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N = 2 EXTREMAL BLACK HOLES

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ABSTRACT

It is shown that extremal magnetic black hole solutions of N = 2 supergravity coupled to vector multiplets X^Λ with a generic holomorphic prepotential $F(X^\Lambda)$ can be described as supersymmetric solitons which interpolate between maximally symmetric limiting solutions at spatial infinity and the horizon. A simple exact solution is found for the special case that the ratios of the X^Λ are real (imaginary), and it is seen that the logarithm of the conformal factor of the spatial metric equals the Kähler potential on the vector multiplet moduli space. Several examples are discussed in detail.

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1 Introduction

Black holes seem to be a never-ending source of surprises. While much has been learned about their behavior, much remains to be understood – even at the classical level. In this paper we study the classical supersymmetric solutions in a general theory with N = 2 supersymmetry. Previous work on this subject can be found in [1, 2]. The solutions appear to have a richer structure than the more thoroughly studied N = 4 case. In section 2 we recall some aspects of N = 2 supergravity. In section 3 the magnetic solutions are described in terms of trajectories in the special geometry of the N = 2 moduli space which terminate at a supersymmetric fixed point at the horizon. In section 4 we find that the equations can be integrated for a restricted but large class of cases. An intriguing relation between the Kähler potential on the moduli space and the metric conformal factor emerges. Some simple examples are worked out in detail in section 5. We do not attain a complete characterization of the classical geometry of N = 2 black holes in this paper, but we hope that our results prove useful for future efforts in this direction.

2 Special Geometry and N = 2 Supersymmetry

We study N = 2 supergravity coupled to n N = 2 vector multiplets in the framework of special geometry [3]–[6]. In this section some formulae that will be needed in the following are recalled. Further details can be found in [4] whose notation we adopt. The supergravity theory is defined in terms of a projective holomorphic section $(X^\Lambda(\phi^i), -\frac{i}{2}F_\Lambda(\phi^i))$, $\Lambda = 0, 1, ..., n$, $i = 1, ..., n$, of an $Sp(2n+2)$ vector bundle over the moduli space parametrized by ϕ^i. (We note that alternate conventions are often employed in which the definition of F_Λ differs by a factor of $2i$.) In some cases the theory can be described in terms of a holomorphic function $F(X)$ of degree two:

$$F_\Lambda(\phi^i) = F_\Lambda(X(\phi^i)) = \frac{\partial}{\partial X^\Lambda} F(X) \ . \tag{1}$$

Given prepotential $F(X)$, or a holomorphic section $(X^\Lambda, -\frac{i}{2}F_\Lambda)$, one can construct the entire scalar and vector parts of the action.

It is convenient to introduce the inhomogeneous coordinates

$$Z^\Lambda = \frac{X^\Lambda(\phi_i)}{X^0(\phi_i)} \ , \qquad Z^0 = 1 \ . \tag{2}$$

We assume $Z^\Lambda(\phi_i)$ to be invertible, so that, in special coordinates, $\frac{\partial Z^\Lambda}{\partial \phi^i} = \delta_i^\Lambda$. In this case the complex scalars $Z^i = \phi^i$ $(i = 1, ..., n)$ represent the lowest component of the n vector multiplets of N = 2 supersymmetry. The Kähler potential determining the metric of these fields is

$$\begin{aligned} K(Z, \bar{Z}) &= 2 \ln |X^0| = -\ln\left(N_{\Lambda\Sigma}(Z, \bar{Z})\, Z^\Lambda \bar{Z}^\Sigma\right) \\ &= -\ln \frac{1}{2}[f(Z) + \bar{f}(\bar{Z}) + \frac{1}{2}(Z^i - \bar{Z}^i)(\bar{f}_i - f_i)] \ , \end{aligned} \tag{3}$$

where $N_{\Lambda\Sigma} = \frac{1}{4}(F_{\Lambda\Sigma} + \bar{F}_{\Lambda\Sigma})$ and $f(Z) = (X^0)^{-2}F(X)$. In the conformal gauge [4]

$$N_{\Lambda\Sigma}X^\Lambda \bar{X}^\Sigma = 1 \ . \tag{4}$$

The graviphoton field strength, as well as the field strengths of the n Abelian vector multiplets, are constructed out of $n+1$ field strengths $\hat{F}^\Lambda_{\mu\nu} = \partial_\mu W^\Lambda_\nu - \partial_\nu W^\Lambda_\mu$. The graviphoton field strength is

$$T^+_{\mu\nu} = \frac{4N_{\Lambda\Sigma}X^\Lambda}{N_{IJ}X^I X^J} \ \hat{F}^{+\Sigma}_{\mu\nu} \ , \tag{5}$$

where the superscript + (-) denotes the (anti)-self-dual part. This defines the central charge of the theory, since it enters into gravitino transformation rules. The vector field strengths which enter the gaugino supersymmetry transformations are

$$\mathcal{F}^{+\Lambda}_{\mu\nu} = \hat{F}^{+\Lambda}_{\mu\nu} - \frac{1}{4}X^\Lambda \, T^+_{\mu\nu} \ . \tag{6}$$

