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PS/AR/Note 88-01
April 1988

STUDY OF SEXTUPOLAR PERTURBATIONS TO THE SECOND POWER IN THE SEXTUPOLE STRENGTH IN LEAR

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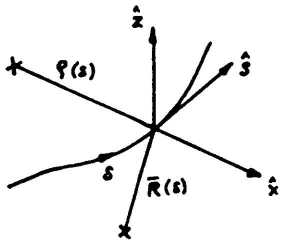
ABSTRACT

The perturbations from the sextupoles used for the chromaticity correction and the excitation of the extraction resonance in LEAR have been studied to the second power in the sextupole strength.

1. INTRODUCTION

To describe the motion of a particle in an accelerator it is customary to use a local coordinate system of the following type :



where R(s) is a reference curve and ρ(s) is the local radius of curvature. We define a local curvature h(s) by :

$$h(s) \equiv 1/\varrho(s) \tag{1}$$

For a circular accelerator we choose the closed orbit for a particle with momentum p_0 as the reference curve. The equations for the transverse motion are then[1] :

$$\boxed{\begin{aligned}
x'' - \frac{x'}{1+hx}(2hx'+h'x) - h(1+hx) &= \frac{q}{(1+\delta)p_0}\sqrt{x'^2+(1+hx)^2+z'^2}\\
&\quad\left\{(1+hx)\left[1+\frac{x'^2}{(1+hx)^2}\right]B_z - z'B_s - \frac{x'z'}{1+hx}B_x\right\}\\
z'' - \frac{z'}{1+hx}(2hx'+h'x) &= \frac{q}{(1+\delta)p_0}\sqrt{x'^2+(1+hx)^2+z'^2}\\
&\quad\left\{x'B_s - (1+hx)\left[1+\frac{z'^2}{(1+hx)^2}\right]B_x + \frac{x'z'}{1+hx}B_z\right\}
\end{aligned}} \tag{2}$$

where a prime denotes differentiation with respect to s and the momentum deviation δ is defined as

$$\delta \equiv \frac{p-p_0}{p_0} \tag{3}$$

where p is the momentum of the particle. B_x, B_s and B_z are local components of the magnetic field and q is the charge of the particle.

The well known linear equations are obtained by using the field expansions

$$\left[\begin{aligned}
\frac{q}{p_0}B_x &= kz\\
\frac{q}{p_0}B_s &= 0\\
\frac{q}{p_0}B_z &= -h + kx
\end{aligned}\right. \tag{4}$$

in eqs (2) and expanding in the coordinates and δ. If we only keep linear terms we find

$$\boxed{\begin{aligned} x'' + (h^2 - k)x &= \delta h \\ z'' + kz &= 0 \end{aligned}} \tag{5}$$

the appearence of δ in eqs (2) leads to chromatic effects. This is normally compensated for by the introduction of sextupoles which add terms in the field expansions eqs (4) of the type

$$\begin{cases} \dfrac{q}{p_0} B_x = Sxz \\ \dfrac{e}{p_0} B_z = \dfrac{1}{2} S \,(x^2 - z^2) \end{cases} \tag{6}$$

where :

$$S_2 = \frac{\partial^2 B_z}{\partial^2 x} = \frac{\partial^2 B_x}{\partial x \partial z}$$

This leads to nonlinear coupled equations of motion. There are several ways to study the behaviour of the nonlinear motion. Either by tracking[2] (numerical simulation) or by direct analytical expressions. Such an analytical description can be found by a Fourier expansion of a Hamiltonian formulation, and keeping only the dominant terms. This leads to the resonance approach[3]. Another way is to use canonical perturbation theory[5,6].

2. PERTURBATIONS FROM SEXTUPOLES

The introduction of the sextupoles for the chromaticity correction drives nonlinear resonances. The nonlinearity of the equations of motion also introduces an amplitude dependent tune shift.

From the application of perturbation theory it is found that sextupoles drive, to first power in the sextupole strength, the amplitude resonances

$$\begin{aligned} Q_x &= p \\ 3Q_x &= p \\ Q_x + 2Q_z &= p \qquad , \; p = 0, \pm 1, \pm 2, \ldots \\ Q_x - 2Q_z &= p \end{aligned} \tag{7}$$

where Q_x, Q_z are the betatron frequencies of the linear motion. To second power in the sextupole strength we also find

$$\begin{aligned} 2Q_x &= p \\ 4Q_x &= p \\ 2Q_z &= p \qquad\qquad p = 0, \pm 1, \pm 2, \ldots \\ 4Q_z &= p \\ 2Q_x + 2Q_z &= p \\ 2Q_x - 2Q_z &= p \end{aligned} \tag{8}$$

These are obtained by adding or subtracting two of the resonances in (7). Note that not all of the combinations appear.

To obtain the perturbations to first power in the sextupole strength one may use the resonance approach or canonical perturbation theory.

The amplitude dependent tune shifts, which are of second power in the sextupole strength, can be calculated by applying canonical perturbation theory to first order, since they are given by the transformed Hamiltonian.

These two approaches also give new invariants of the motion to the second power in the sextupole strength.

3. SECOND ORDER PERTURBATIONS FROM SEXTUPOLES

Canonical perturbation theory can, at least in principle, be extended to any order. This method has however a great disadvantage. The canonical transformation from the new coordinates to the old is implicit.

This problem has been solved by using Lie transforms, to get explicit transformation equations that can be developed recursively to any order[6].

