{dt}\right) = \frac{eV}{2\pi} (\sin \varphi - \sin \varphi_s) . \qquad (1.8)$$

For shortness we introduce

$$R_s = \frac{\beta c}{\omega_s} \qquad (1.9)$$

and our equation reads

$$\frac{d}{dt}\left(\frac{R_s^{2}}{hc^{2}} \; \frac{E}{\eta} \; \frac{d\varphi}{dt}\right) = \frac{eV}{2\pi} \; (\sin \varphi - \sin \varphi_s) \; . \tag{1.10}$$

Let us define

$$\vartheta = \varphi - \varphi_s \tag{1.11}$$

and let us assume ϑ small. It is known that in the transition region the synchrotron oscillations have small amplitudes; thus this approximation is justified. We then have :

$$\frac{d}{dt}\left(\frac{R_s^{2}}{hc^{2}} \; \frac{E}{\eta} \; \frac{d\vartheta}{dt}\right) = \frac{eV}{2\pi} \; \cos\,(\varphi_s)\; \vartheta \; . \tag{1.12}$$

Eq. (1.6) can be written

$$\cos \varphi_s = -|\cos \varphi_s| \;\; \mathrm{sgn}\,(t - t_o) \; . \tag{1.13}$$

Introducing this, we have

$$\frac{d}{dt}\left(\frac{R_s^{2}}{hc^{2}} \; \frac{E}{\eta} \; \frac{d\vartheta}{dt}\right) + \frac{eV}{2\pi} \; |\cos \varphi_s| \; \mathrm{sgn}\,(t - t_o) \; \vartheta = 0 \; . \tag{1.14}$$

From (1.4) we have

$$\frac{d\eta}{dt} = 2\left(\frac{mc^{2}}{E}\right)^{2} \frac{1}{E} \; \frac{dE}{dt} \; . \tag{1.15}$$

We shall assume that this quantity is approximately constant around transition. As η and t are both zero at transition, we have

$$\eta = 2\left(\frac{mc^{2}}{E_{tr}}\right)^{2} \frac{1}{E_{tr}} \left(\frac{dE}{dt}\right)_{tr} t \; . \tag{1.16}$$

This is substituted into (1.14) along with

$$R_s \simeq R_{tr} \; , \quad E \simeq E_{tr} \tag{1.17}$$

and we have :

$$\frac{d}{dt}\left(\frac{1}{t}\frac{d\vartheta}{dt}\right) + 2\,\frac{hc^2}{R_{tr}^{\;2}\,E_{tr}^{\;2}}\left(\frac{mc^2}{E_{tr}}\right)^2\left(\frac{dE}{dt}\right)_{tr}\frac{eV}{2\pi}\,|\cos\varphi_s|\;\mathrm{sgn}\,(t - t_o)\;\vartheta = 0\;. \tag{1.18}$$

For shortness we define a quantity T thus :

$$T^{-3} = 2\,\frac{hc^2}{R_{tr}^{\;2}\,E_{tr}^{\;2}}\left(\frac{mc^2}{E_{tr}}\right)^2\left(\frac{dE}{dt}\right)_{tr}\frac{eV}{2\pi}\,|\cos\varphi_s|\;. \tag{1.19}$$

We remark that defined in this way, T is a positive constant for all t. T has the dimension of time and gives the order of magnitude for the width of the transition region. The value of T for the CPS is given in Appendix 1 .

With this notation the phase equation is written in the form

$$\frac{d}{dt}\left(\frac{1}{t}\frac{d\vartheta}{dt}\right) + \frac{\mathrm{sgn}\,(t - t_o)}{T^3}\;\vartheta = 0\;. \tag{1.20}$$

We now introduce a new dimensionless variable x defined by

$$x = \frac{t}{T}\;. \tag{1.21}$$

We also put

$$x_o = \frac{t_o}{T}\;. \tag{1.22}$$

Our equation now takes the form

$$\frac{d}{dx}\left(\frac{1}{x}\frac{d\vartheta}{dx}\right) + \mathrm{sgn}\,(x - x_o)\;\vartheta = 0 \tag{1.23}$$

or equivalent

$$\frac{d^2\vartheta}{dx^2} - \frac{1}{x}\,\frac{d\vartheta}{dx} + \mathrm{sgn}\,(x - x_o)\;x\,\vartheta = 0\;. \tag{1.24}$$

No machine parameters appear in (1.23) and only one machine parameter, namely T , appears in (1.20). This means that in all synchrotrons exactly the same things happen at transition, only in some synchrotrons things may happen faster than in others.

<u>THE SOLUTION OF THE EQUATION OF MOTION</u>

For $x \neq 0$, $x \neq x_o$ this equation has the following general solution, see ref. [8], p. XX, eqs. (29) and (30) :

$$\vartheta = x \left(B_J \; J_{\frac{2}{3}} (z) + B_N \; N_{\frac{2}{3}} (z) \right) \tag{1.25}$$

where

$$z = \frac{2}{3} \left(\mathrm{sgn}\,(x-x_o)\right)^{\frac{1}{2}} \; x^{\frac{3}{2}} = \frac{2}{3} \left(\mathrm{sgn}\,(x(x-x_o))\right)^{\frac{1}{2}} \; \mathrm{sgn}(x)\,|x|^{\frac{3}{2}} \tag{1.26}$$

and B_J and B_N are constants and $J_{\frac{2}{3}}$ and $N_{\frac{2}{3}}$ are Bessel and Neumann functions of order $\frac{2}{3}$. We shall demand that ϑ and ΔE are continuous functions of t everywhere, especially at $x = 0$ and $x = x_o$. From eqs. (1.2) and (1.16) we see that demanding continuity of ΔE is the same as demanding continuity of $\frac{1}{x} \frac{d\vartheta}{dx}$. At $x=0$ this is a stronger condition than demanding continuity of $\frac{d\vartheta}{dx}$. As $N_{\frac{2}{3}} (z)$ is a discontinuous function of x at $x = 0$, we cannot expect the constants B_J and B_N to have the same values on both sides of $x = 0$, so we may equally well write the solution in the form

$$\vartheta = |x| \left(B_J \; J_{\frac{2}{3}}(z) + B_N \; N_{\frac{2}{3}}(z) \right) \tag{1.27}$$

which is a little more convenient. Also z is a discontinuous function of x at $x = x_o$, so we cannot expect B_J and B_N to have the same values on both sides of $x = x_o$, whether we use (1.27) or (1.25) .

Using the formula

$$N_\nu(z) = \frac{\cos\,(\nu\pi)\; J_\nu(z) - J_{-\nu}(z)}{\sin\,(\nu\pi)} \tag{1.28}$$

the solution (1.27) may be written as

$$\vartheta = |x| \left(B_+ \; J_{\frac{2}{3}}(z) + B_- \; J_{-\frac{2}{3}}(z) \right) , \tag{1.29}$$

with

$$\begin{pmatrix} B_+ \\ B_- \end{pmatrix} = \begin{pmatrix} 1 \; , & -\dfrac{\sin\,(\frac{\pi}{6})}{\cos\,(\frac{\pi}{6})} \\ 0 \; , & \dfrac{1}{\cos\,(\frac{\pi}{6})} \end{pmatrix} \begin{pmatrix} B_J \\ B_N \end{pmatrix} \tag{1.30}$$

or alternatively

$$\begin{pmatrix} B_J \\ B_N \end{pmatrix} = \begin{pmatrix} 1 \,, & -\sin\left(\frac{\pi}{6}\right) \\ 0 \,, & -\cos\left(\frac{\pi}{6}\right) \end{pmatrix} \begin{pmatrix} B_+ \\ B_- \end{pmatrix} \tag{1.31}$$

Both forms (1.27) and (1.29) have certain advantages. The form (1.27) has the advantage that, in the region very far from transition, that is, in the limit of large z , $J_{\frac{2}{3}}(z)$ and $N_{\frac{2}{3}}(z)$ are oscillatory functions with a phase difference relative to each other of $\frac{\pi}{2}$, while the phase difference between $J_{\frac{2}{3}}(z)$ and $J_{-\frac{2}{3}}(z)$ is $\frac{2}{3}\pi$. As a result of this, the total amplitude will be

$$B = \left(B_J^{\,2} + B_N^{\,2}\right)^{1/2} \quad . \tag{1.32}$$

while in terms of B_+ and B_- the expression is more complicated :

$$B = \left(B_+^{\,2} + B_-^{\,2} - 2\sin\left(\tfrac{\pi}{6}\right) B_+ B_-\right)^{1/2} \tag{1.33}$$

(To find the maximum excursions of ϑ , we have to multiply the total amplitude by a slowly time-varying factor.) On the other hand, the form (1.29) has the advantage that $J_{\frac{2}{3}}(z)$ and $J_{-\frac{2}{3}}(z)$ can be expressed in a simple way when z is imaginary, while there is no corresponding simple expression for $N_{\frac{2}{3}}(z)$ when z is imaginary. Consequently, we shall use both forms (1.27) and (1.29) in the subsequent calculation.

We shall define a quantity

$$y = \frac{2}{3} \, |x|^{3/2} \quad . \tag{1.34}$$

y is always real and positive. We have

$$z = \left(\mathrm{sgn}\,(x(x - x_0))\right)^{1/2} \; \mathrm{sgn}\,(x) \cdot y \quad . \tag{1.35}$$

Let us distinguish between the following cases :

$x_o < 0$: The phase is switched too early :

$$x < x_o \qquad z = -y \tag{1.36.1}$$
$$x_o < x < 0 \qquad z = -iy \tag{1.36.2}$$
$$0 < x \qquad z = y \tag{1.36.3}$$

$x_o > 0$: The phase is switched too late :

$$x < 0 \qquad z = -y \tag{1.36.4}$$
$$0 < x < x_o \qquad z = iy \tag{1.36.5}$$
$$x_o < x \qquad z = y \tag{1.36.6}$$

$J_{\frac{2}{3}}(z)$ and $J_{-\frac{2}{3}}(z)$ are real when z is real and positive, elsewhere they are complex, and a real solution for ϑ requires the coefficients B_+ and B_- to be complex too. We shall reformulate the cases where z is negative or imaginary in such a way as to work only with real functions of the real and positive variable y .

First in the region before the phase-jump and before transition, where z is negative (the cases (1.36.1) and (1.36.4)), we shall use the following relation, which is easily proved from the series expansion :

$$B_+ J_\nu(-y) + B_- J_{-\nu}(-y) = A_+ J_\nu(y) + A_- J_{-\nu}(y) \ , \tag{1.37}$$

where

$$A_+ = e^{i\nu\pi} B_+ \tag{1.38.1}$$

$$A_- = e^{-i\nu\pi} B_- \ , \tag{1.38.2}$$

that is, A_+ and A_- are also constants. The relation (1.37) enables us to write the solution in the form

$$\vartheta = -x \left(A_+ J_{\frac{2}{3}}(y) + A_- J_{-\frac{2}{3}}(y)\right) \ , \tag{1.39}$$

for use in the cases (1.36.1) and (1.36.4). In this way we avoid working with complex functions and complex amplitudes to express a real solution.

In case (1.36.5) we use the relation

$$J_\nu(iy) = e^{\frac{1}{2}\nu\pi i} I_\nu(y) \tag{1.40}$$

where $I_\nu(y)$ is a real function for real and positive y . From (1.40) we have

$$B_+ J_\nu(iy) + B_- J_{-\nu}(iy) = A_+'' I_\nu(y) + A_-'' I_{-\nu}(y) \qquad (1.41)$$

with

$$A_+'' = e^{\frac{1}{2}\nu\pi i} B_+ \qquad (1.42.1)$$

$$A_-'' = e^{-\frac{1}{2}\nu\pi i} B_- \qquad (1.42.2)$$

From (1.41) we have

$$\vartheta = x\,(A_+'' I_{\frac{2}{3}}(y) + A_-'' I_{-\frac{2}{3}}(y) \quad , \qquad (1.43)$$

to be used in the case (1.36.5) .