The anti-self-dual vector field strengths $\hat{F}^{-\Lambda}$ are part of the symplectic vector

$$\begin{pmatrix} \hat{F}^{-\Lambda} \\ -2i\bar{\mathcal{N}}_{\Lambda\Sigma}\,\hat{F}^{-\Sigma} \equiv iG^-_\Lambda \end{pmatrix} , \tag{7}$$

where again the expression for the matrix $\mathcal{N}_{\Lambda\Sigma}$ is derived from the prepotential $F(X)$ [4]–[6]. The vector part of the action is then proportional to

$$\text{Re} \ \hat{F}^{-\Lambda}G^-_\Lambda \ , \tag{8}$$

and the graviphoton field strength can be written in the manifestly symplectic form

$$T^-_{\mu\nu} = 2X^\Lambda G^-_{\Lambda\mu\nu} + F_\Lambda \hat{F}^{-\Lambda}_{\mu\nu} \ . \tag{9}$$

The Lagrangian for the scalar components of the vector multiplets is defined by the Kähler potential as

$$g_{i\bar{j}} \, \partial_\mu\phi^i \, \partial_\nu\bar{\phi}^{\bar{j}} \, g^{\mu\nu} \ , \tag{10}$$

where $g^{\mu\nu}$ is the space-time metric, and

$$g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K(\phi,\bar{\phi}) \ . \tag{11}$$

The gravitino supersymmetry transformation law, to leading order in fermi fields, is

$$\delta\psi^\alpha_\mu = 2\nabla_\mu\epsilon^\alpha - \frac{1}{16}\gamma^{\nu\lambda}T^-_{\nu\lambda}\gamma_\mu\epsilon^{\alpha\beta}\epsilon_\beta + iA_\mu\epsilon^\alpha \ , \tag{12}$$

where $\alpha,\beta = 1,2$ are $SU(2)$ indices and $A_\mu = \frac{i}{2}N_{\Lambda\Sigma}[\bar{X}^\Lambda\partial_\mu X^\Sigma - (\partial_\mu\bar{X}^\Lambda)X^\Sigma]$. The gaugino transformation law is

$$\delta\Omega^\Lambda_\alpha = 2\gamma^\mu\nabla_\mu X^\Lambda\epsilon_\alpha + \frac{1}{2}\gamma^{\nu\lambda}\mathcal{F}^{+\Lambda}_{\nu\lambda}\epsilon_{\alpha\beta}\epsilon^\beta + 2i\gamma^\mu A_\mu\epsilon_\alpha \ . \tag{13}$$

BPS states of the N = 2 theory have a mass equal to the central charge z. It follows from the supersymmetry transformation rules that this is simply the graviphoton charge [6]

$$M = |z| = |q_\Lambda^{(e)} X^\Lambda - \frac{i}{2} q_{(m)}^\Lambda F_\Lambda| = e^{K/2} |q_0^{(e)} + q_i^{(e)} Z^i + \frac{i}{2}(q_{(m)}^0 Z^i - q_{(m)}^i) f_i - i q_{(m)}^0 f| \ , \tag{14}$$

where $q^{(e)}$ and $q_{(m)}$ are electric and magnetic charges associated to $i\mathcal{G}$ and $\hat{F}$ and comprise a symplectic vector. Duality transformations of the N = 2 theory correspond to different choices of the symplectic representative $(X^\Lambda, -\frac{i}{2}F_\Lambda)$ of the symplectic geometry.