However, to avoid the machinery of Lie transforms one can also apply time-dependent perturbation theory (variation of constants). Since all the steps are explicit and recursive this can be done by using a symbolic algebraic manipulation system such as REDUCE. This has been done to the second power in the sextupole strength[7,8,9,10] and the result is obtained as FORTRAN functions for first- and second order perturbations of the action, phase and tune. The action J and the angle variable Ψ appear in the horizontal betatron motion as :

$$
\begin{aligned}
x(s) &= \sqrt{2J_x(s)\beta_x(s)}\,\cos\left(\Psi_x(s) + \varphi_x(s)\right) \\
\Psi_x(s) &= \int_{s_0}^{s} \frac{1}{\beta_x(u)}\,du
\end{aligned}
\tag{9}
$$

and similarly for the vertical plane. The action J_x and phase φ_x are constants of motion for the linear motion. It is the perturbation that make them vary with s. We define the perturbations of the action as :

$$
J(s) = J_0\,\left(1 + \Delta_1 j(s) + \Delta_2 j(s)\right) \tag{10}
$$

where $\Delta_1 j$ is linear and $\Delta_2 j$ quadratic in the sextupole strength. J_0 is given by the initial amplitude.

Since the perturbations are oscillating functions of s we choose to study the average values. The perturbations tend to infinity at the resonances so we put a maximum limit to 1. Note that we are doing perturbative calculations so that the expressions are only expected to be a good approximation for perturbations smaller than 1.

We can then plot the average perturbations as functions of the horizontal and vertical tune. To compress both the horizontal- and vertical planes into one plot we plot the function :

$$\Delta J \equiv Min\ \{Max\ [|\langle\Delta_1 j_x(s) + \Delta_2 j_x(s)\rangle|,\ |\langle\Delta_1 j_z(s) + \Delta_2 j_z(s)\rangle],\ 1\} \tag{11}$$

where <> denotes average value.

The expression for the tune shifts contains only terms quadratic in the sextupole strength but are linear in J_x and J_z. They can therefore be written as[11] :

$$\begin{pmatrix} \Delta Q_x \\ \Delta Q_z \end{pmatrix} = \begin{pmatrix} a_{xx} & a_{xz} \\ a_{zx} & a_{zz} \end{pmatrix} \begin{pmatrix} J_x \\ J_z \end{pmatrix} \tag{12}$$

The tune shifts also exhibit resonant behaviour so that they tend to infinity at the first order resonances.

It turns out that to the second power in the sextupole strength :

$$a_{xz} = a_{zx} \tag{13}$$

The expressions for the perturbations of the action J and the phase φ including the s-dependence, can be used for tracking the particles to the second power in the sextupole strength. The frequencies that appear in the betatron motion due to a resonance $n_x Q_x + n_z Q_z = p$ is[12] :

$$(n_x \pm 1)Q_x + n_z Q_z,\ \text{horizontal plane}$$

$$n_x Q_x + (n_z \pm 1)Q_z,\ \text{vertical plane}$$

4. STUDY OF SEXTUPOLE CONFIGURATIONS IN LEAR

The aim here is to study the recent sextupole configuration to the second power in the sextupole strength which has been worked out by M. Chanel. Earlier work may be found in the references[13], [14].

The 18 sextupoles in LEAR are used for :

- chromaticity correction
- excitation of the extraction resonance $3Q_x = 7$
- compensation of the sextupolar resonance $Q_x + 2Q_z = 8$

In appendix A is shown a tune diagram including all resonances for which $|n_x|+|n_z| \leq 4$. The normal working point is :

$$Q_x = 2.305,\ Q_z = 2.725$$
$$\xi_x = 0\ ,\ \xi_z = 0 \tag{14}$$

where ξ is the chromaticity. For extraction

$$Q_x = 2.325,\ Q_z = 2.725$$
$$\xi_x = 0.53,\ \xi_z = 0 \tag{15}$$

with excitation of the extraction resonance by A7=6 (normalized excitation[15]). In appendix B,C are shown contour plots of the average perturbation to second order in the sextupole strength, eq. (11) and the tune shift coefficients in eq. (12).

We also show the motion in the normalized phase space by tracking to the second order in the sextupole strength and by Fourier[16] analysis of the betatron motion. The initial action has been put to

$$J_x = 20 \,.\, 10^{-6}$$
$$J_z = 10 \,.\, 10^{-6} \tag{16}$$

In appendix B two different cases are shown, $Q_x + 2Q_z = 8$ not compensated and compensated for the normal working point (14) when the extraction resonance is not excited. Appendix C shows the similar cases for the extraction working point (15) when the extraction resonance is excited.

5. CONCLUSIONS

For the normal working point we find that without compensation $Q_x + 2Q_z = 8$ is strongly excited. This probably affects the stability of the beam. When we apply the compensation it is found that the resonance nearly disappears. We also find a small excitation of $2Q_x + 2Q_z = 10$ which however is negligible. We find that the tune shifts are strongly decreased by the compensation. From the tracking it is seen that the perturbation is reduced, especially in the vertical plane.

At the extraction working point we also observe the excitation of $Q_x + 2Q_z = 8$. One can conclude that the excitation of the extraction resonance leads to an excitation of $Q_x - 2Q_z = -3$. The resonances $2Q_x + 2Q_z = 10$ and $4Q_z = 11$ are also excited. We expect the motion to be perturbed by $Q_x + 2Q_z \cong 8$, which is confirmed by the tracking. The tune shifts are big without compensation but strongly reduced by the compensation. From the tracking we conclude that in the compensated case there only remains a small perturbation. By looking at the spectra we find that it is mainly due to the excitation of $Q_x - 2Q_z = -3$.