In the case (1.36.2) we use the relation

$$J_\nu(-iy) = e^{-\frac{1}{2}\nu\pi i} I_\nu(y) \qquad (1.44)$$

from which we have

$$B_+ J_\nu(-iy) + B_- J_{-\nu}(-iy) = A_+''' I_\nu(y) + A_-''' I_{-\nu}(y) \qquad (1.45)$$

with

$$A_+''' = e^{-\frac{1}{2}\nu\pi i} B_+ \qquad (1.46.1)$$

$$A_-''' = e^{\frac{1}{2}\nu\pi i} B_- \qquad (1.46.2)$$

From (1.45) we can write the solution in the case (1.36.2) in the form

$$\vartheta = -x\,(A_+''' I_{\frac{2}{3}}(y) + A_-''' I_{-\frac{2}{3}}(y)\,) \qquad (1.47$$

After both transition and phase-jump (the cases (1.36.3) and (1.36.6)), the solution (1.29) is already in terms of real functions of a real variable, so we only need to change the notation to bring it into line with the others. We write A_+' and A_-' for B_+ and B_- and obtain

$$\vartheta = x(A_+' J_{\frac{2}{3}}(y) + A_-' J_{-\frac{2}{3}}(y)\,) \quad . \qquad (1.48)$$

Below we shall list the solutions in the different cases, together with the corresponding expressions for $\frac{1}{x}\frac{d\vartheta}{dx}$, as we shall demand both ϑ and $\frac{1}{x}\frac{d\vartheta}{dx}$ to be continuous at $x = 0$ and at $x = x_o$. We first quote a few useful formulae. From eq. (1.34) we find

$$\frac{dy}{dx} = \operatorname{sgn}(x)\ |x|^{1/2} \quad . \tag{1.49}$$

From ref. [8], p. XVII, eqs. (9) and (10), we have :

$$\frac{d}{dy} J_\nu(y) = J_{\nu-1}(y) - \frac{\nu}{y} J_\nu(y) = -J_{\nu+1}(y) + \frac{\nu}{y} J_\nu(y) \tag{1.50.1}$$

$$\frac{d}{dy} I_\nu(y) = I_{\nu-1}(y) - \frac{\nu}{y} I_\nu(y) = I_{\nu+1}(y) + \frac{\nu}{y} I_\nu(y) \tag{1.50.2}$$

By means of (1.49) and (1.50) we derive :

<u>$x_o < 0$</u> : The phase is shifted too early :

$x < x_o$

$$\vartheta = -x\left(A_+ J_{\frac{2}{3}}(y) + A_- J_{-\frac{2}{3}}(y)\right) \tag{1.51.1}$$

$$\frac{1}{x}\frac{d\vartheta}{dx} = (-x)^{1/2}\left(A_+ J_{-\frac{1}{3}}(y) - A_- J_{\frac{1}{3}}(y)\right) \tag{1.51.2}$$

$x_o<x<0$

$$\vartheta = -x\left(A_+''' I_{\frac{2}{3}}(y) + A_-''' I_{-\frac{2}{3}}(y)\right) \tag{1.51.3}$$

$$\frac{1}{x}\frac{d\vartheta}{dx} = (-x)^{1/2}\left(A_+''' I_{-\frac{1}{3}}(y) + A_-''' I_{\frac{1}{3}}(y)\right) \tag{1.51.4}$$

$0 < x$

$$\vartheta = x\left(A_+' J_{\frac{2}{3}}(y) + A_-' J_{-\frac{2}{3}}(y)\right) \tag{1.51.5}$$

$$\frac{1}{x}\frac{d\vartheta}{dx} = x^{1/2}\left(A_+' J_{-\frac{1}{3}}(y) - A_-' J_{\frac{1}{3}}(y)\right) \tag{1.51.6}$$

$\underline{x_o > 0}$: The phase is shifted too late :

$x < 0$
$$\vartheta = -x \left(A_+ J_{\frac{2}{3}}(y) + A_- J_{-\frac{2}{3}}(y) \right) \tag{1.51.7}$$
$$\frac{1}{x}\frac{d\vartheta}{dx} = (-x)^{\frac{1}{2}} \left(A_+ J_{-\frac{1}{3}}(y) - A_- J_{\frac{1}{3}}(y) \right) \tag{1.51.8}$$

$0<x<x_o$
$$\vartheta = x \left(A_+'' I_{\frac{2}{3}}(y) + A_-'' I_{-\frac{2}{3}}(y) \right) \tag{1.51.9}$$
$$\frac{1}{x}\frac{d\vartheta}{dx} = x^{\frac{1}{2}} \left(A_+'' I_{-\frac{1}{3}}(y) + A_-'' I_{\frac{1}{3}}(y) \right) \tag{1.51.10}$$

$x_o < x$
$$\vartheta = x \left(A_+' J_{\frac{2}{3}}(y) + A_-' J_{-\frac{2}{3}}(y) \right) \tag{1.51.11}$$
$$\frac{1}{x}\frac{d\vartheta}{dx} = x^{\frac{1}{2}} \left(A_+' J_{-\frac{1}{3}}(y) - A_-' J_{\frac{1}{3}}(y) \right) \tag{1.51.12}$$

CONTINUITY OF THE SOLUTION

We now demand continuity of ϑ and of $\frac{1}{x}\frac{d\vartheta}{dx}$ at $x = 0$ and at $x = x_o$. In this way we find relationships between A_+ , A_- and A_+' , A_-' . We shall not be particularily interested in A_+''' , A_-''' and A_+'' , A_-'' .

At $x = 0$, we have $\frac{d\vartheta}{dx} = 0$ irrespective of the values of the coefficients. Therefore $\frac{d\vartheta}{dx}$ is automatically continuous at $x = 0$, whatsoever are the coefficients, and it gives us no new relationship between the coefficients at both sides of $x = 0$ if we demand $\frac{d\vartheta}{dx}$ to be continuous at $x = 0$. However, we shall demand ϑ and ΔE to be continuous functions of x everywhere, especially at $x = 0$ and $x = x_o$. From eqs. (1.2) and (1.16) we see that demanding continuity of ΔE is the same as demanding continuity of $\frac{1}{x}\frac{d\vartheta}{dx}$.

By y_o we shall denote the quantity

$$y_o = \frac{2}{3} |x_o|^{\frac{3}{2}} , \tag{1.52}$$

in accordance with (1.34) . In the following, we shall not write the argument of J's and I's if it is equal to y_o .

Applying now these continuity requirements to the solutions tabulated in eq. (1.51), we get :

$x_0 < 0$: The phase is switched too early :

$$-x_0 \left(A_+ J_{\frac{2}{3}} + A_- J_{-\frac{2}{3}}\right) = -x_0 \left(A_+''' I_{\frac{2}{3}} + A_-''' I_{-\frac{2}{3}}\right) \qquad (1.53.1)$$

$$(-x_0)^{1/2} \left(A_+ J_{-\frac{1}{3}} - A_- J_{\frac{1}{3}}\right) = (-x_0)^{1/2} \left(A_+''' I_{-\frac{1}{3}} + A_-''' I_{\frac{1}{3}}\right) \qquad (1.53.2)$$

$$\lim_{x\to 0-} \left[-x\left(A_+''' I_{\frac{2}{3}}(y) + A_-''' I_{-\frac{2}{3}}(y)\right)\right] = \lim_{x\to 0+} \left[x\left(A_+' J_{\frac{2}{3}}(y) + A_-' J_{-\frac{2}{3}}(y)\right)\right] \qquad (1.53.3)$$

$$\lim_{x\to 0-} \left[(-x)^{1/2}\left(A_+''' I_{-\frac{1}{3}}(y) + A_-''' I_{\frac{1}{3}}(y)\right)\right] = \lim_{x\to 0+} \left[x^{1/2}\left(A_+' J_{-\frac{1}{3}}(y) - A_-' J_{\frac{1}{3}}(y)\right)\right] . \qquad (1.53.4)$$

$x_0 > 0$: The phase is switched too late :

$$\lim_{x\to 0-} \left[-x \left(A_+ J_{\frac{2}{3}}(y) + A_- J_{-\frac{2}{3}}(y)\right)\right] = \lim_{x\to 0+} \left[x \left(A_+'' I_{\frac{2}{3}}(y) + A_-'' I_{-\frac{2}{3}}(y)\right)\right] \qquad (1.53.5)$$

$$\lim_{x\to 0-} \left[(-x)^{1/2}\left(A_+ J_{-\frac{1}{3}}(y) - A_- J_{\frac{1}{3}}(y)\right)\right] = \lim_{x\to 0+} \left[x^{1/2}\left(A_+'' I_{-\frac{1}{3}}(y) + A_-'' I_{\frac{1}{3}}(y)\right)\right] \qquad (1.53.6)$$

$$x_0 \left(A_+'' I_{\frac{2}{3}} + A_-'' I_{-\frac{2}{3}}\right) = x_0 \left(A_+' J_{\frac{2}{3}} + A_-' J_{-\frac{2}{3}}\right) \qquad (1.53.7)$$

$$x_0^{1/2} \left(A_+'' I_{-\frac{1}{3}} + A_-'' I_{\frac{1}{3}}\right) = x_0^{1/2} \left(A_+' J_{-\frac{1}{3}} - A_-' J_{\frac{1}{3}}\right) \quad . \qquad (1.53.8)$$

To simplify eqs. (1.53.3), (1.53.4) , (1.53.5) , (1.53.6) we note that

$$J_\nu(y) = \frac{1}{\Gamma(\nu + 1)} \left(\frac{1}{2} y\right)^\nu + O\left(y^{\nu+2}\right) \qquad (1.54.1)$$

$$I_\nu(y) = \frac{1}{\Gamma(\nu + 1)} \left(\frac{1}{2} y\right)^\nu + O\left(y^{\nu+2}\right) \qquad (1.54.2)$$

From this we find

$$\lim_{x\to 0-} \left[-x\, J_{\frac{2}{3}}(y)\right] = \lim_{x\to 0+} \left[x\, J_{\frac{2}{3}}(y)\right] = \lim_{x\to 0-} \left[-x\, I_{\frac{2}{3}}(y)\right] = \lim_{x\to 0+} \left[x\, I_{\frac{2}{3}}(y)\right] = \lim_{x\to 0-} \left[(-x)^{1/2}\, J_{\frac{1}{3}}(y)\right]$$

$$= \lim_{x\to 0+} \left[x^{1/2}\, J_{\frac{1}{3}}(y)\right] = \lim_{x\to 0-} \left[(-x)^{1/2}\, I_{\frac{1}{3}}(y)\right] = \lim_{x\to 0+} \left[x^{1/2}\, I_{\frac{1}{3}}(y)\right] = 0 \quad , \qquad (1.55.1)$$

while

$$\lim_{x\to 0-}\left[-x\,J_{-\frac{2}{3}}(y)\right] = \lim_{x\to 0+}\left[x\,J_{-\frac{2}{3}}(y)\right] = \lim_{x\to 0-}\left[-x\,I_{-\frac{2}{3}}(y)\right] = \lim_{x\to 0+}\left[x\,I_{-\frac{2}{3}}(y)\right] = \frac{3^{2/3}}{\Gamma(\frac{1}{3})} \quad (1.55.2)$$

and

$$\lim_{x\to 0-}\left[(-x)^{1/2}\,J_{-\frac{1}{3}}(y)\right] = \lim_{x\to 0+}\left[x^{1/2}\,J_{-\frac{1}{3}}(y)\right] = \lim_{x\to 0-}\left[(-x)^{1/2}\,I_{-\frac{1}{3}}(y)\right] = \lim_{x\to 0+}\left[x^{1/2}\,I_{\frac{1}{3}}(y)\right]$$

$$= \frac{3^{1/3}}{\Gamma(\frac{2}{3})} \quad . \qquad (1.55.3)$$

This gives upon insertion in (1.53.3) - (1.53.6) :

$\underline{x_o < 0 :}$

$$A_-''' = A_-' \qquad (1.56.1)$$

$$A_+''' = A_+' \qquad (1.56.2)$$

$\underline{x_o > 0}$

$$A_- = A_-'' \qquad (1.56.3)$$

$$A_+ = A_+''$$

This is substituted into (1.53.1) , (1.53.2) , (1.53.7) , (1.53.8) . We then find :

