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SPACE-CHARGE EFFECT ON QUADRUPOLE OSCILLATION FREQUENCIES FOR LEAR AND PS LATTICES

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Abstract

Quadrupole oscillation frequencies are of practical importance for two reasons: first, they can be measured with an appropriate pickup and used to determine the incoherent space-charge tune-shift rsp. emittance; second, they are responsible for gradient error resonances. It is important to note that they differ from twice the incoherent tune if space charge is present. Our calculations are based on exact solutions of the KV-envelope equations for a given lattice. For small shifts we confirm that smooth approximation formulae (assuming constant focusing) are sufficiently accurate. For large space-charge effects deviations may be significant, depending on the super-period of the lattice. The effect is pronounced for the LEAR lattice, where we find that the horizontal (single-particle) tune is repelled from the integer value 2 even in the absence of any gradient errors. Correspondingly there is a significant effect on the shift of the quadrupole oscillation frequency, which determines the gradient error resonance.

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Quadrupole oscillation frequencies are of practical importance for two reasons: first, they can be measured with an appropriate pickup and used to determine the incoherent space charge tune shift rsp. emittance; second, they are responsible for gradient error resonances. It is important to note that they differ from twice the incoherent tune if space charge is present. Our calculations are based on exact solutions of the KV- envelope equations for a given lattice. For small shifts we confirm that smooth approximation formulae (assuming constant focusing) are sufficiently accurate. For large space charge effects deviations may be significant, depending on the super-period of the lattice. The effect is pronounced for the LEAR lattice, where we find that the horizontal (single-particle) tune is repelled from the integer value 2 even in the absence of any gradient errors. Correspondingly there is a significant effect on the shift of the quadrupole oscillation frequency, which determines the gradient error resonance.

1 Basic Formalism

We first briefly review the method of an exact calculation of the coherent quadrupole frequencies on the basis of the KV envelope equations for $a_{x,s}(s)$, $a_{y,s}(s)$. This takes into account that quadrupole oscillations change the shape of the beam and thus the space charge density.

$$d^2a_x/ds^2 + k_x(s)a_x - \epsilon_x^2/a_x^3 - \frac{qI}{\pi\epsilon_0 Mc^2\beta^3\gamma^3}\frac{1}{a_x+a_y} = 0 \tag{1}$$

$$d^2a_y/ds^2 + k_y(s)a_y - \epsilon_y^2/a_y^3 - \frac{qI}{\pi\epsilon_0 Mc^2\beta^3\gamma^3}\frac{1}{a_x+a_y} = 0 \tag{2}$$

In order to obtain a periodic solution for the matched envelopes one then introduces a small perturbation $\delta a_{x,y}$ and solves the resultant ordinary differential equations. This is most conveniently done numerically by following the Courant-Snyder formalism and determining the phase advance of the quadrupole oscillations per focusing period; at the same time one obtains the corresponding incoherent tune. This procedure is the basis of the numerical calculation in section 3.

2 Smooth Approximation Formulae

For a first orientation it is useful to calculate this shift by assuming constant focusing forces. The necessary derivations have been made for a round beam by Smith [1]. One obtains two frequencies, describing oscillations where the beam remains round (symmetric mode) and one where the beam shape becomes elliptical, but the density unchanged (antisymmetric mode):

$$Q_{2,s} = 2(Q_0 - \frac{1}{2}\Delta Q_{inc}) \tag{3}$$

$$Q_{2,a} = 2(Q_0 - \frac{3}{4}\Delta Q_{inc}) \tag{4}$$

where ΔQ_{inc} is the incoherent tune shift. It is noted that these frequencies differ from $2Q = 2(Q_0 - \Delta Q_{inc})$, which would ignore the change of space charge density due to the quadrupole oscillation. For zero space charge one clearly has twice the betatron frequency.