The final conclusion is therefore that a possible improvement would be to try and find an configuration for the excitation of the extraction resonance without exciting $Q_x - 2Q_z = -3$. Apart from this the configuration seems to be good.

It should be noted that we have neglected the dispersion effects in this analysis. The theory may however be extended to include this contribution.

6. ACKNOWLEDGEMENT

I would like to thank the following persons for many helpful discussions : B. Autin, M. Chanel, R. Giannini, W. Hardt, P. Lefèvre and D. Möhl. I would also like to thank E. Asséo for providing algoritms for the Fourier analysis.

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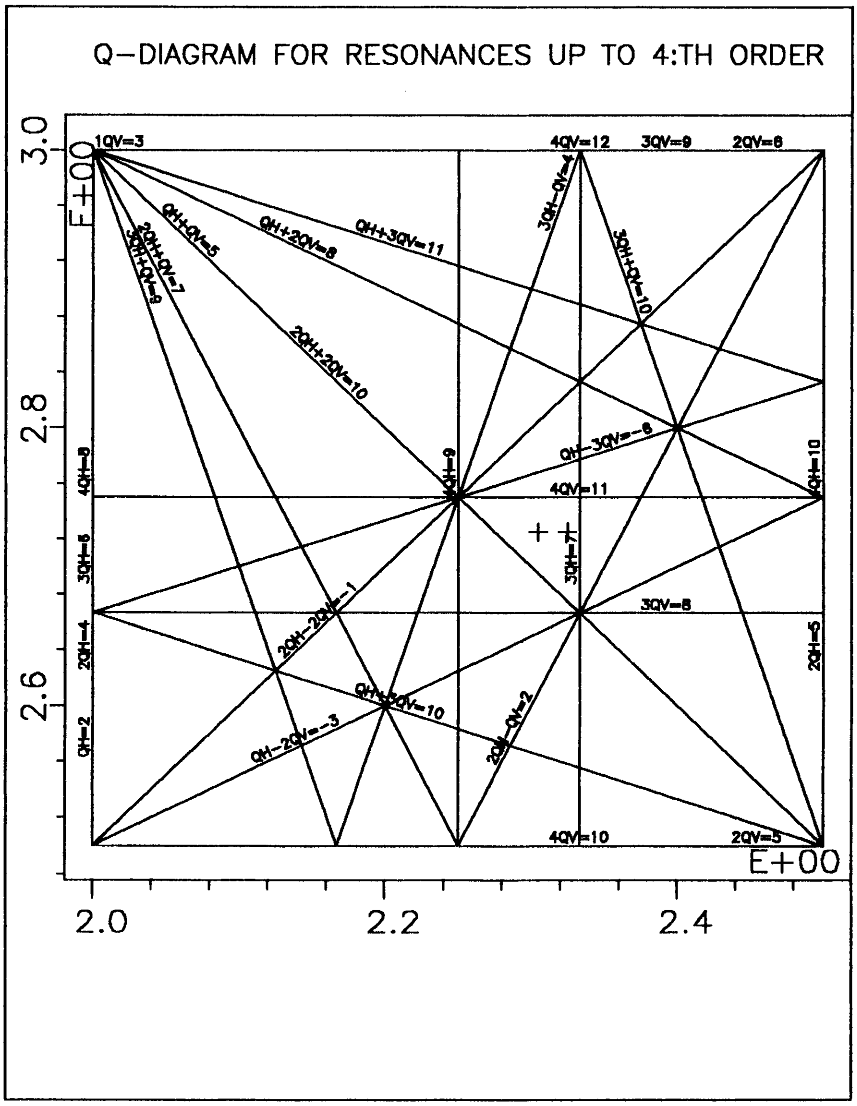
Appendix A