3 Magnetic N = 2 BPS Black Holes

In this section we discuss the general form of the supersymmetric magnetic black hole solutions and their interpretation as interpolating solitons. It has been shown by Tod [2] in general that, for N = 2 theories, a static metric admitting supersymmetries can be put in the form

$$ds^2 = -e^{2U} dt^2 + e^{-2U} d\vec{x}^2 \ . \tag{15}$$

For spherically symmetric black hole solutions U will be a function only of the radial coordinate r. By solving the Bianchi identities $d\hat{F}^\Lambda = 0$ we then find for the radial component of the magnetic field strengths $\hat{F}_r^\Lambda \equiv 2\epsilon_r{}^{\theta\phi} \hat{F}_{\theta\phi}^\Lambda$:

$$\hat{F}_r^\Lambda = \frac{q^\Lambda}{r^2} e^{U(r)} \ . \tag{16}$$

Inserting (15) and (16) into the gravitino transformation law, and demanding that the variation vanish for some choice of ϵ, we derive the following first order differential equation:

$$4U' = -\sqrt{\frac{(\bar{Z}Nq)(ZNq)(\bar{Z}NZ)}{(ZNZ)(\bar{Z}N\bar{Z})}} \, e^U \ , \tag{17}$$

where $U' \equiv \frac{\partial U}{\partial \rho}$, $\rho \equiv 1/r$, and we employ the notation $(ZNq) \equiv Z^\Lambda N_{\Lambda\Sigma}\, q^\Sigma$.

Equation (17) may be viewed as determining U as a function of the moduli fields Z^Λ. A vanishing gaugino transformation further requires that the moduli fields obey

$$(Z^\Lambda)' = -\frac{e^U}{4} \sqrt{\frac{(ZNZ)(\bar{Z}Nq)(\bar{Z}NZ)}{(\bar{Z}N\bar{Z})(ZNq)}} \left(Z^\Lambda q^0 - q^\Lambda\right) \ . \tag{18}$$

Differentiating again with respect to ρ and substituting (17) leads to the second order differential equation

$$(Z^\Lambda)'' - \left(\frac{(ZNq)}{(ZNZ)} + q^0\right) \frac{((Z^\Lambda)')^2}{Z^\Lambda q^0 - q^\Lambda} + \frac{1}{2}\left(\ln \frac{(ZNZ)(\bar{Z}Nq)(\bar{Z}NZ)}{(\bar{Z}N\bar{Z})(ZNq)}\right)' (Z^\Lambda)' = 0 \ . \tag{19}$$

This equation is independent of U and can be viewed as a generalized geodesic equation which describes how Z evolves as one moves into the core of the black hole. Initial conditions for Z are specified at infinity ($\rho = 0$) corresponding to the asymptotic values of the field. The first derivative of Z is then fixed in terms of the charge of the black hole by the supersymmetry constraint (18). Z will then evolve until it runs into a fixed point. It is evident from (18) and (19) that these fixed points are at

$$Z^{\Lambda}_{fixed} = \frac{q^{\Lambda}}{q^0} , \tag{20}$$

where $(Z^{\Lambda})' = 0$. Each fixed point is typically surrounded by a finite basin of attraction. The limiting form of the metric at such a fixed point is found by integrating (17),

$$e^{-U} = \frac{c}{4}\rho , \tag{21}$$

where the constant c is given by

$$c = \sqrt{q^{\Lambda} N_{\Lambda\Sigma}\, q^{\Sigma}} = q^0\, e^{-K(Z^{\Lambda}_{fixed})/2} . \tag{22}$$

This corresponds to the maximally symmetric charged Robinson-Bertotti universe. Thus, as in [7], the extremal black holes may be viewed as solitons which interpolate between maximally symmetric vacua at infinity and the horizon.

The locations of the fixed points (20) depend on the charges but not on the asymptotic values of the moduli fields. Thus if the the asymptotic values of those fields are adiabatically changed, the geometry of the black hole near the horizon remains fixed. Symplectic invariance implies a similar structure for many of the dyonic and electrically charged extremal black holes.

4 Space-Time Geometry from Kähler Geometry

In this section we consider a remarkably simple special class of solutions which exist for a generic prepotential. We will work in a symplectic basis in which $q^0 = 0$ in order to simplify the equations. In such a basis the fixed points Z^{Λ}_{fixed} move to infinite coordinate values. Thus solutions for which the moduli field is constant at the fixed point (which corresponds to the Reissner-Nordström solution) cannot be described in this basis[4]. In the $q^0 = 0$ basis it is straightforward to check that (17) and (18) are solved by

$$e^{2U(\rho)} = e^{K(Z,\bar{Z})-K_{\infty}} , \tag{23}$$

and

$$Z^i = Z^i_{\infty} + \frac{q^i}{4}\rho\, e^{-K_{\infty}/2} , \tag{24}$$

[4]However, by using the manifestly symplectic constraint [6] instead of the superconformal one [4], one can describe the Reissner-Nordström configuration in terms of the Kähler potential, see example 5.4.2

provided that the asymptotic value of Z^i is restricted to obey

$$Z^i_\infty = \bar{Z}^i_\infty \ . \qquad (25)$$

Alternatively we may have

$$Z^i = Z^i_\infty + i\frac{q^i}{4}\rho\, e^{-K_\infty/2} \ , \qquad (26)$$

provided that the asymptotic value of Z^i is restricted to obey

$$Z^i_\infty = -\bar{Z}^i_\infty \ . \qquad (27)$$

The restrictions imply that these solutions exist only at special points in the moduli space where all Z^i are either real or imaginary. More general solutions may be obtained from these by symplectic transformations.

