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LANDAU DAMPING OF COUPLED QUADRUPOLE MODES

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1. INTRODUCTION

A coherent instability of the quadrupolar mode type driven by ions from the residual gas has been observed in the AA[1]. It was successfully cured by tuning both Q-values close to 2.25. One possible explanation of the stabilizing mechanism was given in Ref. 2: at quarter integer tune the stable (m+2Q) and the unstable (n-2Q) modes have the same frequency, so that "fast wave/slow wave cancellation" occurs as discussed in Ref. 2.

In the present note we analyze a different and probably complementary mechanism: working close to the diagonal $Q_h = Q_v$, a large tune spread in one plane is coupled into the other transverse plane and contributes to Landau damping there.

In Part I below we study the broad behavior of coupled quadrupole modes and derive general conditions for Landau damping. To simplify the discussion we then specialize (Part 2) to the "symmetric case" assuming round beams having the same betatron frequencies and the same frequency spreads in the two transverse planes, except for the horizontal (Δ_h) and the vertical (Δ_v) Q-spread of the $\bar{p}$-beam.

We arrive at a stability condition which clearly exhibits the coupling of the stabilizing frequency spread from one plane into the other. In fact, we find that the "inverse average" spread Δ_{av} obtained from

$$\Delta_{av}^{-1} = \frac{1}{2}\left(\hat{\Delta}_h^{-1} + \hat{\Delta}_v^{-1}\right)$$

enters into the stability criterion. Thus if the spread in one plane is small (but finite) a large spread in the other plane contributes to damping.

2. GENERAL CONSIDERATIONS

The envelope equations of motion of the antiproton beam coupled with an ion pocket, taking into account both $\bar{p}$-$\bar{p}$, $\bar{p}$-ion and ion-ion forces, were derived in Ref. 2. Linearized for small deviations (ξ, η) from a stable solution (a_0, b_0) of the beam dimensions, they have the form,

$$
\begin{aligned}
\Omega_0^{-2}\ddot{\xi} + 4Q_h^2\xi &= k_1Q_c^2\overline{\xi}_i + k_2Q_c^2\overline{\eta}_i - k_1Q_{sc}^2\overline{\xi} - k_2Q_{sc}^2\overline{\eta} \\
\Omega_0^{-2}\ddot{\xi}_i + 4q^2\xi_i &= k_1q_c^2\overline{\xi} + k_2q_c^2\overline{\eta} - k_1q_{sc}^2\overline{\xi}_i - k_2q_{sc}^2\overline{\eta}_i \\
\Omega_0^{-2}\ddot{\eta} + 4Q_v^2\eta &= k_3Q_c^2\overline{\eta}_i + k_2Q_c^2\overline{\xi}_i - k_3Q_{sc}^2\overline{\eta} - k_2Q_{sc}^2\overline{\xi} \\
\Omega_0^{-2}\ddot{\eta}_i + 4q^2\eta_i &= k_3q_c^2\overline{\eta} + k_2q_c^2\overline{\xi} - k_3q_{sc}^2\overline{\eta}_i - k_2q_{sc}^2\overline{\xi}_i
\end{aligned}
\qquad (1)
$$

Here, the subscript i distinguishes the variables related to the ions. The ion frequencies are denoted with small case q and the $\overline{\mathrm{p}}$ frequencies with capital Q; Q_c, Q_{sc}, q_c and q_{sc} are the coupling constants, normalized to the antiproton beam revolution frequency Ω_0; $Q_{h,v}^2 = Q_{0h,v}^2 + Q_c^2 - Q_{sc}^2$, $q^2 = q_c^2 - q_{sc}^2$, $Q_{0h,v}$ being the betatron wave numbers, including space charge as discussed in Ref. 2; $k_1 = \left(2\hat{a} + \hat{b}\right)/\left(\hat{a} + \hat{b}\right)$, $k_2 = \hat{a}/\left(\hat{a} + \hat{b}\right)$, $k_3 = \left(\hat{a}/\hat{b}\right)\left(a + 2\hat{b}\right)/\left(\hat{a} + \hat{b}\right)$, with $\hat{a}$ and $\hat{b}$ the horizontal and vertical beam radii at equilibrium.