The case of oscillations of a beam with elliptic cross section in constant focusing was treated by Hardt [2]. Assuming different focusing constants and emittances the distinction between symmetric and antisymmetric is inappropriate. One finds the following expressions for two modes, which have preferential directions x, rsp. y:

$$Q_{2,x} = 2\left(Q_{0,x} - (\frac{3}{4} - \frac{1}{4}\frac{a_x/a_y}{1 + a_x/a_y}) \times \Delta Q_{inc,x}\right) \tag{5}$$

$$Q_{2,y} = 2\left(Q_{0,y} - (\frac{3}{4} - \frac{1}{4}\frac{a_y/a_x}{1 + a_y/a_x}) \times \Delta Q_{inc,y}\right) \tag{6}$$

We note that these approximations have been derived under the asumption that the space charge tune shift is small compared with the difference in machine tunes in x and y. It is noted from Eqs. 5,6 that for not too different beam size in x, y one has a good approximation by

$$Q_{2,x} \approx 2(Q_{0,x} - 0.625 \times \Delta Q_{inc,x}) \tag{7}$$

and the same for y. Comparing this with the expressions Eq. 3,4 one recognizes the rule that the quadrupolar shifts 3/4 and 1/2 (times the incoherent shift) of round beams are pushed towards the arithmetic mean 5/8 for non-round beams.

It should be noted, however, that the smooth approximation is invalidated for larger space charge shifts in the case of lattices, which have a systematic stop-band of tune at a nearby integer value. This is, for instance, the case for the usual LEAR lattice when Q_x approaches 2, as will be seen below.

3 Numerical Calculation

A detailed calculation must be based on a numerical solution of the Eq. 1,2. Matched solutions are obtained for a superperiod of the lattice, which also yields the single particle phase advance, hence the incoherent betatron tune. All focusing gradients are first determined in the absence of space charge and kept the same in the presence of space charge. With space charge new matching conditions for a periodic solution are searched.

As a next step the phase advance of small perturbations of the envelopes (i.e. two eigenmodes) is determined, which gives the two desired quadrupole oscillation tunes. In the present version of the program bending magnets are ignored.

Results are presented for equal emittances in both transverse planes. As they depend only on I/ϵ, we use this quantity as a measure of the space charge effect in units $\frac{mA}{mm\ mrad}$, where the emittance is unnormalized. We have chosen the energy of 4.2 MeV/u; for different energies I/ϵ has to be scaled accordingly.

3.1 LEAR - Lattice

As a first example we consider a superperiod of the LEAR lattice with an FD and a DF doublet and a length of 19.635 m. The zero space charge phase advance is assumed 208.35° horizontally and 235.8° vertically, hence four such superperiods yield tunes of $Q_{0,x} = 2.315$ (horizontal) and $Q_{0,y} = 2.62$ (vertical). We find that with increasing space charge (i.e. beyond the values of usual operation) the beta functions for the matched solution (keeping the lattice unchanged) become strongly sensitive to space charge, with the minimum value decreasing and the maximum value increasing (see Fig. 1). Q_x is repelled from the integer value 2, which is apparently due to a 180° stopband of the superperiod. Hence, the actual tuneshift is apparently smaller than calculated from the smooth approximation formulae.

In Fig. 2 we show the corresponding frequencies. The incoherent betatron tunes $Q_{x,y}$ are determined consistently taking into account the variation of beta functions. For comparison we also plot the respective values of the incoherent tune shifts in smooth approximation (constant focusing, i.e. Laslett's formula). It is noted that the differences becomes appreciable if the incoherent tune shift is of the order of 0.1 or larger. It is also noted that the quadrupole tune crosses the integer value 5 for $I/\epsilon \approx 2.75$ (close to where incidentially the smooth approximation incoherent tune crosses the value $Q_x = 2$); this would lead to a gradient error resonance in an imperfect lattice. Clearly, the amplitude of such a resonance is limited by the fact that the quadrupole frequency changes with space charge density and de-tuning is expected [1]. This crossing should not be confused with the half integer crossing of the incoherent tune $2Q_y = 5$, which occurs at much lower intensity and has no relevance for gradient errors.

3.2 PS - Lattice

As next example we consider a superperiod of the PS lattice, which is composed of two half-superperiods (hsupp, - hsupp). The zero space charge phase advance per superperiod is assumed 224.64° horizontally and 226.62° vertically, hence 10 such superperiods yield tunes of $Q_{0,h} = 6.24$ and $Q_{0,v} = 6.295$. It is noted that this lattice is quite insensitive to space charge due to the small phase advance per elementary FD-DF focusing cell (approximately 45°) and the large number of such elementary cells (see Fig. 3).

