CERN/MPS/LIN 69-16
25.7.1969.

Paper submitted to the 7th International Conference on High Energy Accelerators, Yerevan 1969.

BEAM DYNAMICS IN A PROTON LINAC WITH SPACE CHARGE

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1. INTRODUCTION

In the design and exploitation of modern high current proton linacs, it is essential for understanding the dynamics of the accelerated beam to take into account the strong repulsive forces due to space charge. Because of the complicated nature of this type of interaction, analytical computations do not seem to be possible unless many simplifying but unrealistic hypotheses are made.

On the other hand, the availability of bigger and faster electronic computers has rendered possible the use of sophisticated numerical calculations. As a consequence, in the recent past, several computer programs capable of describing the motion of dense beams of charged particles have been written in various Laboratories.

The purpose of this paper is to present two such programs†, MAPRO1 and MAPRO2, written as result of a collaboration between CERN and Saclay, and to show some of the results which can be obtained from such powerful and flexible tools. The outputs of these programs, which have been produced in connection with the CERN 3 MeV experimental linac [1], are compared between themselves. The emittance growths, matching techniques and transverse longitudinal coupling effects are presented with reference to similar results previously obtained by Mrs. R. Chasman [2], [3].

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† A program somewhat similar to MAPRO2 is being developed at the University of Rennes, France, by B. Houssais.

2. EQUATIONS OF MOTION

The principle of the method consists in transferring the six coordinates defining each particle in phase space ($x,x' = (dx/dz)$, $y,y' = (dy/dz)$, the cynetic energy w and φ) over finite intervals of the order of the cell length as distinct from small step integration techniques.

In particular for each cell of the accelerator, the six coordinates are evaluated :

i) at the end of the first quadrupole,
ii) at the dynamical centre of the cell*,
iii) at the beginning of the second quadrupole,
iv) at the end of the cell (which is the middle of the second quadrupole and the beginning of the next cell).

The set of formulae used to transfer the particle coordinates across a gap are given in [4]; a modified version of these equations, more suitable for computer use, can be found in [5]. The 'true' trajectory of a proton is replaced by an 'equivalent' one, where changes in momentum-like coordinates are impulsive and therefore effectively separated from changes in position-like coordinates.

Several coefficients appear in the equations of motion to take into account the shape of the electric field (transit time factor and similar parameters). All these quantities have been calculated using a computer program CLAS [6] which solves numerically Maxwell's equations in each cell.

The following simplifying hypotheses have been made in the present calculations :

i) the motion is supposed to be non-relativistic. Therefore magnetic field effects have been ignored.
ii) no beam loading effects have been taken into account.
iii) quadrupole misalignments and fringing field effects have been neglected.
iv) neighbouring walls and bunches are ignored in the space charge calculation.

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* The dynamical centre of the cell is defined by the point reached by the axial synchronous particle $\tau/2$ seconds after having entered the cell, if τ is the RF period.

3. THE SPACE CHARGE CALCULATION

According to the general philosophy of these calculations the space charge action is taken into account by giving, twice per cell, an impulsive change to the velocity of each of the N particles forming the beam. This is done in the middle of the drift-tubes and in the dynamical centre of the cells. To compute the space charge forces, one needs to know the three dimensional shape of the bunch. This is done by interpreting $\frac{\beta\lambda}{2\pi}\varphi$ as a longitudinal distance (implicit is a change of the independent variable from z to t). Details are given in [10].

If F_x and F_z are the overall transverse and longitudinal forces acting on a proton due to the presence of the other (N-1) charged particles, the corresponding variations of x' and w are :

$$\Delta w = F_z \cdot \ell \qquad (1)$$

$$\Delta x' = \frac{\ell}{m_o \beta^2 c^2} (F_r - x' F_z) \qquad (2)$$

where ℓ is either the gap or the drift-tube length*, and the other symbols have their usual meaning. An analogous formula applies for y.

It is the way of computing F_x (F_y) and F_z which differs in the two programs MAPRO1 and MAPRO2.

3.1. MAPRO1

In MAPRO1 a particle to particle interaction method is used : the space charge force acting at any point is simply the sum of the Coulomb forces due to all the other particles in the bunch.

Two problems must be solved to render this method really workable :

i) the computation time must be kept within reasonable limits,

ii) particle collisions, which would give rise to infinitely big forces, must be avoided.

The solution which has been adopted in MAPRO1 is similar to the one originally developed in [7].

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* This will make it easier to add image effects in the future.

The space volume around a bunch is enclosed in a cage, divided into a great number of small cubic cells (up to 27 000 in MAPR01) and the force between any two particles is taken as if they fell in the centre of two such cells. If the two cells coincide, the force is set to zero.

In this manner, having discretized the positions, the number of possible space charge forces becomes finite and they can be precalculated and stored in the computer memory before one starts tracing the protons through the linac. Thereafter, during the motion of the beam, the interaction between any two particles is obtained by selecting the convenient approximate force from the entries in the space charge table. This can be done in a very fast manner, particularly if assembler language routines are used.

Also, the symmetry of the beam with respect to the x-z and y-z planes and the reciprocal character of the Coulomb force are exploited to speed up the calculation. For 500 particles actually traced (representative of 2000 since to each particle three images with respect to the z axis can be associated) and 37 space charge computations the CDC 6600 Central Processor (CP) time was 4.5 minutes.

Particle collisions are also avoided using this method, because the minimum distance between two particles which do not occupy the same cell is equal to the side of a cell. A convenient value for this quantity was found to be $\tilde{-}$ 1 mm.

If the beam is traced over a very wide range of energies and its geometrical dimensions vary by a great amount, one might have to recompute the space charge force table several times using different sizes for the cage and the cells. Anyway this was not the case for the 3 MeV linac. The CDC 6600 CP time for one such computation is $\tilde{-}$ 2 seconds.

