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SOME EXAMPLES OF ENERGY FLOW IN THE ALVAREZ STRUCTURE

I. The Model

A reasonable model of the Alvarez structure, as used to accelerate protons in 2π-mode in the low-β region, is either one of the following circuits:

Fig. 1

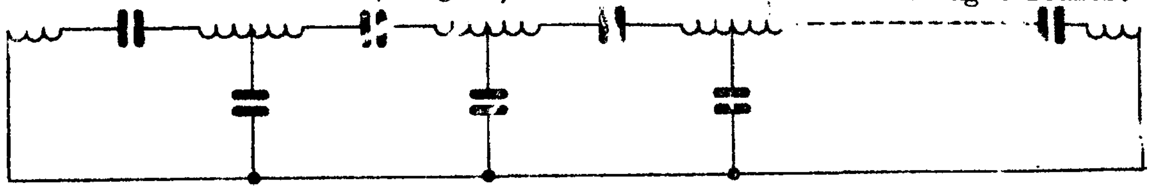
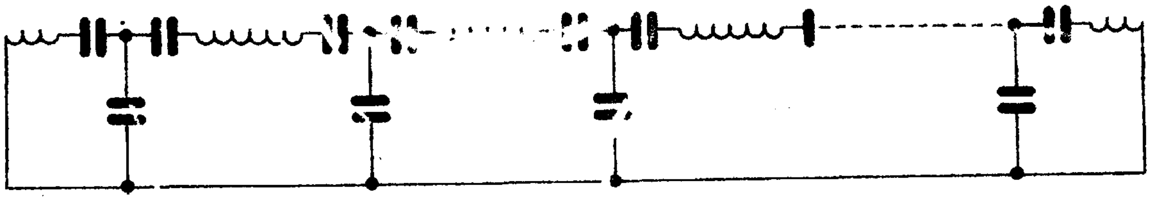


Fig. 2



For the purpose of this note there is no difference between them.

We take as starting point a structure without losses. This is because we should like to make a distinction between the necessity of energy flow and the means of energy flow. Given that power can be fed into the structure at one point (for example, at one end) there are several processes which may result in the necessity of energy flow:

- Copper losses.
- Beam loading.
- Transient situations, typically the build-up of stored energy.

On the other hand the means of energy flow, i.e. the property of the structure which enables it to transport energy from one end to the other, seems likely to exist even in a lossless structure. If so, it is reasonable to study it first in the approximation where losses are neglected.

In these models the earth line is supposed to represent the outer wall of the tank. The series capacitors represent the accelerating gaps with their longitudinal electric field, including the fringing field as far as it extends, so in principle right out to the cylindrical wall. The series inductors represent

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the inductance of the "central conductor" which is the stack of drift-tubes and gaps *) on the basis that the current returns by the outer cylindrical wall, so their stored energy represents that of the whole ϕ-component magnetic field, from the axis to the outer wall. The shunt capacitors obviously represent the capacitance between the drift-tubes and the outer wall: the drift-tube support stems, which are also a shunt element, are as usual neglected in the first analysis and looked at later.

The operating mode is called 2π because of the way in which the structure is used to accelerate particles. From the point of view of all electrical field considerations it is more convenient to consider it as mode zero. When questions of flatness and tilt, energy propagation and transients, etc. are studied, one has to bring other modes into consideration, but they are usually all rather near to zero**), for typically one is considering modes with a few half-wavelengths in the whole length of a tank which contains some twenty or more cells. For the behaviour of the modes near zero it is convenient and sufficiently accurate to treat the structure as continuous, i.e. tocalculate in the limit of very short cells.

We shall therefore let L represent the inductance per unit length, $\frac{1}{C}$ the series reciprocal-capacitance per unit length, and C_s the shunt capacitance per unit length. We let V be the potential on the drift-tube centres (or gap centres) with respect to ground and I the current from left to right through the drift-tubes and gaps. We adopt the usual engineering convention ***) that these quantities depend on time by way of an unwritten factor $e^{j\omega t}$, so that one has

$$\frac{\partial}{\partial t} = j\omega \quad . \tag{1}$$

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*) For inductance purposes the longitudinal current in the drift tubes must be regarded as crossing the gap from one drift-tube to the next in the form of the electric displacement D .

**) Technically, a mode near zero is a mode in which the phaseshift per cell is small compared with a radian.

***) Some writers or wave propagation use $e^{-j\omega t}$ for the time factor; with this convention one must write $-j$ for j in (1) and in everything that follows.

The wave equation for our structure is then evidently

$$\frac{dV}{dx} = -(j\omega L + \frac{1}{j\omega C})\ I \qquad (2a)$$

$$\frac{dI}{dx} = -\ j\omega C_s .\ V \qquad (2b)$$

where x is distance measured from left to right and I is the current flowing from left to right in the top of the circuit and returning by the earth line.

I is more interesting *) than V for the structure we are looking at, so the second-order equation that we derive from (2) is

$$\frac{d^2I}{dx^2} = -\ \omega\ C_s\ (\omega L - \frac{1}{\omega C})\ I \qquad (3)$$

In the rest of this chapter we shall list the solutions of (3) appropriate to the situation where we either have an infinite length of the structure or have not yet decided on the boundary conditions at its ends.

There are the well known solutions of form coskx, sinkx (standing waves) and e^{+jkx}, e^{-jkx} (travelling waves) where

$$k^2 = C_s(\omega^2 L - \frac{1}{C}) \qquad (4)$$

provided k^2 is positive.