As is usual in Landau damping calculations we start solving the equations with the assumption that the averaged variables (generically designated by $\overline{\alpha}$) oscillate according to:

$$\overline{\alpha} = \overline{\alpha}_0 e^{i(ns/R - \nu\Omega_0 t)}$$

where n = 1, 2, 3, ... is the mode number.

We regard the terms on the right-hand side of Eqs. (1) as driving forces. We can ignore the steady-state solutions of the homogeneous equations (oscillatory motion with frequencies $2Q_h$, $2Q_v$ and 2q) and concentrate on the particular solutions, which describe the coherent motion. These solutions have the same time and space dependence as the driving terms.

We assume also that the particles of the main beam (i.e. the antiprotons) and the ions have a spread in momentum and in incoherent betatron amplitudes, a and b, with a certain distribution $N(p, a^2, b^2)$ depending linearly on p and quadratically on a and b such that

$$\overline{\alpha} = \iiint \alpha N_\alpha\left(p, a^2, b^2\right) dp\, da^2\, db^2 \ .$$

We can then solve (1) with respect to the variables ξ, η, ξ_i and η_i, for individual particles and integrate them after multiplication by the respective distribution functions, in order to obtain the beam average. We neglect the variation of the numerators in the integrals, and we obtain

$$
\begin{cases}
\left(I_\xi^{-1} + k_1Q_{sc}^2\right)\overline{\xi} + k_2Q_{sc}^2\overline{\eta} - Q_c^2\left(k_1\overline{\xi}_i + k_2\overline{\eta}_i\right) = 0 \\
\left(I_{\xi_i}^{-1} + k_1q_{sc}^2\right)\overline{\xi}_i + k_2q_{sc}^2\overline{\eta}_i - q_c^2\left(k_1\overline{\xi} + k_2\overline{\eta}\right) = 0 \\
\left(I_\eta^{-1} + k_3Q_{sc}^2\right)\overline{\eta} + k_2Q_{sc}^2\overline{\xi} - Q_c^2\left(k_3\overline{\eta}_i + k_2\overline{\xi}_i\right) = 0 \\
\left(I_{\eta_i}^{-1} + k_3q_{sc}^2\right)\overline{\eta}_i + k_2q_{sc}^2\overline{\xi}_i - q_c^2\left(k_3\overline{\eta} + k_2\overline{\xi}\right) = 0
\end{cases}
\qquad (2)
$$

where

$$\begin{cases} I_{\xi,\eta} & = \iiint \dfrac{N_{h,v}\left(p,a^2,b^2\right)dp\,da^2\,db^2}{4Q_{h,v}^2 - \left[v - n(\Omega/\Omega_0)\right]^2} \\ I_{\xi_i,\eta_i} & = \iiint \dfrac{N_{h_i,v_i}\left(p,a^2,b^2\right)dp\,da^2\,db^2}{4q^2 - v^2} \end{cases} \quad (3)$$

and $\Omega = (1/R)(ds/dt)$ is the angular velocity.

The combined effect of the three spreads in the above integrals is difficult to deal with, but the analysis is very much simplified if one assumes that the spreads are uncorrelated. In this case, the distribution functions become the product of three independent functions. Using the results of Ref. 5, which states that the effective spread for damping is mainly the largest of the spreads, we consider only the effect of this one.

