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NON-LINEAR BETATRON OSCILLATIONS

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The equations of motion of a particle oscillating in a non-linear magnetic field are solved using a time-dependent perturbation theory extended to the second order. The theory is applied to the design of correction fields which limit the amplitude growth of the oscillation. The correction schemes are verified by numerical integration of the equations of motion ("particle tracking") and discussed on two special machines: the Berkeley Advanced Light Source and the CERN Antiproton Collector for which experimental data have been collected.

1. INTRODUCTION

The optical properties of an accelerator or of a storage ring are usually defined in a hierarchical way. There is first the determination of the ideal trajectory which passes through the center of the quadrupoles where there is no magnetic field and which closes onto itself after one turn, this is the *central orbit* which is a consequence of the bending field distribution around the ring and gives its general shape to the machine: a circle, an oval, a racetrack, etc. Then come the focusing properties about the central orbit; they concern the oscillatory motion of the particles in a monochromatic beam whose energy corresponds to the central orbit and determine the number of betatron oscillations per turn, the *tune*, and the beam envelope via the *β function*. In a real beam, the particles have a certain energy distribution; the orbits and the focusing or *chromatic* properties of the *off-momentum* particles such as they result from the dipole and quadrupole fields are usually not acceptable; they have to be controlled using non-linear fields and especially *sextupolar* fields which vary quadratically with the position of the particle with respect to the central orbit. A secondary effect of the non-linear fields is the alteration of the motion for the particles with a large oscillation amplitude. It reduces the number of particles which can be injected or stored in the machine. It is especially severe for the superconducting machines, the large acceptance storage rings and the new generation of synchrotron light sources. A traditional cure has consisted of analyzing the motion into its Fourier components and suppressing the most harmful component yet keeping unchanged the chromatic properties (*resonance* method) [1]. However, modern machines are built in such a way that the *operation point*, the couple of the horizontal and vertical tunes, is far from any known dangerous resonance. The analysis and the compensation of the non-linear distortions must then be treated in the frame of a more general theory.

We use here a first and second-order theory of the general equations of motion. The order 0 is the linear theory (section 2) in which we introduce the main notations and we recall that the phase space particle trajectories are circles for the horizontal and vertical motions when the phase space is normalized. One transforms then the cartesian coordinates (x, y, p_x, p_y) into the polar coordinates (J_x, J_y, μ_x, μ_y) where J and μ are the action and the angle variables respectively. The reason for this change of coordinates lies in the fact that the integrated value of $\mu/2\pi$ over one turn is the betatron tune and that the action angle is directly related to the beam size. The magnetic field (section 3) is described by a scalar potential which is directly analyzed in terms of the action-angle variables. The first-order differential system of the four coupled equations of motion (section 4) is derived from the Hamiltonian of the motion. This system is solved by successive approximations using the method of *variation of the constants* [2]. The constants are the initial values of the action and angle variables. At the first order, action and angle are replaced in the general system by their linear expressions and the integration is straightforward. At the second order, the first-order expressions of J and μ are injected into the differential system; the result is written in the form of double integrals which express the fact that the total effect of the non-linear fields is no longer a superposition of the individual fields as is the case in the first-order theory but that the non-linear fields are now correlated between them.

For closed machines (section 5), the condition of periodicity of the magnetic structure must be manifest. As this case is of paramount importance in practice, the integrals are written explicitly using the simplest models of integration. At the first order (section 5.1), the *thin-lens* approximation which consists of concentrating the field into a magnet of zero length using δ - functions is very convenient. At the second order (section 5.2), the thin-lens approximation cannot be used and the simplest model consists of assuming a constant value for the β function and a linear variation of the betatron phase with respect to the longitudinal coordinate inside a magnet. The various types of correlation are analyzed in order to give as much insight as possible into the physical problem. They apply to the coupling of a field with itself (section 5.2.1), to the correlation between a given field and the same field at the next turn (section 5.2.2) and to the correlation between two different fields (section 5.2.3). A technical difficulty appears in the evaluation of each type of correlation in the form of *secular* terms, those terms which repeat themselves identically at each turn and lead to an unphysical growth of the oscillation; fortunately, when all the correlations are summed up, these terms cancel out. In the formalism which is developed here, the motion is completely described as the superposition of a periodic motion and of an oscillation about the periodic motion; this feature contrasts with other treatments [3] which are deliberately restricted to the periodic solution of the equations of motion. Our purpose being to design a machine with the maximum aperture, there is no reason to neglect the oscillatory part of the motion. In order to test the validity of the theory, one can compare scatter plots of a section of the phase space based on the analytical formulae with purely numerical non-perturbative techniques. However, the main interest of analytical formulae lies in the prescription of correction schemes (section 6).

The compensation method applies to the correction of the systematic non-linear distortions created by the chromaticity correction elements. It is illustrated in the case of a hadron machine, the CERN Antiproton Collector [4], and of a lepton machine, the Berkeley Advanced Light Source [5] and it is shown that a substantial improvement in machine aperture can be obtained at the cost of a rather modest correction scheme implemented in regions where the orbit position is independent of the particle momentum.

2. LINEAR THEORY

In a particle accelerator, the shape of the beam is determined by the focusing system [6] which is composed of linear and non-linear magnetic fields. The linear elements are the *quadrupoles* whose field components are linear functions of the transverse coordinates (x,y). In these fields, the particles are submitted to restoring forces proportional to the distance of the particle from the magnetic axis where the field is zero and the transverse coordinates have their origin. The equations of the transverse motion of a particle are then:

$$\frac{d^2x}{ds^2} + K_x(s)\,x = 0 \qquad \frac{d^2y}{ds^2} + K_y(s)\,y = 0 \tag{1}$$

The curvilinear coordinate s is taken along the reference trajectory which passes through the centre of the quadrupoles. In contrast with the harmonic oscillator equation, the focusing strength K is not constant but a function of s. One can still nevertheless write the solution to the differential equations in a phase-amplitude form [1]:

$$x = \sqrt{2\beta_x(s)\,J_x}\cos[\,\mu_x(s) + \phi_x\,] \qquad y = \sqrt{2\beta_y(s)\,J_y}\cos[\,\mu_y(s) + \phi_y\,] \tag{2}$$

By back-substitution of x (or y) into the differential equations and identification with respect to the sine and cosine terms, it turns out that β and μ are related by

$$\mu(s) = \int_0^s \frac{d\sigma}{\beta(\sigma)} \tag{3}$$

and that the derivative of β which is traditionally defined via the coefficient

$$\alpha = -\frac{1}{2}\frac{d\beta}{ds} \tag{4}$$

satisfies the equation

$$\frac{d\alpha}{ds} = K\beta - \frac{1+\alpha^2}{\beta} \tag{5}$$

For brevity, the subscripts are omitted when the relations between physical parameters like β, μ, J, ϕ, ... or the statements about them are valid for the horizontal and the vertical plane as well. The four constants of the motion are the actions J and the angles ϕ. The Hamiltonian of the motion which has the form

$$H = \frac{1}{2}[\,p_x^2 + p_y^2 + K_x\,x^2 + K_y\,y^2\,] \tag{6}$$

with (x, y) and $(p_x = dx/ds, p_y = dy/ds)$ as conjugate variables becomes

$$H_l = \frac{J_x}{\beta_x} + \frac{J_y}{\beta_y} \tag{7}$$

with $(\mu_x + \phi_x, \mu_y + \phi_y)$ and (J_x, J_y) as new position and momentum variables. As our purpose is essentially to determine the evolution of the beam envelope in the presence of non-linear fields, the equations of motion will be derived from a Hamiltonian of the same type as H_l. The non-linear Hamiltonian is deduced from H_l by addition of a potential term V which has to be written in the angle-action variables.

