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CLOSED ORBIT AMPLITUDES NEAR STOP-BAND

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1. Introduction

In a synchrotron, the distortion of the closed orbit by magnetic kicks may be affected by gradient imperfections. However, as long as the working point for the betatron number Q_o is not close to an imperfection stop-band this effect is usually small.

The situation is less clear when Q_o approaches the stop-band since then, even for small amounts of imperfections, the amplitude function β tends to infinity at certain points.

In the present note it is shown that for an h-harmonic gradient imperfection and for Q_o approaching $\frac{h}{2}$:

a) when h is even, the closed orbit amplitude goes more rapidly to infinity for the perturbed machine than for the unperturbed one;

b) when h is odd, on the contrary, the closed orbit amplitude is not affected at first approximation.

Furthermore, for h even, the influence on the closed orbit is only noticeable when Q_o is within a few stopband widths of $\frac{h}{2}$ and in a region where the closed orbit amplitude, in the absence of imperfection, is going to infinity anyhow.

One may therefore conclude that the influence of gradient imperfections on the closed orbit amplitude when Q_o varies around the working point, is negligible in practice (*).

2. Method

The method has been described in a note of H.G.Hereward (private communication).

Let us consider the perturbed machine and let

$$_k y_{\hat{\beta}}$$

be the displacement of the closed orbit at a high β point, for a unit kick in a point K (**). The r.m.s. displacement of the closed orbit for a unit kick in point K is then obtained by the formula

$$x_k = < {}_k y^2_{\hat{\beta}} >^{\frac{1}{2}} \tag{2.1}$$

where $< {}_k y^2_{\hat{\beta}} >$ indicates an arithmetic mean over the high β points.

For a random distribution of kicks the r.m.s. displacement will then be given, per unit kick, by

$$x = < {}_k x^2 >^{\frac{1}{2}} \tag{2.2}$$

averaging this time over the points K.

(*) Some similar results were obtained by E.Keil in 1967 (private communication), using the Monte Carlo method of AR/Int.SG/65-3.

(**) By definition, the unit kick does not affect the displacement y of the particle but causes its derivative $\frac{dy}{ds}$ to be augmented by one unit, at point K.

The value of x is a measure for the perturbed machine of the closed orbit for a random distribution of kicks. This value may be computed once the model for the unperturbed machine, a mechanism for the introduction of the gradient imperfection and a set of representative points K have been chosen. The choice of the high β points to evaluate the distortion is justified if one is interested in the influence on the aperture requirements all round the machine: at the high β points the size of the beam is greatest and, furthermore, for an imperfection stop-band one does not know in advance what the phase of the perturbation will be.

The x values for different Q_o approaching $\frac{h}{2}$ may then be compared to some classical estimate x_e without perturbation.

In this note the unperturbed machine has M cells and the gradient imperfections are produced by $2h$ equally spaced point-lenses, equal in strength but alternating in sign. Since we are interested in the maximum effect of the imperfection the lenses have been placed in mid-F points of the unperturbed machine. It follows that M is a multiple of $2h$.

(2.3) $$M = 2\ h\ N$$

and that there are N cells of the unperturbed machine between two consecutive lenses.

The set of mid-D and mid-F points of the unperturbed machine has been chosen as representative points K For symmetry reason it is sufficient to consider all mid-D and mid-F between two consecutive perturbing lenses.

Typically one has chosen $N = 15$ and
$h = 12$ which gives the integer case $Q_o \to 6$
$h = 13$ which gives the half-integer case $Q_o \to 6.5$.

The cell of the non-perturbed machine is characterized by its shift μ_o and the values β_{do} and β_{fo} of its amplitude function β_o at the mid-D and mid-F points of the cell. One has still to decide how β_{do} and β_{fo} will depend upon Q_o or μ_o where

(2.4) $$Q_o = \frac{M.\mu_o}{2}$$

Since one has to satisfy the relation

(2.5) $$\frac{1}{\beta_{fo}} \frac{2\ Lh}{M} \leqslant (\mu_o = \int_{cell} \frac{ds}{\beta_o}) \leqslant \frac{1}{\beta_{do}} \frac{2\ Lh}{M}$$

where L is the distance between two consecutive perturbing lenses there is some reason to choose the dependence laws given by