A final approximation (slow-wave approximation) leads us to expand the denominators in (3), to keep only the largest terms when $v \sim n\Omega/\Omega_0 - 2Q_{h,v}$ and $v \sim 2q$, and to linearize the result in terms of the largest spreading parameters,

$$\begin{aligned} I_{\xi,\eta} &\cong -\frac{1}{4Q_{h,v}^0}\int_{-\infty}^{+\infty} \frac{N\left(x_{1,2}\right)dx_{1,2}}{u_{h,v}\left(x_{1,2}\right) - v} \\ I_{\xi_i,\eta_i} &\cong \frac{1}{4q^0}\int_{-\infty}^{+\infty} \frac{N\left(x_{3,4}\right)dx_{3,4}}{2q\left(x_{3,4}\right) - v} \end{aligned} \quad (4)$$

where x_k, k = 1, 2, 3, 4, can be any of the spreading parameters p, a^2 or b^2, and $u_h(x_{1,2}) = n[\Omega(x_{1,2})/\Omega_0] - 2Q_{h,v}(x_{1,2})$.

Linearizing $u_{h,v}(x_{1,2})$ and $q(x_{3,4})$ with respect to the spreading parameters, and assuming semi-circular distributions of the form

$$N(x) = \begin{cases} \dfrac{2\sqrt{\hat{x}^2 - x^2}}{\pi\hat{x}^2}, & for |x| \le \hat{x} \\ 0 & , for |x| \ge \hat{x} \end{cases}$$

we obtain (Ref. 6)

$$\begin{cases} I_{\xi,\eta}^{-1} & = 2Q_{h,v}^0\left[v - \left(n - 2Q_{h,v}^0\right) + i\hat{\Delta}_{h,v}\right] \\ I_{\xi_i,\eta_i}^{-1} & = -2q^0\left[v - 2q^0 + i\hat{\Delta}_{h_i,v_i}\right] \end{cases} \quad (5)$$

where the spreads are given by

$$\begin{cases}\hat{\Delta}_{h,v} &= \sqrt{\Delta_{h,v}^2 - \left[v - \left(n - 2Q_{h,v}^0\right)\right]^2} \\ \hat{\Delta}_{h_i,v_i} &= \sqrt{\Delta_{h_i,v_i}^2 - \left(v - 2q^0\right)^2}\end{cases}$$

with

$$\begin{cases}\Delta_{h,v} &= 2\hat{x}_{1,2}\left|\dfrac{\partial Q_{h,v}}{\partial x_{1,2}} - \dfrac{n}{2\Omega_0}\dfrac{\partial\Omega}{\partial x_{1,2}}\right|_0 \\ \Delta_{h_i,v_i} &= 2\hat{x}_{3,4}\left|\dfrac{\partial q_{h,v}}{\partial x_{3,4}}\right|_0\end{cases} \tag{6}$$

With results (5), and assuming that the dependence of $\hat{\Delta}_{h,v}$ and $\hat{\Delta}_{h_i,v_i}$ on v can be neglected, we obtain the following system of equations:

$$\begin{cases}(v-a_{11})\overline{\xi} + a_{13}\overline{\eta} - a_{12}\overline{\xi}_i - a_{14}\overline{\eta}_i = 0 \\ (v-a_{22})\overline{\xi}_i - a_{24}\overline{\eta}_i + a_{21}\overline{\xi} + a_{23}\overline{\eta} = 0 \\ (v-a_{33})\overline{\eta} + a_{31}\overline{\xi} - a_{34}\overline{\eta}_i - a_{32}\overline{\xi}_i = 0 \\ (v-a_{44})\overline{\eta}_i - a_{42}\overline{\xi}_i + a_{41}\overline{\xi} + a_{43}\overline{\eta} = 0\end{cases} \tag{7}$$