$\underline{x_o < 0 :}$

$$\begin{pmatrix} J_{\frac{2}{3}} , & J_{-\frac{2}{3}} \\ J_{-\frac{1}{3}} , & -J_{\frac{1}{3}} \end{pmatrix} \begin{pmatrix} A_+ \\ A_- \end{pmatrix} = \begin{pmatrix} I_{\frac{2}{3}} , & I_{-\frac{2}{3}} \\ I_{-\frac{1}{3}} , & I_{\frac{1}{3}} \end{pmatrix} \begin{pmatrix} A_+' \\ A_-' \end{pmatrix} \qquad (1.57.1)$$

$\underline{x_o < 0 :}$

$$\begin{pmatrix} I_{\frac{2}{3}} , & I_{-\frac{2}{3}} \\ I_{-\frac{1}{3}} , & I_{\frac{1}{3}} \end{pmatrix} \begin{pmatrix} A_+ \\ A_- \end{pmatrix} = \begin{pmatrix} J_{\frac{2}{3}} , & J_{-\frac{2}{3}} \\ J_{-\frac{1}{3}} , & -J_{\frac{1}{3}} \end{pmatrix} \begin{pmatrix} A_+' \\ A_-' \end{pmatrix} \qquad (1.57.2)$$

We solve for A_+' , A_-' :

$\underline{x_o < 0 :}$

$$\begin{pmatrix} A_+' \\ A_-' \end{pmatrix} = D \begin{pmatrix} A_+ \\ A_- \end{pmatrix} \qquad (1.58.1)$$

$\underline{x_o > 0 :}$

$$\begin{pmatrix} A_+' \\ A_-' \end{pmatrix} = E \begin{pmatrix} A_+ \\ A_- \end{pmatrix} \qquad (1.58.2)$$

with

$$D = \begin{pmatrix} I_{\frac{2}{3}} , & I_{-\frac{2}{3}} \\ I_{-\frac{1}{3}} , & I_{\frac{1}{3}} \end{pmatrix}^{-1} \begin{pmatrix} J_{\frac{2}{3}} , & J_{-\frac{2}{3}} \\ J_{-\frac{1}{3}} , & -J_{\frac{1}{3}} \end{pmatrix} \quad x_o < 0 \tag{1.59.1}$$

$$E = \begin{pmatrix} J_{\frac{2}{3}} , & J_{-\frac{2}{3}} \\ J_{-\frac{1}{3}} , & -J_{\frac{1}{3}} \end{pmatrix}^{-1} \begin{pmatrix} I_{\frac{2}{3}} , & I_{-\frac{2}{3}} \\ I_{-\frac{1}{3}} , & I_{\frac{1}{3}} \end{pmatrix} \quad x_o > 0 \quad . \tag{1.59.2}$$

Eqs. (1.58) give the amplitudes of $J_{\frac{2}{3}}(y)$ and $J_{-\frac{2}{3}}(y)$ after transition in terms of the corresponding amplitudes before transition. The matrices D and E we shall call the transformation matrices.

PROPERTIES OF THE TRANSFORMATION MATRICES

The transformation matrices D and E describe how the synchrotron oscillations are influenced by passing the transition with different timing errors x_o of the phase-jump. Note that the argument of all these Bessel functions is y_o , which is defined so as to be positive whether the phase-jump is early or late; one should therefore not be surprised that the difference between the early and late cases appears as different formulae for the two matrices, D and E respectively. We shall investigate some of the properties of the transformation matrices.

Using the relations (ref. [8], p. XXVIII, eqs (19) , (20)

$$J_\nu(x)\ J_{1-\nu}(x) + J_{-\nu}(x)\ J_{\nu-1}(x) = \frac{2 \sin \nu \pi}{\pi x} \tag{1.60.1}$$

$$I_\nu(x)\ I_{1-\nu}(x) - I_{-\nu}(x)\ I_{\nu-1}(x) = \frac{2 \sin \nu \pi}{\pi x} \tag{1.60.2}$$

we can show that

$$|D| = |E| = 1 \tag{1.61}$$

It is readily seen that with timing errors x_o of the same amplitude (and opposite signs) in the two cases, so that the y_o's are the same, then the matrices D and E are each other's inverses :

$$D(-|x_o|)\ E\ (|x_o|) = E\ (|x_o|)\ D\ (-|x_o|) = \binom{10}{01} \ . \tag{1.62}$$

By developing in series all the J's and I's involved, we calculate the matrices to first order in x_o, which will be useful in Chapter 3. We find :

$$D = \begin{pmatrix} 1 + 0\,(x_o^3) \;, & \frac{2}{\pi}\, 3^{1/3} \cos(\frac{\pi}{6})\, (\Gamma(\frac{2}{3}))^2\, x_o + 0\,(x_o^4) \\ 0\,(x_o^5) & ,\; 1 + 0\,(x_o^3) \end{pmatrix}, \; x_o < 0 \quad (1.63.1)$$

$$E = \begin{pmatrix} 1 + 0\,(x_o^3) \;, & \frac{2}{\pi}\, 3^{1/3} \cos(\frac{\pi}{6})\, (\Gamma(\frac{2}{3}))^2\, x_o + 0\,(x_o^4) \\ 0\,(x_o^5) & ,\; 1 + 0\,(x_o^3) \end{pmatrix}, \; x_o > 0 \,. \quad (1.63.2)$$

This shows that

$$\lim_{x_o \to 0-} D = \lim_{x_o \to 0+} E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,. \qquad (1.64$$

We shall rewrite our solution in the Bessel-Neumann from, (1.27), instead of the form (1.29). Before transition the solution shall be

$$\vartheta = |x|\, (A_J\, J_{\frac{2}{3}}(y) + A_N\, N_{\frac{2}{3}}(y)) \quad , \quad \begin{matrix} x < 0 \\ x < x_o \end{matrix} \qquad (1.65)$$

and after transition

$$\vartheta = |x|\, (A_J'\, J_{\frac{2}{3}}(y) + A_N'\, N_{\frac{2}{3}}(y)) \,, \quad \begin{matrix} x > 0 \\ x > x_o \end{matrix} \,. \qquad (1.66)$$

From (1.30) and (1.31) we have

$$\begin{pmatrix} A_J' \\ A_N' \end{pmatrix} = d \begin{pmatrix} A_J \\ A_N \end{pmatrix} \quad \text{for } x_o < 0 \qquad (1.67.1)$$

$$\begin{pmatrix} A_J' \\ A_N' \end{pmatrix} = e \begin{pmatrix} A_J \\ A_N \end{pmatrix} \quad \text{for } x_o > 0 \qquad (1.67.2)$$

with

$$d = \begin{pmatrix} 1 \ , & -\sin\left(\frac{\pi}{6}\right) \\ 0 \ , & -\cos\left(\frac{\pi}{6}\right) \end{pmatrix} \qquad D \begin{pmatrix} 1 \ , & -\dfrac{\sin(\frac{\pi}{6})}{\cos(\frac{\pi}{6})} \\ 0 \ , & -\dfrac{1}{\cos(\frac{\pi}{6})} \end{pmatrix} \quad , x_o < 0 \tag{1.68.1}$$

$$e = \begin{pmatrix} 1 \ , & -\sin\left(\frac{\pi}{6}\right) \\ 0 \ , & -\cos\left(\frac{\pi}{6}\right) \end{pmatrix} \qquad E \begin{pmatrix} 1 \ , & -\dfrac{\sin(\frac{\pi}{6})}{\cos(\frac{\pi}{6})} \\ 0 \ , & -\dfrac{1}{\cos(\frac{\pi}{6})} \end{pmatrix} \quad , x_o > 0 \ . \tag{1.68.2}$$

The new transformation matrices d and e have properties very similar to D and E. (They are, in fact, connected with the old ones by a similarity transformation). It is easily shown that

$$|d| = |e| = 1 \tag{1.69}$$

and that

$$d(-|x_o|)\ e(|x_o|) = e(|x_o|)\ d(-|x_o|) = \binom{10}{01} \tag{1.70}$$

By straight forward substitution we find that

$$d = \begin{pmatrix} 1 + O(x_o^3) \ , & -\frac{2}{\pi} 3^{1/3} \left(\Gamma(\frac{2}{3})\right)^2 x_o + O(x_o^3) \\ O(x_o^5) \ , & 1 + O(x_o^3) \end{pmatrix} \quad , x_o < 0 \tag{1.71.1}$$

$$e = \begin{pmatrix} 1 + O(x_o^3) \ , & -\frac{2}{\pi} 3^{1/3} \left(\Gamma(\frac{2}{3})\right)^2 x_o + O(x_o^3) \\ O(x_o^5) \ , & 1 + O(x_o^3) \end{pmatrix} \quad , x_o > 0 \ . \qquad (1.71.2$$

The numerical value of the coefficient is

$$-\frac{2}{\pi} 3^{1/3} \left(\Gamma(\tfrac{2}{3})\right)^2 = -1.683 \tag{1.72}$$

We see that

$$\lim_{x_o \to 0-} d = \lim_{x_o \to 0+} e = \binom{10}{01} \ . \tag{1.73}$$

To facilitate the subsequent discussion, let us introduce the symbols A and ψ, defined thus

$$A_J = A \cos\psi \tag{1.74.1}$$

$$A_N = A \sin\psi \tag{1.74.2}$$

and in a corresponding way

$$A_J' = A' \cos\psi' \tag{1.75.1}$$

$$A_N' = A' \sin\psi' \tag{1.75.2}$$

Let us regard the asymptotic region well before or well after transition. We then use the following asymptotic formulae (ref. [9], p. 1563) :

$$J_\nu(y) \approx \left(\frac{2}{\pi y}\right)^{1/2} \cos\left(y - \frac{1}{2}\pi\left(\nu + \frac{1}{2}\right)\right) \tag{1.76.1}$$

$$N_\nu(y) \approx \left(\frac{2}{\pi y}\right)^{1/2} \sin\left(y - \frac{1}{2}\pi\left(\nu + \frac{1}{2}\right)\right) \tag{1.76.2}$$

For shortness we put

$$\delta = \frac{1}{2}\pi\left(\frac{2}{3} + \frac{1}{2}\right) = \frac{7}{12}\pi \ . \tag{1.77}$$

For $x \to -\infty$ eq. (1.65) may be written as

$$\vartheta \approx |x|\left(A \cos\psi\left(\frac{2}{\pi y}\right)^{1/2} \cos(y-\delta) + A \sin\psi\left(\frac{2}{\pi y}\right)^{1/2} \sin(y-\delta)\right). \tag{1.78}$$

We substitute (1.34) and have

$$\vartheta \approx \left(\frac{3}{\pi}\right)^{1/2} A\, |x|^{1/4} \cos(y - \delta - \psi) \ , \quad x \to -\infty \ . \tag{1.79}$$

Correspondingly, eq. (1.66) is written as

$$\vartheta \approx \left(\frac{3}{\pi}\right)^{1/2} A' |x|^{1/4} \cos(y - \delta - \psi') \ , \quad x \to \infty \ . \tag{1.80}$$

These asymptotic solutions are a good approximation provided one is sufficiently far from transition. In this connection "sufficiently far" means

$$|t| >> T = 1.85 \text{ ms} \ . \tag{1.81}$$

(see Appendix 1). On each side of transition there is a considerable interval of time which is far enough away for this condition to be valid, yet near enough for the approximation

$$E_s \approx E_{tr} \tag{1.82}$$

which we introduced at (1.16) to (1.18), to be good also. See ref. [6] .

From (1.79) and (1.80) we see that in the region well before or well after transition, A or A' may be regarded as a total amplitude while ψ and ψ' may be regarded as a phase angle. See eqs. (1.32) . The particles with some constant A and all possible values of ψ can be taken as the outline of a bunch in equilibrium, approaching transition; then the dependance of A' and ψ' on ψ describes completely what deformation the bunch has suffered after passing through the transition region.