Thus the space-time metric is

$$ds^2 = e^{K(Z^i,\bar{Z}^i=Z^i)-K_\infty}\, dt^2 - e^{-K(Z^i,\bar{Z}^i=Z^i)+K_\infty}\, d\vec{x}^2 \ , \qquad (28)$$

where each Z^i solves a three-dimensional harmonic equation and is given in eq. (24) or (26). Hence the logarithm of the spatial conformal factor is identified with the moduli space Kähler potential.

5 Examples of $\mathbf{N \geq 2}$ BPS states

5.1 Calabi-Yau magnetic black holes

$$F = i d_{ABC}\, \frac{X^A X^B X^C}{X^0} \ . \qquad (29)$$

We consider pure imaginary Z^A and real d_{ABC} (corresponding to the classical Calabi-Yau moduli space)

$$e^{-K(Z,\bar{Z})} = -2 d_{ABC}\, \mathrm{Im}Z^A\, \mathrm{Im}Z^B\, \mathrm{Im}Z^C, \qquad (30)$$

where $\mathrm{Im}Z^A = \frac{1}{2i}(Z^A - \bar{Z}^A) = \mathrm{Im}(Z^A)_\infty + \frac{q^A_{(m)}}{r}\, e^{-K_\infty/2}$.

$$ds^2 = \left(\frac{d_{ABC}\, \mathrm{Im}Z^A\, \mathrm{Im}Z^B\, \mathrm{Im}Z^C}{[d_{ABC}\, \mathrm{Im}Z^A\, \mathrm{Im}Z^B\, \mathrm{Im}Z^C]_\infty}\right)^{-1} dt^2 - \left(\frac{d_{ABC}\, \mathrm{Im}Z^A\, \mathrm{Im}Z^B\, \mathrm{Im}Z^C}{[d_{ABC}\, \mathrm{Im}Z^A\, \mathrm{Im}Z^B\, \mathrm{Im}Z^C]_\infty}\right) d\vec{x}^2 \ . \qquad (31)$$

5.2 Massive and massless $\mathbf{\frac{SU(1,n)}{SU(n)}}$ supersymmetric white holes

$$F(X^0, X^1) = (X^0)^2 - (X^i)^2 \ , \qquad e^{-K(Z,\bar{Z})} = 1 - |Z^i|^2 \ . \qquad (32)$$

Here Z^i are real:

$$Z^i = Z^i_\infty + \frac{q^i}{r}\, e^{-K_\infty/2} \ , \tag{33}$$

$$ds^2 = \left(\frac{1-|Z|^2}{1-|Z_\infty|^2}\right)^{-1} dt^2 - \left(\frac{1-|Z|^2}{1-|Z_\infty|^2}\right) d\vec{x}^2 \ . \tag{34}$$

In particular in the simplest case of $i=1$ we get

$$g_{tt} = g_{rr}^{-1} = \left(1 - \frac{2Z_\infty\, q}{r}\, e^{K_\infty/2} - \frac{q^2}{r^2}\right)^{-1} . \tag{35}$$

To satisfy the supersymmetry bound we require

$$M = -\frac{Z_\infty\, q}{\sqrt{1-|Z_\infty|^2}} \geq 0 \ , \tag{36}$$

and the geometry is

$$g_{tt} = g_{ii}^{-1} = \left(1 - \frac{2M}{r} - \frac{q^2}{r^2}\right)^{-1} . \tag{37}$$

These configurations are non-trivial in the limit when the ADM mass tends to zero. It will be interesting to understand if or when such states can arise in a physical theory. The limit $M \to 0$ can be achieved via $Z_\infty \to 0$. In this limit the scalar field becomes inversely proportional to the radius r,

$$Z(r) = \frac{q}{r} \ . \tag{38}$$

For N = 4 BPS states the analogous massless states have been studied recently [8]–[10]. The configuration exhibits a repulsive (i.e. antigravitating) singularity and was referred to as a supersymmetric white hole [9]. The difference between the metric (37) for N= 2 and the corresponding metric obtained in [8]–[9] for N = 4 is that $g_{tt}^{N=4} = (g_{tt}^{N=2})^{1/2}$. This does not change the repulsive nature of the singularity.