(2.6) $$\beta_{do} = \frac{L}{N\mu_o} \eta_d \qquad \text{with} \qquad \eta_d \leqslant 1$$

(2.7) $$\beta_{fo} = \frac{L}{N\mu_o} \eta_f \qquad \text{with} \qquad \eta_f \geqslant 1$$

The dimensionless constants η_d and η_f together with μ_o determine the model of the unperturbed synchrotron. The constant gradient model is for instance obtained in making

$$\eta_d = \eta_f = 1$$

The amount of gradient imperfection is fixed by the strength value of the perturbing lenses and a measure of the imperfection is the modulation it produces on the β function at the lenses locations

(2.8) $$\delta = \frac{\beta_f - \beta_d}{\beta_f + \beta_d}$$

Typically one has chosen a strength value producing

$$\delta = 0.05 \quad (\text{a} \pm 5\% \ \beta\text{-modulation})$$

at

$$Q_o = \frac{h}{2} - 0.25$$

3. Computation and Results

Two different algorithms may be used to compute the measure x of the closed orbit distortion.

The first uses an analytical approach and the formulae(*)

(3.1) $$_k y_f = \frac{\beta_f^{\frac{1}{2}} \beta_k^{\frac{1}{2}}}{2 \sin \pi Q} \cos \alpha_{kf}$$

(3.2) $$\alpha_{kf} = \int_K^F \frac{ds}{\beta} - \pi Q$$

where $_k y_f$ is the closed orbit displacement in a focusing perturbing lens location for a kick in a point K, the β and Q values referring to the perturbed machine.

Averaging $_k y_f^2$ over these focusing lenses locations one finds $_k x$

(3.3) $$_k x = < {_k y_f^2} >^{\frac{1}{2}}$$

and averaging

(*) From now on we shall assume that the high β points fall on the focusing perturbing lenses located in the mid-F points. This assumption is valid if Q_o is less than $\frac{h}{2}$.

and averaging ${}_k x^2$ over the set of points K one gets, per unit kick, for a random distribution of kicks

$$x = \langle {}_k x^2 \rangle^{\frac{1}{2}} \tag{3.4}$$

One obtains after simplification the formulae

$$ {}_k x = \frac{\beta_f^{\frac{1}{2}} \beta_k^{\frac{1}{2}}}{2\sqrt{2}\,|\sin \pi Q|} \left(1 + t\cos 2\xi_k\right)^{\frac{1}{2}} \tag{3.5}$$

$$ x = \frac{\beta_f^{\frac{1}{2}} \langle \beta_k^{\frac{1}{2}} \rangle}{2\sqrt{2}\,|\sin \pi Q|} \left(1 + t\,\frac{\langle \beta_k \cos 2\xi_k \rangle}{\langle \beta_k \rangle}\right)^{\frac{1}{2}} \tag{3.6}$$

where

$$ t = \frac{\sin h\mu}{h \sin \mu} \qquad \left(\mu = \frac{2\pi Q}{h} \right) \tag{3.7}$$

$$ \xi_k = \int_{\text{Dlens}}^{\text{next K}} \frac{ds}{\beta} \tag{3.8}$$

(= phase shift through the interval DK).

Formula (3.6) is suitable for numerical computation if μ as well as β may be computed for the set of points K .

The second algorithm computes step by step the displacements ${}_k y_f$ using the matrices through half-periods of the unperturbed machine together with the matrices relative to the perturbing lenses.

The displacement vector of the closed orbit in K for a unit amplitude kick in the same K is given by

(3.9)
$$
{}_k\begin{pmatrix} y_o \\ y'_o \end{pmatrix} = (I - R_k)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

where I is the unit matrix and R_k the one-revolution matrix from just after the kick to just before it.

The displacement vector in a focusing lens location for a kick in K is then given by

(3.10)
$$
{}_k\begin{pmatrix} y \\ y' \end{pmatrix}_f = F^{p-1} \; F_k \; {}_k\begin{pmatrix} y \\ y'_o \end{pmatrix}_k \qquad p = 1,\dots,h
$$

where F is the matrix through the interval between two consecutive focusing lenses and F_k the matrix from the point K to the next focusing lens location. The values for ${}_k y_f$ are obtained from (3.10) and introduced in (3.3).