We substitute (1.74) and (1.75) into (1.67) and have

$$\frac{A'}{A} \cdot \begin{pmatrix} \cos \psi' \\ \sin \psi' \end{pmatrix} = d \begin{pmatrix} \cos \psi \\ \sin \psi \end{pmatrix} \qquad x_o < 0 \tag{1.83.1}$$

$$\frac{A'}{A} \cdot \begin{pmatrix} \cos \psi' \\ \sin \psi' \end{pmatrix} = e \begin{pmatrix} \cos \psi \\ \sin \psi \end{pmatrix} \quad , \quad x_o > 0 \ . \tag{1.83.2}$$

These equations give the amplitude ratio $\frac{A'}{A}$ and the new phase angle ψ' as functions of the old phase angle ψ . We want to find the maximum of $\frac{A'}{A}$ under variation of ψ , and the corresponding values of ψ and ψ' .

We shall use this notation for the elements of the matrix d :

$$d = \begin{pmatrix} d_1 \ , & d_2 \\ d_3 \ , & d_4 \end{pmatrix} \ . \tag{1.84}$$

From (1.70) it follows that

$$e = \begin{pmatrix} d_4 \ , & -d_2 \\ -d_3 \ , & d_1 \end{pmatrix} \ . \tag{1.85}$$

Because d has unity determinant, eq. (1.69), it must be possible to express it in the form

$$d = \begin{pmatrix} \cos\alpha' , & -\sin\alpha' \\ \sin\alpha' , & \cos\alpha' \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \cos\alpha , & \sin\alpha \\ -\sin\alpha , & \cos\alpha \end{pmatrix}, \quad a > 1 \tag{1.86}$$

From (1.70) we then have

$$e = \begin{pmatrix} \cos\alpha & , -\sin\alpha \\ \sin\alpha & , \cos\alpha \end{pmatrix} \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \cos\alpha' , & \sin\alpha' \\ -\sin\alpha' , & \cos\alpha' \end{pmatrix}, \quad a > 1 \tag{1.87}$$

Eq. (1.86) is fulfilled if (see ref. [10], p. 9) :

$$a = \frac{1}{2}\left((d_1^2 + d_2^2 + d_3^2 + d_4^2 + 2)^{1/2} + (d_1^2 + d_2^2 + d_3^2 + d_4^2 - 2)^{1/2} \right), \tag{1.88}$$

$$\mathrm{tg}\,\alpha = \frac{a^2\, d_3 + d_2}{a^2\, d_1 - d_4} = \frac{a^2\, d_4 - d_1}{a^2\, d_2 + d_3} \tag{1.89}$$

(the two distinct values of α that satisfy this are both solutions)

or

$$\left\{ \begin{array}{l} \mathrm{tg}\, 2\alpha = 2\, \dfrac{d_1\, d_3 + d_2\, d_4}{(d_1^2 + d_2^2) - (d_3^2 + d_4^2)} \\ \cos(2\alpha) \cdot [(d_1^2 + d_2^2) - (d_3^2 + d_4^2)] > 0 \end{array} \right. , \tag{1.90}$$

$$\mathrm{tg}\,\alpha' = \frac{a^2\, d_2 + d_3}{a^2\, d_1 - d_4} = \frac{a^2\, d_4 - d_1}{a^2\, d_3 + d_2} \tag{1.91}$$

(the two distinct values of α' that satisfy this are both solutions

or

$$\left\{ \begin{array}{l} \mathrm{tg}\, 2\alpha' = 2\, \dfrac{d_1\, d_2 + d_3\, d_4}{(d_1^2 + d_3^2) - (d_2^2 + d_4^2)} \\ \cos(2\alpha') \cdot [(d_1^2 + d_3^2) - (d_2^2 + d_4^2)] > 0 \end{array} \right. . \tag{1.92}$$

It can be shown that a^2 is the largest eigenvalue of $(\tilde{d}\, d)$ and of $(d\, \tilde{d})$.

With the representations (1.86) and (1.87) for d and e , equations (1.83) take the simple forms

$$\frac{A'}{A} \cdot \begin{pmatrix} \cos(\psi' - \alpha') \\ \sin(\psi' - \alpha') \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \cos(\psi - \alpha) \\ \sin(\psi - \alpha) \end{pmatrix}, \quad x_0 < 0 \qquad (1.93.1)$$

$$\frac{A_\prime}{A} \cdot \begin{pmatrix} \cos(\psi' - \alpha) \\ \sin(\psi' - \alpha) \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \cos(\psi - \alpha') \\ \sin(\psi - \alpha') \end{pmatrix}, \quad x_0 > 0 . \qquad (1.93.2)$$

From this we find :

For $\underline{x_0 < 0}$:

$$\left(\frac{A_\prime}{A}\right)_{max} = a \text{ with } \psi = \alpha , \quad \psi' = \alpha' , \qquad (1.94.1)$$

For $\underline{x_0 > 0}$:

$$\left(\frac{A_\prime}{A}\right)_{max} = a \text{ with } \psi = \alpha' + \frac{\pi}{2} , \quad \psi' = \alpha + \frac{\pi}{2} . \qquad (1.94.2)$$

From these equations we see that two timing errors of equal magnitude and opposite signs give the same maximum amplitude blow-up, as conjectured by Johnsen [6] .

Up to this point, our formulae are valid to all orders in the timing error t_0 . By substituting t_0 , we can compute the blow-up and the relevant phases accurately. We shall now derive the corresponding first-order formulae, because this is needed in Chapter 3 . To find a , α , and α' , we now substitute (1.71.1) into (1.88) - (1.92) :

$$a = 1 + \frac{3^{1/3}}{\pi} \left(\Gamma\left(\tfrac{2}{3}\right)\right)^2 |x_0| + O(x_0^{\,2}) \qquad (1.95.1)$$

$$\alpha = \frac{\pi}{4} + n\pi + \frac{3^{1/3}}{2\pi} \left(\Gamma\left(\tfrac{2}{3}\right)\right)^2 |x_0| + O(x_0^{\,2}) \qquad (1.95.2)$$

$$\alpha' = \frac{\pi}{4} + n\pi - \frac{3^{1/3}}{2\pi} \left(\Gamma\left(\tfrac{2}{3}\right)\right)^2 |x_0| + o(x_0^{\,2}) \qquad (1.95.3)$$

Introducing numerical values, we have :

$$a = 1 + 0.842 \; |x_o| + 0 \, (x_o{}^2) \tag{1.96.1}$$

$$\alpha = \frac{\pi}{4} + n \pi + 0.421 \; |x_o| + 0 \, (x_o{}^2) \tag{1.96.2}$$

$$\alpha' = \frac{\pi}{4} + n \pi - 0.421 \; |x_o| + 0 \, (x_o{}^2) \quad . \tag{1.96.3}$$

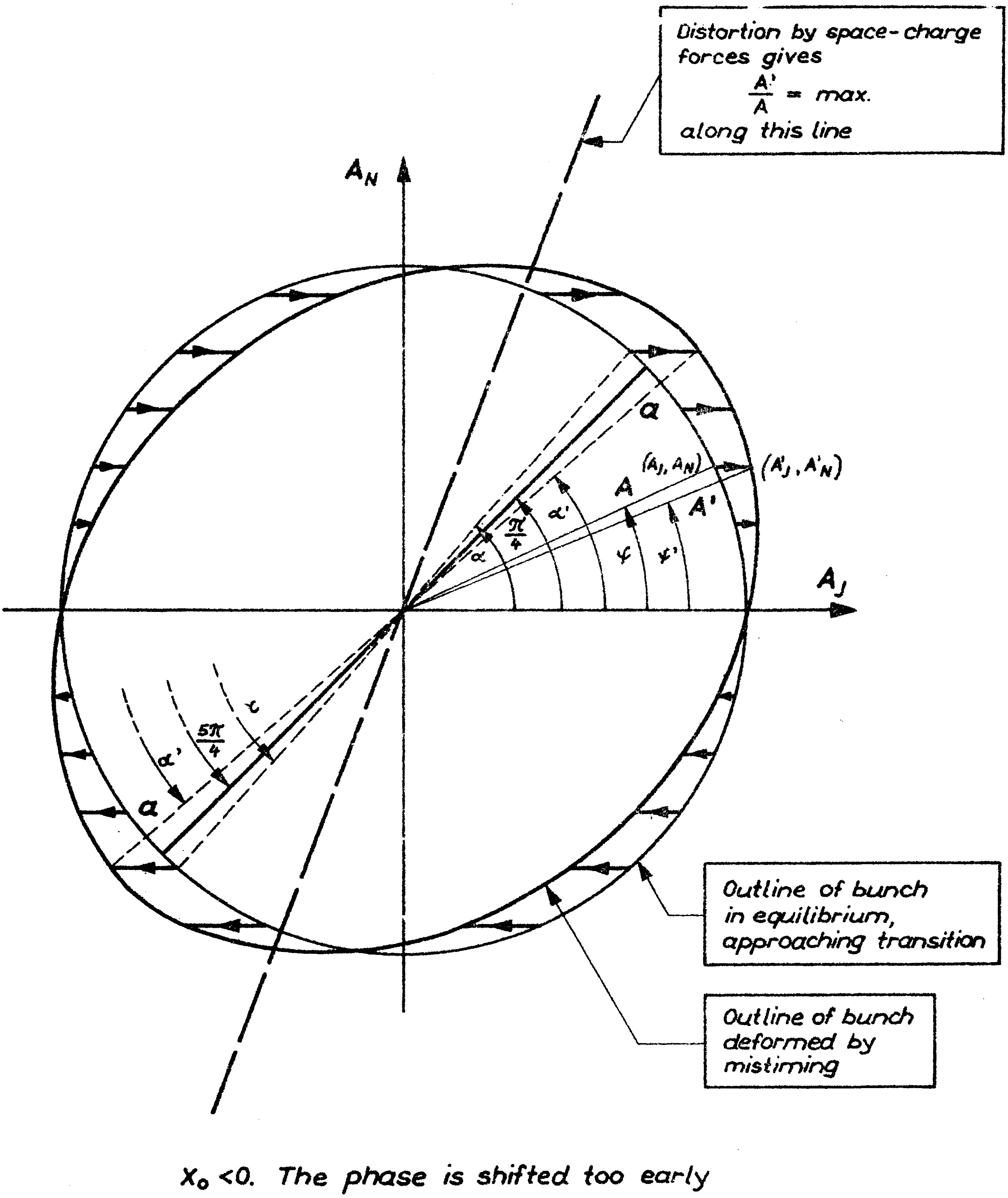
Introducing $T = 1.85$ ms (Appendix 1), we have :

$$a = 1 + 0.455 \frac{|t_o|}{\text{ms}} + 0 \, (t_o{}^2) \tag{1.97.1}$$

$$\alpha = \frac{\pi}{4} + n \pi + 0.227 \frac{|t_o|}{\text{ms}} + 0 \, (t_o{}^2) \tag{1.97.2}$$