It is obviously convenient to define

$$\omega_o = \sqrt{\frac{1}{L\,C}} \qquad (5)$$

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*) The accelerating voltage between adjacent drift-tubes is associated with I rather than V .

and then we have:

$$\text{Waves are possible with } \kappa^2 = LC_s(\omega^2 - \omega_o^2) \quad \text{provided } \omega > \omega_o \tag{6}$$

If ω is lower than this "resonant" or "cut-off" ω_o, the well known solutions are the evanescent modes with longitudinal space-dependance $e^{+\alpha x}$ and $e^{-\alpha x}$, or $\cosh\alpha x$ and $\sinh\alpha x$, and one has:

$$\text{Evanescant modes with } \alpha^2 = LC_s(\omega_o^2 - \omega^2) \quad \text{provided } \omega < \omega_o. \tag{7}$$

When we let ω and ω_o be equal, we can take the solutions we have and put $k = 0$ or $\alpha = 0$, and find that I must then be constant (independent of x). Appeal to (2b) shows that then $V = 0$ and the shunt capacitance C_s takes no part in the process:

$$\text{"Cut-off mode" with } I = \text{constant}, V = 0, \quad \text{provided } \omega = \omega_o \tag{8}$$

It may be remarked that with $\omega \neq \omega_o$ we always got solutions in pairs, in the special case $\omega = \omega_o$ these have given us only one solution: there must be another because (3) is a second order equation. This is a well-known mathematical phenomenon and we shall have to come back to it.

II. Mode Spacing

The simplest useful question about the dispersion properties of a standing-wave accelerator structure is the question of mode spacing for a tank of length ℓ. The usual metallic end plates admit the cut-off mode solution at $\omega = \omega_o$, the next one admitted will be of form $I = \cos k_1 x$, if we take the origin of x at one end of the tank, with *)

$$k_1 \ell = \pi \tag{9}$$

Putting $\omega^2 - \omega_o^2 \cong 2\,\omega_o \delta\omega$, the spacing to this first mode is, from (6), evidently

$$\delta\omega_1 \cong \frac{k_1^2}{2\,\omega_o L C_s} = \frac{\pi^2}{2\,\omega_o \ell^2} \; \frac{1}{L\,C_s} \tag{10}$$

If we look for the best mode-spacing at given ω_o and tank-length ℓ, or for the maximum tank length subject to a practical lower limit on tolerable mode spacing, the only quantity at our disposal is $L\,C_s$, so one should ask roughly what this quantity is likely to be, and what determines it. Consider the case where the drift tubes are cylinders, short compared with their diameter, with short capacitive gaps between them. It is then reasonable to treat this cylindrical stack, so far as the transverse fields involved in L and C_s are concerned, as the inner conductor of a coaxial line. In this case, independently of the dimensions, one has

$$L\,C_s = \mu_o \varepsilon_o = c^{-2} \tag{11}$$

where c is the velocity of light, so that

$$\delta\,\omega_1 = \frac{\pi^2 \; c^2}{2\,\omega_o\,\ell^2} \tag{12}$$

or perhaps move conveniently

$$\frac{\delta\,\omega_1}{\omega_o} = \frac{1}{8}\,\frac{\lambda_o^2}{\ell^2} \quad \text{or} \quad \frac{\pi^2}{2}\,\frac{\bar{\lambda}_o^2}{\ell^2} \tag{13}$$

where λ_o is the resonant free-space wavelength $2\pi\, c/\omega_o$ and $\bar{\lambda}_o$ is c/ω_o.

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*) For the other higher modes $k_n \ell = n\,\pi$. (10')

The validity of (11) and the influence of the stems are discussed a little more in Appendix I . In what follows we shall assume in any case the approximate equality

$$L\,C_s \cong c^{-2} \qquad (11^*)$$

so that (12) and (13) are also approximately correct.

III. Group Velocity and Energy Velocity for the travelling Waves

One does not use an Alvarez structure in the region $\omega > \omega_o$, where travelling waves are possible, but we shall look briefly at this region first as it is here that simple concepts of group-velocity and energy-velocity hold.

We already have the dispersion equation (6) :

$$k^2 = L\,C_s\,(\omega^2 - \omega_o^2) \qquad (15)$$

or, if one prefers to have ω given as function of k :

$$\omega = \left[\omega_o^2 + \frac{k^2}{LC_s}\right]^{1/2} \qquad (16)$$

The phase velocity is ω/k, the group velocity v_g is $d\omega/dk$, which from (16) gives

$$v_g = \frac{k}{LC_s}\left[\omega_o^2 + \frac{k^2}{LC_s}\right]^{-1/2} \qquad (17)$$

The diagram corresponding to (16) for an Alvarez structure suitable for accelerating 50 MeV protons is shown in Figure III.