TUNE DIAGRAM



Appendix B

PERTURBATIONS FOR THE NORMAL TUNE

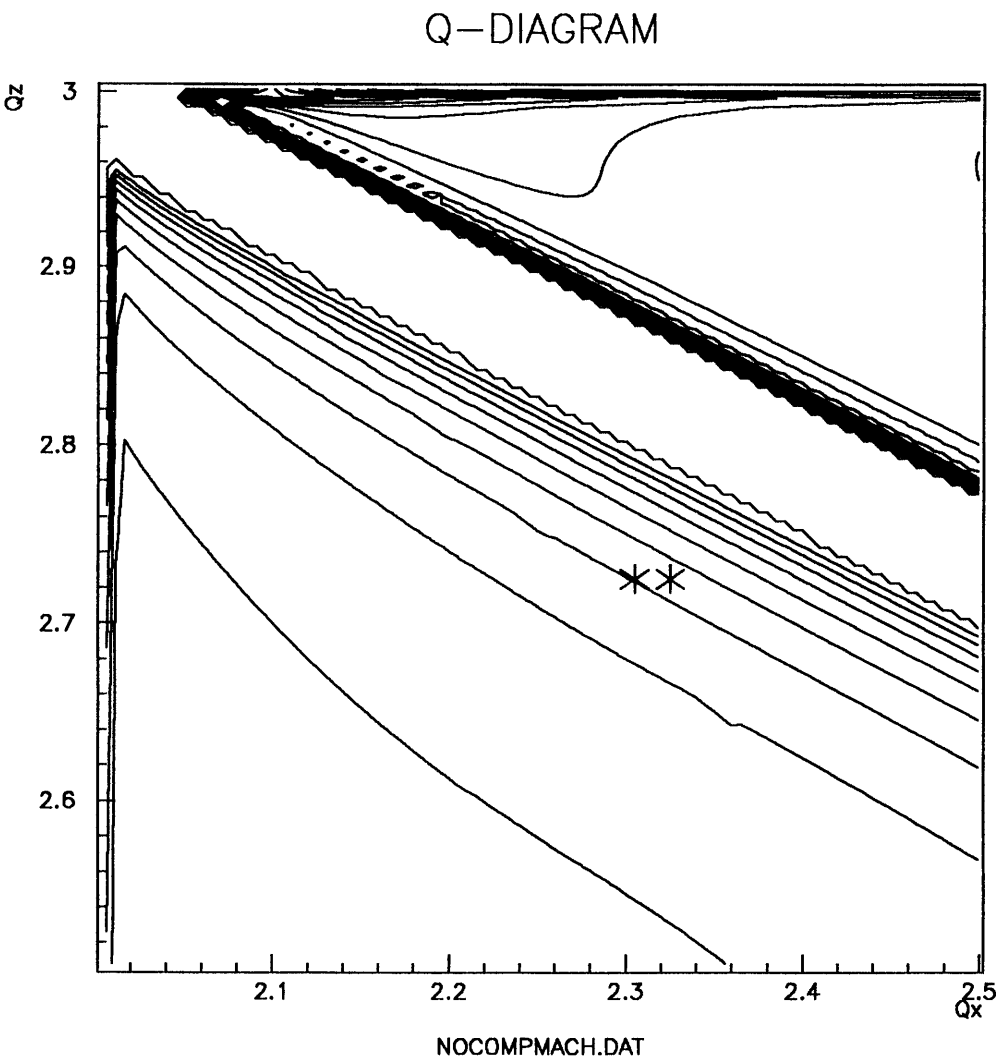


Figure 2: Perturbations in the action, Q + 2Q = 8 not compensated.