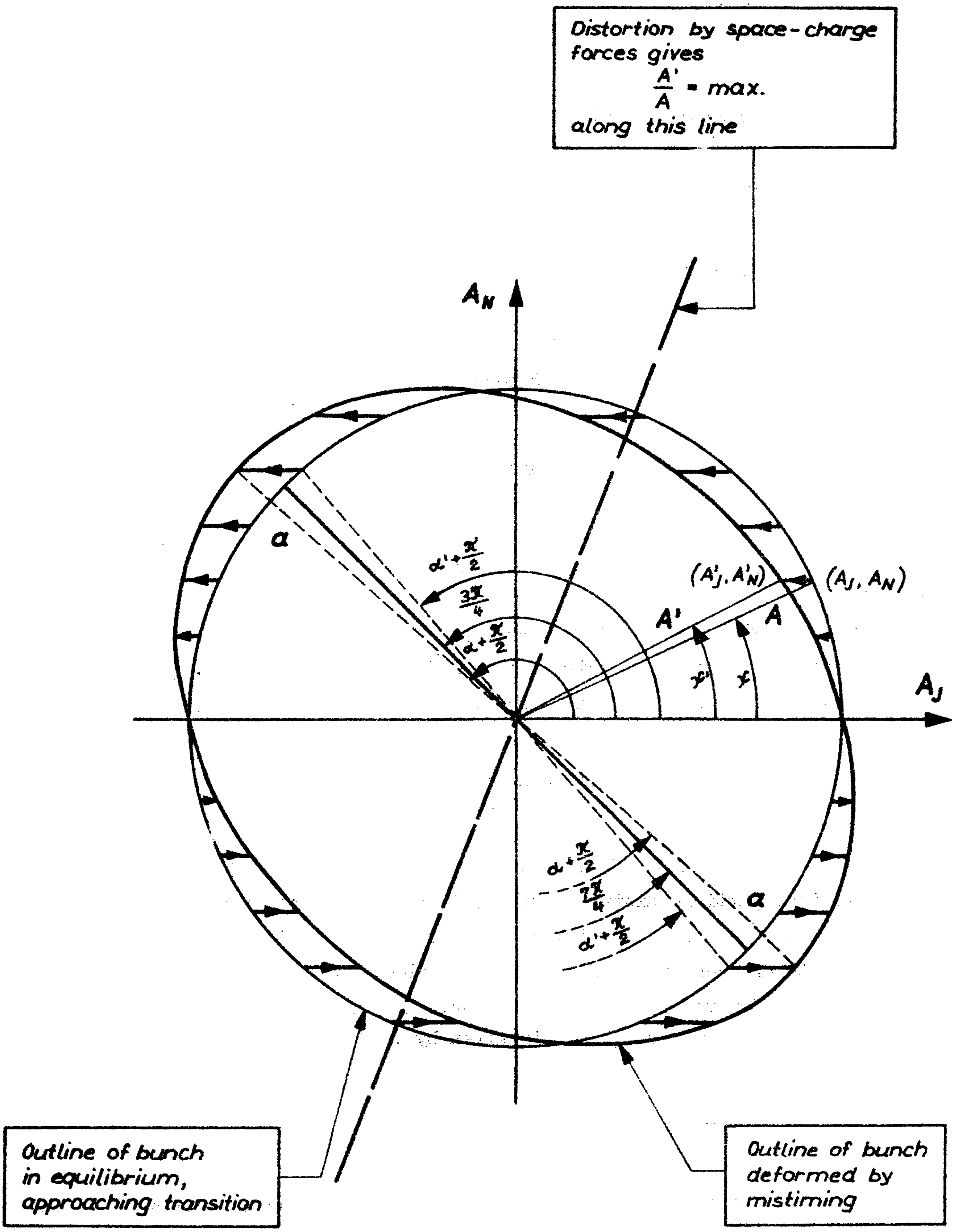
$$\alpha' = \frac{\pi}{4} + n \pi - 0.227 \frac{|t_o|}{\text{ms}} + 0 \, (t_o{}^2) \quad . \tag{1.97.3}$$

An illustration of these results is found in Fig. 1 and Fig. 2.



$X_0 < 0$. The phase is shifted too early

Fig. 1



$X_0 > 0$. The phase is shifted too late

Fig. 2

2. LONGITUDINAL SPACE-CHARGE FORCES, NO TIMING ERROR

THE EQUATION OF MOTION

We shall now modify our equations to include space-charge forces. Let us consider a bunch of peak amplitude $\hat{\vartheta}$ and of elliptical shape in the phase plane. The bunch contains N/h particles. We shall assume that the centre of the bunch is at the phase stable point $\varphi = \varphi_s$, $\vartheta = 0$ (this is not always the case). The energy gain per turn is then

$$eV \sin \varphi + e E_\vartheta \cdot 2 \pi R \tag{2.1.1}$$

instead of

$$eV \sin \varphi \tag{2.1.2}$$

as it was in the absence of space-charge forces. Here, E_ϑ denotes the component in the azimuthal direction due to space-charge. Eq. (1.1), valid when there are no space-charge forces, must now be replaced by

$$\frac{d}{dt} \left(\frac{\Delta E}{\omega_s}\right) = \frac{eV}{2\pi} (\sin \varphi - \sin \varphi_s) + e E_\vartheta R \quad . \tag{2.1.3}$$

To evaluate E_ϑ, we assume that (see ref. [11], p. 61) the longitudinal particle distribution within a bunch is of parabolic form with a peak of $NB^{-1}/ 2 \pi R$ particles per unit length. This implies that the half-length of each bunch is $(3/4)\, B\, \ell$, where ℓ is the distance between bunches. B denotes the bunching factor. Let x be the distance from the centre of the bunch. Then E_ϑ is given by :

$$E_\vartheta = - \frac{16}{9\pi} \frac{g_o\, e\, N}{4 \pi \epsilon_o \gamma^2 R} \frac{x}{B^3 \ell^2} \tag{2.2.1}$$

where the definition of g_o is that $4 \pi \epsilon_o / g_o$ is the capacitance per unit length between the beam and the vacuum chamber.

We then substitute

$$h = \frac{2\pi R}{\ell} \quad \Rightarrow \quad \ell = \frac{2\pi R}{h} \tag{2.2.2}$$

$$\vartheta = \frac{x}{R} h \quad \Rightarrow \quad x = \frac{\vartheta R}{h} \tag{2.2.3}$$

$$\hat{\vartheta} = \frac{\frac{3}{4} B \ell}{R} h \quad \Rightarrow \quad B\ell = \frac{4}{3} \frac{\hat{\vartheta} R}{h} \tag{2.2.4}$$

and find

$$E_\vartheta = -\frac{2}{3} \frac{h g_o e N}{4 \pi \epsilon_o \gamma^2 R^2} \frac{\vartheta}{\hat{\vartheta}^3} . \tag{2.2.5}$$

Here, $\hat{\vartheta}$ denotes half the bunch length. In using (2.2) we are taking only the linear term in the longitudinal space-charge force. This may or may not be a good approximation, depending on the distribution of particle density. The distribution which gives precisely a linear space-charge force is one in which phase-space density falls as one goes from the centre to the boundary of the bunch, and this is more realistic than assuming a uniform density in phase-space.

Thus

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_s}\right) = \frac{eV}{2\pi} (\sin \varphi - \sin \varphi_s) - \frac{3}{2} \frac{h g_o e^2 N}{4\pi \epsilon_o \gamma^2 R} \frac{\vartheta}{\hat{\vartheta}^3} . \tag{2.3}$$

Eq. (1.2) is still valid. But using (2.3) instead of (1.1), we, this time, find

$$\frac{d}{dt}\left(\frac{R_s^2}{hc^2} \frac{E}{\eta} \frac{d\varphi}{dt}\right) + \left(\frac{eV}{2\pi} \left|\cos \varphi_s\right| \operatorname{sgn}(t-t_o) + \frac{3}{2} \frac{h g_o e^2 N}{4\pi \epsilon_o \gamma^2 R} \frac{1}{\hat{\vartheta}^3}\right) \vartheta = 0 . \tag{2.4}$$

Eq. (2.4) now replaces (1.14). The only difference is the new term inside the bracket to the right.

We now define a quantity η_{sc}, being the ratio between the space-charge forces and the R.F. focusing forces. (We remark that this is not precisely the same quantity as that denoted by η in the 300 GeV report [11], pp. 60-61. η is defined to be the ratio between the space-charge forces and the R.F. focusing forces, calculated from a bunch shape and bunching factor in which the effects of this field has been neglected.)

$$\eta_{sc} = \frac{\frac{3}{2} \; \frac{h \, g_o \, e^2 \, N}{4\pi \, \epsilon_o \, \gamma^2 \, R} \; \frac{1}{\hat{\vartheta}^3}}{\frac{eV}{2\pi} \, |\cos \varphi_s| \, \mathrm{sgn}\,(t - t_o)} \tag{2.5}$$

This is substituted together with the expression (1.16) for η and we have

$$\frac{d}{dt}\left(\frac{1}{t}\frac{d\vartheta}{dt}\right) + \frac{\mathrm{sgn}\,(t - t_o)}{T^3}\,\vartheta = -\,\eta_{sc}\,\frac{\mathrm{sgn}\,(t - t_o)}{T^3}\,\vartheta\,, \tag{2.6}$$

the constant T being given by (1.19) .

Again we introduce x and x_o given by (1.21) and (1.22) . We now find the equation of motion

$$\frac{d}{dx}\left(\frac{1}{x}\frac{d\vartheta}{dx}\right) + \mathrm{sgn}\,(x - x_o)\,\vartheta = -\,\eta_{sc}\,\mathrm{sgn}\,(x - x_o)\,\vartheta \tag{2.7.1}$$

or equivalent

$$\frac{d^2\vartheta}{dx^2} - \frac{1}{x}\frac{d\vartheta}{dx} + \mathrm{sgn}\,(x - x_o)\,x\,\vartheta = -\,\eta_{sc}\,\mathrm{sgn}\,(x - x_o)\,x\,\vartheta\,, \tag{2.7.2}$$

which replaces (1.23) and (1.24) . The only difference between (1.24) and (2.7.2) is the term with η_{sc} on the right hand side of (2.7.2). This term is a very complicated one, as it does not only contain the coordinate ϑ of the particle in question, but also the coordinates of all the other particles in the bunch. These other coordinates enter through $\hat{\vartheta}$ in η_{sc}.

THE FIRST APPROXIMATION

We shall solve eq. (2.7) in the first approximation only : We regard the term on the right hand side as a perturbation and evaluate this term from the zeroth order approximation, that is, from the solution of eq. (1.24). Thereafter, we solve this approximate equation by "The method of variation of parameters", see e.g. ref. [12] , p. 193. "The method of variation of parameters" is an exact method, hence the only source of error is the first order approximation. Our solution will be correct only to the first power in the number of particles N . The expression which is found upon substitution of the zeroth order solution into the right hand side of eq. (2.7.2) we shall denote for shortness by $f(x)$. We thus have to solve an equation of the form

$$\frac{d^2\vartheta}{dx^2} - \frac{1}{x}\frac{d\vartheta}{dx} + \operatorname{sgn}(x - x_1)\, x\, \vartheta = f(x) \; . \qquad (2.8)$$

The solution of (1.24) is given by eq. (1.51) . We can always write this solution in the form

$$\vartheta_o(x) = A_1\, \vartheta_1(x) + A_2\, \vartheta_2(x)$$

where $\vartheta_1(x)$ and $\vartheta_2(x)$ are two independant solutions of (1.24) , valid in the whole range $< -\infty , +\infty >$, A_1 and A_2 being constants. Then the solution of (2.8) is

$$\vartheta(x) = a_1(x)\, \vartheta_1(x) + a_2(x)\, \vartheta_2(x) \qquad (2.10)$$

with

$$a_1(x) = -\int^{x} \frac{\vartheta_2(x')}{W\,(\vartheta_1(x')\,,\,\vartheta_2(x')\,,\,x')}\; f\,(x')\, dx' \qquad (2.11.1)$$

$$a_2(x) = \int^{x} \frac{\vartheta_1(x')}{W\,(\vartheta_1(x')\,,\,\vartheta_2(x')\,,\,x')}\; f\,(x')\, dx' \; . \qquad (2.11.2)$$

The Wronskian is defined as

$$W\,(\vartheta_1(x')\,,\,\vartheta_2(x')\,,\,x') = \vartheta_1(x')\,\frac{d\vartheta_2(x')}{dx'} - \frac{d\vartheta_1(x')}{dx'}\,\vartheta_2(x') \; . \qquad (2.11.3)$$

SPECIALIZATION : NO TIMING ERROR

It is possible to carry through this programme in the general case $x_o \neq 0$. However, the functions $\vartheta_1(x)$ and $\vartheta_2(x)$ will be complicated functions, composed partly of J's and partly of I's (see eq. (1.51)) . Such complications will also arise in the evaluation of $f(x)$. If we introduce the approximation

$$x_o = 0 \quad , \qquad (2.12)$$

that is, no timing error of the phase jump, great simplification arise in the calculation. By a simple trick we shall be able to treat timing errors to the first power of x_o in Chapter 3 , and this is exactly the accuracy needed for the subsequent calculations on compensation of the space-charge effect by a deliberate timing error.