Both algorithms have been used. The first one gives a better insight of the influence of the different parameters but need simple structures for the non-perturbed machine. The constant gradient model has for instance been studied using this algorithm.

The behaviour of the closed orbit near stopband may be described by the quantity x and by the correcting factor

(3.11)
$$
f = \frac{x}{x_e}
$$

where

(3.12) $$x_o = \frac{\beta_{of}^{\frac{1}{2}}(\beta_{od} + \beta_{of})^{\frac{1}{2}}}{2\sqrt{2}\,|\sin\pi Q_o|}$$

is a classical estimate for the value x for the machine without imperfection.

The value x is therefore obtained in multiplying the estimated value x_e by the correcting factor f .

Near stopband, the values x, x_e and f are best expressed as a function of

$$\varepsilon = \frac{q_o}{q_s}$$

where

$$q_o = \frac{h}{2} - Q_o$$

and where q_s is the half-width of the stop-band around $Q_o = \frac{h}{2}$. Since it may be shown that q_s is proportional to δ , i.e. to the β-modulation produced by the gradient imperfection, the resulting curve

$$f = f(\varepsilon)$$

will be largely independent of the amount of imperfection introduced. Typical results are shown in

<u>Table 1</u> for $Q_o \rightarrow 6$ (h = 12)
<u>Table 2</u> for $Q_o \rightarrow 6.5$ (h = 13)

Two cases are shown for the unperturbed machine:

- a constant gradient model $(\eta_f = \eta_d = 1)$
- an alternating gradient model $(\eta_f = 2.0,\ \eta_d = 0.5)$.

The amount of imperfection was choosen to produce a $\pm$ 5% β-modulation at the lenses for $q_o = 0.25$. This gave a half stop-band width of $q_s = 0.01$.

Besides these two cases, runs with similar results were made for different values for η_f and η_d, different β-modulation factors and different harmonic number h.

For h even, figure 3 summarizes the results obtained for the cases having been studied. It gives versus ε, the correcting factor to be applied to the classical estimate for the closed orbit distortion when introducing gradient imperfections. For instance the correction is

+ 20% at $6\ q_s$

+ 50% at $3\ q_s$

+100% at $2\ q_s$

In the upper left corner, a table gives the order of magnitude of q_s versus the β-modulation factor.

For h odd, table 2 shows that the classical estimate does not need more than two or three percent correction near the half-odd integer.

4. Conclusions

When approaching an imperfection stop-band corresponding to an integer (h even) the closed orbit amplitude tends towards infinity as expected, but more rapidly than in the absence of gradient imperfection. This is quantitatively shown in figure 3. However, since the closed orbit of the unperturbed machine is going to infinity anyhow (see x_e in table 1) the working point for Q_o in practice is never in a region where the correcting factor f is noticeable. This case is therefore not of great interest.

The main purpose of the study was to look at the behaviour of the closed orbit when Q_o was approaching an imperfection stop-band corresponding to half an odd-integer (h odd). Table 2 and similar cases show that <u>the closed orbit distortion due to a random distribution of magnetic kicks is not affected by the type of gradient imperfections considered</u>.

5. Acknowledgements

I should like to thank H.G. Hereward for his constant help throughout this work.

<u>Distribution</u>: (open)