In Fig. 4 corresponding frequencies are shown, where we have only plotted the fractional tunes. Differences from the smooth approximation values are negligible in the range of intensities considered. It is also noted that the predictions from the approximate expressions Eq. 5,6,7 are well satisfied even if the space charge tune shift is not small compared with the difference in machine tunes.

According to Fig. 4 the quadrupolar frequency crosses an integer (i.e. 12) for I/ϵ somewhat above the crossing of the incoherent tune. Attention should be paid in case of slow

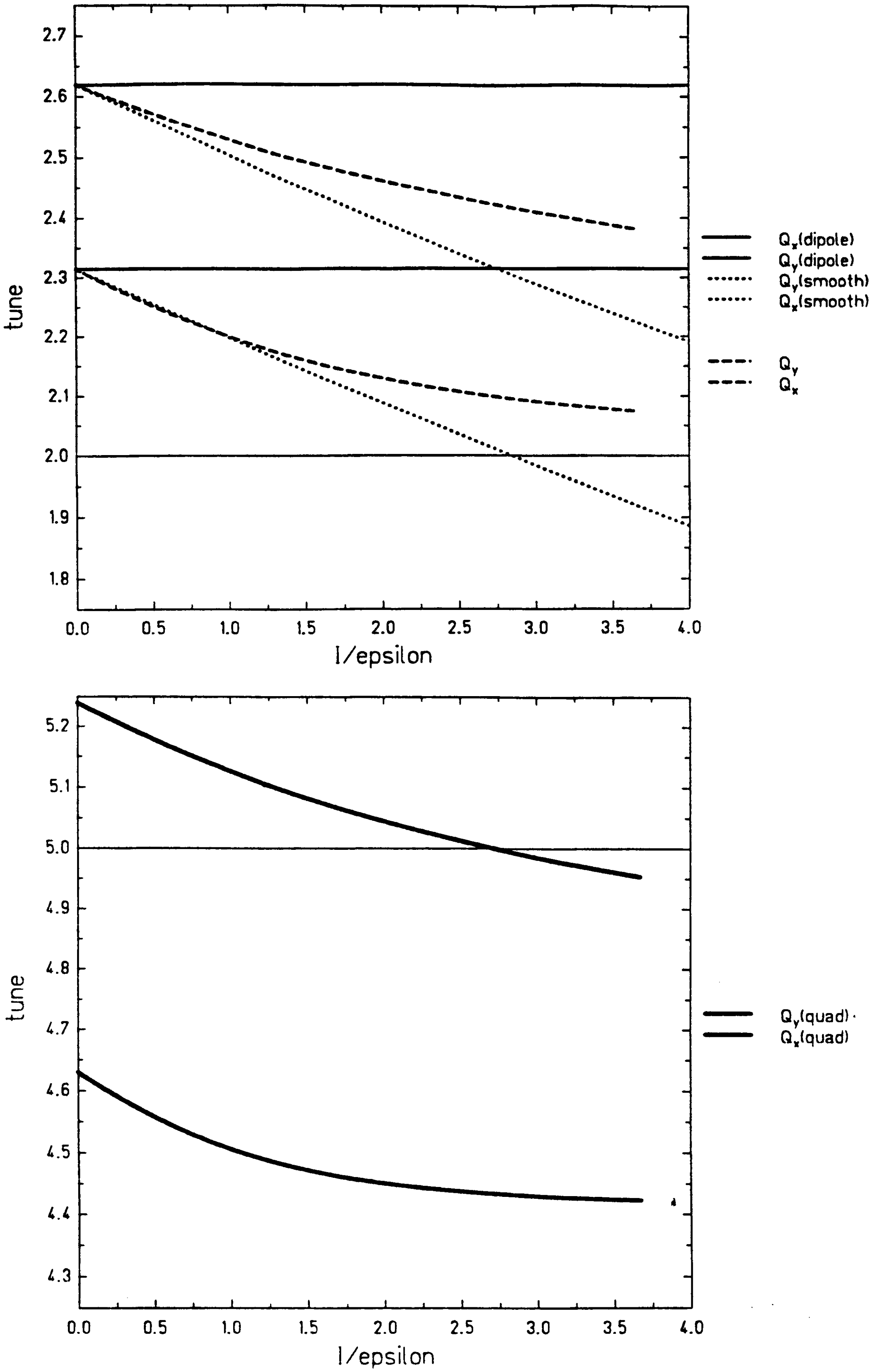

Figure 2: Top: dipole tunes and incoherent tunes (for real lattice as well as in smooth approximation); bottom: quadrupole tunes; calculated for LEAR lattice as function of I/ϵ ($\epsilon_h = \epsilon_v$).