Assuming (2.12), the solution of the unperturbed eq. (1.40) is

$$\vartheta_o = |x| \left(A_J \; J_{\frac{2}{3}}(y) + A_N \, N_{\frac{2}{3}}(y)\right) . \tag{2.13}$$

This solution is valid for all x . The index o indicates the zeroth order approximation.

Far from transition but still inside the range of validity of our approximation (1.16) , we have from eq. (1.79) :

$$\hat{\vartheta}_o \approx \left(\frac{3}{\pi}\right)^{1/2} \; A \; |x|^{1/4} \quad . \tag{2.14}$$

$\hat{\vartheta}_o$ denotes half the bunch length derived from the zeroth order approximation. A is the total amplitude for a particle whose phase orbit outlines a bunch.

From (2.13) we shall derive a generally valid expression for the bunch length. We have shown (1.79) that in the region well before or well after transition, letting ψ run through $0 \rightarrow 2\pi$ while keeping A constant, describes the outline of a possible bunch. But as phase orbits never cross each other, this must always be the case. By putting

$$\frac{d\vartheta_o}{d\psi} = 0 \tag{2.15}$$

we find

$$\vartheta_{o_{max}} = -\vartheta_{o_{min}} = A \; |x| \; \left[\left(J_{\frac{2}{3}}(y)\right)^2 + \left(N_{\frac{2}{3}}(y)\right)^2\right]^{1/2} , \tag{2.16}$$

and therefore

$$\hat{\vartheta}_o = A \; |x| \; \left[\left(J_{\frac{2}{3}}(y)\right)^2 + \left(N_{\frac{2}{3}}(y)\right)^2\right]^{1/2} . \tag{2.17}$$

Developing in series we have

$$\hat{\vartheta}_o \; (x = 0) = \frac{3^{2/3}}{\pi} \; \Gamma\left(\frac{2}{3}\right) A \; . \tag{2.18}$$

Using the zeroth approximation $\hat{\vartheta}_o$ for the bunch length $\hat{\vartheta}$, η_{sc} is approximated by η as defined in the 300 GeV report [11], pp. 60-61 :

$$\eta = \frac{\frac{3}{2}\ \frac{h\ g_o\ e^2\ N}{4\pi\ \epsilon_o\ \gamma^2\ R}}{\frac{eV}{2\pi}\ |\cos\ \varphi_s|}\ \frac{\text{sgn}\ (x)}{\hat{\vartheta}_o{}^3}\ . \tag{2.19}$$

For the perturbing function we have

$$f(x) = -\ \eta\ \text{sgn}\ (x)\ x\ \vartheta_o\ . \tag{2.20}$$

The strength of the space-charge forces we shall describe by specifying the value of η immediately after transition. This quantity we shall denote by η_o .

$$\eta_o = \eta\ (x = 0+)\ . \tag{2.21}$$

At any time x we have

$$\eta = \eta_o\ \left(\frac{\hat{\vartheta}_o\ (0)}{\hat{\vartheta}_o\ (x)}\right)^3\ \text{sgn}\ (x) \tag{2.22}$$

and substitution of (2.17) and (2.18) yields

$$\eta = \eta_o\ \frac{9}{\pi^3}\left(\Gamma(\tfrac{2}{3})\right)^3\left[(J_{\frac{2}{3}}(y)\ )^2 + (N_{\frac{2}{3}}(y)\ )^2\right]^{-3/2}\ \frac{\text{sgn}\ (x)}{|x|^3}\ . \tag{2.23}$$

We then substitute (2.23) and (2.13) into (2.20) and finally have

$$f(x) = -\ \frac{9}{\pi^3}\left(\Gamma(\tfrac{2}{3})\right)^3\ \frac{1}{x}\ \frac{A_J\ J_{\frac{2}{3}}(y) + A_N\ N_{\frac{2}{3}}(y)}{[(J_{\frac{2}{3}}(y)\ )^2 + (N_{\frac{2}{3}}(y)\ )^2]^{3/2}}\ \eta_o\ . \tag{2.24}$$

SOLUTION OF THE EQUATION OF MOTION

We now have to solve the equation of motion (2.8) with $f(x)$ given by (2.24). According to (2.10) - (2.11) the solution is

$$\vartheta = |x| \left(j(x)\, J_{\frac{2}{3}}(y) + n(x)\, N_{\frac{2}{3}}(y) \right) \tag{2.25}$$

with the functions $j(x)$ and $n(x)$ given by

$$j(x) = - \int^{x} \frac{|x'|\, N_{\frac{2}{3}}(y')}{W(|x'|\, J_{\frac{2}{3}}(y'),\ |x'|\, N_{\frac{2}{3}}(y'),\ x')} f(x')\, dx' + C_J \tag{2.26.1}$$

$$n(x) = \int^{x} \frac{|x'|\, J_{\frac{2}{3}}(y')}{W(|x'|\, J_{\frac{2}{3}}(y'),\ |x'|\, N_{\frac{2}{3}}(y'),\ x')} f(x')\, dx' + C_N\ . \tag{2.26.2}$$

The lower integration limits are both arbitrary. C_J and C_N are arbitrary constants. We shall take the two lower integration limits equal to the same number x_1. We have

$$W(|x'|J_{\frac{2}{3}}(y'),\ |x'|N_{\frac{2}{3}}(y'),\ x') = \frac{dy'}{dx'} W(|x'|J_{\frac{2}{3}}(y'),\ |x'|N_{\frac{2}{3}}(y'),\ y')\ . \tag{2.27}$$

We then work out the last Wronskian and find

$$W(|x'|J_{\frac{2}{3}}(y'),\ |x'|N_{\frac{2}{3}}(y'),\ y') = |x'|^2\, W(J_{\frac{2}{3}}(y'),\ N_{\frac{2}{3}}(y'),\ y')\ . \tag{2.28}$$

From ref. [9], p. 1564 or ref. [8], p. XIV we take :

$$W(J_{\frac{2}{3}}(y'),\ N_{\frac{2}{3}}(y'),\ y') = \frac{2}{\pi y'}\ . \tag{2.28}$$

Consequently

$$W(|x'|\, J_{\frac{2}{3}}(y'),\ |x'|\, N_{\frac{2}{3}}(y'),\ x') = \frac{2}{\pi} \frac{|x'|^2}{y'} \frac{dy'}{dx'}\ . \tag{2.30}$$

This is substituted into (2.26) :

$$j(x) = -\int_{x_1}^{x} \frac{\pi}{2} \frac{y'}{|x'|} \frac{dx'}{dy'} N_{\frac{2}{3}}(y')\ f(x')\ dx' + C_J \tag{2.31.1}$$

$$n(x) = \int_{x_1}^{x} \frac{\pi}{2} \frac{y'}{|x'|} \frac{dx'}{dy'} J_{\frac{2}{3}}(y')\ f(x')\ dx' + C_N \tag{2.31.2}$$

We then substitute f(x) from (2.24) :

$$j(x) = \frac{9}{2\pi^2}\left(\Gamma\left(\frac{2}{3}\right)\right)^3 \eta_o \int_{x_1}^{x} \frac{y'}{x' \cdot |x'|} \frac{dx'}{dy'} N_{\frac{2}{3}}(y') \frac{A_J J_{\frac{2}{3}}(y') + A_N N_{\frac{2}{3}}(y')}{\left[(J_{\frac{2}{3}}(y'))^2 + (N_{\frac{2}{3}}(y'))^2\right]^{3/2}}\ dx' + C_J \tag{2.32.1}$$

$$n(x) = -\frac{9}{2\pi^2}\left(\Gamma\left(\frac{2}{3}\right)\right)^3 \eta_o \int_{x_1}^{x} \frac{y'}{x' \cdot |x'|} \frac{dx'}{dy'} J_{\frac{2}{3}}(y') \frac{A_J J_{\frac{2}{3}}(y') + A_N N_{\frac{2}{3}}(y')}{\left[(J_{\frac{2}{3}}(y'))^2 + (N_{\frac{2}{3}}(y'))^2\right]^{3/2}}\ dx' + C_N . \tag{2.32.2}$$

y is given in terms of x by eq. (1.34). From this we easily work out that

$$\frac{y'}{x' \cdot |x'|} \frac{dx'}{dy'} = \frac{2}{3} \frac{1}{|x'|} \quad . \tag{2.33}$$

We demand ϑ to coincide with ϑ_o at $x = x_1$:

$$j(x_1) = A_J \tag{2.34.1}$$

$$n(x_1) = A_N \tag{2.34.2}$$

(see eqs. (2.13) and (2.25)). This gives

$$C_J = A_J \tag{2.35.1}$$

$$C_N = A_N \tag{2.35.2}$$

We then substitute (2.33) and (2.35) into (2.32) and have

$$j(x) = A_J + \frac{3}{\pi^2}\left(\Gamma\left(\tfrac{2}{3}\right)\right)^3 \eta_0 \int_{x_1}^{x} \frac{1}{|x'|} N_{\frac{2}{3}}(y') \frac{A_J J_{\frac{2}{3}}(y') + A_N N_{\frac{2}{3}}(y')}{[(J_{\frac{2}{3}}(y'))^2 + (N_{\frac{2}{3}}(y'))^2]^{3/2}} \, dx' \qquad (2.36.1)$$

$$n(x) = A_N - \frac{3}{\pi^2}\left(\Gamma\left(\tfrac{2}{3}\right)\right)^3 \eta_0 \int_{x_1}^{x} \frac{1}{|x'|} J_{\frac{2}{3}}(y') \frac{A_J J_{\frac{2}{3}}(y') + A_N N_{\frac{2}{3}}(y')}{[(J_{\frac{2}{3}}(y'))^2 + (N_{\frac{2}{3}}(y'))^2]^{3/2}} \, dx' \; . \qquad (2.36.2)$$

y' is an even function of x' ; therefore the integrands in (2.36) are even functions of x' . Let us take two times x_1 and x which are symmetrioal with respect to transition $x = 0$. We then have :

$$j(x) = A_J + \frac{6}{\pi^2}\left(\Gamma\left(\tfrac{2}{3}\right)\right)^3 \eta_0 \int_{0}^{x} \frac{1}{x'} N_{\frac{2}{3}}(y') \frac{A_J J_{\frac{2}{3}}(y') + A_N N_{\frac{2}{3}}(y')}{[(J_{\frac{2}{3}}(y'))^2 + (N_{\frac{2}{3}}(y'))^2]^{3/2}} \, dx' \qquad (2.37.1)$$

$$n(x) = A_N - \frac{6}{\pi^2}\left(\Gamma\left(\tfrac{2}{3}\right)\right)^2 \eta_0 \int_{0}^{x} \frac{1}{x'} J_{\frac{2}{3}}(y') \frac{A_J J_{\frac{2}{3}}(y') + A_N N_{\frac{2}{3}}(y')}{[(J_{\frac{2}{3}}(y'))^2 + (N_{\frac{2}{3}}(y'))^2]^{3/2}} \, dx' \; . \qquad (2.37.2)$$

for

$$x_1 = -x \; . \qquad (2.38)$$

In the integrands (2.37) x' is always positive and we then have

$$y' = \frac{2}{3} x'^{3/2} \qquad (2.39)$$

which gives

$$\frac{1}{x'} dx' = \frac{2}{3} \frac{1}{y'} dy' \; . \qquad (2.40)$$

Substitution yields

$$j(x) = A_J + \frac{4}{\pi^2}\left(\Gamma\left(\tfrac{2}{3}\right)\right)^3 \eta_0 \int_{0}^{y} N_{\frac{2}{3}}(y') \frac{A_J J_{\frac{2}{3}}(y') + A_N N_{\frac{2}{3}}(y')}{[(J_{\frac{2}{3}}(y'))^2 + (N_{\frac{2}{3}}(y'))^2]^{3/2}} \frac{1}{y'} \, dy' \qquad (2.41.1)$$

$$n(x) = A_N - \frac{4}{\pi^2}\left(\Gamma\left(\tfrac{2}{3}\right)\right)^3 \eta_0 \int_{0}^{y} J_{\frac{2}{3}}(y') \frac{A_J J_{\frac{2}{3}}(y') + A_N N_{\frac{2}{3}}(y')}{[(J_{\frac{2}{3}}(y'))^2 + (N_{\frac{2}{3}}(y'))^2]^{3/2}} \frac{1}{y'} \, dy' \; . \qquad (2.41.2)$$