where

$$\begin{cases}a_{11} = n - 2Q_h^0 - \dfrac{k_1 Q_{sc}^2}{2Q_h^0} - i\hat{\Delta}_h, & a_{12} = \dfrac{k_1 Q_c^2}{2Q_h^0}, & a_{13} = \dfrac{k_2 Q_{sc}^2}{2Q_h^0}, & a_{14} = \dfrac{k_2 Q_c^2}{2Q_h^0} \\ a_{22} = 2q^0 + \dfrac{k_1 q_{sc}^2}{2q^0} - i\hat{\Delta}_{h_i}, & a_{21} = \dfrac{k_1 q_c^2}{2q^0}, & a_{23} = \dfrac{k_2 q_c^2}{2q^0}, & a_{24} = \dfrac{k_2 q_{sc}^2}{2q^0} \\ a_{33} = n - 2Q_v^0 - \dfrac{k_3 Q_{sc}^2}{2Q_v^0} - i\hat{\Delta}_v, & a_{31} = \dfrac{k_2 Q_{sc}^2}{2Q_v^0}, & a_{32} = \dfrac{k_2 Q_c^2}{2Q_v^0}, & a_{34} = \dfrac{k_3 Q_c^2}{2Q_v^0} \\ a_{44} = 2q^0 + \dfrac{k_3 q_{sc}^2}{2q^0} - i\hat{\Delta}_{v_i}, & a_{41} = \dfrac{k_2 q_c^2}{2q^0}, & a_{42} = \dfrac{k_2 q_{sc}^2}{2q^0}, & a_{43} = \dfrac{k_3 q_c^2}{2q^0}\end{cases}$$

To solve system (7), it is necessary to solve the determinant equation:

$$\begin{vmatrix} v-a_{11} & -a_{12} & a_{13} & -a_{14} \\ a_{21} & v-a_{22} & a_{23} & -a_{24} \\ a_{31} & -a_{32} & v-a_{33} & -a_{34} \\ a_{41} & -a_{42} & a_{43} & v-a_{44}\end{vmatrix} = 0 \tag{8}$$

or, what amounts to the same, to find the eigenvalues of the matrix

$$A = \begin{vmatrix} a_{11} & a_{12} & -a_{13} & a_{14} \\ -a_{21} & a_{22} & -a_{23} & a_{24} \\ -a_{31} & a_{32} & a_{33} & a_{34} \\ -a_{41} & a_{42} & -a_{43} & a_{44} \end{vmatrix} \tag{9}$$

For our purpose, it is not necessary to find explicit expressions for the eigenvalues, but only to find conditions leading to Landau damping, i.e. leading to eigenvalues in the lower half-plane Im(ν) < 0 (as we assumed solutions of the form $e^{-i\nu\Omega t}$).

Equation (8) is a polynomial equation of 4th order,

$$\nu^4 + a_1\nu^3 + a_2\nu^2 + a_3\nu + a_4 = 0 \tag{10}$$

where

$$a_1 = -a_{11} - a_{22} - a_{33} - a_{44}$$

$$a_2 = a_{11}(a_{22}+a_{33}+a_{44}) + a_{12}a_{21} - a_{13}a_{31} + a_{14}a_{41} + a_{22}a_{33} + a_{23}a_{32} + a_{22}a_{44} - a_{24}a_{42} + a_{33}a_{44} + a_{34}a_{43}$$

etc.

The problem of finding the necessary and sufficient conditions in the coefficients of a polynomial such that all its roots lie in a certain half-plane (Hurwitz problem[4]) was studied for complex coefficients by E. Frank[3]. A theorem (theorem 3.2, Ref. 3), adapted to our case, states that a polynomial equation of the 4th order, as (10), has all its zeros in the lower half-plane Im(ν) < 0, if, and only if, the determinants Δ_k (k = 1, 2, 3, 4) obtained by taking the first 2k-1 rows and columns from

$$\begin{vmatrix} q_1 & q_2 & q_3 & q_4 & 0 & 0 & 0 \\ 1 & p_1 & p_2 & p_3 & p_4 & 0 & 0 \\ 0 & q_1 & q_2 & q_3 & q_4 & 0 & 0 \\ 0 & 1 & p_1 & p_2 & p_3 & p_4 & 0 \\ 0 & 0 & q_1 & q_2 & q_3 & q_4 & 0 \\ 0 & 0 & 1 & p_1 & p_2 & p_3 & p_4 \\ 0 & 0 & 0 & q_1 & q_2 & q_3 & q_4 \end{vmatrix}$$

are all positive. Here, $p_k = \mathrm{Re}(a_k)$ and $q_k = \mathrm{Im}(a_k)$.