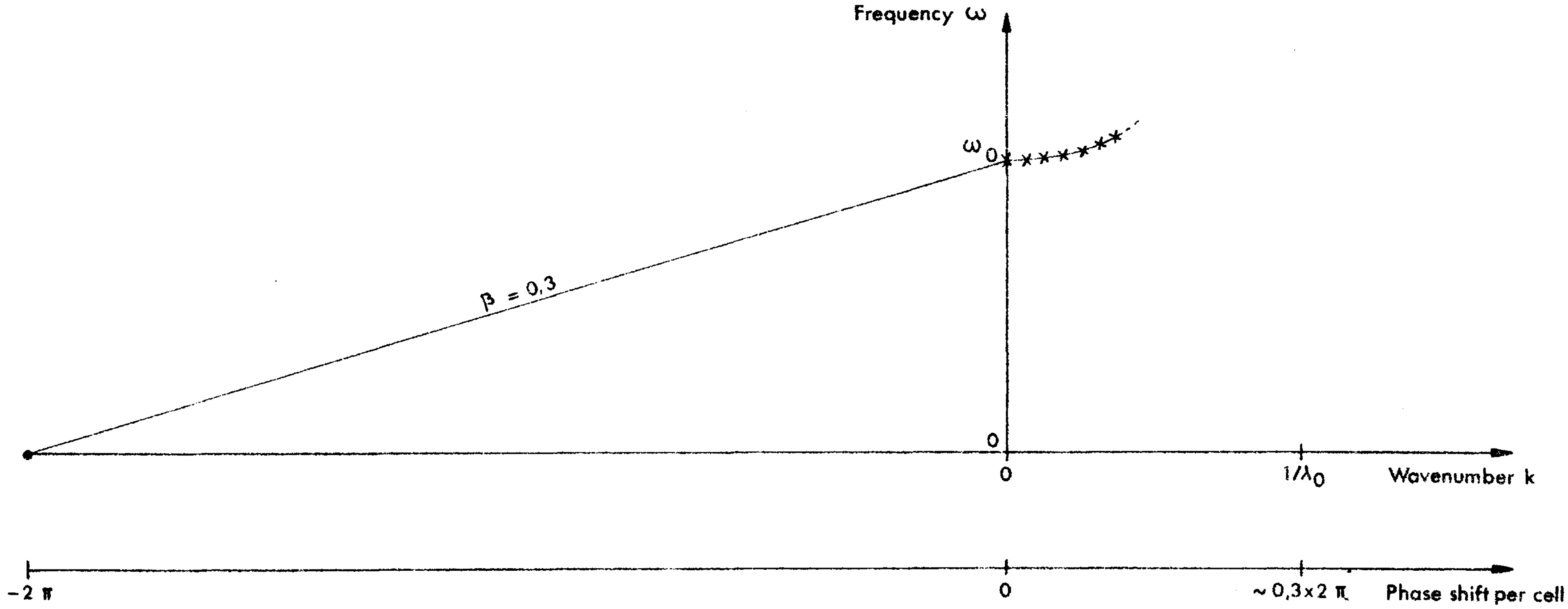
Frequency ω
ω0
0
β = 0,3
0
1/λ0
Wavenumber k
−2 π
0
~0,3x2 π
Phase shift per cell

Fig.3

This figure is drawn with scales such that a ratio of one for ordinate to abscissa corresponds to the velocity of light, and has been based on the assumption that LC_s is in fact given by c^{-2}, (11). The first few modes of a 12 metre long tank are indicated by crosses, and the region where our close-pitch approximation can be expected to break down is indicated by not drawing the dispersion curve beyond a certain point *). The phase velocity of $\beta \approx 0.3$, used by the protons, is obtained by interpreting the zero mode as a 2π mode, and is also indicated in the figure.

In the region $k \neq 0$, where we have some group velocity, it is of interest to compare it with the energy velocity, defined as the ratio

$$v_e = \frac{\text{Energy flow}}{\text{Energy stored per unit length}} \tag{18}$$

It is convenient to take both quantities as time-averages and to use the symbols $\hat{V}$ and $\hat{I}$ to refer to peak values.

We may take the case of a forward-travelling wave with

$$I = A \quad e^{-jkx} , \tag{19}$$

then (2b) gives us

$$V = A \; \frac{k}{\omega \, C_s} \; e^{-jkx} \tag{20}$$

And at any x the energy flow is

$$\frac{1}{2} \hat{I} \, \hat{V} = \frac{k}{2\omega \, C_s} \; A^2 \tag{21}$$

where the factor $\frac{1}{2}$ appears from the time-averaging.

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*) Though in fact it would not be much trouble to calculate the curve on into this region, allowing for the finite pitch of the structure, using the model of Fig.I or Fig.II.

The stored energy per unit length is

Transverse electric : $$\frac{1}{4} C_s \hat{V}^2 = \frac{k^2}{4\,\omega^2 C_s} A^2 \qquad (22)$$

Transverse magnetic: $$\frac{1}{4} L \hat{I}^2 = \frac{1}{4} L A^2 \qquad (23)$$

Longitudinal electric: $$\frac{1}{4} \frac{1}{C} \frac{\hat{I}^2}{\omega^2} = \frac{1}{4} \frac{1}{\omega^2 C} A^2 \qquad (24)$$

where the numerical factors arise as $1/2$ from the time-averaging and $1/2$ from the simple stored-energy formulae $\frac{1}{2} C V^2$ etc.

Hence

$$v_e = \frac{2\,k/\omega\, C_s}{k^2/\omega^2 C_s + L + 1/C\,\omega^2} \qquad (25)$$

For comparison with (17) we use (5) to eliminate C and (15) to eliminate ω, obtaining

$$v_e = \frac{k}{LC_s} \left[\omega_o^2 + \frac{k^2}{LC_s} \right]^{-1/2} \qquad (26)$$

This is just the same as the expression we found for group velocity, (17), so the group velocity and energy velocity are the same, and they both vanish at $\omega = \omega_o$, $k = o$, which is where we in fact use an Alvarez structure.

One should not be too surprised at the fact that the energy velocity, defined by (18), is zero at $\omega = \omega_o$. For we have calculated for a lossless structure, and by taking one given frequency we have effectively cut ourselves off from any consideration of transient behaviour: with neither losses nor transients there is nowhere for power to go. Our definition of energy velocity (18) relates to the velocity with which the energy is in fact moving, not to the energy transport capabilities of the structure. It is therefore quite reasonable that this v_e should depend on circumstances as well as on the structure.

Some further remarks on this point are in Appendix II.