3.2. MAPR02

In MAPR02, only for the sake of computing space charge forces, the basic assumption is made that the particle distribution in real space can be approximated by an 'equivalent' Gaussian density function. The criterion for equivalency is that the two distributions must have the same second moments with respect to cartesian coordinate planes with origin in the barycentre of the bunch.

Setting :

$$\rho = \rho_0 . e^{-\frac{1}{2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right)} \qquad (3)$$

where ρ_0 is a number of particles/meter^3, x,y and z are coordinates in the frame of reference defined above, the 'equivalent' Gaussian will satisfy the equation :

$$\int_{-\infty}^{\infty}\int\int x^2 \rho \, dxdydz = (2\pi)^{3/2} a^3 bc \rho_0 = \sum_i^{4N} x_i^2 \qquad (4)$$

where N is the number of particles effectively traced, with similar results for y and z.

Also, because the total amount of charge in the bunch must be the same :

$$\int_{-\infty}^{\infty}\int\int \rho \, dxdydz = 4N \qquad (5)$$

Solving (4) and (5) one obtains :

$$\rho_0 = \frac{4}{(2\pi)^{3/2}} \frac{N}{abc} \; ; \; a^2 = \frac{\sum_i^N x_i^2}{N} \; ; \; b^2 = \frac{\sum_i^N y_i^2}{N} \; ; \; c^2 = \frac{\sum_i^N z_i^2}{N} \; . \qquad (6)$$

Therefore, knowing the r.h.s. of Eqs. (6), one can easily compute the parameters of an 'equivalent' Gaussian distribution.

The electric field, at all points of space, is then given by [8] :

$$E_x = \frac{Q.abc.\rho_0}{2\epsilon_0} . x . \int_0^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{x^2}{a^2+t} + \frac{y^2}{b^2+t} + \frac{z^2}{c^2+t} \right)}}{(a^2+t)\sqrt{(a^2+t)(b^2+t)(c^2+t)}} \, dt \qquad (7)$$

where Q is equal to the total charge of a bunch divided by 4N. Analogous formulae hold for E_y and E_z. The integrals on the r.h.s. of Eq. (7) can be evaluated quickly and precisely enough (within 1% in most practical cases) by using Gauss' numerical integration method with ten points.

The practical sequence of operations performed by MAPRO2, which is designed to treat up to 5000 particles (representative of 20 000), to trace the bunch through the n-th cell is as follows :

i) starting from the middle of the n-th drift-tube the particles are allowed to move, one by one, to the dynamical centre of the n-th cell.

ii) the three sums appearing in Eqs.(6) are incremented as each particle arrives at the centre of the gap. Thus at the end of step i) they are known.

iii) the particles are now traced through the complete cell, until the middle of drift-tube (n+1). In passing through the centre of the gap an impulsive correction is given to the velocities, calculated from Eqs. (1) and (2). The space charge force is computed from Eqs.(6) and (7).

iv) from the knowledge of the coordinates in the middle of drift-tube (n+1), the required second order moments and thus the space charge velocity corrections can be computed at this same section. The sequence is then repeated for the next cell.

The fact that only overall quantities like moments of distribution are needed to compute space charge forces, makes it possible to store, in the computer memory, the six particle coordinates only at one section at a time. This is important when so many particles are traced.

The corresponding time for MAPR02 for the same case as for MAPR01 is 34 seconds, i.e. about 1/8 the time.

4. THE INITIAL FILLING

For the sake of comparison all the results given below (except for Fig. 6) were computed using exactly the same 'matched' initial filling as described in [2] . That is a uniform random filling of a four-dimensional hyperellipsoid in x,x',y,y' and an independent random filling (also uniform) of an ellipse in w-φ plane was employed. However, a more realistic way of simulating the emittance of the pre-injector output beam was developed. Details are given in [9].

A program, BUNCH [7], capable of tracing a beam through the buncher to the linac, was available so that it was only necessary to find a reasonable way of populating the four-dimensional phase space at a section just before the buncher. Thereafter, the output of BUNCH in six-dimensional phase space could be used. This simplified the problem a great deal.

To start up BUNCH, the real distribution was approximated by a four-dimensional Gaussian (normal) probability density function (p.d.f.) which best fitted the six marginal distributions (projections) onto the planes of any two of the four variables x,x',y,y'. These projections are available experimentally and could be measured. The 'maximum likelihood' criterion was used for the choice of the Gaussian p.d.f.

Consider one of the two dimensional projections. The result of measurements will essentially be the amount of current falling into each bin of a mesh covering the plane. Now this same quantity can be calculated as function of the ten (as yet unknown) parameters of the distribution. Indicating with m_i the measured value of current into a bin and with t_i the theoretical expected one, in the case of a normal p.d.f., the criterion used reduces to minimizing with respect to the parameters of the distribution, the sum :

$$\chi = \sum_i \frac{(m_i - t_i)^2}{\xi_i^2} = \min$$

where ξ_i is the standard deviation of the error (which is also supposed to be normally distributed) on the i-th measurement and the sum is extended to all bins of all planes.

Note that, although for each plane the t_i's involve only three parameters, the minimization must be done for all planes at the same time because the ten parameters are linked together by the condition that the covariance matrix of the four-dimensional distribution be positive definite.

5. RESULTS AND COMMENTS

5.1. All the results given herewith refer to the CERN 3 MeV experimental linac [1]. It is a strong focusing (++-- structure) 18 cell machine which reproduces almost exactly the first part of the CPS injector. In the calculations the quadrupole gradients have been kept to their theoretical values, originally computed neglecting space charge effects. 500 particles (representative of 2000) have been used throughout the computation except for the results of Fig. 5 where 4500 particles were traced. The injection energy is 500 KeV.