The first condition is always verified, since

$$\Delta_1 = q_1 = \mathrm{Im}(a_1) = \mathrm{Im}(-a_{11} - a_{22} - a_{33} - a_{44}) = \hat{\Delta}_h + \hat{\Delta}_{h_i} + \hat{\Delta}_\nu + \hat{\Delta}_{\nu_i} > 0$$

The other three conditions are very hard to check, given the complicated expressions we would obtain for Δ_2, Δ_3 and Δ_4. We therefore try to obtain only sufficient conditions for Landau damping.

A theorem by Gershgorin[7] states that all the eigenvalues of an arbitrary nxn matrix (a_{ij}) are located in the union of the circular disks defined by

$$\left|\nu - a_{ii}\right| \le \sum_{\substack{j=1 \\ j \ne i}}^{n} \left|a_{ij}\right| \qquad i = 1,2,\ldots,n$$

and also in the union of the disks defined by

$$\left|\nu - a_{jj}\right| \le \sum_{\substack{i=1 \\ i \ne j}}^{n} \left|a_{ij}\right| \qquad j = 1,2,\ldots,n$$

Thus we assure that all the eigenvalues of our matrix (9) are in the lower half-plane if we have all these disks in the same half-plane. For this, it is enough that

$$-\mathrm{Im}(a_{ii}) > \sum_{\substack{j=1 \\ j \ne i}}^{n} \left|a_{ij}\right| \qquad i = 1,2,\ldots,n$$

or

$$-\mathrm{Im}(a_{jj}) > \sum_{\substack{i=1 \\ j \ne i}}^{n} \left|a_{ij}\right| \qquad j = 1,2,\ldots,n$$

i.e. that the absolute value of the (negative) imaginary part of their centres be bigger than their radii.

In our case, after substitution of the a_{ij}, we obtain the following conditions,

$$\begin{cases} \hat{\Delta}_h > \dfrac{1}{2Q_h^0}\left[k_2 Q_{sc}^2 + (k_1 + k_2) Q_c^2\right] \\ \hat{\Delta}_{h_i} > \dfrac{1}{2q^0}\left[k_2 q_{sc}^2 + (k_1 + k_2) q_c^2\right] \\ \hat{\Delta}_v > \dfrac{1}{2Q_v^0}\left[k_2 Q_{sc}^2 + (k_2 + k_3) Q_c^2\right] \\ \hat{\Delta}_{v_i} > \dfrac{1}{2q^0}\left[k_2 q_{sc}^2 + (k_2 + k_3) q_c^2\right] \end{cases} \tag{11}$$

or

$$\begin{cases} \hat{\Delta}_h > \dfrac{1}{2}\left[k_2 \dfrac{Q_{sc}^2}{Q_v^2} + (k_1+k_2)\dfrac{q_c^2}{q^0}\right] \\ \hat{\Delta}_{h_i} > \dfrac{1}{2}\left[k_2 \dfrac{q_{sc}^2}{q^0} + \left(\dfrac{k_1}{Q_h^0} + \dfrac{k_2}{Q_v^0}\right) Q_c^2\right] \\ \hat{\Delta}_v > \dfrac{1}{2}\left[k_2 \dfrac{Q_{sc}^2}{Q_h^0} + (k_2+k_3)\dfrac{q_c^2}{q^0}\right] \\ \hat{\Delta}_{v_i} > \dfrac{1}{2}\left[k_2 \dfrac{q_{sc}^2}{q^0} + \left(\dfrac{k_3}{Q_v^0} + \dfrac{k_2}{Q_h^0}\right) Q_c^2\right] \end{cases} \tag{12}$$