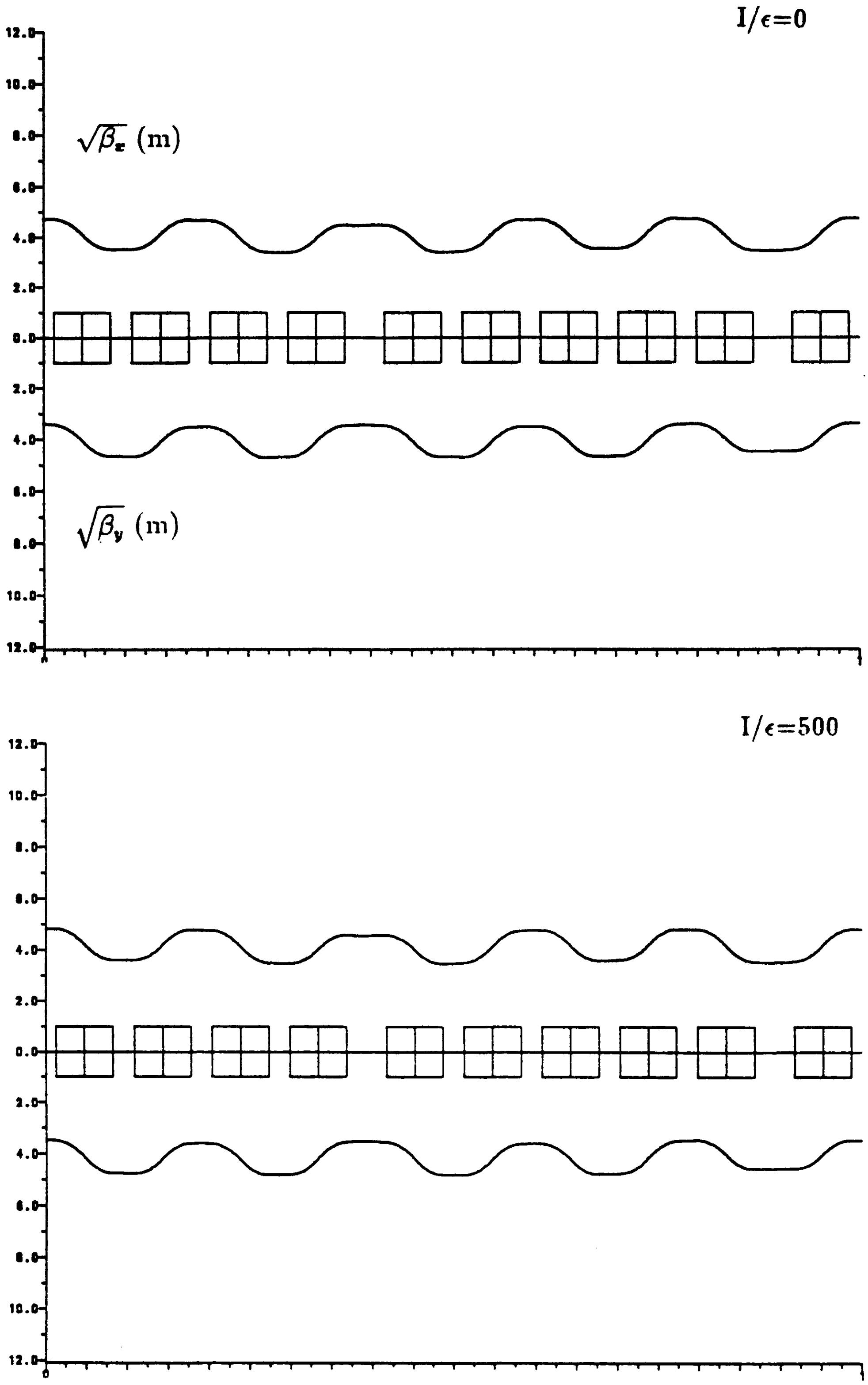

Figure 3: PS superperiod for different I(mA)/ϵ (mm mrad) ($\epsilon_h = \epsilon_v$) and 1 GeV