Figs. 1 and 2 summarize the results obtained with the two programs. The following should be noted:

i) Qualitatively the same type of output emittance vs. input emittance curves as given by Mrs. R. Chasman [2] has been found.

ii) MAPR02 yields emittance growths some 20% higher than those given by MAPR01. This is an acceptable agreement given the differences between the two programs. Discretizing distances in MAPR01 probably diminishes space charge forces. The discrepancy on beam envelopes seems of the order of 10%.

iii) No precise comparison can be done with the results of [2] and [3] because of the differences between the two linacs.

Fig. 3 shows emittance growths vs. input brightnesses. In Fig. 4 output vs. input current is displayed, in the hypothesis of an input emittance proportional to the current. Higher output currents can probably be achieved by increasing the quadrupole gradients.

A typical output of MAPR02 is shown in Fig. 5. In Fig. 6 the initial radial and longitudinal emittances, obtained using BUNCH program (cf. Section 4) are given.

5.2. As in [2], the importance of injecting a matched beam, particularly for minimizing emittance growths, has been verified. The matched initial dimensions of the bunch, as given by Eqs. 1a and 1b of [2], seem to fit rather well for longitudinal energy spread Δw of order of 20 KeV but they deteriorate for smaller values of Δw.

Fig. 1 shows that the output transverse emittance of a beam, for a given current, never decreases below a certain threshold, whatever the input emittance is. On the other hand, there is also an upper limit due to the presence of drift-tubes. Therefore, it seems that, for a fixed current, the output emittance must be comprised between two limits which become closer for higher intensities.

The transverse emittance growth can be considerably reduced by decreasing the longitudinal emittance area (cf. Fig. 1). This was already observed in [3].

6. CONCLUSIONS

A new method of computing space charge forces based on the representation of the particle distribution by means of an analytical model has been introduced. This basic hypothesis has been tested by comparing the results obtained with those of a program which used a more general approach.

For a given number of representative particles, the basic advantage of the method is to speed up the computation time. Conversely, for a fixed computation time, roughly an order of magnitude more particles can be traced. This is particularly useful :

i) to obtain density marginal distribution onto one phase space axis. With fewer particles there is a considerable noise on the statistics : attempts with 500 particles gave rather poor results,

ii) to trace isodensity curves on the various two-dimensional projections.

Finally, concerning the comparison with the results of [2] and [3], the fact that calculations carried out completely independently, using different approaches, gave similar results, seems to suggest that these numerical techniques are trustworthy.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the constant encouragement and help that they received from Dr. P. Lapostolle during the course of this work. Many thanks are also due to Dr. D.J. Warner for several helpful discussions and suggestions and to Dr. H.G. Hereward who revised and improved the manuscript.