2. FULLY SYMMETRIC CASE, EXCEPT FOR ANTIPROTON FREQUENCY SPREAD

We now study the simpler case of round beams with the same betatron frequencies and the same spreads in both transverse planes, except for the antiproton spread, i.e. we assume $\hat{a} = \hat{b}$ (round beam), $k_1 = k_3 = 3/2$, $k_2 = 1/2$, $Q_h = Q_v = Q$, $\hat{\Delta}_{h_i} = \hat{\Delta}_{v_i} = \Delta_i$, but $\hat{\Delta}_h \neq \hat{\Delta}_v$. We use the following notation:

$$\Delta = \left(\hat{\Delta}_h + \hat{\Delta}_v\right)/2, \qquad d = \left(\hat{\Delta}_h - \hat{\Delta}_v\right)/2$$

$$\Delta Q_{sc} = Q_{sc}^2/2Q, \qquad \Delta Q_c = Q_c^2/2Q$$

$$\Delta q_{sc} = q_{sc}^2/2q, \qquad \Delta q_c = q_c^2/2q$$

and we obtain, for Eqs (7):

$$\begin{cases} \left[\nu + i(\Delta + d) - n + 2Q + \dfrac{3}{2}\Delta Q_{sc}\right]\bar{\xi} + \dfrac{1}{2}\Delta Q_{sc}\bar{\eta} - \dfrac{1}{2}\Delta Q_c\left(3\bar{\xi}_i + \bar{\eta}_i\right) = 0 \\ \left[\nu + i(\Delta - d) - n + 2Q + \dfrac{3}{2}\Delta Q_{sc}\right]\bar{\eta} + \dfrac{1}{2}\Delta Q_{sc}\bar{\xi} - \dfrac{1}{2}\Delta Q_c\left(3\bar{\eta}_i + \bar{\xi}_i\right) = 0 \\ \left[\nu + i\Delta_i - 2q - \dfrac{3}{2}\Delta q_{sc}\right]\bar{\xi}_i - \dfrac{1}{2}\Delta q_{sc}\bar{\eta}_i + \dfrac{1}{2}\Delta q_c(3\bar{\xi} + \bar{\eta}) = 0 \\ \left[\nu + i\Delta_i - 2q - \dfrac{3}{2}\Delta q_{sc}\right]\bar{\eta}_i - \dfrac{1}{2}\Delta q_{sc}\bar{\xi}_i + \dfrac{1}{2}\Delta q_c(3\bar{\eta} + \bar{\xi}) = 0 \end{cases}$$

By summing and subtracting the equations for the antiprotons (the first two equations), and for the ions (the other two), we obtain:

$$\begin{cases}[\nu+i\Delta-n+2(Q+\Delta Q_{sc})](\overline{\xi}+\overline{\eta})+id(\overline{\xi}-\overline{\eta})-2\Delta Q_c(\overline{\xi}_i+\overline{\eta}_i)=0\\ [\nu+i\Delta-n+2Q+\Delta Q_{sc}](\overline{\xi}-\overline{\eta})+id(\overline{\xi}+\overline{\eta})-\Delta Q_c(\overline{\xi}_i-\overline{\eta}_i)\quad=0\end{cases} \tag{13.a}$$

$$\begin{cases}[\nu+i\Delta_i-2(q+\Delta q_{sc})](\overline{\xi}_i+\overline{\eta}_i)+2\Delta q_c(\overline{\xi}+\overline{\eta})=0\\ [\nu+i\Delta-2q-\Delta q_{sc}](\overline{\xi}_i-\overline{\eta}_i)+\Delta q_c(\overline{\xi}-\overline{\eta})\quad=0\end{cases} \tag{13.b}$$