T A B L E 1

For a β-modulation of ±5% at $q_o = 0.25$

		Constant gradient model $n_f = 1.0$ $n_d = 1.0$			Alternating gradient model $n_f = 2.0$ $n_d = 0.5$		
ε	q_o	x	x_e	f	x	x_e	f
1.15	.0138	41.939	5.217	8.039	65.505	8.248	7.942
2.18	.0263	5.155	2.741	1.881	8.046	4.333	1.857
3.21	.0388	2.754	1.863	1.478	4.296	2.945	1.458
4.24	.0513	1.883	1.414	1.332	2.936	2.236	1.313
5.27	.0638	1.434	1.142	1.256	2.234	1.806	1.237
6.28	.0764	1.160	.960	1.209	1.807	1.517	1.191
7.30	.0889	.976	.829	1.178	1.519	1.311	1.159
8.31	.1014	.844	.731	1.155	1.313	1.156	1.136
9.32	.1139	.745	.655	1.137	1.158	1.036	1.118
10.32	.1264	.668	.594	1.124	1.038	.940	1.104
11.31	.1389	.606	.545	1.113	.942	.862	1.093
12.31	.1515	.556	.504	1.103	.864	.797	1.084
13.30	.1640	.515	.470	1.096	.799	.743	1.076
14.28	.1765	.480	.440	1.089	.744	.696	1.069
15.26	.1890	.450	.415	1.083	.698	.657	1.063
16.24	.2015	.424	.394	1.078	.658	.622	1.058
17.21	.2141	.402	.375	1.074	.624	.592	1.053
18.17	.2266	.383	.358	1.070	.594	.566	1.049
19.14	.2391	.366	.343	1.066	.568	.543	1.045
20.09	.2516	.351	.331	1.063	.545	.523	1.042
21.05	.2641	.338	.319	1.060	.524	.505	1.039
22.00	.2766	.327	.309	1.057	.506	.488	1.037
22.94	.2892	.316	.300	1.055	.490	.474	1.034
23.88	.3017	.307	.292	1.052	.476	.461	1.032
24.82	.3142	.299	.285	1.050	.464	.450	1.031
25.75	.3267	.292	.278	1.048	.453	.440	1.029

$Q_o \rightarrow 6$

$(h = 12)$

$\delta = 0.05$

$N = 15$

$L = 1$

$\varepsilon = q_o/q_s$ $\quad q_s$ = half width of stop band; $\quad q_o = \frac{h}{2} - Q_o$

$f = x/x_e$ $\quad x$ = r.m.s. closed orbit distortion with perturbation

x_e = r.m.s. closed orbit distortion without perturbation

T A B L E 2

For a β-modulation of ±5% at q_o = 0.25

ε	q_o	Constant gradient model $n_f = 1.0$ $n_d = 1.0$			Alternating gradient model $n_f = 2.0$ $n_d = 0.5$		
		x	x_e	f	x	x_e	f
1.14	.0138	±.230	.226	1.020	±.361	.357	1.010
2.18	.0263	±.231	.227	1.021	±.363	.359	1.011
3.20	.0388	±.233	.228	1.021	±.365	.361	1.012
4.23	.0513	±.235	.230	1.022	±.368	.363	1.013
5.25	.0638	±.237	.232	1.022	±.372	.367	1.014
6.27	.0763	±.240	.234	1.023	±.376	.371	1.015
7.28	.0889	±.243	.237	1.023	±.381	.375	1.016
8.29	.1014	±.246	.241	1.024	±.387	.381	1.017
9.30	.1139	±.251	.245	1.024	±.394	.387	1.018
10.30	.1264	±.255	.249	1.025	±.401	.394	1.019
11.30	.1389	±.260	.254	1.025	±.409	.401	1.020
12.29	.1514	±.266	.259	1.026	±.418	.410	1.021
13.28	.1639	±.272	.265	1.026	±.428	.420	1.021
14.27	.1765	±.279	.272	1.027	±.440	.430	1.022
15.25	.1890	±.287	.280	1.027	±.452	.442	1.023
16.22	.2015	±.296	.288	1.027	±.466	.456	1.023
17.20	.2140	±.306	.297	1.028	±.482	.470	1.024
18.17	.2265	±.317	.308	1.028	±.499	.487	1.024
19.13	.2390	±.329	.320	1.029	±.518	.505	1.025
20.09	.2516	±.342	.333	1.029	±.539	.526	1.025
21.05	.2641	±.358	.347	1.029	±.563	.549	1.025
22.01	.2766	±.375	.364	1.030	±.590	.576	1.025
22.96	.2891	±.394	.383	1.030	±.621	.605	1.026
23.90	.3016	±.417	.404	1.030	±.656	.639	1.026
24.84	.3141	±.442	.429	1.031	±.696	.678	1.026
25.78	.3266	±.472	.457	1.031	±.742	.723	1.026

$Q_o \to 6.5$

(h = 13)