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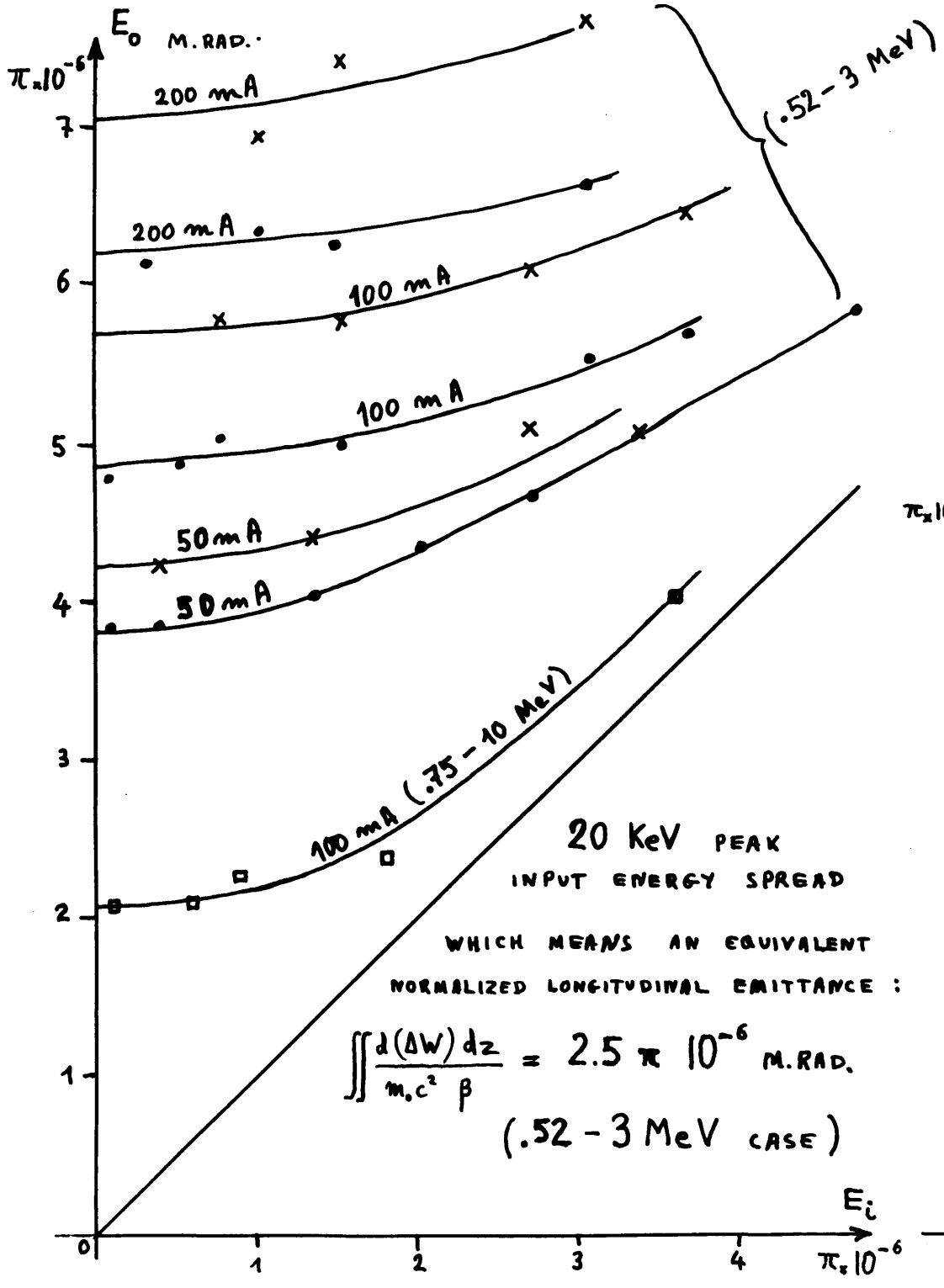
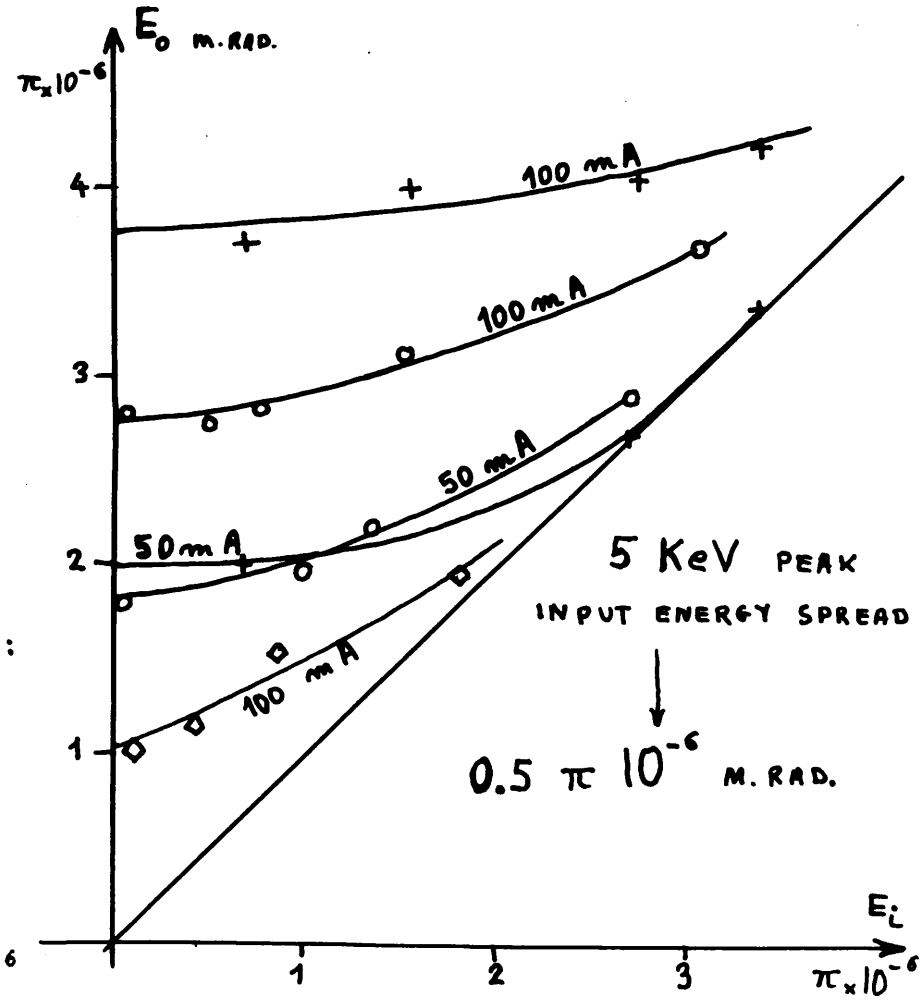
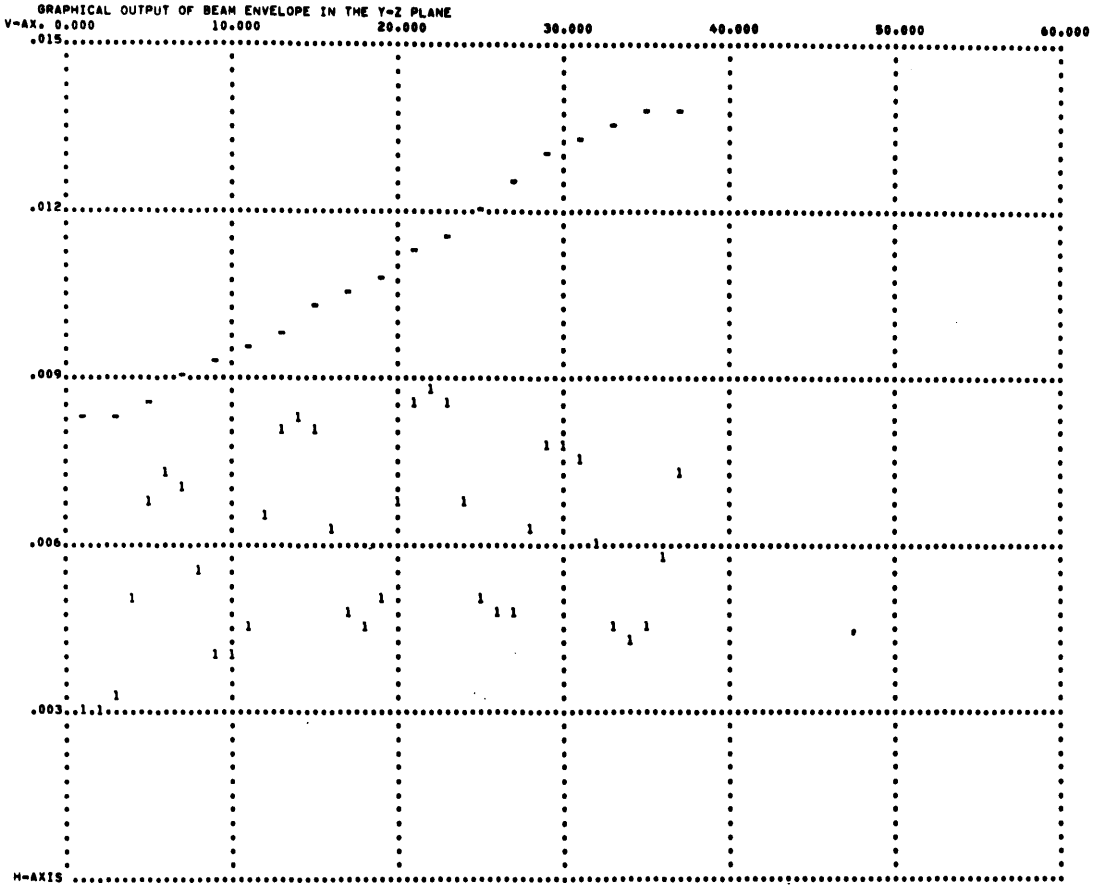
FIGURE 1

OUTPUT NORMALIZED EMITTANCE VS. INPUT NORMALIZED EMITTANCE.