3. SCALAR POTENTIAL OF THE MAGNETIC FIELD

The scalar potential of a magnetic multipole of order m is

$$v = \frac{g^{(m-2)}}{m!} \mathrm{Re}(x + i y)^m \tag{8}$$

where $g^{(m-2)}$ is the (m-2)- derivative of the vertical component of the magnetic field gradient with respect to x. The order m may be defined from 1, $g^{(-1)}$ is then the *dipole* component and $g^{(o)}$ the quadrupole component, but we shall only consider fields corresponding to values of m larger than 2: $g^{(1)}$, $g^{(2)}$, ... are then the *sextupolar, octupolar,* ... components. In particle dynamics, the potential comes in the Hamiltonian of the motion in the form

$$V = \frac{e}{p} v \tag{9}$$

where e is the charge and p the momentum of the particle. It is therefore convenient to use the derivatives of the focusing strength

$$K^{(m-2)} = \frac{e}{p} g^{(m-2)} \tag{10}$$

By substituting the expressions of the betatron motion into (8), the expression of the potential V becomes

$$V = \sum_{i=0}^{I(m/2)} \sum_{j=0}^{I(m/2)-i} \sum_{k=0}^{i} C_{ijk} [\cos \mu^+_{ijk} + \cos \mu^-_{ijk}] \tag{11}$$

where I is the integer part function and

$$C_{ijk} = K^{(m-2)} \frac{(J_x\beta_x)^{\frac{m}{2}-i} (-J_y\beta_y)^i}{(2)^{\frac{m}{2}-1} j!\, k!\, (m-2i-j)!\, (2i-k)!} \tag{12}$$

$$\mu^{\pm}_{ijk} = \overline{m^{\pm}_{ijk}} \cdot \overline{\mu} \tag{13}$$

with

$$\overline{\mu} = (\mu_x + \phi_x \, , \mu_y + \phi_y) \tag{14}$$

$$\overline{m^{\pm}_{ijk}} = (m_{xij} \, , \pm m_{yik}) \tag{15}$$

$$m_{xij} = m - 2(i + j) \tag{16}$$

$$m_{yik} = 2(i - k) \tag{17}$$

Special cases occur when either i or m-2i are zero, then m_y or m_x are zero and the two cosines are the same in the expression of V which must be re-written

$$V = \sum_{i=0}^{I(m/2)} \sum_{j=0}^{I(m/2)-i} \sum_{k=0}^{i} C_{ijk} \cos \mu^{+}_{ijk} \tag{18}$$

These cases correspond to the combinations

$$\begin{aligned} (i, j, k) &= (0, j<\tfrac{m}{2}, 0) \\ (i, j, k) &= (\tfrac{m}{2}, 0, k<\tfrac{m}{2}) \end{aligned} \tag{19}$$

When m is even, m_x and m_y can be zero simultaneously, a constant term then appears at the right hand side of V which is always defined within an an additive constant anyhow; the combinations of the type

$$(i, j, k) = (i, \tfrac{m}{2} - i, i) \tag{20}$$

have thus simply to be eliminated.

4. EQUATIONS OF MOTION IN THE ACTION-ANGLE VARIABLES

When a non-linear field is introduced, J and ϕ become functions of s and the equations of motion are solved using a method of *variation of constants* also called sometimes *time dependent perturbation* method. The non-linear Hamiltonian is

$$H_1 = \frac{J_x}{\beta_x} + \frac{J_y}{\beta_y} + V \tag{21}$$

The equations of motion are then

$$\frac{dJ}{ds} = -\frac{\partial H_1}{\partial(\mu+\phi)} = -\frac{\partial V}{\partial(\mu+\phi)}$$

$$\frac{d(\mu+\phi)}{ds} = \frac{\partial H_1}{\partial J} = \frac{1}{\beta} + \frac{\partial V}{\partial J} \tag{22}$$

Because of the relation (3) between μ and β, the second equation can be simplified and the system re-written

$$\frac{dJ}{ds} = -\frac{\partial V}{\partial \phi}$$

$$\frac{d\phi}{ds} = \frac{\partial V}{\partial J} \tag{23}$$

or, more explicitly,

$$\frac{dJ_x}{ds} = \sum_{i=0}^{I(\frac{m}{2})} \sum_{j=0}^{I(\frac{m}{2})-i} \sum_{k=0}^{i} C_{ijk} m_{xij} [\sin \mu_{ijk}^{+} + \sin \mu_{ijk}^{-}]$$

$$\frac{dJ_y}{ds} = \sum_{i=0}^{I(\frac{m}{2})} \sum_{j=0}^{I(\frac{m}{2})-i} \sum_{k=0}^{i} C_{ijk} m_{yik} [\sin \mu_{ijk}^{+} - \sin \mu_{ijk}^{-}] \tag{24}$$

$$\frac{d\phi_x}{ds} = \sum_{i=0}^{I(\frac{m}{2})} \sum_{j=0}^{I(\frac{m}{2})-i} \sum_{k=0}^{i} \frac{\partial C_{ijk}}{\partial J_x} [\cos \mu_{ijk}^{+} + \cos \mu_{ijk}^{-}]$$

$$\frac{d\phi_y}{ds} = \sum_{i=0}^{I(\frac{m}{2})} \sum_{j=0}^{I(\frac{m}{2})-i} \sum_{k=0}^{i} \frac{\partial C_{ijk}}{\partial J_y} [\cos \mu_{ijk}^{+} + \cos \mu_{ijk}^{-}] \tag{25}$$

This set of equations can be solved by successive approximations.

4.1 First iteration

The first-order perturbation of the angle-action variables (J,ϕ) is obtained by assuming (J,ϕ) to be constant and equal to (J_0,ϕ_0) at the right hand side of the equations. In order to have an explicit form which can be generalized at the second iteration, let us define the vectors

$$\overline{A_x^{(1)}} = (C_{i,j,k}, C_{i,j,k})$$

$$\overline{A_y^{(1)}} = (C_{i,j,k}, -C_{i,j,k}) \tag{26}$$

$$\overline{F_x^{(1)}} = \left(\frac{\partial C_{ijk}}{\partial J_x}, \frac{\partial C_{ijk}}{\partial J_x} \right)$$

$$\overline{F_y^{(1)}} = \left(\frac{\partial C_{ijk}}{\partial J_y}, \frac{\partial C_{ijk}}{\partial J_y} \right) \qquad (27)$$

$$\overline{\Psi^{(1)}} = (\sin \mu^{+}_{ijk} , \sin \mu^{-}_{ijk})$$

$$\overline{\Phi^{(1)}} = (\cos \mu^{+}_{ijk} , \cos \mu^{-}_{ijk}) \qquad (28)$$

The set of equations integrated over a length s takes the form

$$J_{x1} = \sum_{i=0}^{I(m/2)} \sum_{j=0}^{I(m/2)-i} \sum_{k=0}^{i} m_{xij} \int_0^s \overline{A_x^{(1)}}.\overline{\Psi^{(1)}}\, d\sigma$$

$$J_{y1} = \sum_{i=0}^{I(m/2)} \sum_{j=0}^{I(m/2)-i} \sum_{k=0}^{i} m_{yik} \int_0^s \overline{A_y^{(1)}}.\overline{\Psi^{(1)}}\, d\sigma \qquad (29)$$

$$\phi_{x1} = \sum_{i=0}^{I(m/2)} \sum_{j=0}^{I(m/2)-i} \sum_{k=0}^{i} \int_0^s \overline{F_x^{(1)}}.\overline{\Phi^{(1)}}\, d\sigma$$

$$\phi_{y1} = \sum_{i=0}^{I(m/2)} \sum_{j=0}^{I(m/2)-i} \sum_{k=0}^{i} \int_0^s \overline{F_y^{(1)}}.\overline{\Phi^{(1)}}\, d\sigma \qquad (30)$$