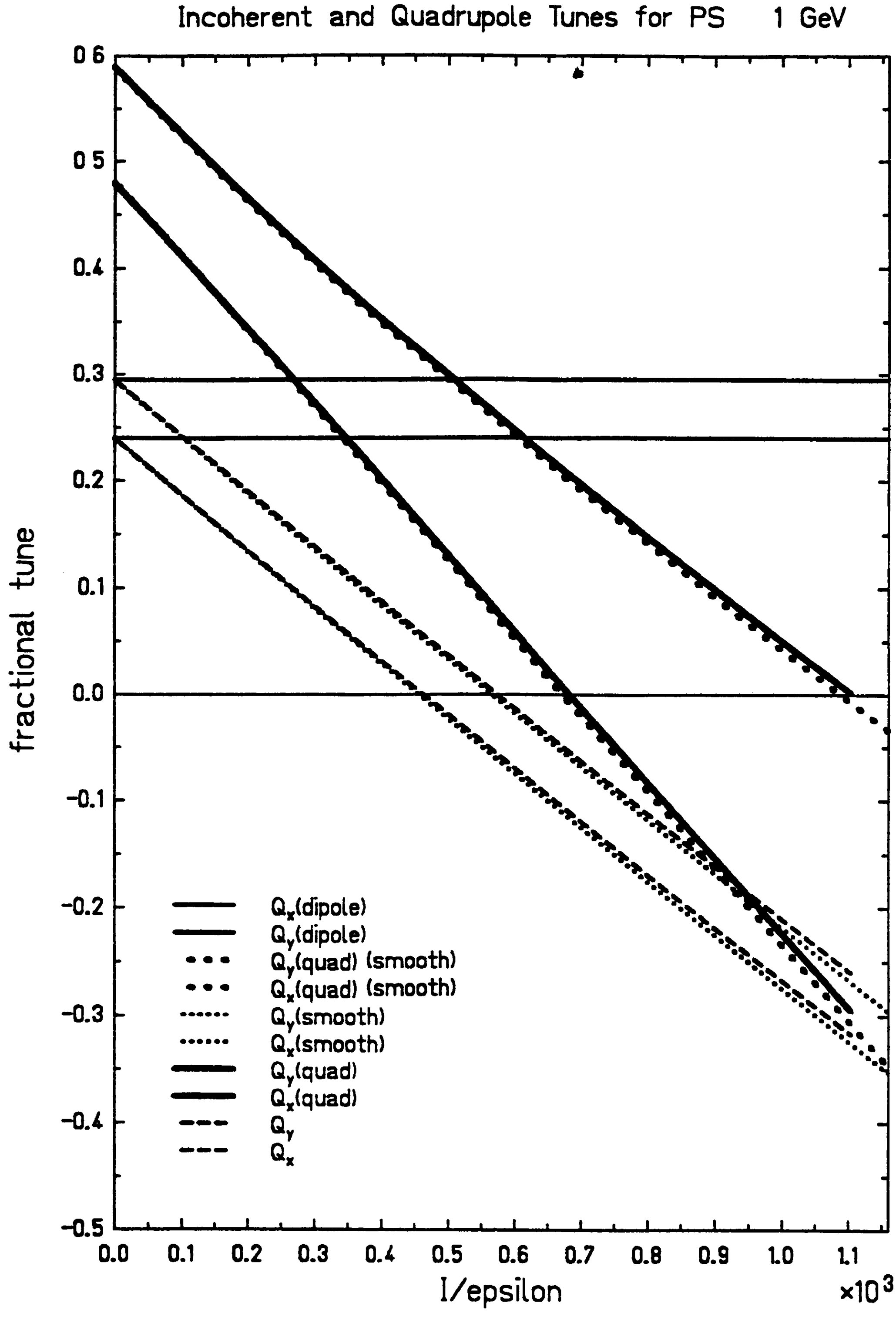

Figure 4: Tunes for PS superperiod as function of I/ϵ ($\epsilon_h = \epsilon_v$)