- • MAPRO1 20 KeV
- o MAPRO1 5 KeV
- x MAPRO2 20 KeV
- \+ MAPRO2 5 KeV
- □ R. CHASMAN 20 KeV (ACCORDING TO [2])
- ◇ R. CHASMAN 5 KeV (ACCORDING TO [3])







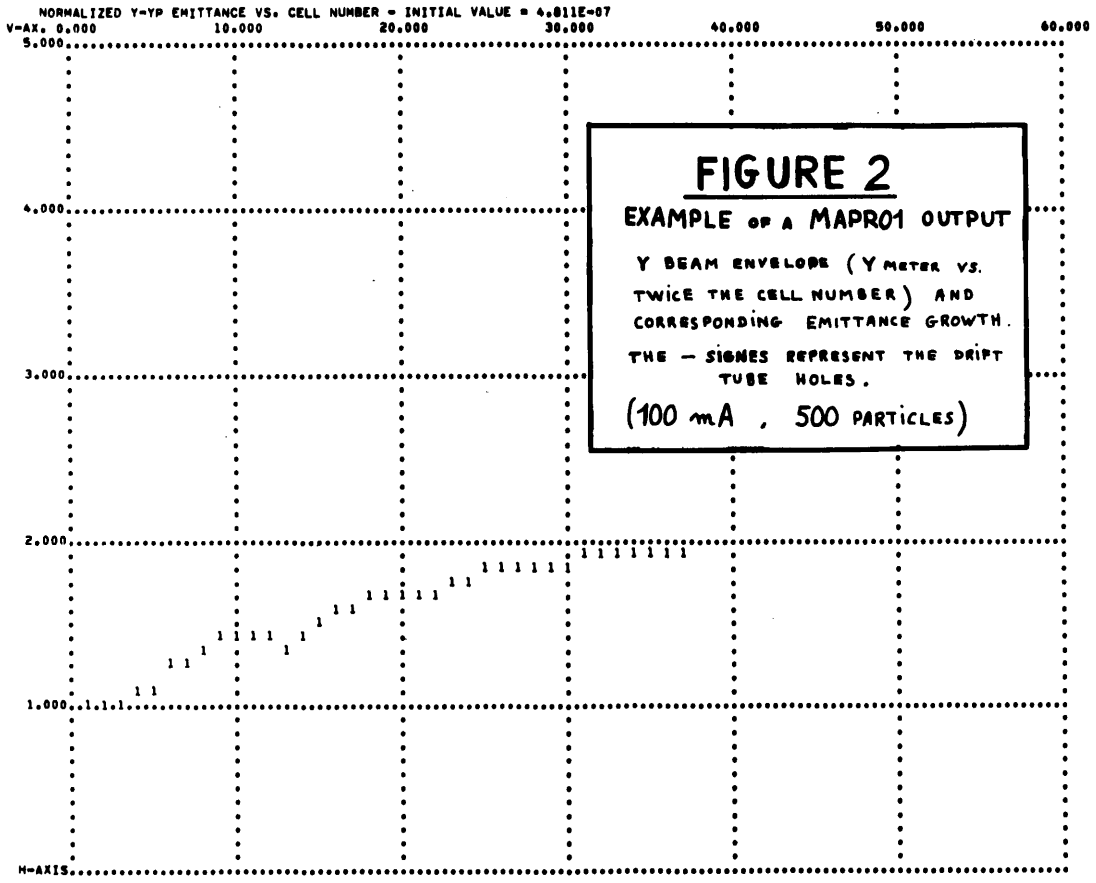


FIGURE 2

EXAMPLE OF A MAPR01 OUTPUT

Y BEAM ENVELOPE (Y METER VS. TWICE THE CELL NUMBER) AND CORRESPONDING EMITTANCE GROWTH.

THE — SIGNES REPRESENT THE DRIFT TUBE HOLES.

(100 mA , 500 PARTICLES)