$a_{xx} = 12.2 \quad a_{xz} = 262$

$a_{zx} = 262 \quad a_{zz} = 93.0$

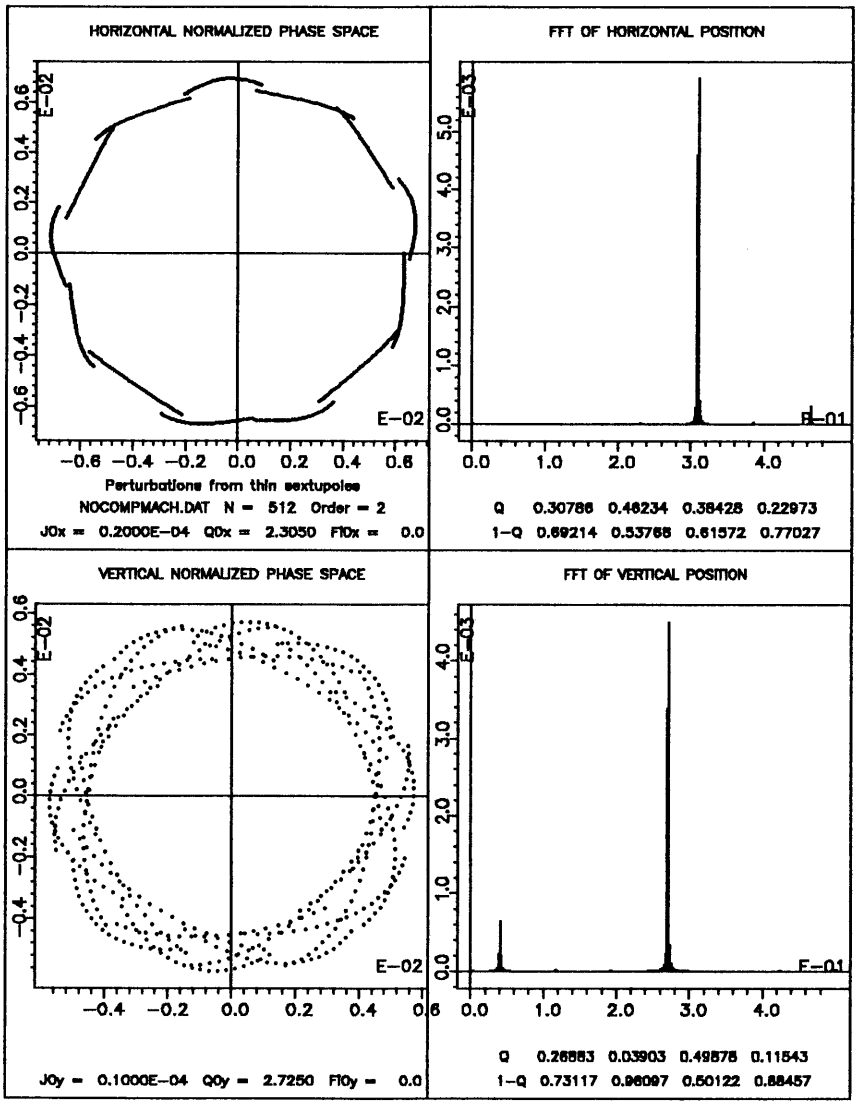


Figure 3: Tracking at the normal working point, Q + 2Q = 8 not compensated

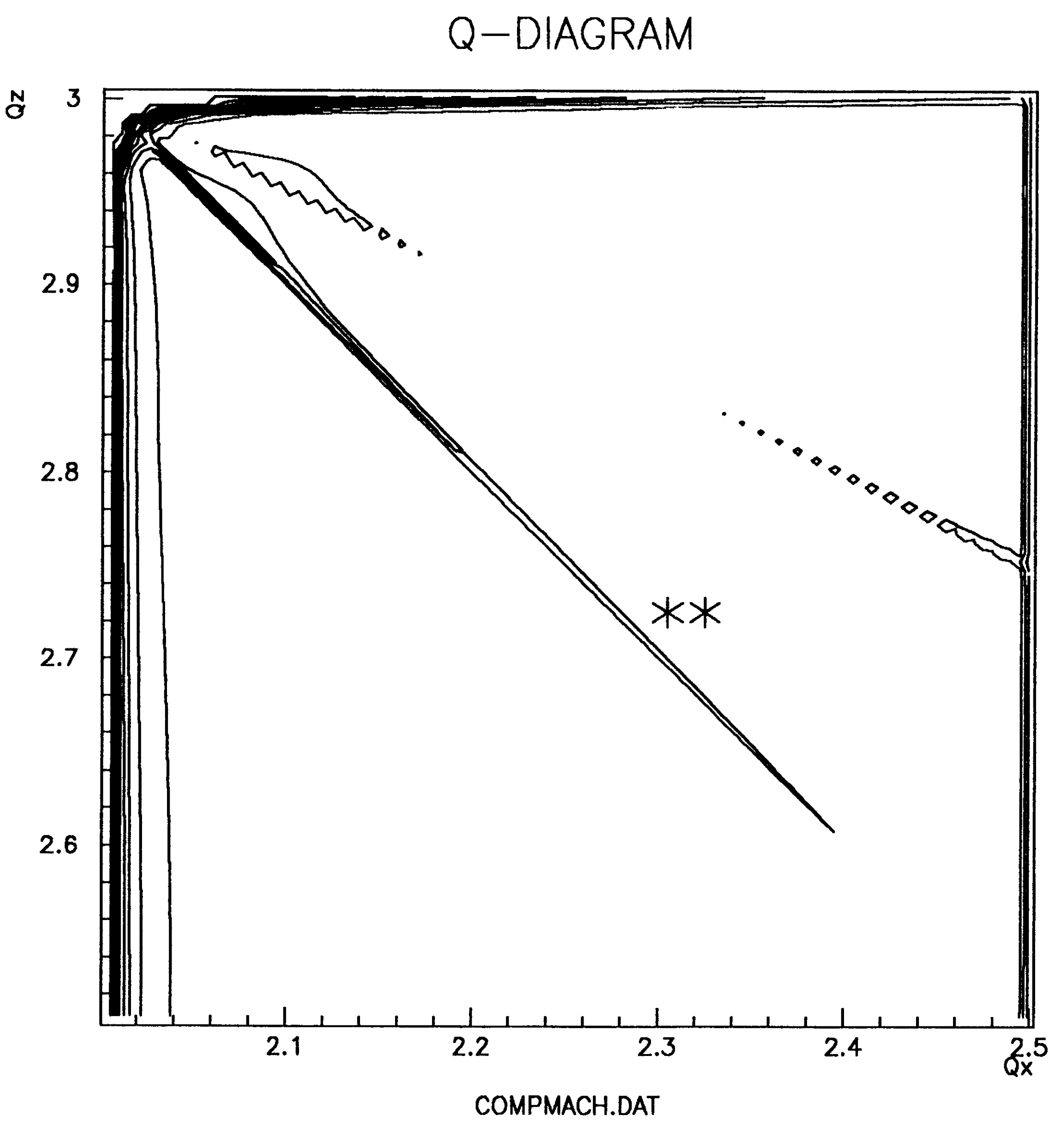


Figure 4: Perturbations in the action, Q + 2Q = 8 compensated.