4.2 Second iteration

At the second order, the coefficients C and the trigonometric functions that we call T for brevity have to be expanded as functions of J and ϕ respectively:

$$C_{ijk} = C_{0ijk} + \frac{\partial C_{0ijk}}{\partial J_x} J_{x1} + \frac{\partial C_{0ijk}}{\partial J_y} J_{y1}$$

$$T_{ijk} = T_{0ijk} + \frac{\partial T_{0ijk}}{\partial \phi_x} \phi_{x1} + \frac{\partial T_{0ijk}}{\partial \phi_y} \phi_{y1} \qquad (31)$$

The differential equations for the second-order terms of the action variables are thus

$$\frac{dJ_{x2}}{ds} = \sum_{i=0}^{I(m/2)} \sum_{j=0}^{I(m/2)-i} \sum_{k=0}^{i} m_{xij} \left[\left(\frac{\partial C_{0ijk}}{\partial J_x} J_{x1} + \frac{\partial C_{0ijk}}{\partial J_y} J_{y1} \right) T_{x0ijk} + \left(\frac{\partial T_{x0ijk}}{\partial \phi_x} \phi_{x1} + \frac{\partial T_{x0ijk}}{\partial \phi_y} \phi_{y1} \right) C_{0ijk} \right]$$

$$\frac{dJ_{y2}}{ds} = \sum_{i=0}^{I(m/2)} \sum_{j=0}^{I(m/2)-i} \sum_{k=0}^{i} m_{yik} \left[\left(\frac{\partial C_{0ijk}}{\partial J_x} J_{x1} + \frac{\partial C_{0ijk}}{\partial J_y} J_{y1} \right) T_{y0ijk} + \left(\frac{\partial T_{y0ijk}}{\partial \phi_x} \phi_{x1} + \frac{\partial T_{y0ijk}}{\partial \phi_y} \phi_{y1} \right) C_{0ijk} \right] \qquad (32)$$

Due to the products $J_{x1} T_{x0ijk}$,...the multiplication of the trigonometric functions gives rise to a new generation of modes defined by the eight characteristic arguments

$$
\begin{aligned}
\psi_1 &= \mu^{+}_{ijk}(s) + \mu^{+}_{i1j1k1}(\sigma) \\
\psi_2 &= \mu^{+}_{ijk}(s) - \mu^{+}_{i1j1k1}(\sigma) \\
\psi_3 &= \mu^{-}_{ijk}(s) + \mu^{-}_{i1j1k1}(\sigma) \\
\psi_4 &= \mu^{-}_{ijk}(s) - \mu^{-}_{i1j1k1}(\sigma) \\
\psi_5 &= \mu^{+}_{ijk}(s) + \mu^{-}_{i1j1k1}(\sigma) \\
\psi_6 &= \mu^{+}_{ijk}(s) - \mu^{-}_{i1j1k1}(\sigma) \\
\psi_7 &= \mu^{-}_{ijk}(s) + \mu^{+}_{i1j1k1}(\sigma) \\
\psi_8 &= \mu^{-}_{ijk}(s) - \mu^{+}_{i1j1k1}(\sigma)
\end{aligned}
\tag{33}
$$

and the four types of coefficients

$$
\begin{aligned}
b_1 &= \frac{1}{2} m_{x\,i1\,j1} \frac{\partial C_{0ijk}(s)}{\partial J_x} C_{0\,i1\,j1\,k1}(\sigma) \\
b_2 &= \frac{1}{2} m_{x\,ij} \frac{\partial C_{0\,i1\,j1\,k1}(\sigma)}{\partial J_x} C_{0\,ijk}(s) \\
b_3 &= \frac{1}{2} m_{y\,i1\,j1} \frac{\partial C_{0ijk}(s)}{\partial J_y} C_{0\,i1\,j1\,k1}(\sigma) \\
b_4 &= \frac{1}{2} m_{y\,ik} \frac{\partial C_{0\,i1\,j1\,k1}(\sigma)}{\partial J_y} C_{0\,ijk}(s)
\end{aligned}
\tag{34}
$$

For a given set of indices (i, j, k), (i1, j1, k1), the eight modes are independent but, among all the combinations of indices, a repetition of the modes may occur. As the coefficients C are the products of independent functions of (β_x, β_y) and (J_x, J_y), the function of (β_x, β_y) in the coefficients b is the same for a given combination of indices. The arguments ψ always appear in cosine functions and one can form the 8-vector

$$
\overline{\Psi^{(2)}} = [\cos \psi_i] \quad i = 1,8
\tag{35}
$$

The functions of the action variables can also be written as 8-vectors whose components are linear combinations of the b coefficients

$$
\begin{aligned}
\overline{A_x^{(2)}} &= (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \\
\overline{A_y^{(2)}} &= (a_1, a_2, a_3, a_4, -a_5, -a_6, -a_7, -a_8)
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
a_1 &= -b_1 + b_2 - b_3 + b_4 \\
a_2 &= b_1 + b_2 + b_3 + b_4 \\
a_3 &= -b_1 + b_2 + b_3 + b_4 \\
a_4 &= b_1 + b_2 - b_3 + b_4 \\
a_5 &= -b_1 + b_2 - b_3 - b_4 \\
a_6 &= b_1 + b_2 + b_3 - b_4 \\
a_7 &= -b_1 + b_2 + b_3 - b_4 \\
a_8 &= b_1 + b_2 - b_3 - b_4
\end{aligned}
\tag{37}
$$

The second-order perturbation of the action variables integrated over a length L have then the compact expression which is the expected generalized form of the expressions derived at the first iteration

$$J_{x2} = \sum_{i=0}^{I(m/2)} \sum_{j=0}^{I(m/2)-i} \sum_{k=0}^{i} m_{xij} \int_0^L ds \sum_{i1=0}^{I(m/2)} \sum_{j1=0}^{I(m/2)-i1} \sum_{k1=0}^{i1} \int_0^s \overline{A_x^{(2)}} . \overline{\Psi^{(2)}} d\sigma$$

$$J_{y2} = \sum_{i=0}^{I(m/2)} \sum_{j=0}^{I(m/2)-i} \sum_{k=0}^{i} m_{yik} \int_0^L ds \sum_{i1=0}^{I(m/2)} \sum_{j1=0}^{I(m/2)-i1} \sum_{k1=0}^{i1} \int_0^s \overline{A_y^{(2)}} . \overline{\Psi^{(2)}} d\sigma \qquad (38)$$

5. ACTION - ANGLE PERTURBATIONS IN A PERIODIC LATTICE

When the particle performs revolutions inside a ring, the solution to the equations of motion must reflect the periodic structure of the magnetic lattice. A ring has an intrinsic n-fold symmetry ($n \geq 1$) and is said to be made of n *superperiods*; a superperiod is an assembly of more elementary repetitive elements: the *periods* are also called the *cells* of the lattice. The C coefficients are the same for all the superperiods and at the position of a given element the linear phase is $\mu+2(n-1)\pi Q$ with n=1, ..., N in the lattice unfolded over N superperiods. The purpose of this section is to show how the full integration over the N superperiods can be limited to an integration over a single superperiod.