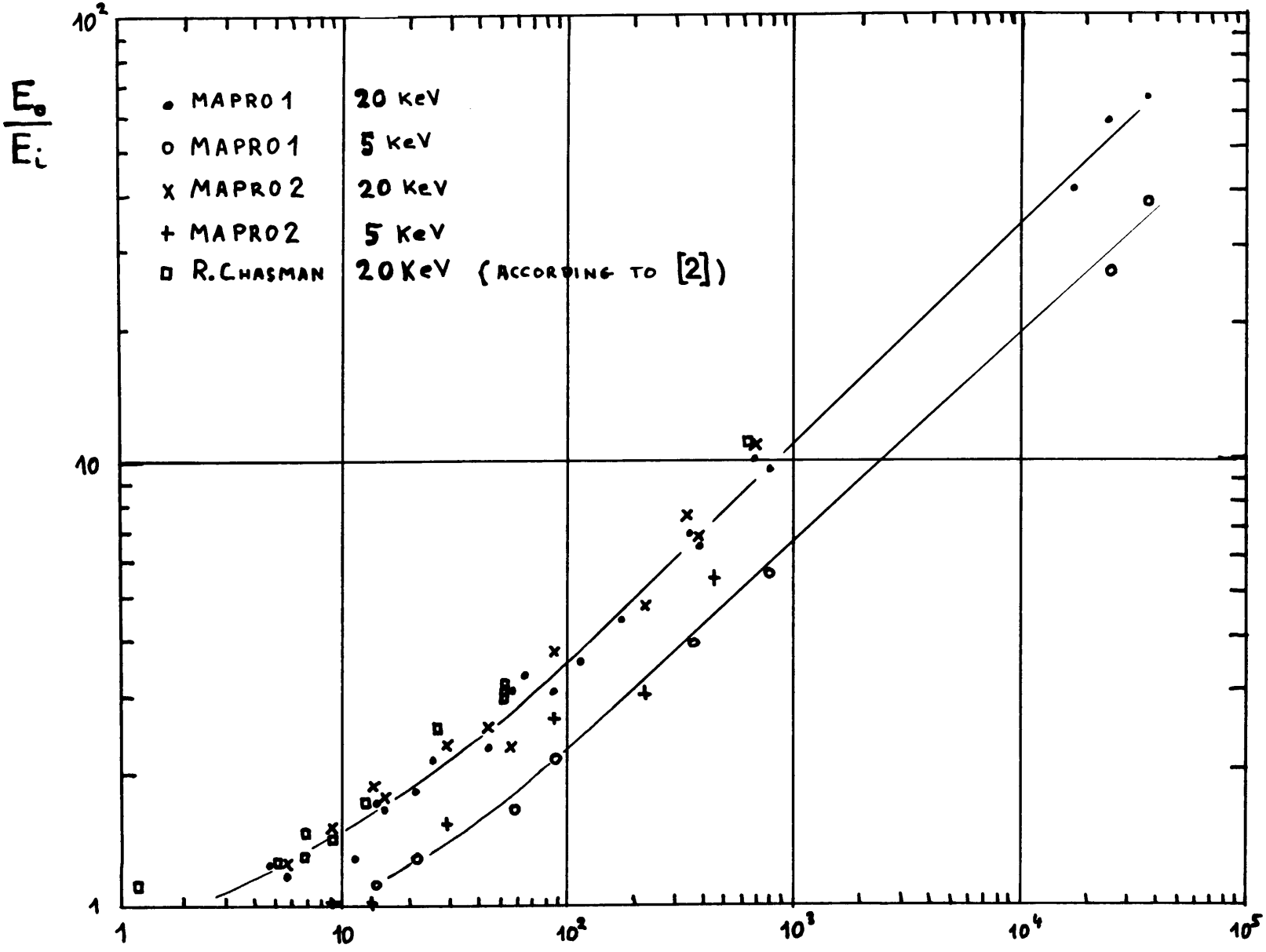


FIGURE 3

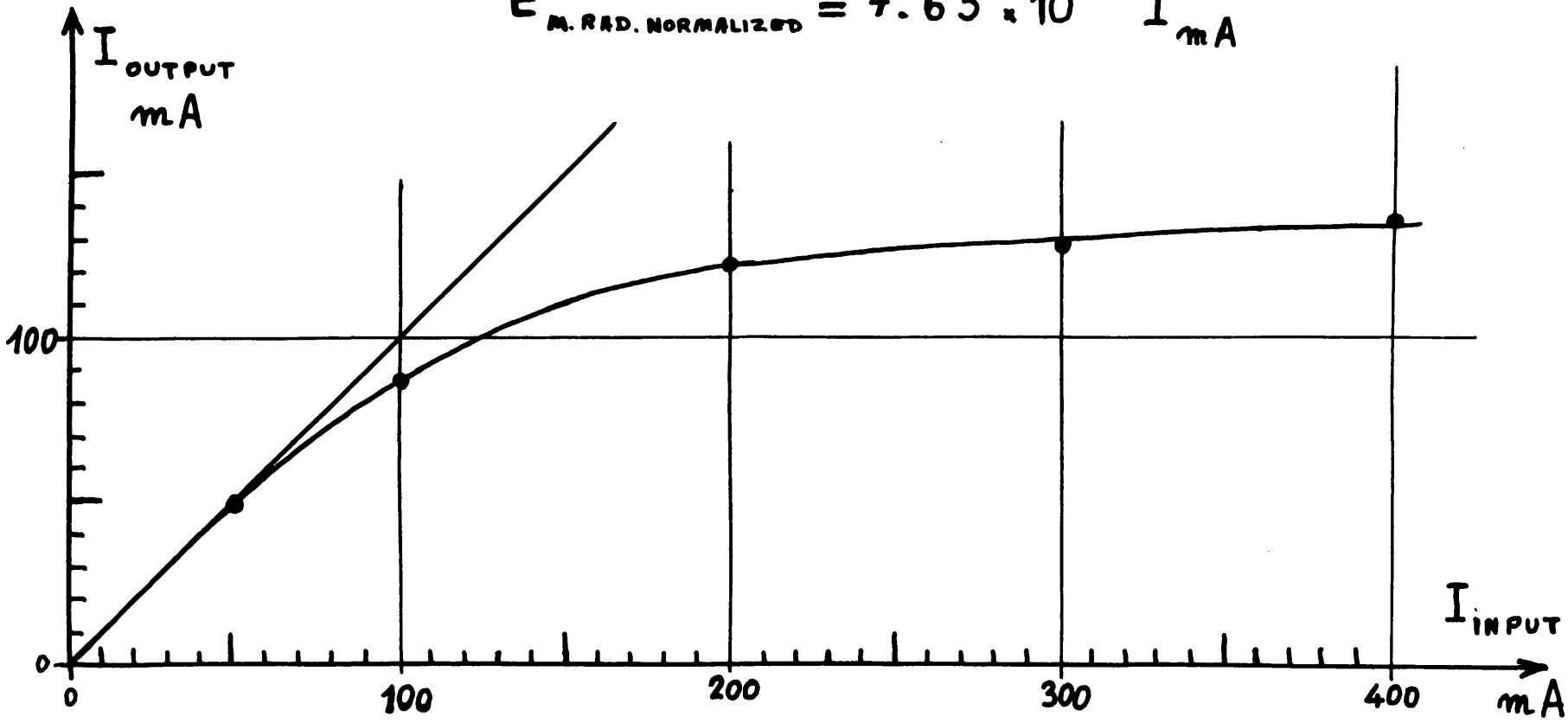
TRANSVERSE EMITTANCE GROWTH $\frac{E_o}{E_i}$ VS. BRIGHTNESS B