IV. Energy Transport at zero-mode.

1. Resistive loading.

With our model circuit at zero mode, $\omega = \omega_o$, $k = o$, the current I is constant along the whole length of the structure, and flows from and to ground in the end-plates at the two ends, with no voltage (with respect to ground) on the drift-tube centres or gap-centres. The simplest way of modifying this model in such a way that power flows is to open the right-hand end connection and insert a small resistor R, open the left-hand end connection and insert a low-impedance voltage source. In first approximation I is unchanged, power $\frac{1}{2}\hat{I}^2R$ is transported, a voltage $V = IR$ appears at gap-centres and drift-tube centres along the whole length (producing a radial electric field, which combines with the existing ϕ-direction magnetic field to produce a Poynting vector $[E_\wedge H]_x$ which integrates to VI).

This first-approximation thus indicates that we can make as little or as much energy flow as we like, according to the value R of the loss-resistor that is inserted : to find out whether any distortion of the pattern is produced and whether there is any upper limit to the possible energy flow it is necessary to look at the exact solution, and for this we shall need the other cut-off frequency solution of (3), that exists in addition to (8), as already mentioned on page 4.

At $\omega = \omega_o$, (3) becomes

$$\frac{d^2 I}{d\,x^2} = 0 \tag{28}$$

And the general solution is

$$I = A + B\,x \tag{28a}$$

where A and B are arbitrary constants (in general complex, because the two solutions can occur with any phase relationship).

Then using (2b)

$$V = \frac{-1}{j\,\omega_o\,C_s}\;B \tag{29}$$

The "A" solution (B = 0) is the one we already listed, (8). A certain amount of the "B" solution obviously has to be introduced if our boundary condition at one end of the structure is inconsistent with $V = 0$.

We suppose that the structure is powered in some way at the near end and that the power has somewhere to go in the form of a resistor R at the far end. There is a certain convenience in taking the far end to be at $x = o$ and the near end at $x = -\ell$.

The boundary condition imposed by R at $x = o$ is then, from (28a) and (29) :

$$R = \frac{V_{(x=o)}}{I_{(x=o)}} = \frac{-1}{j\,\omega_o\,C_s}\;\frac{B}{A} \tag{30}$$

We therefore put

$$B = -j\,\omega_o\,C_s\,A\,R \tag{31}$$

and the solution becomes

$$\begin{aligned} I_{(x)} &= A\,(1 - j\,\omega_o C_s R\,x) \\ V_{(x)} &= A\,R \end{aligned} \tag{32}$$

Values at the driven end, $x = -\ell$ are

$$\begin{aligned} I_{(-\ell)} &= A\,(1 + j\,\omega_o C_s R\,\ell) \\ V_{(-\ell)} &= A\,R \end{aligned} \tag{33}$$

To obtain the energy flow we can assume A to be real ; then

$$\text{Energy flow} = \frac{1}{2} V_{(x)} \quad \mathrm{Re}\, I_{(x)} = \frac{1}{2} A^2 R . \tag{34}$$

This is independent of x and evidently equal to the power dissipated in R and to the power delivered by the driving source.

We can make the energy flow as big as we like at given A , by increasing the loss resistor R .

Evidently from a practical point of view it may be of importance that the flow of energy (into a resistor at the far end) causes a distortion of the field pattern in the tank, as shown by the $-j\, \omega_o C_s R\, x$ term in I . Since this distorting term is pure imaginary, and so in quadrature to the main term, it represents mainly (especially if it is relatively small) a phase shift of amount

$$\phi = \omega_o C_s R \ell \tag{35}$$

between the two ends of the tank.

Its influence on the amplitude is small unless this ϕ is at least an appreciable fraction of a radian.

In some circumstances it may be interesting to know what energy-velocity is available if phase shift ϕ is allowed only up to a certain value. The energy flow is given by (34). It is

$$\frac{1}{2} A^2 R .$$

To avoid unpleasantly complicated expressions we consider the energy density in the approximation that ϕ is small, so that the term proportional to R in (33) is neglected and we have

$$I_{(x)} \cong A$$

$$V_{(x)} \cong 0$$

and from (22), (23), (24) we find the energy density to be approximately

$$\frac{1}{2} A^2 L ,$$

giving the energy velocity

$$v_e \cong \frac{R}{L} = \frac{\phi}{\omega_o C_s L \ell} \tag{36}$$

If we suppose, as we have shown to be plausible:

$$C_s L \cong c^{-2}$$

we get

$$v_e \cong \phi \, c \frac{\lambda_o}{\ell} \tag{37}$$

Evidently $c \lambda_o / \ell$ is a sort of "standard" energy velocity for this structure in the sense that if you accept 10^{-2} radians phase shift you can have 1 o/o of $c \lambda_o / \ell$ as energy velocity.

Note that it is dependent on the tank length ℓ.

For CPS linac tank III ℓ is about 12 m and λ_o is 0.24 m so we get

$$\frac{c \lambda_o}{\ell} = \frac{1}{50} c \tag{38}$$

A figure which can be calculated from this "standard" energy velocity is the phase shift necessary to carry the energy in the steady state for a tank of length ℓ and losses that result in a certain Q . We have:

$$\frac{\text{Power input}}{\text{Stored energy}} = \frac{\omega_o}{Q} \tag{39}$$

Consequently one has near the input end of the structure

$$v_e = \frac{\text{Energy flow}}{\text{Stored energy per unit length}} = \frac{\omega_o \ell}{Q} \tag{40}$$

If we continue to take a model in which all the losses are concentrated at the far end, this v_e will exist along the whole length of the tank, and we have from (37)

$$\phi \cong v_e \frac{\ell}{\lambda_o c} = \frac{\omega_o \ell}{Q} \cdot \frac{\ell}{\lambda_o c}$$

$$= \frac{1}{Q} \left(\frac{\ell}{\lambda_o} \right)^2 \tag{41}$$

For an order-of-magnitude example one may take $Q \sim 50{,}000$, $\ell/\lambda_o \sim 50$ and obtain a ϕ of 1/20 radian or 3^o, which is small but by no means negligible.*)

One would be inclined to guess that the phase-shift given by (41) would be halved if the losses are distributed evenly along the length, and this is readily confirmed (Appendix III). Also it is possible, for a tank of given length, to reduce this effect by a factor of 4 by feeding from the middle rather than from one end.