$a_{xx} = 8.0 \quad a_{xz} = 13.7$

$a_{zx} = 13.7 \quad a_{zz} = -28.4$

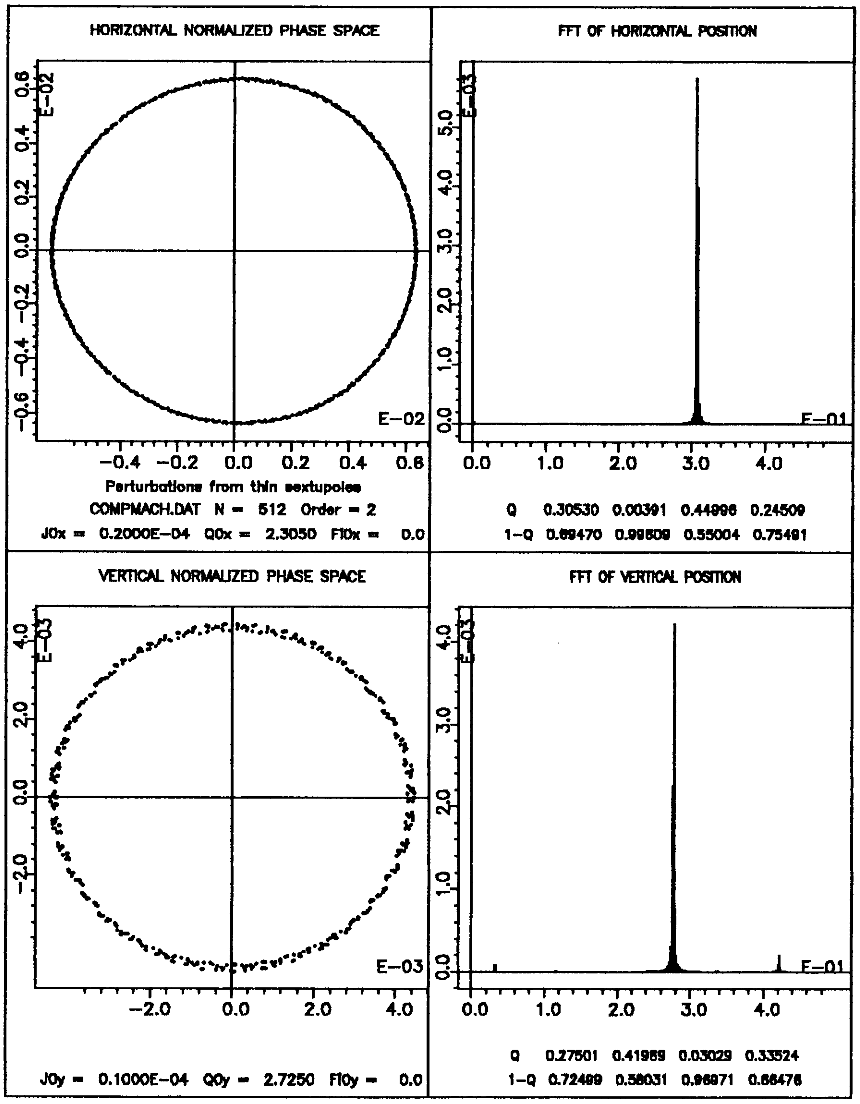


Figure 5: Tracking at the normal working point, Q + 2Q = 8 compensated

Appendix C

PERTURBATIONS AT EXTRACTION

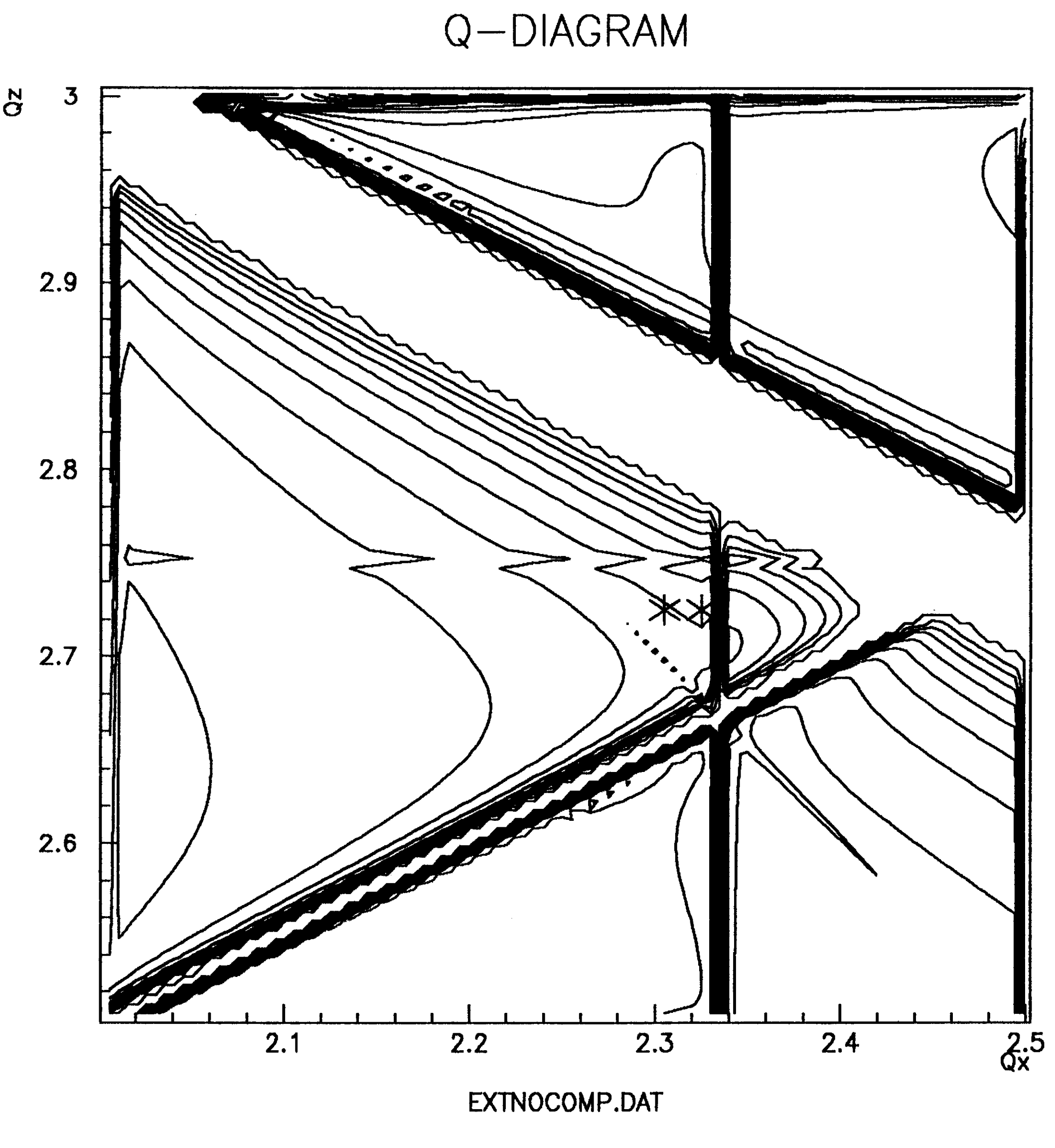


Figure 6: Perturbations in the action, Q + 2Q = 8 not compensated.