5.1 First-order perturbation

As the perturbing fields are zero everywhere but at magnet locations, the following type of substitutions can be performed in the solutions (29)

$$\int_0^s \overline{A^{(1)}} . \overline{\Psi^{(1)}} d\sigma = \sum_{i_p=1}^{p} \int_0^l \overline{A_{i_p}^{(1)}} . \overline{\Psi_{i_p}^{(1)}} d\sigma \qquad (39)$$

where p is the number of magnets. The length l may vary from magnet to magnet and the integral can be calculated exactly. The analytical integration is useful for applications which require accurate results. It is, however, very heavy and it is much simpler to describe the essential features of the particle oscillation using the *thin-lens* approximation which consists of concentrating all the field into a δ-function (Fig.1). The ∫ symbol thus disappears from the formulae provided the rule

$$\int_0^s \overline{A^{(1)}} . \overline{\Psi^{(1)}} d\sigma = \sum_{i_p=1}^{p} \overline{A_{i_p}^{(1)}} . \overline{\Psi_{i_p}^{(1)}} \, l \qquad (40)$$

is applied.

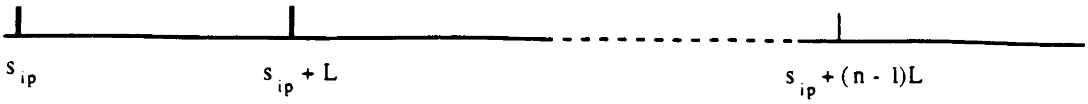


Fig. 1 Unfolded lattice with thin lenses in a first-order perturbation

By inspection of the expression (28), it can be seen that the cumulative effect of a perturbing element i_p over N superperiods is found by evaluating geometric series of the type

$$S = \sum_{n=1}^{N} \exp i\, \overline{m^{\pm}_{ijk}} . (\overline{\mu} + (n-1) 2\pi \overline{Q}) \tag{41}$$

where

$$\overline{Q} = (Q_x , Q_y) \tag{42}$$

The perturbation of the action at the end of the N-th superperiod is thus obtained from the expressions (29) and (30) where the integration is limited to the length L of a superperiod and the vector $\Psi^{(1)}$ is given by

$$\overline{\Psi^{(1)}} = \overline{\Psi^{(1)}_0} + \overline{\Psi^{(1)}_N} \tag{43}$$

with

$$\overline{\Psi^{(1)}_0} = \left(\frac{\cos \overline{m^{+}_{ijk}}.(\overline{\mu} - \pi \overline{Q})}{2 \sin \overline{m^{+}_{ijk}}.\pi \overline{Q}} , \frac{\cos \overline{m^{-}_{ijk}}.(\overline{\mu} - \pi \overline{Q})}{2 \sin \overline{m^{-}_{ijk}}.\pi \overline{Q}} \right) \tag{44}$$

$$\overline{\Psi^{(1)}_N} = \left(\frac{\cos \overline{m^{+}_{ijk}}.(\overline{\mu} + (2N-1)\pi \overline{Q})}{2 \sin \overline{m^{+}_{ijk}}.\pi \overline{Q}} , \frac{\cos \overline{m^{-}_{ijk}}.(\overline{\mu} + (2N-1)\pi \overline{Q})}{2 \sin \overline{m^{-}_{ijk}}.\pi \overline{Q}} \right) \tag{45}$$

The phase components contain a *stationary* term, independent of N, which can also be obtained as the periodic solution of the equations of motion and a *time-dependent* term superimposed on the previous one. Moreover, as one could expect, when the frequency of one of the oscillation modes becomes an integer, a *resonance* condition

$$\overline{m^{\pm}_{ijk}} . \overline{Q} = r \tag{46}$$

is fulfilled and the perturbation theory is no longer valid. This is the famous problem of the *small denominators* which is at the heart of KAM theory [7].

5.2 Second-order perturbation

In the previous section, the action distortion has been calculated at the end of a superperiod by a straight-forward addition of the elementary perturbations. At the second order, it is necessary to evaluate the distortion at every element inside a superperiod (Fig. 2).

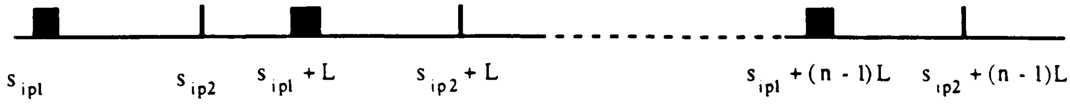


Fig. 2 Unfolded lattice in a second-order perturbation

Let us consider two elements located at s_1 and s_2 with s_2 downstream of s_1. In the double integration of (38), the perturbation at s_1 is calculated first and s_2 plays the role of σ, then the perturbation is calculated at s_2 and s_1 plays the role of σ. It is clear that these two correlations must have a similar structure since the vectors A are the same and the relative phases can only differ by their sign. To calculate the double integral, we shall group the two correlations between elements located at s_1 and s_2 in a single term, the integral over σ is thus extended to the elements downstream to s only so that its lower and upper limits are s and s+L respectively. The*thin-lens* model can be used in *almost* all cases. For correlations between different magnets (*cross-correlation terms)* or between images of a magnet in all the superperiods, one can indeed write

$$\int_0^L \delta(s-s_{i_p})\,ds \int_0^s \overline{A^{(2)}}\,.\,\overline{\Psi^{(2)}}\,\delta(\sigma-\sigma_{i_{pl}})\,d\sigma = \sum_{i_p=1}^{p}\sum_{i_{pl}=1}^{i_p-1} \overline{A^{(2)}}\,.\,\overline{\Psi^{(2)}}\,|^2 \qquad (47)$$

but, for the correlation of a magnet with itself (*self-correlation terms*), the double integral must be calculated differently using an explicit form for $\Psi^{(2)}$.

5.2.1 *Self correlation*

By inspection of the structure of the arguments ψ_i , it turns out that the components of $\Psi^{(2)}$ are

$$\begin{aligned} \Psi_i^{(2)} &= \mathrm{Re}\, e^{i\,[\mu_{ijk}^{\varepsilon_1}(s) + \varepsilon_2 \mu_{1jlkl}^{\varepsilon_1}(\sigma)]} \quad i = 1,2,3,4 \\ \Psi_i^{(2)} &= \mathrm{Re}\, e^{i\,[\mu_{ijk}^{\varepsilon_1}(s) + \varepsilon_2 \mu_{1jlkl}^{-\varepsilon_1}(\sigma)]} \quad i = 5,6,7,8 \end{aligned} \qquad (48)$$

where ε_1 and ε_2 are either +1 or -1. For a single magnet, the simplest approximation which can be used beyond the *thin-lens* model assumes that the length is small enough that β can be considered as constant and therefore μ as a linear function of s according to (3)

$$\begin{aligned} \mu(s) &= \mu(s_{i_p}) + \frac{s}{\beta_{i_p}} \\ \mu(\sigma) &= \mu(s_{i_p}) + \frac{\sigma}{\beta_{i_p}} \end{aligned} \qquad (49)$$