2. Transient propagation, sinusoidal.

Evidently energy has to be transported in transient situations : for example, to raise the level of the far end of the tank stored energy is required and this must, losses or no losses, flow down the tank from the feeding end. So one would expect phase shifts or delays or other forms of pattern distortion to be associated with any transient situation.

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*) Comparing (41) with (13) one finds that the condition for ϕ to be small is of the same form as the condition that the mode spacing $\delta\, \omega_1$ should be large compared with the mode width ω_o/Q .

We continue to work in the no-losses approximation, as it seems reasonable to suppose that, to a fair approximation at least, these problems of transient-propagation are independent of small losses.

The most elementary way of getting some information about behaviour under transient conditions is to use our knowledge of behaviour at frequencies on either side of ω_o, (6) and (7), to construct a modulated situation out of side-bands.

Let us suppose that, by some means unspecified, we can modulate the level of the near end (i.e. loop end) of the tank according to

$$I = (1 + m \cos \omega_m t)\, e^{j\omega_o t} = e^{j\omega_o t} + \frac{m}{2} e^{j(\omega_o + \omega_m)t} + \frac{m}{2} e^{j(\omega_o - \omega_m)t} \qquad (42)$$

With the well-known rotating vector diagram:

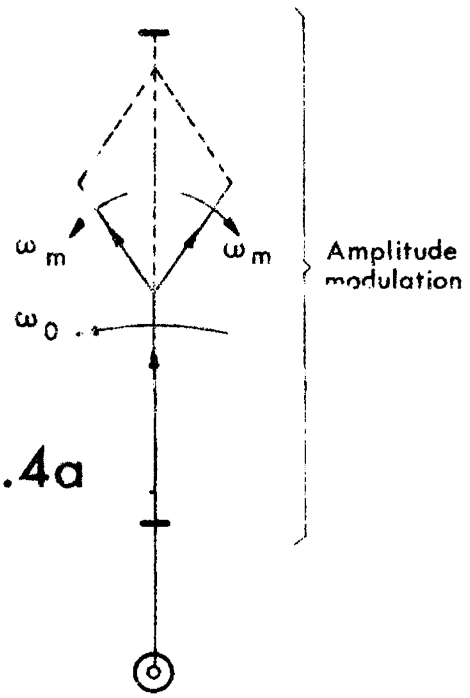

Fig.4a

The boundary condition at the far end of the tank is $V = o$ and therefore $\frac{dI}{dx} = 0$, for all three frequencies, so the centre frequency ω_o will be flat along the length of the tank, $\omega_o + \omega_m$ will be of cosine form with peak at the far end, and $\omega_o - \omega_m$ will be a hyperbolic cosine with its minimum at the far end.

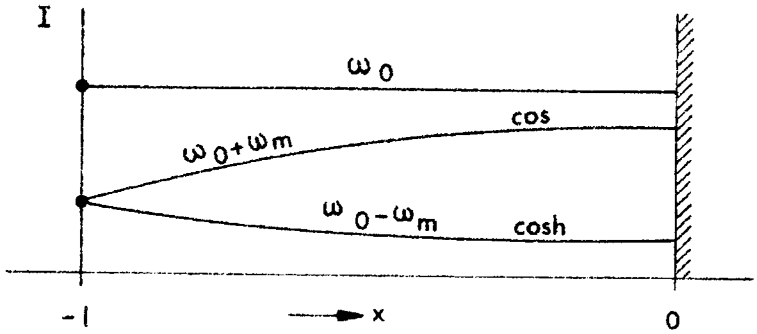

Fig.4b

At the far end, the vector diagram becomes

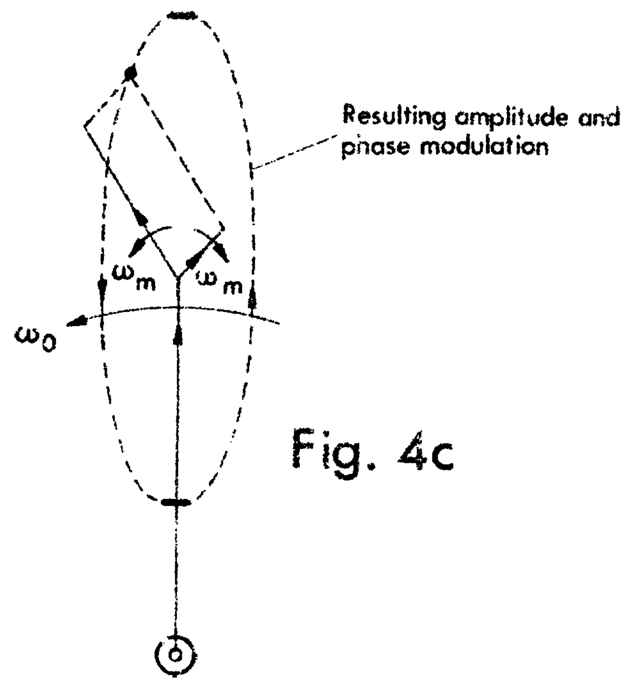

Fig. 4c

We have supposed ω_m small compared with the spacing to the next mode.