DEFINED AS : $B = \frac{2 \cdot I}{E^2}$ $\left(\text{UNITS} : \frac{mA}{(mm \cdot mRAD)^2} \right)$

FIGURE 4

OUTPUT CURRENT VS. INPUT CURRENT ,
THE RADIAL EMITTANCE BEING TAKEN
PROPORTIONAL TO THE INPUT CURRENT

$$E_{\text{M. RAD. NORMALIZED}} = 7.65 \times 10^{-9} \, I_{mA}$$



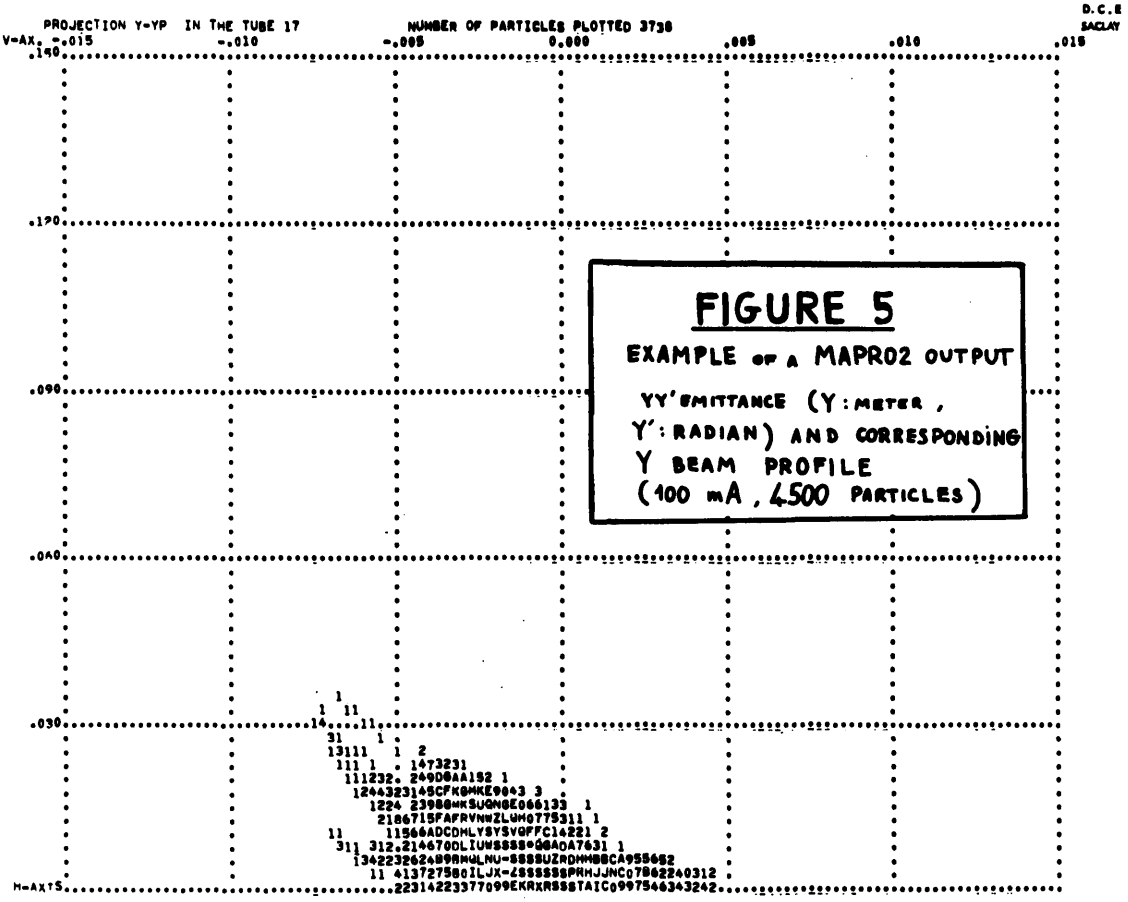
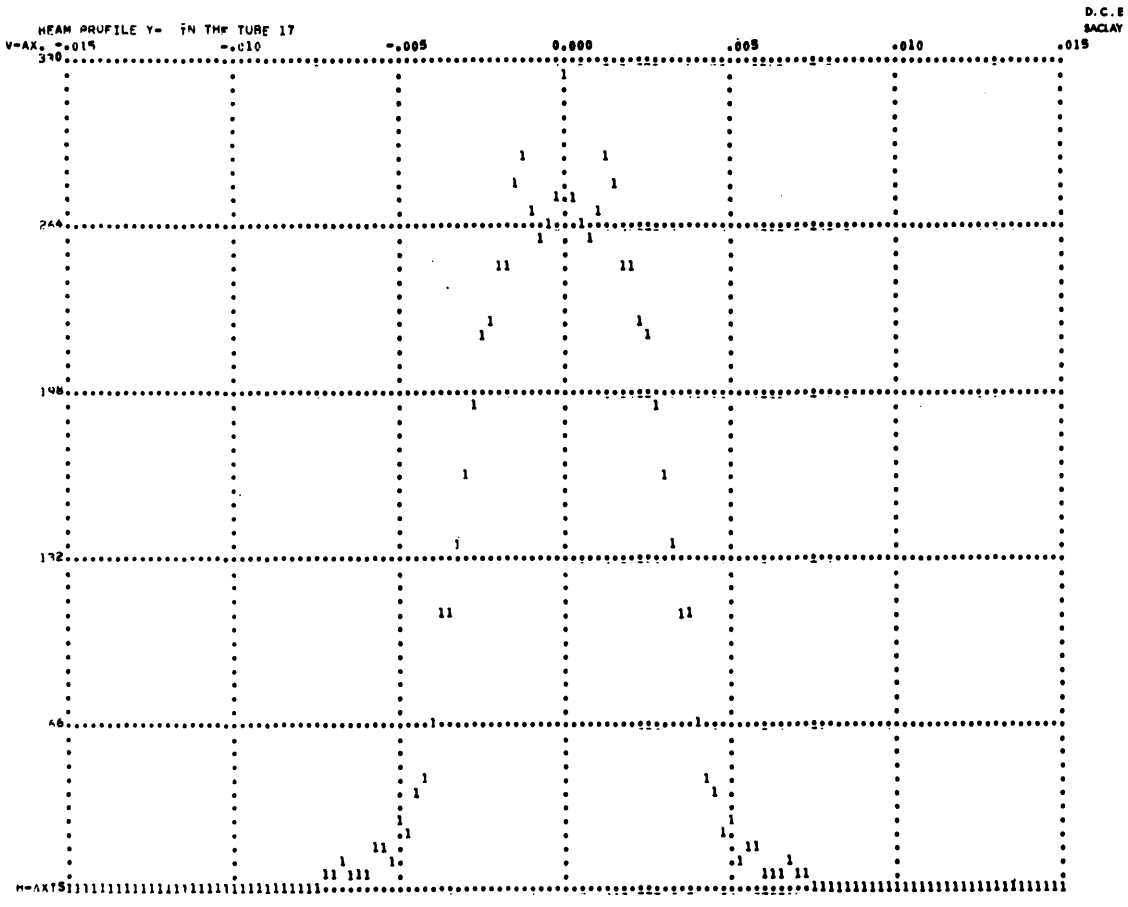