It is then elementary to show that the contribution of the self-correlation term to the second-order perturbation of the action in one superperiod is

$$\frac{1}{2}\left|\overline{A_1^{(2)}} \cdot \overline{\Psi_1^{(2)}}\right|^2$$

whatever ε_1 or ε_2 may be; the vectors are evaluated at the beginning of the element. When the perturbation is summed over N superperiods (Fig. 3), it is useful to define the quantities

$$Q^{\pm}_{ijk} = \overline{m^{\pm}_{ijk}} \cdot \overline{Q} \qquad (50)$$

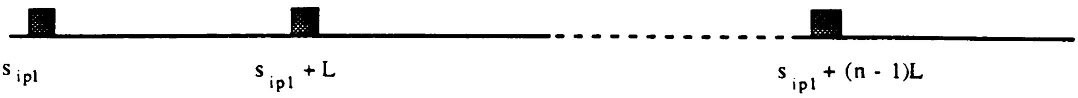


Fig. 3 Cumulative effect of single elements in a second-order perturbation

The contribution from the self-correlation terms can then be written

$$I_1 = \frac{1}{2}\left|\overline{A_1^{(2)}} \cdot \overline{\Psi_1^{(2)}}\right|^2 \sum_{n=1}^{N} \mathrm{Re}\, e^{i\,2\pi\left(Q^{\varepsilon_1}_{ijk} + \varepsilon_2 Q^{\pm\varepsilon_1}_{i1j1k1}\right)(n-1)} \qquad (51)$$

For some combinations of modes that we call *subtractive* and note with the superscript "-", the exponent vanishes if the betatron tunes fulfill the condition

$$Q^{\varepsilon_1}_{ijk} + \varepsilon_2\, Q^{\pm\varepsilon_1}_{i1j1k1} = 0 \qquad (52)$$

which can also be applied to the betatron phases

$$\mu^{\varepsilon_1}_{ijk}(s_{ip}) + \varepsilon_2\, \mu^{\pm\varepsilon_1}_{i1j1k1}(s_{ip}) = 0 \qquad (53)$$

I_1 takes the value

$$I_1^- = \frac{1}{2}\left|\overline{A_1^{(2)-}} \cdot \overline{\Psi_1^{(2)-}}\right|^2 \qquad (54)$$

the components of $\Psi^{(2)-}$ being simply

$$\Psi^{(2)-}_{1i} = N \qquad (55)$$

The other combinations are said to be *additive*, they are noted with the superscript "+" and their contribution is

$$I_1^+ = \frac{1}{2}\overline{A_1^{(2)+}} \cdot \overline{\Psi_1^{(2)+}} |^2 \tag{56}$$

the components of $\Psi^{(2)+}$ being given by

$$\Psi^{(2)+}_{1i} = \Psi^{(2)+}_{0,1i} + \Psi^{(2)+}_{N,1i} \tag{57}$$

$$\Psi^{(2)+}_{0,1i} = \frac{\sin(\pi Q^{\varepsilon_1}_{ijk} - \mu^{\varepsilon_1}_{ijk}(s_{ip}) + \varepsilon_2(\pi Q^{\pm\varepsilon_1}_{i_1j_1k_1} - \mu^{\pm\varepsilon_1}_{i_1j_1k_1}(s_{ip})))}{2\sin\pi(Q^{\varepsilon_1}_{ijk} + \varepsilon_2 Q^{\pm\varepsilon_1}_{i_1j_1k_1})} \tag{58}$$

$$\Psi^{(2)+}_{N,1i} = -\frac{\sin(\pi(2N-1)Q^{\varepsilon_1}_{ijk} + \mu^{\varepsilon_1}_{ijk}(s_{ip}) + \varepsilon_2(\pi(2N-1)Q^{\pm\varepsilon_1}_{i_1j_1k_1} + \mu^{\pm\varepsilon_1}_{i_1j_1k_1}(s_{ip})))}{2\sin\pi(Q^{\varepsilon_1}_{ijk} + \varepsilon_2 Q^{\pm\varepsilon_1}_{i_1j_1k_1})} \tag{59}$$

The expression of I_1 is puzzling because it contains terms which increase with N, the *secular* terms, and this is in contradiction with the principle of energy conservation. We shall see that the correlations between images of a same element in the various superperiods have also secular terms which, fortunately, cancel the previous ones.

5.2.2 Correlations between an element and its images in the superperiods

The distribution of the images is shown in Fig. 4.

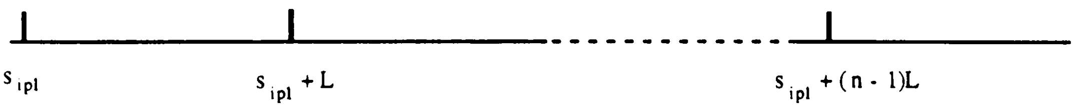


Fig. 4 Images of a single element in the superperiods

We have to consider the new contribution to the action distortion

$$I_2 = \overline{A_1^{(2)}} \cdot \overline{\Psi_2^{(2)}} |^2 \tag{60}$$

where the components of Ψ_2 are

$$\Psi^{(2)}_{2i} = \sum_{n=1}^{N} \mathrm{Re}\, e^{i[\mu^{\varepsilon_1}_{ijk}(s_{ip}) + (n-1)2\pi Q^{\varepsilon_1}_{ijk} + \varepsilon_2 \mu^{\pm\varepsilon_1}_{i_1j_1k_1}(s_{ip})]}[1 + \ldots + e^{i(n-2)\varepsilon_2 2\pi Q^{\pm\varepsilon_1}_{i_1j_1k_1}}] \tag{61}$$

The decomposition made for I_1 can be repeated for I_2 and it turns out that

$$I_2^- = -I_1^- + \frac{1}{2}\overline{A_1^{(2)-}} \cdot \overline{\Psi_2^{(2)-}} |^2 \tag{62}$$

$$I_2^+ = \frac{1}{2}\overline{A_1^{(2)+}} \cdot \overline{\Psi_2^{(2)+}} |^2 \tag{63}$$

with

$$\Psi^{(2)-}_{2i} = \left(\frac{\sin \pi N Q^{\varepsilon_1}_{ijk}}{\sin \pi Q^{\varepsilon_1}_{ijk}}\right)^2 \tag{64}$$

$$\Psi^{(2)+}_{2i} = \Psi^{(2)+}_{0,2i} + \Psi^{(2)+}_{N,2i} \tag{65}$$

$$\Psi^{(2)+}_{0,2i} = -\frac{\cos\left[\mu^{\varepsilon_1}_{ijk} + \varepsilon_2\, \mu^{\pm\varepsilon_1}_{i1j1k1} - \pi\left(Q^{\varepsilon_1}_{ijk} + \varepsilon_2\, Q^{\pm\varepsilon_1}_{i1j1k1}\right)\right]}{2 \sin \pi Q^{\pm\varepsilon_1}_{i1j1k1}\ \sin \pi Q^{\varepsilon_1}_{ijk}} + \frac{\cos\left[\mu^{\varepsilon_1}_{ijk} + \varepsilon_2\, \mu^{\pm\varepsilon_1}_{i1j1k1} - \pi\left(Q^{\varepsilon_1}_{ijk} + 2\varepsilon_2\, Q^{\pm\varepsilon_1}_{i1j1k1}\right)\right]}{2 \sin \pi Q^{\pm\varepsilon_1}_{i1j1k1}\ \sin \pi\left(Q^{\varepsilon_1}_{ijk} + \varepsilon_2\, Q^{\pm\varepsilon_1}_{i1j1k1}\right)} \tag{66}$$

$$\Psi^{(2)+}_{N,2i} = \frac{\cos\left[\mu^{\varepsilon_1}_{ijk} + \varepsilon_2\, \mu^{\pm\varepsilon_1}_{i1j1k1} + \pi\left((2N-1) Q^{\varepsilon_1}_{ijk} + \varepsilon_2\, Q^{\pm\varepsilon_1}_{i1j1k1}\right)\right]}{2 \sin \pi Q^{\pm\varepsilon_1}_{i1j1k1}\ \sin \pi Q^{\varepsilon_1}_{ijk}} - \frac{\cos\left[\mu^{\varepsilon_1}_{ijk} + \varepsilon_2\, \mu^{\pm\varepsilon_1}_{i1j1k1} - \pi\left((2N-1) Q^{\varepsilon_1}_{ijk} + 2\,\varepsilon_2\,(N-1) Q^{\pm\varepsilon_1}_{i1j1k1}\right)\right]}{2 \sin \pi Q^{\pm\varepsilon_1}_{i1j1k1}\ \sin \pi\left(Q^{\varepsilon_1}_{ijk} + \varepsilon_2\, Q^{\pm\varepsilon_1}_{i1j1k1}\right)} \tag{67}$$

For the subtractive modes, the secular terms in (62) cancel those found in the previous section and the phase vector (64) is independent of the position of the element.