The main features of the situation at the far end of the tank can be seen from this vector diagram:

1) The end of the resultant describes an ellipse, so there is phase modulation as well as amplitude modulation.

2) The amplitude modulation is roughly as big as it is at the loop end *) - propagation along the length of the tank does not significantly smooth out the modulation.

3) The amplitude-modulation at the far end of the tank is in phase with that at the near end. Since group velocity normally shows up as a progressive phase-shift of any modulation, this means that the group velocity is apparently infinite.

4) The phase-modulation at the far end is in such a sense that it lags the near end of the tank during times when the amplitude is increasing, i.e. times when energy is flowing in the positive direction.

In principle any sort of time-dependent behaviour can be built up out of sidebands and studied by this method, but it is rather cumbersome, so we next look at the possibility of handling the frequency ω_o, modulated in a quite general way.

3. Transient propagation, general

We go back to our equation (3) for the structure and generalise it to apply any form of time-dependence by writing $\frac{\partial}{\partial t}$ in place of $j\omega$:

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*) $1/\cos k\ell \approx 1 + k^2\ell^2/2! +$ fourth order terms. . .
$1/\cosh k\ell \approx 1 - k^2\ell^2/2! +$ "
sum $\approx 2 + 0 +$ "

$$\frac{\partial^2 I}{\partial x^2} = C_s L \frac{\partial^2}{\partial t^2} I + \frac{C_s}{C} I \tag{43}$$

Now let us suppose that

$$I = F(x,t) \quad e^{j\omega_o t} \tag{44}$$

where F is assumed to be only slowly varying with time (compared with the $e^{j\omega t}$ factor). In general F is complex, so it is capable of representing changes of amplitude or changes of phase of the basic ω_o signal.

We put (44) into (43) and obtain (using the definition (5) of ω_o):

$$F'' = L C_s (\ddot{F} + 2 j\omega_o \dot{F}) \tag{45}$$

where we use the primes to indicate differentiation with respect to x and the dots for differention with respect to time.

Our assumption that F is slowly varying means that it is reasonable to neglect $\ddot{F}$ in comparison with $j \omega_o \dot{F}$, at least in a first approach to the problem. So we have

$$F'' = 2 j L C_s \omega_o \dot{F} \tag{46}$$

or alternatively

$$\dot{F} = D F'' \tag{46a}$$

where D is given by

$$D = 1/(2jL C_s \omega_o) \tag{46b}$$

This is the well-known equation for conduction of heat (or for diffusion, etc.) in one dimension, with only the difference that the diffusivity D is in our case a pure imaginary. It is worth listing the most important similarities to a heat-conduction problem and the most important differences.

1) Physical constants and the parameters of the hardware only appear in the form of the group of factors that make up D. One may see from (46a) that D must have the dimensions

$$\left[L^2 \quad T^{-1}\right]$$

and from (46b) one may verify that it has. These are not the dimensions of a velocity.

2) The equation satisfies a scaling law. If some solution F is known, then one may increase all distances by a factor say r and all times*) by r^2 and one again has a solution.

A special case of this is the following: take a semi-infinite tank with the feeding loop at this end and put onto this loop some R.F. whose amplitude is time dependent, but with a form of time dependence which is unchanged by a change of time-scale. A delta-function or step-function, at $t = o$, are practical examples. After say 1 µs the x-dependence of F wil have a certain form; then at 4 µs the x-dependence will have the same form but with all lengths doubled, and in a certain sense it will be possible to say that the signal has travelled twice as far down the tank.

3) Evidently disturbances whose variation with x or with time is rapid move around as though with a high group velocity, those whose variation is slow appear to move with a low group velocity.

4) In the conduction of heat, and in diffusion, where we have the equation (46a) with D real, the transport of heat or material is always associated with a gradient of temperature or concentration. In our case with D imaginary one can expect a gradient in the phase of F, rather than in its magnitude, to be associated with transport of signals.

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*) Excluding, of course, the time $\frac{1}{\omega_o}$ and the distance $\lambda\!\!\!^{-}_o$.

5) In principle any of the known solutions to the one-dimensional heat conduction problem can be taken over as a solution of our problem, only by taking account of the fact that D is imaginary. In practice this does not seem to be very profitable, at least for the two examples below.

(i) One well-known solution to the heat conduction equation is the spreading Gaussian. With a delta-function or most other sufficiently concentrated initial distributions of temperature and with infinitely remote boundaries one finds a solution

$$F = A\, t^{-1/2}\, e^{-\frac{x^2}{4\,D\,t}} \qquad (47$$

With D real this is a curve of Gaussian form with width increasing proportionally to $t^{1/2}$ and peak height decreasing like $t^{-1/2}$, the area (proportional to total heat) being constant.

With D imaginary (47) becomes a useless expression: it does not tend to a delta-function, or other plausible form of idealised initial condition, at small t, and as x increases its amplitude does not decrease but the rapidity of its oscillations (with x or with t) increases without limit.