$a_{xx} = 31.2 \quad a_{xz} = 310$

$a_{zx} = 310 \quad a_{zz} = 131$

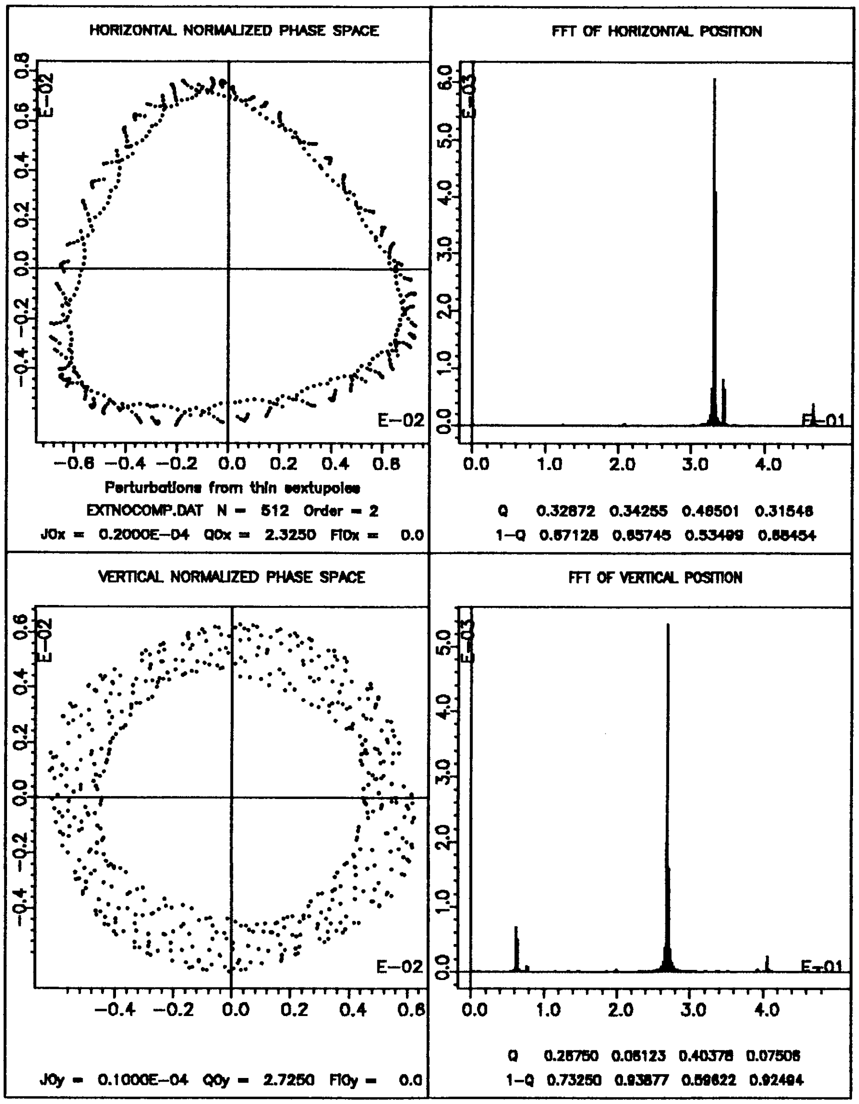


Figure 7: Tracking at extraction, Q + 2Q = 8 not compensated

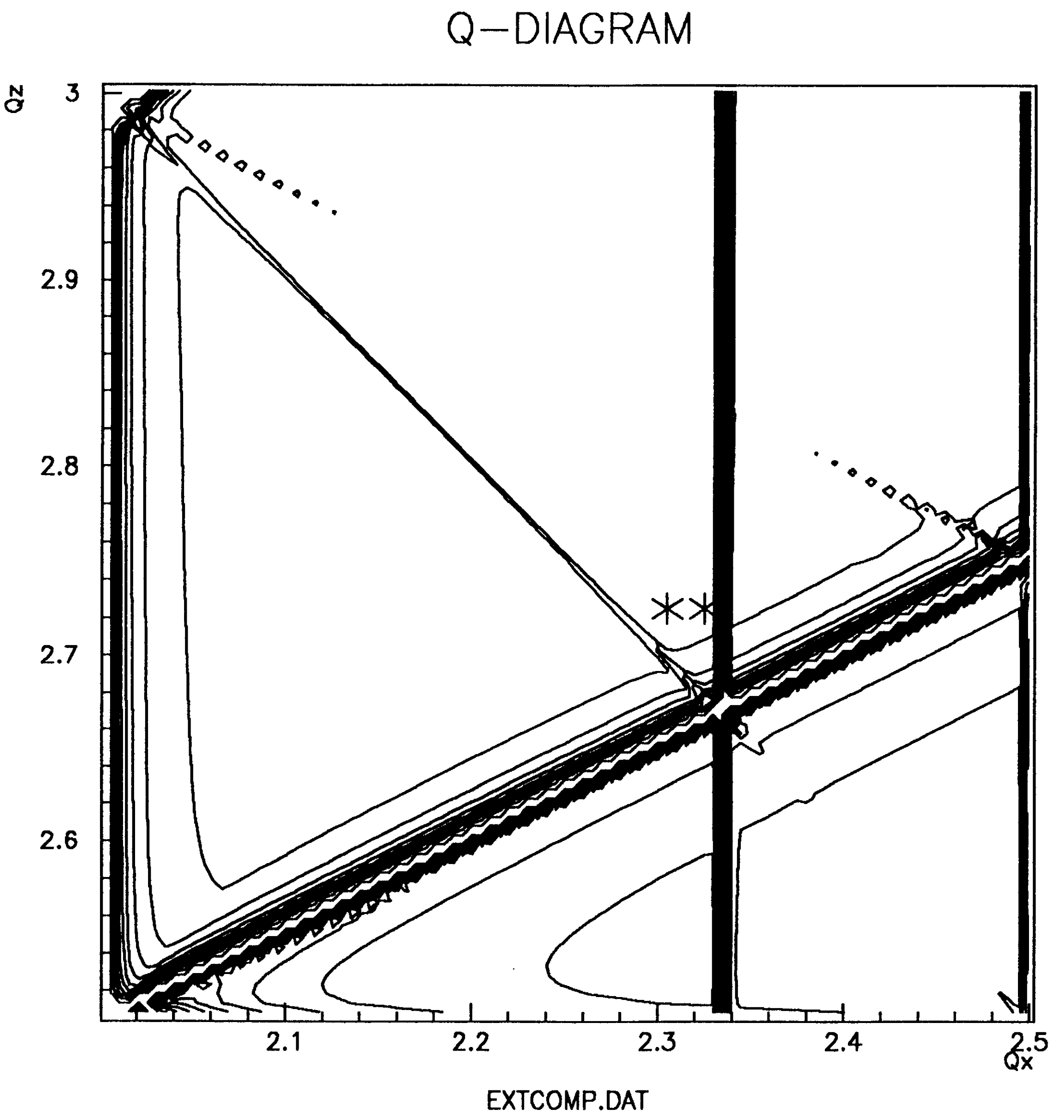