FIGURE 5

EXAMPLE OF A MAPRO2 OUTPUT

YY' EMITTANCE (Y: METER, Y': RADIAN) AND CORRESPONDING Y BEAM PROFILE (100 mA, 4500 PARTICLES)



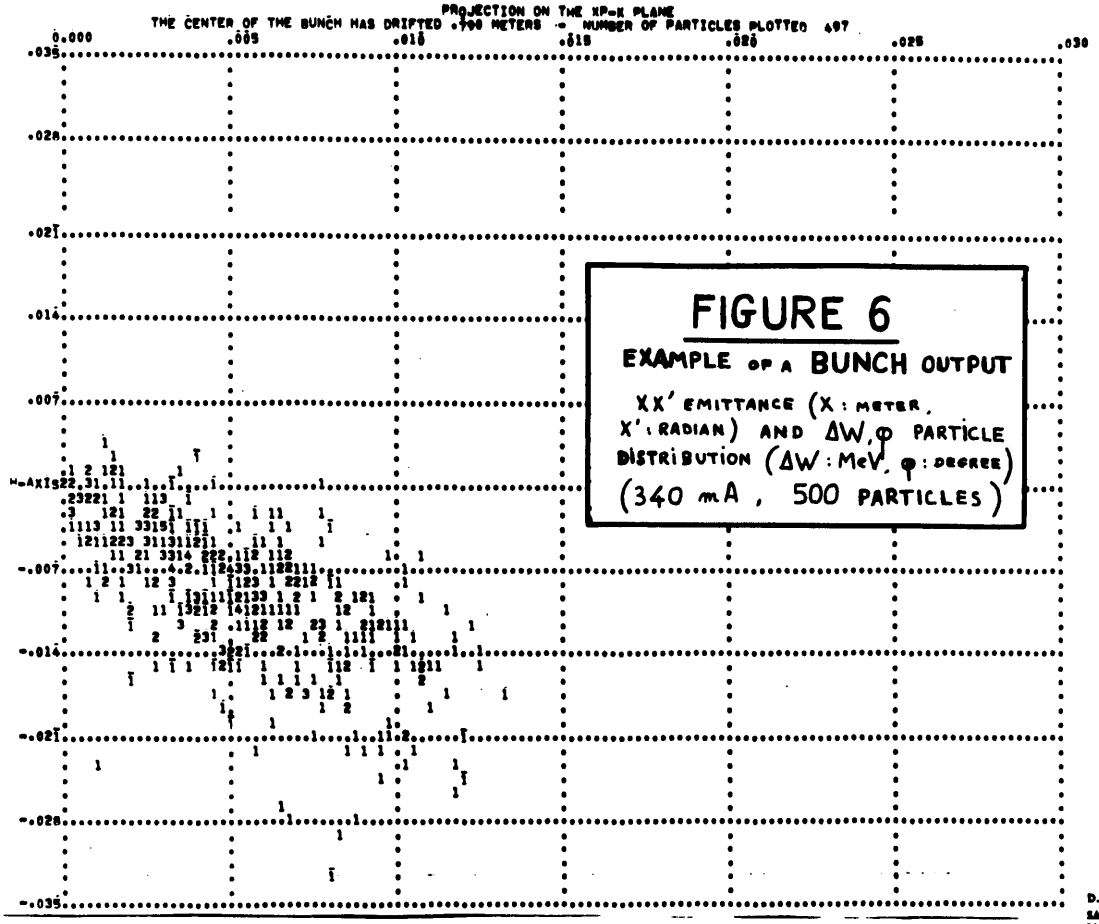


FIGURE 6

EXAMPLE OF A BUNCH OUTPUT

XX' EMITTANCE (X : METER, X' : RADIAN) AND ΔW, φ PARTICLE DISTRIBUTION (ΔW : MeV, φ : DEGREE)

(340 mA , 500 PARTICLES)