We use the ion equations (13.b) to express the ion variables in terms of the antiproton ones,

$$\overline{\xi}_i\pm\overline{\eta}_i=-\frac{m_\pm\Delta q_c}{\nu+i\Delta_i-2q-m_\pm\Delta q_{sc}}(\overline{\xi}\pm\overline{\eta}) \tag{14}$$

where $m_-=1$ and $m_+=2$. Introduced into the antiproton equations (13.a), this leads to:

$$\begin{aligned}&A_1(\overline{\xi}+\overline{\eta})-id(\overline{\xi}-\overline{\eta})=0\\ &A_1(\overline{\xi}-\overline{\eta})-id(\overline{\xi}+\overline{\eta})=0\end{aligned} \tag{15}$$

where

$$\begin{cases}A_1\ =\Delta\nu-i\Delta+\dfrac{4\Delta Q_c\Delta q_c}{\Delta\nu_i-i\Delta_i}\\ A_2\ =\Delta\nu-i\Delta+\Delta Q_{sc}+\dfrac{\Delta Q_c\Delta q_c}{\Delta\nu_i-i\Delta_i-\Delta q_{sc}}\\ \Delta\nu=[n-2(Q+\Delta Q_{sc})]-\nu\equiv(n-2\tilde{Q})-\nu\\ \Delta\nu_i=2(q+q_{sc})-\nu\equiv2\tilde{q}-\nu\end{cases}$$

In order to be solvable, the determinant of (15) must yield $A_1A_2+d^2=0$, which is reducible to a fourth order polynomial equation, analogue to (8) or (10). In the fully symmetric case ($d=0$), we obtain two independent quadratic equations ($A_1=0$, $A_2=0$) for the 4 roots.

In the present case, assuming small d, we use the following approximation: we introduce the roots of $A_1=0$ into A_2, treating it as a constant, to obtain a quadratic equation and solve this one; we proceed in a similar way to obtain the second pair of roots, by taking the roots of $A_2=0$ and introducing them into $A_1=0$.

Near the symmetric mode frequency (where $A_1=0$), we have, in the worst case $\Delta\nu=\Delta\nu_i$ and $\Delta\Delta_i=4\Delta Q_c\Delta q_c$. We insert this into A_2 and obtain the improved symmetric mode equation:

$$\Delta\nu-i\Delta+\frac{4\Delta Q_c\Delta q_c}{\Delta\nu_i-i\Delta_i}+\alpha=0 \tag{16}$$

where

$$\alpha = \frac{d^2}{-i\Delta + \Delta Q_{sc} - \Delta Q_c \Delta q_c / (i\Delta_i + \Delta q_{sc})}$$

We note that[2]

$$\Delta Q_{sc} = \Delta Q_c / \gamma^2 f, \qquad \Delta q_{sc} = \Delta q_c \cdot f$$

If the neutralization factor f = N_i/N is in the range

$$\frac{1}{\gamma^2} << f << 1$$

($1/\gamma^2$ = 0.07 for the AA) we have $\Delta Q_{sc} \ll \Delta Q_c$ and $\Delta q_{sc} \ll \Delta q_c$, and it seems reasonable to assume $\Delta Q_{sc} \ll \Delta$ and $\Delta q_{sc} \ll \Delta_i$.

Assuming these conditions (for values of f outside that range, a different treatment must be done), and the worst case $\Delta\nu_i = \Delta\nu$ (full ion-antiproton resonance) Eq. (16) becomes a quadratic equation in $\Delta\nu$:

$$(\Delta\nu)^2 + i(\delta - \Delta - \Delta_i)\Delta\nu + \Delta_i(\delta - \Delta) + 4\Delta Q_c \Delta q_c = 0 \qquad (17)$$

where

$$\delta = \frac{\Delta_i d^2}{\Delta\Delta_i - \Delta Q_c \Delta q_c} \qquad (18)$$