For simplicity we introduce the notation :

$$I(J^2) = \int_0^y \frac{(J_{\frac{2}{3}}(y'))^2}{[(J_{\frac{2}{3}}(y'))^2 + (N_{\frac{2}{3}}(y'))^2]^{3/4}} \; \frac{1}{y'} \; dy' \tag{2.42.1}$$

$$I(JN) = \int_0^y \frac{J_{\frac{2}{3}}(y') \,.\, N_{\frac{2}{3}}(y')}{[(J_{\frac{2}{3}}(y'))^2 + (N_{\frac{2}{3}}(y'))^2]^{3/2}} \; \frac{1}{y'} \; dy' \tag{2.42.2}$$

$$I(N^2) = \int_0^y \frac{(N_{\frac{2}{3}}(y'))^2}{[(J_{\frac{2}{3}}(y'))^2 + (N_{\frac{2}{3}}(y'))^2]^{3/2}} \; \frac{1}{y'} \; dy' \tag{2.42.3}$$

The amplitudes of $J_{\frac{2}{3}}(y)$ and $N_{\frac{2}{3}}(y)$ at the time x after transition, for a particle with initial amplitudes A_J and A_N respectively at the time $-x$ (see (2.34.1) , (2.34.2)), we shall denote by A_J' and A_N' :

$$A_J' = j(x) \tag{2.43.1}$$

$$A_N' = n(x) \tag{2.43.2}$$

With the notation (2.42) we have

$$\begin{pmatrix} A_J' \\ A_N' \end{pmatrix} = f \begin{pmatrix} A_J \\ A_N \end{pmatrix} \tag{2.44}$$

with

$$f = \begin{pmatrix} 1 + \frac{4}{\pi^2}\left(\Gamma(\frac{2}{3})\right)^3 \eta_o \; I(JN) \;, & \frac{4}{\pi^2}\left(\Gamma(\frac{2}{3})\right)^3 \eta_o \; I\,(N^2) \\ -\frac{4}{\pi^2}\left(\Gamma(\frac{2}{3})\right)^3 \eta_o \; I(J^2) \;, & 1 - \frac{4}{\pi^2}\left(\Gamma(\frac{2}{3})\right)^3 \eta_o \; I\,(JN) \end{pmatrix} \tag{2.45}$$

PROPERTIES OF THE TRANSFORMATION MATRIX

According to Liouville's theorem the determinant of such a transformation matrix should always be unity, as phase space can never be diluted or compressed, only distorted. By direct computation from (2.45) we find :

$$|f| = 1 - \left[\frac{4}{\pi^2}\left(\Gamma(\tfrac{2}{3})\right)^3 \eta_o\right]^2 \left[\left(I(JN)\right)^2 - I(J^2)\, I(N^2)\right] \qquad (2.46)$$

This is correctly unity as far as the first order in η_o . In the general case of an arbitrary x this differs from unity to the second order in η_o. This is just what we should expect : our first approximation theory will only be correct to the first power of η_o.

Because $|f| \neq 1$ we cannot write f in the way we wrote d and e in (1.86) and (1.87). By introducing total amplitudes and phase angles according to (1.74) - (1.75) we have from (2.44) and (2.45) :

$$\begin{aligned}\left(\frac{A'}{A}\right)^2 = &\left[1 + \frac{1}{2}\left(\frac{4}{\pi^2}\,(\Gamma(\tfrac{2}{3}))^3\ \eta_o\right)^2 \left\{\left(I(N^2)\right)^2 + \left(I(J^2)\right)^2 + 2\left(I(JN)\right)^2\right\}\right] \\ &+ \left[2\cdot\frac{4}{\pi^2}\left(\Gamma(\tfrac{2}{3})\right)^3 \eta_o\ I(JN) + \frac{1}{2}\left(\frac{4}{\pi^2}(\Gamma(\tfrac{2}{3}))^3\ \eta_o\right)^2\left\{\left(I(J^2)\right)^2 - \left(I(N^2)\right)^2\right\}\right]\cdot \cos 2\psi \\ &+ \left[\frac{4}{\pi^2}\left(\Gamma(\tfrac{2}{3})\right)^3 \eta_o\left\{I(N^2) - I(J^2)\right\} + \left(\frac{4}{\pi^2}(\Gamma(\tfrac{2}{3}))^3\ \eta_o\right)^2 I(JN)\left\{I(N^2) - I(J^2)\right\}\right]\cdot \sin 2\psi\end{aligned} \qquad (2.47)$$

But as our theory is only correct to the first power in η_o we may write to this accuracy :

$$\frac{A'}{A} = 1 + \frac{2}{\pi^2}\left(\Gamma(\tfrac{2}{3})\right)^3 \eta_o\left[2\, I(JN)\ \cos 2\psi + \left(I\ (N^2) - I(J^2)\right)\ \sin 2\psi\right]. \qquad (2.48)$$

Put for shortness :

$$I_1 = 2\, I\ (JN) \qquad (2.49.1)$$

$$I_2 = I\ (N^2) - I(J^2) \qquad (2.49.2)$$

and we have with this notation :

$$\frac{A'}{A} = 1 + \frac{2}{\pi^2} \left(\Gamma(\tfrac{2}{3})\right)^3 \eta_o \left[I_1 \cos 2\psi + I_2 \sin 2\psi \right] \tag{2.50}$$

The maximum of this expression with respect to variation of ψ is

$$\left(\frac{A'}{A}\right)_{max} = 1 + \frac{2}{\pi^2} \left(\Gamma(\tfrac{2}{3})\right)^3 \eta_o \left(I_1^{\,2} + I_2^{\,2}\right)^{1/2} \tag{2.51}$$

with

$$\sin 2\psi = + \frac{I_2}{(I_1^{\,2} + I_2^{\,2})^{1/2}} \quad . \tag{2.52.1}$$

$$\cos 2\psi = + \frac{I_1}{(I_1^{\,2} + I_2^{\,2})^{1/2}} \tag{2.52.2}$$

The integrals $I(N^2)$ and $I(J^2)$, which appear in the off-diagonal elements of the transformation matrix (2.45), both diverge when $x \to \infty$. However, the integrals I_1 and I_2 , which we use in (2.50) and subsequent formulae, can be shown to converge when $x \to \infty$; consequently all quantities we need can be calculated to first order in η_o without divergence difficulties at large x . Terms of second order in η_o , such as the one in the determinant (2.46), are of no meaning even for small x . We have computed I_1 and I_2 numerically in the limit $x \to \infty$ and obtained

$$I_1 = -1.005 \tag{2.53.1}$$

$$I_2 = 0.954 \tag{2.53.2}$$

Substitution yields

$$\left(\frac{A'}{A}\right)_{max} = 1 + 0.697 \ \eta_o \tag{2.54}$$

with

$$\psi = 68^\circ + N \cdot 180^\circ \ , \quad N \text{ integer} \ . \tag{2.55}$$

This result is illustrated in Figs. 2 and 3 .

3. LONGITUDINAL SPACE-CHARGE FORCES AND TIMING ERROR OF THE PHASE-JUMP

THE TRANSFORMATION MATRIX

In the case of longitudinal space-charge forces but no timing error of the phase-jump, the transformation matrix is (2.45)

$$f = \begin{pmatrix} 1 + f_1\ \eta_0 + O(\eta_0{}^2) & , & f_2\ \eta_0 + O(\eta_0{}^2) \\ f_3\ \eta_0 + O(\eta_0{}^2) & , & 1 - f_1\ \eta_0 + O(\eta_0{}^2) \end{pmatrix} \tag{3.1}$$

with

$$f_1 = \frac{4}{\pi^2} \left(\Gamma(\tfrac{2}{3})\right)^3 \quad I(JN) \tag{3.2.1}$$

$$f_2 = \frac{4}{\pi^2} \left(\Gamma(\tfrac{2}{3})\right)^3 \quad I(N^2) \tag{3.2.2}$$

$$f_3 = -\frac{4}{\pi^2} \left(\Gamma(\tfrac{2}{3})\right)^3 \quad I(J^2) \tag{3.2.3}$$

In the case of no longitudinal space-charge forces but a timing error x_o of the phase-jump, the transformation matrix is (1.71)

$$d = \begin{pmatrix} 1 + O\,(x_o{}^3) & , & d_2\ x_o + O\,(x_o{}^3) \\ O\,(x_o{}^5) & , & 1 + O\,(x_o{}^3) \end{pmatrix} \qquad \text{for } x_o < 0 \tag{3.3.1}$$

and

$$e = \begin{pmatrix} 1 + O\,(x_o{}^3) & , & e_2\ x_o + O\,(x_o{}^3) \\ O\,(x_o{}^5) & , & 1 + O\,(x_o{}^3) \end{pmatrix} \qquad \text{for } x_o < 0 \tag{3.3.2}$$

with

$$d_2 = e_2 = -\frac{2}{\pi}\ 3^{1/3} \left(\Gamma(\tfrac{2}{3})\right)^2 = -1.683 \quad . \tag{3.4}$$

If we now have both effects occurring at the same time, we shall call the resulting transformation matrix g . g must be some kind of product of the matrices d and f or of e and f . As is easily shown by computation, d and f do not commute, nor do e and f , but they commute to the first power in η_o and x_o . As our calculation of space-charge effects is correct only to the first power in η_o , this is sufficient. Neglecting higher order terms we have for $x_o < 0$ as well as for $x_o > 0$:

$$g = \begin{pmatrix} 1 + f_1 \eta_o & , & f_2 \eta_o + e_2 x_o \\ f_3 \eta_o & , & 1 + f_1 \eta_o \end{pmatrix} . \tag{3.5}$$

By the same technique as was used for the transformation matrix f we have from (3.5) :

$$\max_{\psi} \left(\frac{A'}{A}\right) = 1 + \eta_o \left[f_1^{\,2} + \left(\frac{f_2 + f_3}{2} + \frac{e_2}{2} \frac{x_o}{\eta_o}\right)^2 \right]^{1/2} \tag{3.6}$$

with

$$\sin 2\psi = + \frac{\frac{f_2 + f_3}{2} + \frac{e_2}{2} \frac{x_o}{\eta_o}}{\left[f_1^{\,2} + \left(\frac{f_2 + f_3}{2} + \frac{e_2}{2} \frac{x_o}{\eta_o}\right)^2 \right]^{1/2}} \tag{3.7.1}$$

$$\cos 2\psi = + \frac{f_1}{\left[f_1^{\,2} + \left(\frac{f_2 + f_3}{2} + \frac{e_2}{2} \frac{x_o}{\eta_o}\right)^2 \right]^{1/2}} . \tag{3.7.2}$$

OPTIMUM TIMING OF THE PHASE-JUMP

From (3.6) we see that in order to make $(A'/A)_{max}$ as small as possible, we should switch the phase at a time x_o as to fulfil the equation

$$\frac{f_2 + f_3}{2} + \frac{e_2}{2} \frac{x_o}{\eta_o} = 0 \tag{3.8}$$

which yields

$$x_o = - \frac{f_2 + f_3}{e_2} \eta_o . \tag{3.9}$$

With this value of x_o we have from (3.7)

$$\sin 2\psi = 0 \quad , \quad \cos 2\psi = \mathrm{sgn}\,(f_1) . \tag{3.10}$$

From (3.21) and (2.53.1) we have $f_1 < 0$ and hence

$$\cos 2\psi = -1 \tag{3.11}$$

Therefore

$$2\psi = \pi + N \cdot 2\pi \tag{3.12.1}$$

or

$$\psi = \frac{\pi}{2} + N \cdot \pi \quad . \tag{3.12.2}$$

We substitute (3.2) and (3.4) into (3.9) and have

$$x_o = \frac{2}{3^{1/3}\,\pi} \, \Gamma(\tfrac{2}{3}) \quad I_2 \quad \eta_o \quad . \tag{3.13}$$