(ii) For a tank of finite length and simple boundary conditions at the ends it is natural to consider expressing the x-dependence of F in the form of a Fourier series. One may recall that Fourier analysis was invented in connection with the heat-conduction equation. Let us take the boundary condition $F' = 0$ at the tank ends, and expand in longitudinal modes of which the n'th depends on x like

$$\cos k_n x \qquad (48)$$

$$\text{with} \quad k_m = \frac{n\,\pi}{\ell} \qquad (49)$$

This gives us

$$\frac{F''_n}{F_n} = - k_n^2 \tag{50}$$

Substitute in (46a)

$$\frac{\dot{F}_n}{F_n} = - D\, k_n^2 \tag{51}$$

The time-dependence of the solution is therefore of form

$$e^{- D\, k_n^2\, t} \quad , \tag{52}$$

and with D given by (46b), this gives

$$e^{j \frac{k_n^2\, t}{2 L C_s \omega_o}} \tag{53}$$

In heat conduction the various longitudinal Fourier terms die away with various timeconstants, the short wavelength ones faster, and one readily gets a qualitative feeling for the way in which a temperature distribution becomes flatter and smoother and tends to a very simple asymptotic form; now we see from (53) that in our problem with imaginary D there is no such decay: the F's of the longitudinal modes just go round and round in the complex plane, and in fact (53) is not really anything new, it is just the old formula for the mode spectrum (10'), derived in a curious and round-about way.

Although this result is discouraging and we shall not pursue the Fourier method further, it seems probable that the straightforward and reliable method of finding out in detail what happens when one starts to build-up an empty tank is to make the expansion in terms of a complete set of orthogonal normal modes appropriate to a specified boundary condition at the loop end of the tank.

An elementary way of getting something useful out of the equation (46) is to solve it by successive approximation.

We put

$$F'' = \frac{1}{D} \dot{F} \tag{55}$$

and we note that the usual solutions in which all questions of energy-transport etc. are disregarded are ones where F'' is assumed zero while $\dot{F}$ is taken to be whatever the pulsed nature of the RF supply dictates: such a solution is compatible with (55) only if $1/D$ is neglected. In other words, neglecting questions of energy transport is equivalent to assuming infinite diffusivity.

Thus the appropriate method of successive-approximation is to expand F in ascending powers of $1/D$:

$$F = F_o + \frac{1}{D} F_1 + \frac{1}{D^2} F_2 + \ldots \tag{56}$$

Substitute this into (55) and equate terms of the same order:

$$F_o'' = 0$$

$$F_1'' = \dot{F}_o \tag{57}$$

$$F_2'' = \dot{F}_1$$

etc.

Since we are neglecting losses, it is reasonable to look at the situation with a linear rise, as happens in the early part of an RF pulse before the finite Q makes the envelope level off. So we put

$$F_o = A\,t$$

independent of x

Therefore $$F_1'' = A$$

The general solution of this is

$$F_1 = \frac{A}{2} x^2 + Bx + C \qquad (60)$$

We can put C equal to zero, because any constant (independent of t and of x) part of F can be absorbed into F_o by a suitable choice of the origin of t.

Taking the far end of the tank as being at x = o, and the boundary condition there as F' = 0, we have also B = 0, and the solution is

$$F = F_o + \frac{1}{D} F_1 + \ldots = A(t + \frac{1}{2D} x^2)$$

$$= A(t + j\,L\,C_s \omega_o x^2) \qquad (61)$$

It seems reasonable to suppose that this solution is not good at very small t*), but is approximately correct when the second term is small compared with the first.

When the second term is relatively small it can be interpreted as a phase shift which falls like t^{-1} :

$$\frac{L\,C_s \omega_o\, x^2}{t} \qquad (62)$$

On the approximation $L\,C_s \approx c^{-2}$, this is

$$\frac{1}{\lambda_c}\ \frac{x^2}{t}$$

E.g. for C.P.S. tank II or III $x^2 = 144\ m^2$

$\lambda = 0.24$ m

c = 300 m/μs

t = say 20 μs

and one obtains 0.1 radian.

---

*) What is wrong with (61) is that it does not satisfy the initial condition, F(x,o) = 0 at all x, that we want. If we attempt to use F_o = At H(t) where H is the unit step function, in place of (58), then the neglected $\ddot{F}$ term is infinite at t = 0 and one is no closer to the real physical problem.

As to the sign of this phase-shift, we can see from (61) that the far end of the tank lags the loop end, in the same way as we found at (33) for propagation into a resistor and at (44) page 16 for sinusoidal modulation at times when the energy flow is positive.

Both the sign and the approximate magnitude of this predicted phase shift agree with preliminary measurements made on the CPS linac, tank III*).

It is clear that the detailed behaviour of the amplitude as a function of x in the early part of the RF pulse must be more complicated than any of the examples that we have looked at. In particular, an accurate solution has to agree with the fact that no signal can travel faster than the velocity of light.

V. Summary

1. The general phase-gradient formula

An interesting fact appears if we compare the phase-gradients that we have found in the lumped-resistor load case (33), the distributed losses case of Appendix III, the sinusoidal modulation case (42), and the transient build-up (62). One finds a phase-gradient given by

$$\frac{\partial \phi}{\partial x} \cong - \frac{1}{v\,\tau_0} v_e \,, \qquad (63)$$

where v_e is the local value of energy velocity, whether this energy flow is into losses or into a region of increasing stored energy.

If the result is so general one would expect to be able to demonstrate it from the wave equation of the structure, (2), and the formulae for energy flow and energy density. An outline of such a proof is given in Appendix IV. We are therefore justified in assuming that (63) will also hold when the energy flow is occasioned by beam loading.

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*) Private communication of C.S.Taylor, December 1964.

2. Conclusions

Longitudinal energy flow in an Alvarez structure produces a longitudinal gradient of RF phase. The numerical coefficient may be estimated from considerations of a rough model, or obtained from the dispersion characteristics of the structure if they are known. Then one may calculate the upper limit on usable tank length dictated by phase-difference considerations under given conditions.