For this equation, we want that all the solutions $\Delta\nu$ have $\mathrm{Im}(\Delta\nu) = -\mathrm{Im}(\nu) > 0$. Again, we use E. Frank's[3] results, and define

$$p_1 = \mathrm{Re}[i(\delta - \Delta - \Delta_i)] = 0$$

$$q_1 = \mathrm{Im}[i(\delta - \Delta - \Delta_i] = \delta - \Delta - \Delta_i$$

$$p_2 = \mathrm{Re}[\Delta_i(\delta - \Delta) + 4\Delta Q_c \Delta q_c] = \Delta_i(\delta - \Delta) + 4\Delta Q_c \Delta q_c$$

$$q_2 = \mathrm{Im}[\Delta_i(\delta - \Delta) + 4\Delta Q_c \Delta q_c] = 0$$

to obtain

$$\Delta_1 = q_1$$

$$\Delta_2 = \begin{vmatrix} q_1 & q_2 & 0 \\ 1 & p_1 & p_2 \\ 0 & q_1 & q_2 \end{vmatrix} = \begin{vmatrix} q_1 & 0 & 0 \\ 1 & 0 & p_2 \\ 0 & q_1 & 0 \end{vmatrix} = -p_2 q_1^2$$

In order to have $\mathrm{Im}(\Delta\nu) > 0$, we must have $\Delta_1 < 0$ and $\Delta_2 > 0$, i.e.

$$\delta < \Delta + \Delta_i$$
$$\Delta_i(\delta - \Delta) + 4\Delta Q_c \Delta q_c < 0 \tag{19}$$

The second condition includes the first one, since it states that

$$\delta < \Delta - \frac{4\Delta Q_c \Delta q_c}{\Delta_i} \Rightarrow \delta < \Delta + \Delta_i$$

and we concentrate on the second one. We define the new variable $y = \Delta_i\Delta_{av}/\Delta Q_c\Delta q_c$, the "inverse average spread" being defined by

$$\Delta_{av}^{-1} = \frac{\Delta}{\hat{\Delta}_h\hat{\Delta}_v} = \frac{1}{2}\left(\hat{\Delta}_h^{-1} + \hat{\Delta}_v^{-1}\right) \tag{20}$$

With this new variable, and substituting $d^2 = (\Delta - \Delta_{av})\Delta$, we can re-write the second condition in (19) as

$$\frac{y^2 - 5y + 4\Delta_{av}/\Delta}{y - \Delta_{av}/\Delta} > 0 \tag{21}$$

In analogy to the dipole modes which require $\Delta_i\Delta > \Delta Q_c\Delta q_c \Leftrightarrow y > \Delta_{av}/\Delta$, condition (21) leads to the following stability condition for quadrupole modes:

$$y > \frac{5}{2}\left(1 + \sqrt{1 - \frac{16\Delta_{av}}{25\Delta}}\right) \tag{22}$$

(the other solution of (21) $(5/2)\left[1 - \sqrt{1 - (16\Delta_{av}/25\Delta)}\right] < y < \Delta_{av}/\Delta$, is forbidden by the stability of dipole modes).

Since for the r.h.s. of (22) we have

$$\frac{5}{2}\left(1 + \sqrt{1 - \frac{16\Delta_{av}}{25\Delta}}\right) < 5 \quad ,$$

we can use the simpler condition $y > 5$, or, using the definition of y,

$$\Delta_i\Delta_{av} > 5\Delta Q_c\Delta q_c. \tag{23}$$

4. DISCUSSION

Equation (23) is our main result. It shows that the average spread as defined by Eq. (20) enters when the tune is close to the diagonal $Q_h = Q_v$, with tolerances probably given by the tune shifts ΔQ_c and or the spreads Δ_h, Δ_v. In the AA the vertical tune spread is considerably larger than the horizontal one. This might explain why the observed instability which occurs usually in the horizontal plane is reduced or absent for $Q_h \rightarrow Q_v$. Experiments with equal tunes, different from the quarter integer, could be useful to disentangle the fast wave-slow wave stabilization of Ref. 2 from the "Q-spread coupling" discussed in the present note.

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