Figure 8: Perturbations in the action, Q + 2Q = 8 compensated.

$$a_{xx} = 22.4 \qquad a_{xz} = 0.25$$

$$a_{zx} = 0.25 \qquad a_{zz} = -17.2$$

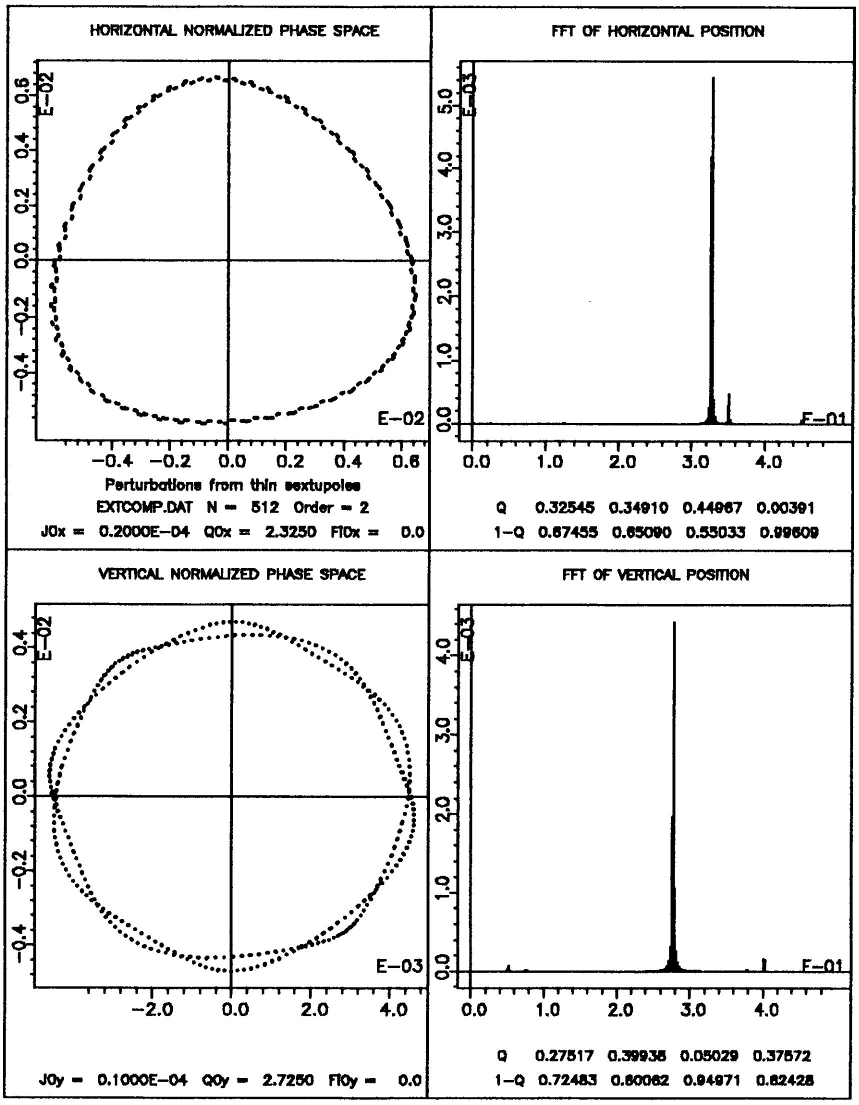


Figure 9: Tracking at extraction, Q + 2Q = 8 compensated