And from (3.6) :

$$\min_{x_o} \max_{\psi} \left(\frac{A'}{A}\right) = 1 + |f_1| \; \eta_o = 1 + \frac{2}{\pi^2}\left(\Gamma(\tfrac{2}{3})\right)^3 \; |I_1| \quad \eta_o \quad . \tag{3.14}$$

Substitution of (2.53) finally gives

$$\min_{\dot{x}_o} \max_{\psi} \left(\frac{A'}{A}\right) = 1 + 0.506 \quad \eta_o \tag{3.15}$$

with

$$\psi = \frac{\pi}{2} + N\,\pi \tag{3.16}$$

and

$$x_o = 0.570 \quad \eta_o \quad . \tag{3.17.1}$$

With $T = 1.85$ ms we find

$$t_o = 1.06 \quad \eta_o \text{ ms} \quad . \tag{3.17.2}$$

This shall be compared with a measurement done by Y. Baconnier [17] on the CPS at an intensity of $N = 0.55 \times 10^{12}$. As $\eta_o = 1.3$ at $N = 10^{12}$, the above intensity gives $\eta_o = 0.55 \times 1.3 = 0.72$. Substitution of this η_o into (3.17.2) gives

$$(t_o)_{theor} = 0.76 \text{ ms} \quad . \tag{3.17.3}$$

The experiment gave

$$(t_o)_{exp} = 2.5 \text{ ms} \quad . \tag{3.17.4}$$

This is somewhat excessive, but there is agreement in sign and in order of magnitude. Both theory and experiment must be considered as rather inaccurate.

This improvement obtained by optimum timing of the phase-jump is not very great. As one may see from Figs. 1 and 2, this can be explained by the fact that neither positive nor negative x_0 deforms the bunch in a direction which is ideal for correcting the deformation associated with space-charge. With a deformation independently adjustable in "phase" and magnitude [14] it would be possible (to the accuracy of the present theory) to correct perfectly. Such a possibility will be examined in a later report.

ARTIFICIAL BLOW-UP

Let A and A′ be the total amplitudes before and after transition of a particle whose phase orbit outlines a bunch. We then have :

$$A' = A \; (1 + k_1 \; \eta_0) \tag{3.18}$$

where

$$k_1 = 0.506 \tag{3.19}$$

if we perform the phase-jump as to minimize bunch distortion, and

$$k_1 = 0.697 \tag{3.20}$$

if we perform the phase-jump at transition, $x_0 = 0$. From (2.18) , (2.19), (2.21) we have

$$\eta_0 = k_2 \; N \; A^{-3} \tag{3.21}$$

with

$$k_2 = \frac{1}{9}\left(\frac{\pi}{\Gamma(\frac{2}{3})}\right)^3 \frac{\frac{3}{2}\;\frac{h\; g_0\; e^2}{4\pi\;\epsilon_0\;\gamma^2\; R}}{\frac{eV}{2\pi}\;|\cos\;\varphi_s|} \; . \tag{3.22}$$

Such a rapid decrease of η_0 with A suggests that it may be profitable to blow-up the bunches, by beam gymnastics and filamentation well before transition, in order to reduce this space-charge effect. From (3.21) and (3.18) we have

$$A' = A + k_1 \; k_2 \;\; N \; A^{-2} \; . \tag{3.23}$$

This is a minimum, for variation of A alone, at

$$\frac{dA'}{dA} = 1 - 2\,k_1\,k_2\;N\,A^{-3} = 0 \tag{3.24}$$

which gives

$$A = (2\,k_1\,k_2\,N)^{1/3} \tag{3.25}$$

$$A' = \frac{3}{2}\,A = \frac{3}{2}\,(\,2\,k_1\,k_2\,N)^{1/3} \tag{3.26}$$

$$\eta_0 = \frac{1}{2k_1} = 0.988 \qquad \text{(using (3.19))}\quad . \tag{3.27}$$

This is interpreted in the following way : if the bunches before transition are shorter than given by (3.25), then one may reduce the final maximum amplitude by blowing-up the bunches before transition. The ideal bunch length is one which gives the η_0 of (3.27), and is such as to give a space-charge blow-up ratio A'/A at transition of $3/2$. The resulting bunch length after transition is proportional to the cube root of the intensity, and so is the final average phase-space density in the bunch :

$$\frac{N}{A'^2} \sim N^{1/3}\quad . \tag{3.28}$$

It should perhaps be emphasized that we have taken k_2 as constant in the above optimization. Changing the bunch length at transition by using a different strength of R.F. focusing, $eV|\cos\varphi_s|$, would not give the same result.

THE LONGITUDINAL SPACE-CHARGE LIMIT

Let us first assume that we do no artificial blow-up of the bunches before entering transition. The formula (3.18) applies :

$$A' = A\,(1 + k_1\,\eta_0) \tag{3.29}$$

From Regenstreif [13], p. 34, we see that we can tolerate a blow-up of the synchrotron oscillations by a factor 5 without losing particles longitudinally out of the bucket :

$$\left(\frac{A'}{A}\right)_{max} = 5\quad . \tag{3.30}$$

One can hardly justify the use of the first-order perturbation theory in this region, but it is the only theory we have.

With

$$k_1 = 0.506 \tag{3.31}$$

as given by (3.19), we have

$$(\eta_o)_{max} = 7.9 \quad . \tag{3.32}$$

Since, with a given bunch size, η_o is proportional to the intensity N, and

$$\eta_o = 1.3 \quad \text{for } N = 10^{12} \tag{3.33}$$

(see Appendix 1), we have :

$$\boxed{\left(N_{max}\right)_{\text{no blow-up}} = 6.1 \times 10^{12}} \quad . \tag{3.34}$$

This figure is valid for the present R.F. system of the CPS.

Let us then assume that an optimal blow-up has been performed before transition. Just after trapping the bunch size is described by an A having the same value as in the present CPS, some time later the bunches are shaken and A is increased to A_{int}, and finally at transition the space-charge forces give a further blow-up and the final value is A'. The permissible total blow-up factor (artificial + space-charge) is again taken to be 5 :

$$\frac{A'_{max}}{A} = 5 \quad . \tag{3.35}$$

We have from (3.26) :

$$N_{max} = \frac{1}{2\, k_1\, k_2} \left(\frac{2}{3}\, A'_{max}\right)^3 \quad . \tag{3.36}$$

We solve (3.21) for k_2, substitute and have

$$N_{max} = \frac{N}{\eta_o} \frac{1}{2\, k_1} \left(\frac{2}{3} \frac{A'_{max}}{A}\right)^3 , \tag{3.37}$$

with N , η_o , and A , being the quantities valid for the present CPS. Taking again

$$\eta_o = 1.3 \qquad \text{for } N = 10^{12} \quad , \tag{3.38}$$

we have

$$\boxed{\left(N_{max} \right)_{\text{optimum blow-up}} = 2.8 \times 10^{13}} \quad . \tag{3.39}$$

At last we investigate by which amount we can increase phase-space density at $N = 10^{12}$ by performing an optimum blow-up before transition. At present, with no blow-up, we have (3.18) :

$$A'_{\text{no blow-up}} = A \left(1 + k_1 \eta_o\right) \quad . \tag{3.40}$$

With optimum blow-up we have (3.26) :

$$A'_{\text{optimum blow-up}} = \frac{3}{2} \left(2 k_1 k_2 N\right)^{1/3} \quad . \tag{3.41}$$

We solve (3.21) for k_2 and substitute :

$$A'_{\text{optimum blow-up}} = A \cdot \frac{3}{2} \left(2 k_1 \eta_o\right)^{1/3} \quad , \tag{3.42}$$

with η_o and A being the quantities valid for the present CPS. With

$$k_1 = 0.506 \quad , \quad \eta_o = 1.3 \tag{3.43}$$

we have

$$A'_{\text{no blow-up}} = 1.66 \; A \tag{3.44}$$

$$A'_{\text{optimum blow-up}} = 1.65 \; A \tag{3.45}$$

and the improvement factor in phase-space density is

$$\left(\frac{A'_{\text{no blow-up}}}{A'_{\text{optimum blow-up}}} \right)^2 = 1.01 \quad . \qquad (3.46$$

This improvement is very small, as we are already very close to the optimum :

$$\eta_o = 1.3 \qquad (3.47.1)$$

while, from (3.27)

$$(\eta_o)_{optimum} = 0.988 \quad . \qquad (3.47.2)$$

CONCLUSIONS

The longitudinal space-charge forces deform the bunch in an undesirable way when passing transition. Some of this deformation can be counteracted by performing the phase-jump slightly after transition. The longitudinal space-charge limit can be increased from 6.1×10^{12} to 2.8×10^{13} by blowing-up the bunches artificially before passing transition. Such an artificial blow-up is undesirable from the point of view of the ISR because it gives a dilutation of phase-space, and we should investigate more refined tricks which might be done with the R.F. system [14] to avoid the blow-up. Even with no such tricks the phase-space is blown up by a smaller factor than the increase of intensity, so the density of particles in phase-space increases.

It must be remembered that all these results are valid only as far as the first approximation is valid : the infinite matrix elements which we found in Chapter 2 make us more cautious than usual on this point. Probably the theory gives the right order of magnitude for the various phenomena predicted, but as we have sometimes used it also where it predicts fairly large blow-up ratios A'/A (which should be small for the first approximation to be valid), the results should not be taken too seriously as far as accuracy is concerned.

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APPENDIX 1

Our quantitative results depend on the values of the quantity T, defined by (1.19), and η_o, defined by (2.19) and (2.21). This in its turn can be taken as dependent on $\hat{\vartheta}_o(0)$, the bunch half-length at transition as calculated neglecting space-charge, and the coefficient k_3 :

$$\eta_o = \frac{k_3}{\hat{\vartheta}_o^{\,3}(0)} \tag{A.1}$$

$$k_3 = \frac{3}{2}\,\frac{\dfrac{h\ g_o\ e^2\ N}{4\pi\ \epsilon_o\ \gamma^2\ R}}{\dfrac{eV}{2\pi}\ |\cos\ \varphi_s|} \tag{A.2}$$

A fuller report on the evaluation of these quantities will appear [15], here we shall only give the results and put on record the parameters and assumptions on which they are based.

Our formulae for η, (2.19), (A.2), are consistent with those used in the 300 GeV report [11]. The early papers [3] and [4] have an extra factor $8/3\pi$, due to linearising the force in a less satisfactory way.

Basic CPS machine parameters we can take from Regenstreif [13], with the exception of the following :

	At injection	At transition		
$\dot{B}$	1.428	1.316	Ts^{-1}	(A.3)
V	135	135	kV	

The R.F. volts per turn is not very accurately known, but the above values are certainly better than the nominal ones. Substituting in (1.19) we find

$$T = 1.85 \text{ ms .} \tag{A.4}$$

To obtain $\hat{\vartheta}_o(0)$ we assumed a full bucket at injection and conservation of area in the phase-plane thereafter. This gave

$$\hat{\vartheta}_0(0) \quad = \quad 0.127 \quad \text{radian} \tag{A.5}$$

It must be remarked that the measured [16] bunch length at transition is a factor 1.5 greater than this ,

$$\hat{\vartheta}(0) \quad = \quad 0.19 \quad \text{measured at} \quad N = 0.8 \times 10^{12} \quad , \tag{A.6}$$

this factor is higher than one would expect on the present theory from the influence of space-charge forces up to the instant of transition, but this may not be significant.

Estimating g_0 , from the theoretical beam dimensions, to be 4.5, we calculate

$$k_3 \quad = \quad 2.68 \times 10^{-3} \quad \text{at} \quad N = 10^{12} \tag{A.7}$$

giving finally

$$\eta_0 \quad = \quad 1.3 \quad \text{at} \quad N = 10^{12} \tag{A.8}$$

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