5.2.3 *Cross-correlations*

For the interaction between elements located at two different positions (Fig. 5) in the superperiod, the calculation can be conducted in the same way as in the previous case.

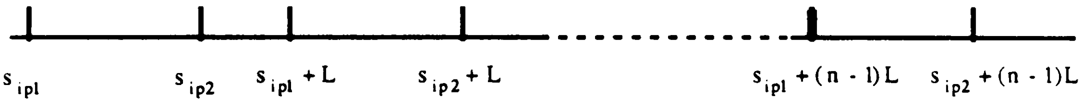


Fig 5 Cross-correlations between distinct elements

The contribution to the action distortion is then

$$I_3 = \overline{A^{(2)}_{12}} \cdot \overline{\Psi^{(2)}_{3}}\,|^2 \tag{68}$$

where the phase vector is of the form

$$\overline{\Psi^{(2)}_{3}} = \overline{\Psi^{(2)}_{12}} + \overline{\Psi^{(2)}_{21}} \tag{69}$$

with the components

$$\Psi^{(2)}_{12i} = \sum_{n=1}^{N} \mathrm{Re}\, e^{i\,[\,\mu^{\varepsilon_1}_{ijk}(s_{ip1}) + (n-1)2\pi Q^{\varepsilon_1}_{ijk} + \varepsilon_2\, \mu^{\pm\varepsilon_1}_{i1j1k1}(s_{ip2})\,]}[\,1 + \ldots + e^{i\,(n-2)\,\varepsilon_2 2\pi\, Q^{\pm\varepsilon_1}_{i1j1k1}}\,] \tag{70}$$

$$\Psi^{(2)}_{21i} = \sum_{n=1}^{N} \mathrm{Re}\, e^{i\,[\,\mu^{\varepsilon_1}_{ijk}(s_{ip2}) + (n-1)2\pi Q^{\varepsilon_1}_{ijk} + \varepsilon_2\, \mu^{\pm\varepsilon_1}_{i1j1k1}(s_{ip1})\,]}[\,1 + \ldots + e^{i\,(n-1)\,\varepsilon_2 2\pi\, Q^{\pm\varepsilon_1}_{i1j1k1}}\,] \tag{71}$$

After summation and addition of the two types of correlations, one finds that the secular terms cancel out and that the residual contribution of the subtractive combinations is

$$I_3^- = \overline{A^{(2)-}_{12}}\,.\,\overline{\Psi^{(2)-}_3}\,|^2 \tag{72}$$

with

$$\Psi^{(2)-}_{3i} = \left(\frac{\sin N\pi Q^{\varepsilon_1}_{ijk}}{\sin \pi Q^{\varepsilon_1}_{ijk}}\right)^2 \cos\left(\mu^{\varepsilon_1}_{ijk}(s_{ip2}) - \mu^{\varepsilon_1}_{ijk}(s_{ip1})\right) \tag{73}$$

It is interesting to note that, thanks to the relation (53), a single set of modes characterized by the indices (i, j, k) is sufficient to calculate this type of cross-correlation modes. For the additive modes, I_3 becomes

$$I_3^+ = \overline{A^{(2)+}_{12}}\,.\,\overline{\Psi^{(2)+}_3}\,|^2 \tag{74}$$

with

$$\Psi^{(2)+}_{3i} = \Psi^{(2)+}_{0,3i} + \Psi^{(2)+}_{N,3i} \tag{75}$$

$$\begin{aligned}
\Psi^{(2)+}_{0,3i} = {} & \frac{1}{4\sin \pi\varepsilon_2 Q^{\pm\varepsilon_1}_{i1j1k1}\ \sin\pi(Q^{\varepsilon_1}_{ijk} + \varepsilon_2 Q^{\pm\varepsilon_1}_{i1j1k1})} \\
& \Big\{ \cos\big(\mu^{\varepsilon_1}_{ijk}(s_{ip1}) + \varepsilon_2\mu^{\pm\varepsilon_1}_{i1j1k1}(s_{ip2}) - \pi\,(Q^{\varepsilon_1}_{ijk} + 2\varepsilon_2 Q^{\pm\varepsilon_1}_{i1j1k1})\big) \\
& + \cos\big(\mu^{\varepsilon_1}_{ijk}(s_{ip2}) + \varepsilon_2\mu^{\pm\varepsilon_1}_{i1j1k1}(s_{ip1}) - \pi\, Q^{\varepsilon_1}_{ijk}\big) \Big\} \\
& - \frac{1}{4\sin \pi\varepsilon_2 Q^{\pm\varepsilon_1}_{i1j1k1}\ \sin\pi\, Q^{\varepsilon_1}_{ijk}} \\
& \Big\{ \cos\big(\mu^{\varepsilon_1}_{ijk}(s_{ip1}) + \varepsilon_2\mu^{\pm\varepsilon_1}_{i1j1k1}(s_{ip2}) - \pi\,(Q^{\varepsilon_1}_{ijk} + \varepsilon_2 Q^{\pm\varepsilon_1}_{i1j1k1})\big) \\
& + \cos\big(\mu^{\varepsilon_1}_{ijk}(s_{ip2}) + \varepsilon_2\mu^{\pm\varepsilon_1}_{i1j1k1}(s_{ip1}) - \pi\,(Q^{\varepsilon_1}_{ijk} + \varepsilon_2 Q^{\pm\varepsilon_1}_{i1j1k1})\big) \Big\}
\end{aligned} \tag{76}$$

$$\Psi^{(2)-}_{N,3i} = - \frac{1}{4 \sin \pi\varepsilon_2 Q^{\pm\varepsilon_1}_{i1j1k1} \sin \pi(Q^{\varepsilon_1}_{ijk} + \varepsilon_2 Q^{\pm\varepsilon_1}_{i1j1k1})}$$

$$\left\{ \cos (\mu^{\varepsilon_1}_{ijk}(s_{ip1}) + \varepsilon_2\mu^{\pm\varepsilon_1}_{i1j1k1}(s_{ip2}) + \pi ((2N-1)Q^{\varepsilon_1}_{ijk} + 2(N-1)\varepsilon_2 Q^{\pm\varepsilon_1}_{i1j1k1})) \right.$$

$$\left. + \cos (\mu^{\varepsilon_1}_{ijk}(s_{ip2}) + \varepsilon_2\mu^{\pm\varepsilon_1}_{i1j1k1}(s_{ip1}) + \pi ((2N-1)Q^{\varepsilon_1}_{ijk} + 2N\varepsilon_2 Q^{\pm\varepsilon_1}_{i1j1k1})) \right\}$$

$$+ \frac{1}{4 \sin \pi\varepsilon_2 Q^{\pm\varepsilon_1}_{i1j1k1} \sin \pi Q^{\varepsilon_1}_{ijk}}$$

$$\left\{ \cos (\mu^{\varepsilon_1}_{ijk}(s_{ip1}) + \varepsilon_2\mu^{\pm\varepsilon_1}_{i1j1k1}(s_{ip2}) + \pi ((2N-1)Q^{\varepsilon_1}_{ijk} - \varepsilon_2 Q^{\pm\varepsilon_1}_{i1j1k1})) \right.$$

$$\left. + \cos (\mu^{\varepsilon_1}_{ijk}(s_{ip2}) + \varepsilon_2\mu^{\pm\varepsilon_1}_{i1j1k1}(s_{ip1}) + \pi ((2N-1)Q^{\varepsilon_1}_{ijk} - \varepsilon_2 Q^{\pm\varepsilon_1}_{i1j1k1})) \right\} \qquad (77)$$

These formulae complete the treatment of second-order perturbations. They are not easy to read but the complication inherent to high-order perturbations is less and less untractable with symbolic codes such as Macsyma, Reduce or *Mathematica* [12] and it is left as an exercise to tabulate the resonances generated by a sextupolar field and their driving terms.