5.3 $\frac{\mathbf{SO(2,1)}}{\mathbf{SO(2)}} \times \frac{\mathbf{SO(2,n)}}{\mathbf{SO(2)\times SO(n)}}$ BPS states

Here again we consider pure imaginary Z.

$$F = -i\,\frac{X^s}{X^0}\left((X^1)^2 - \sum_{a=2}^{n}(X^a)^2\right) , \qquad e^{-K(Z,\bar{Z})} = 2\mathrm{Im}Z^s\left((\mathrm{Im}Z^1)^2 - \sum_{a=2}^{n}(\mathrm{Im}Z^a)^2\right) . \tag{39}$$

The metric of the BPS configuration defined by the Kähler potential is given by

$$g_{tt} = g_{ii}^{-1} = \left[\frac{\mathrm{Im}Z^s\left((\mathrm{Im}Z^1)^2 - (\mathrm{Im}Z^a)^2\right)}{\left[\mathrm{Im}Z_s\left((\mathrm{Im}Z^1)^2 - (\mathrm{Im}Z^a)^2\right)\right]_\infty}\right]^{-1} . \tag{40}$$

The configurations presented in this example may contain both types of supersymmetric states, those with attractive singularities and those with the repulsive ones, depending on the choice of the parameters describing the harmonic functions

$$Z^i = Z^i_\infty + i \frac{q^i_{(m)}}{r} e^{-K_\infty/2} , \qquad i = \{s, 1, a = 2, \ldots, n\}. \tag{41}$$

Note that this example actually provides one of the particular choices of Calabi-Yau magnetic black holes.

5.4 N=4, 2 pure supergravity black holes from Kähler geometry perspective

Here we will analyse some previously known supersymmetric black hole solutions in the framework of the manifestly symplectic formalism [6]. The Kähler potential as different from the conformal gauge (4) is given by

$$e^{-K(X,\bar{X})} = \bar{X}^\Lambda N_{\Lambda\Sigma} X^\Sigma . \tag{42}$$

5.4.1 SL(2,Z) axion-dilaton dyons

It is instructive to analyse the known $SL(2,Z)$-invariant axion-dilaton dyonic black hole solution [11] of N = 4 supergravity as a particular solution of N = 2 supergravity coupled to an N = 2 vector multiplet. These solutions have complex moduli. The prepotential for this theory is $F(X) = 2X^0 X^1$. The holomorphic section includes

$$\begin{pmatrix} X^\Lambda \\ -\frac{i}{2} F_\Lambda \end{pmatrix} \Longrightarrow \begin{pmatrix} X^0 \\ -iX^1 \end{pmatrix} , \quad \begin{pmatrix} X^1 \\ -iX^0 \end{pmatrix} . \tag{43}$$

For the axion-dilaton black hole X^0 and X^1 can be identified with two complex harmonic functions $\mathcal{H}_1$, $\mathcal{H}_2$ as follows:

$$X^1 = \mathcal{H}_1(\vec{x}) , \qquad X^0 = i\mathcal{H}_2(\vec{x}) . \tag{44}$$

The Kähler potential is

$$e^{-K(X,\bar{X})} = i\,(\bar{\mathcal{H}}_1 \mathcal{H}_2 - \bar{\mathcal{H}}_2 \mathcal{H}_1) = g_{tt}^{-1}(\vec{x}) , \tag{45}$$

in agreement with the metric of the axion-dilaton black hole found in [11].

5.4.2 Reissner-Nordström solution

The prepotential for pure $N = 2$ supergravity is $F(X) = (X^0)^2$. The Kähler potential of this theory in the manifestly symplectic formalism [6] is given by

$$e^{-K(X,\bar{X})} = X^0 \bar{X}^0 = V^{-1}(\vec{x}) \bar{V}^{-1}(\vec{x}) = e^{-2U(\vec{x})} , \tag{46}$$

where $X^0 = V^{-1}(\vec{x})$ is a real (imaginary) harmonic function for electric (magnetic) Reissner-Nordström extremal supersymmetric black hole [1, 2].

In conclusion, we have found a simple relation between the special geometry describing the couplings of scalars and vectors in extended locally supersymmetric theories and the space-time geometry of the black-hole-type solutions in these theories. It is likely that more general solutions with complex moduli will be found in this framework.

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