Although we have not solved the problem of the longitudinal variations of amplitude associated with transient conditions, what we have done tends to suggest that they are small except under conditions where the phase shifts are intolerably large.

H.G.Hereward

Distribution: (open)

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Appendix I

The value of $\dot{L}\, C_s$

The relationship (11) :

$$L\, C_s = c^{-2}$$

is mostfamiliar for coaxial lines, but is not special to a coaxial cylinder geometry: any two-conductor system with transverse cross-section independent of the longitudinal coordinate x satisfies the relationship; that is why such a structure always has a transmission-line mode which propagates with the velocity of light.

One can suppose the fact that our central conductor is broken into drift tubes and gaps will have a similar effect on L and C_s as has the corrugation of the inner conductor of a coaxial. Such corrugation always increases $L\, C_s$; sometimes one regards it as being fins or irises which increase the shunt capacity, sometimes as slots which add inductance to L . Thus one would tend to expect

$$L\, C_s > c^{-2}$$

for an Alvarez structure.

The drift-tubes support stems will have capacitance to ground which is to be added to C_s , but they also act as parallel inductive path which will reduce the effective value of C_s by $1/\omega^2 L_s$. When the length of the stems is of the order of $\lambda_0/4$ it is difficult to say which of these effectd will predominate.

It has been pointed out *) that one can determine the curvature of the dispersion diagram for the structure, in the neighbourhood of cut-off, by measuring the frequencies of the first few longitudinal modes of a given tank. To know this curvature is to know the effective value of $L\, C_s$ and is probably sufficient for all practical propagation problems.

*) Private communication from P.Lapostolle.

Appendix II

Some remarks on energy velocity.

It is worth elaborating a little our remark at the end of Chapter III to the effect that it is reasonable to find v_e dependent on circumstances, as well as on the structure. In the calculation of energy velocity for the travelling wave, $\omega > \omega_o$, case, we did not specifically introduce any losses or transients; what we did was to select, at (19), a pure forward travelling wave: we could perfectly well have taken a standing wave, a backward travelling wave, or any mixture; and obtained, instead of the expression (26), respectively zero, minus (26), or any value between -(26) and +(26) . One can of course say that we were interested in the energy velocity for the pure forward travelling wave, so that is what we calculated, but one can also link up this choice with the physical situation. The pure forward-travelling-wave situation can be realised physically in three ways:

(1) At some moment $t = o$ we start launching a wave at one end of the structure. We make the structure sufficiently long, and examine the situation at a time sufficiently soon after $t = o$, that the reflected wave has not yet arrived back from the far end. It does not seem too artificial to describe this situation as one which is "100 o/o transient".

(2) We introduce loss into the structure, keeping this loss small enough that it has negligible effect on the properties of any moderate length of structure, and then we take the structure very, very, long, so that the wave reflected from the far end is of negligible amplitude compared with the forward one. This is a situation of "100 o/o loss".

(3) The far end of the structure is terminated by a matched load. A matched load is one which absorbs the forward wave without producing a reflected one: i.e., it absorbs 100 o/o of the incident power, so again we have a situation of "100 o/o loss".

Thus our derivation of (26) has some concealed implications aoub about the situation with respect to losses or transients, and gives a maximal value of energy velocity, associated with "100 o/o loss" or "100 o/o transient" situations.

These easy methods of setting up a "100 o/o loss" or "100 o/o transient" situation are not available at $\omega = \omega_o$, because there the structure does not propagate travelling waves, so we are forced to look at somewhat more complicated situations.

Appendix III

Distributed losses

We introduce distributed losses corresponding to a certain value of Q by replacing $j\omega L$ in our structure, and in the wave equation (2), by

$$j\,\omega\,L + R \tag{65}$$

and giving R the value $\omega_o L/Q$. (66)

It is true that the skin-effect does not produce a purely resistive term, nor one which is independent of frequency, but if we confine ourselves to frequencies near ω_o we have in (65) and (66) an adequate model for the distributed losses. Making the corresponding change in the second-order wave equation (3), and putting $\omega = \omega_o$, we get

$$\frac{d^2 I}{dx^2} = -j\,\frac{C_s L\,\omega_o^2}{Q}\,I$$

Note the pure imaginary coefficient, which gives wavenumbers k with equal real and imaginary parts. The solution appropriate to a metallic wall boundary condition at the far end, x = 0, of the tank is

$$I = A\,(e^{\alpha x} + e^{-\alpha x})\,/2$$

$$\text{where } \alpha = (1-j)\left(\frac{C_s L\,\omega_o^2}{2Q}\right)^{1/2}$$

If the tank is not too long we may expand the exponentials to obtain

$$I = A\left(1 - j\ \frac{C_s L\, \omega_o^2}{2\,Q}\ x^2 + \ldots\right) \quad ,$$

and in a tank of length ℓ this gives just half the phase-shift given by (41).

Appendix IV.

Energy flow and phase-gradient

The time-averaged energy flow is given by $\frac{1}{2}$ I V, and energy density, if one is close enough to cut-off to put $\omega = \omega_0$ and to neglect the transverse electric energy, is given by $\frac{1}{2}$ L I^2 ; see (23) and (24). Thus

$$v_e = \frac{V}{L\,I} \quad .$$

We use this to eliminate V in the wave-equation (2b), and obtain

$$\frac{dI}{dx}/I = - j\omega \quad C_s L \quad v_e$$

which, if $C_s L \cong c^{-2}$ and $\omega \cong \omega_0$, gives

$$- j \, \frac{1}{c \, \lambda\!\!\!^{-}_0} \; v_e$$

in agreement with (63).