6. COMPENSATION SCHEMES FOR AMPLITUDE DEPENDENT DISTORTIONS

As already mentioned in the introduction, systematic non-linear fields are introduced to act on the chromatic properties of the machine and it is the amplitude dependent distortions tthey produce that we want to correct. Their coupling with the off-momentum particles depends on the distance between the off-momentum closed orbit and the central orbit via a quantity called the *orbit dispersion*. If one cannot design a chromaticity correction system self compensated [8] with respect to the non-linear distortions, and this is often the case save in very large storage rings, the correcting fields are to be placed in dispersionless straight sections. The method which will be described [9] can be applied in any circumstance but, for clarity, we shall assume that momentum and amplitude dependent effects are decoupled.

6.1 Method

There are a few preliminary remarks which are very important in the design of a correction scheme. First, all the distortions depend not only on the initial amplitude but also on the initial phase of the particle oscillation via the betatron phase advance μ (Eq. 14). A correction scheme must obviously be valid for all the particles in the beam. In order for it to be phase independent, the cosine and sine components of the oscillation have to be corrected simultaneously; in other terms, a *vector* correction has to be performed. A simplification occurs when the machine superperiod has a specular symmetry; then, for two symmetric elements, the driving term has the same amplitude and the associated phases are $\mathbf{m}.(-\pi\mathbf{Q}+\mu)$ and $\mathbf{m}.(-\pi\mathbf{Q}+(2\pi\ \mathbf{Q}-\mu))$ and thus opposite so that the sine components vanish. Amplitude independence is obtained by using a correcting field of the same nature as the chromaticity field. Let us note that a correction independent of the initial phase is also independent of the number of turns so that it is sufficient to take the periodic terms into consideration; this means that, when a scatter plot of the action or angle variables is observed at some position around the machine, both the average value and the oscillation of the variable about its average value are reduced.

Another aspect of the correction concerns the observation of the distortion. If the scheme respects the periodicity of the ring, it seems that a single observation point at the end of the superperiod is sufficient provided it is not a node of the oscillation. For a given chromatic field, the indices m_{xij} (Eq. 16) and m_{yik} (Eq. 17) of the oscillation modes are determined. For instance, the modes of a sextupolar field are characterized by the four couples (1,0), (3,0), (1,2), (1,-2). The cosine and sine components of each mode are then calculated to the first order at the observation point for the action distortion using the relations (26), (29), (39) or (40) and (44), they form a known vector **b**. In the case of a sextupole, **b** has 8 or 4 independent components, depending on the superperiod symmetry:

$$\mathbf{b} = \sum_{\substack{\text{chromaticity}\\ \text{sextupoles}}} K'l\sqrt{2J_x\beta_x}\begin{pmatrix} \left(J_x\beta_x - J_y\beta_y\right)\cos\left(\mu_x - \mu_{x0}\right) \\ J_x\beta_x\cos 3\left(\mu_x - \mu_{x0}\right) \\ J_y\beta_y\cos\left(\mu_x - \mu_{x0} + 2\left(\mu_y - \mu_{y0}\right)\right) \\ J_y\beta_y\cos\left(\mu_x - \mu_{x0} - 2\left(\mu_y - \mu_{y0}\right)\right) \end{pmatrix} \qquad (78)$$

The four other components are obtained by replacing cos by sin. The constant factors have been omitted. All the variables are characteristic of the sextupoles with the exception of the phase terms subscripted with 0 which are referred to the observation point.

The correction multipoles, in number n, are located at every possible place. Each corrector is characterized by a vector calculated with the same expressions as those used for the chromaticity multipoles but with a unit strength (K'l=1). Each correction vector is a column in the correction matrix **A**. The correction vector **x** is made of the n unknown correction strengths and the residual vector to be minimized is

$$\mathbf{r} = \mathbf{A}\,\mathbf{x} + \mathbf{b} \qquad (79)$$

A good minimization is obtained with the MICADO program [10] which selects the most efficient correctors in an iterative process, the number of correctors being equal to the order of the iteration. It is theoretically possible to get **r** zero but an exact cancellation of the first-order terms usually leads to a catastrophic over compensation of the non-linear distortions.

If a second-order calculation turns out to be necessary, the relation (79) is still valid if **A** is considered as a funcion of **x**. The correction multipoles selected in the first iteration are maintained but their strengths are re-calculated by adjoining to the first-order expressions the second-order expressions calculated in section 5.2 and minimizing the norm of **r** with respect to its components.

The last step consists of testing the correction scheme using a numerical tracking program which provides scatter plots of the motion and permits the quality of the correction to be assessed. In these programs, the equations of motion are integrated numerically [11] in the non-linear fields and the solution is not limited by the order of a perturbative treatment.

6.2 Applications

The method which has just been outlined was applied to the CERN Antiproton Collector (ACOL) and to the Berkeley Advanced Light Source (ALS). An antiproton beam does not radiate and the goal is to make the beam envelope respect the size defined by the linear emittance as close as possible. In a synchrotron light source, the electrons or the positrons are ultra relativistic and radiate significantly; in the process of photon emission, a particle can reach a very large ampliude, typically more than ten times the standard beam width, and the criterion is rather to get the maximum *dynamic aperture*. The dynamic aperture is defined as the set of all the couples of initial transverse coordinates (x,y) for which the particle oscillation is stable.

6.2.1 *Antiproton Collector*

The ring has a twofold superperodicity and each superperiod has a specular symmetry. A quadrant is represented in Fig. 6. The horizontal and vertical variations of the linear betatron tune with the momentum and the quadratic momentum dependence of the orbit in the long straight sections are controlled by three families of sextupoles (SF1, SF2, SD) in the arcs . The sextupolar field is superimposed on the linear field of the quadrupoles and obtained with a special design of the pole profiles.