δ = 0.05

N = 15

L = 1

$\varepsilon = q_o/q_s$ q_s = half width of step band; $q_o = \frac{h}{2} - Q_o$

$f = x/x_e$ x = r.m.s. closed orbit distortion with perturbation

x_e = r.m.s. closed orbit distortion without perturbation

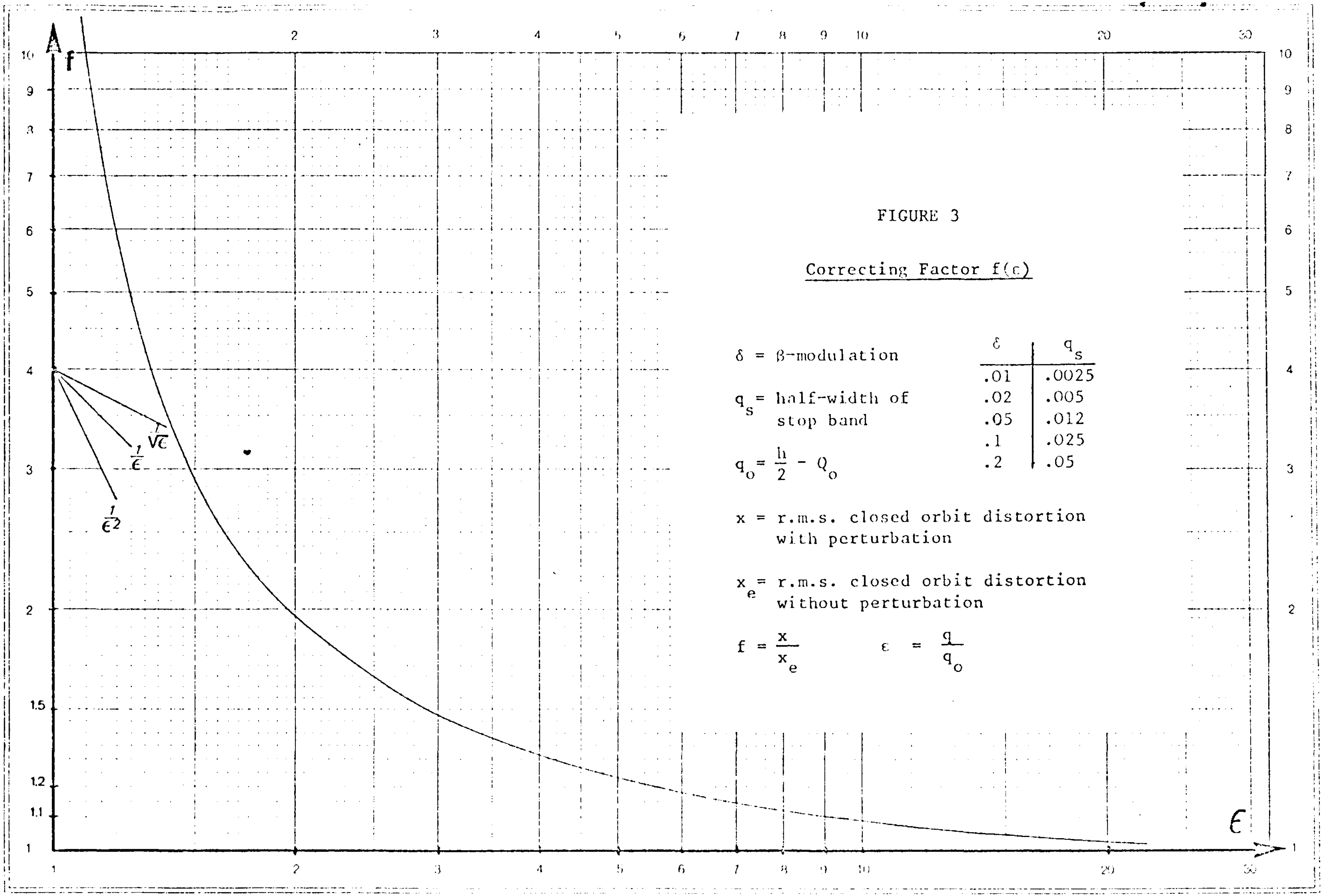

FIGURE 3

Correcting Factor f(ε)

δ = β-modulation

q_s = half-width of stop band

$q_o = \frac{h}{2} - Q_o$

x = r.m.s. closed orbit distortion with perturbation

x_e = r.m.s. closed orbit distortion without perturbation

$f = \frac{x}{x_e}$ $\epsilon = \frac{q}{q_o}$

δ	q_s
.01	.0025
.02	.005
.05	.012
.1	.025
.2	.05