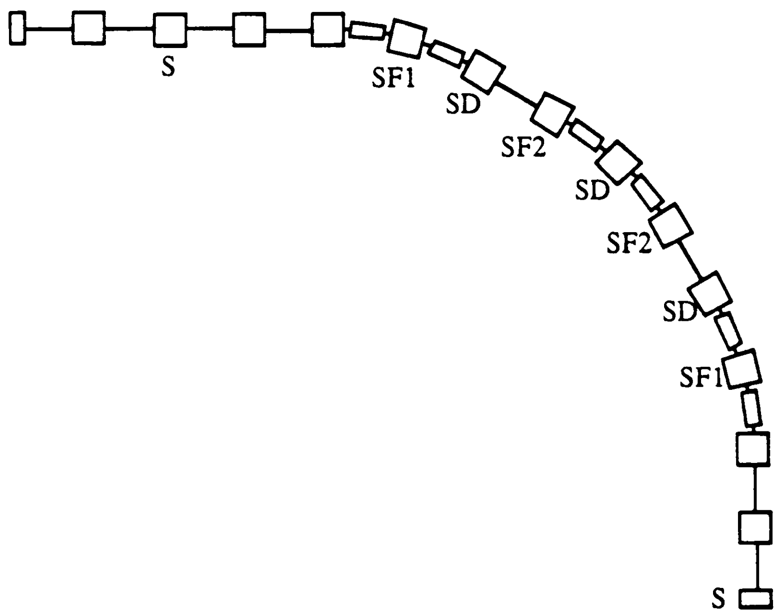
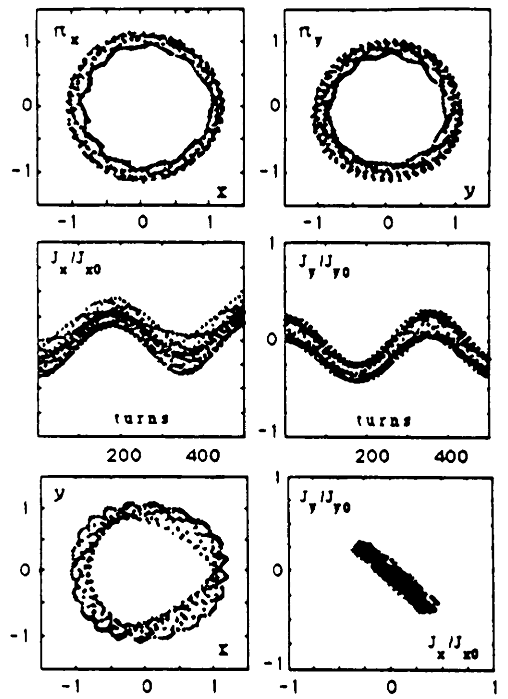


Fig. 6 Quadrant of ACOL

The correction sextupolar fields were first distributed all along the straight sections and computations made with MICADO and tested with a numerical tracking program showed that a sufficient correction (Fig. 7) could be obtained with the sextupoles S produced by pole face windings (Fig. 8).



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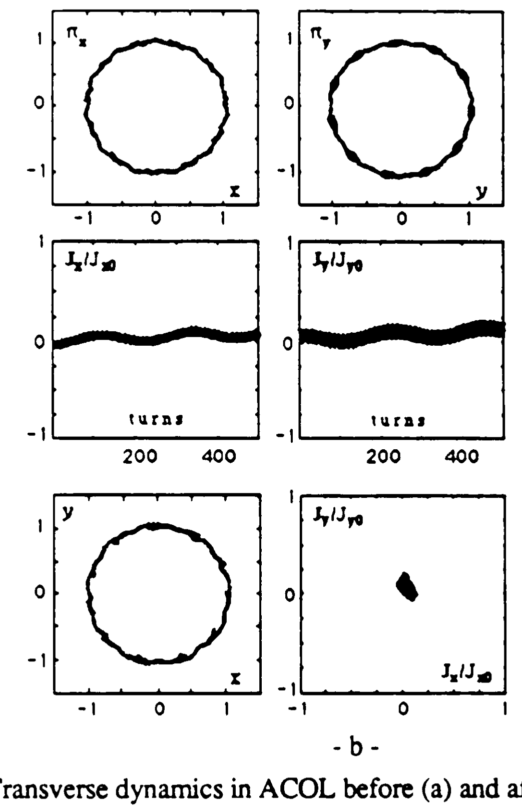


- b -

Fig. 7 Transverse dynamics in ACOL before (a) and after (b) correction

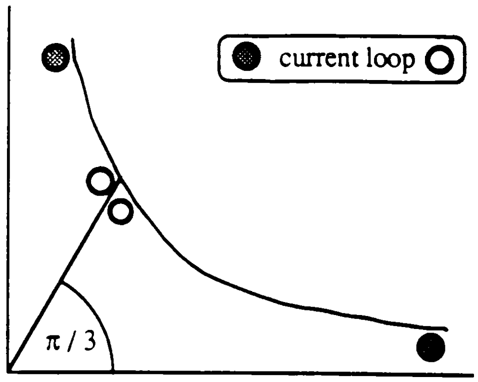


Fig. 8 Configuration of the sextupolar conductors on the pole of a quadrupole

The experimental evidence of the importance of this type of correction should be given by comparing the number of antiprotons injected into the ring without and with sextupolar correction but this has not been possible until now because the antiproton beam had insufficient emittance. However, two measurements prove the validity of the method:

i) the non-linear coupling observed when the horizontal and vertical betatron tunes are equal almost disappears after correction (Fig. 9);

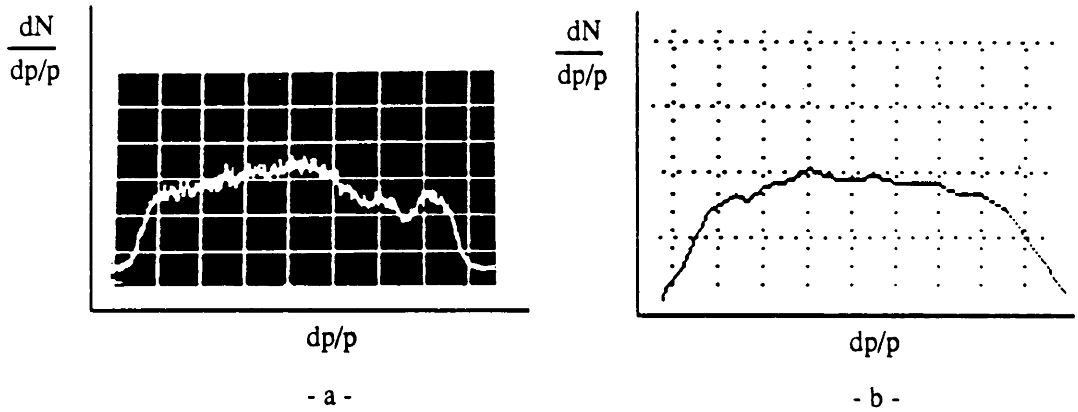


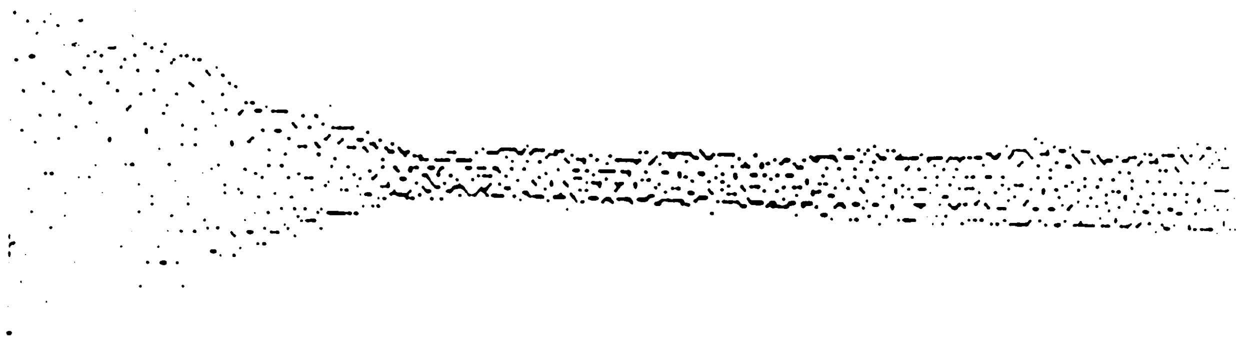
Fig. 9 Beam density distribution before (a) and after (b) sextupolar correction

ii) the motion of the bunch center of gravity resulting from a large injection error both in the horizontal and vertical planes (see F. Willeke, these proceedings) is much more regular after correction (Fig. 10).

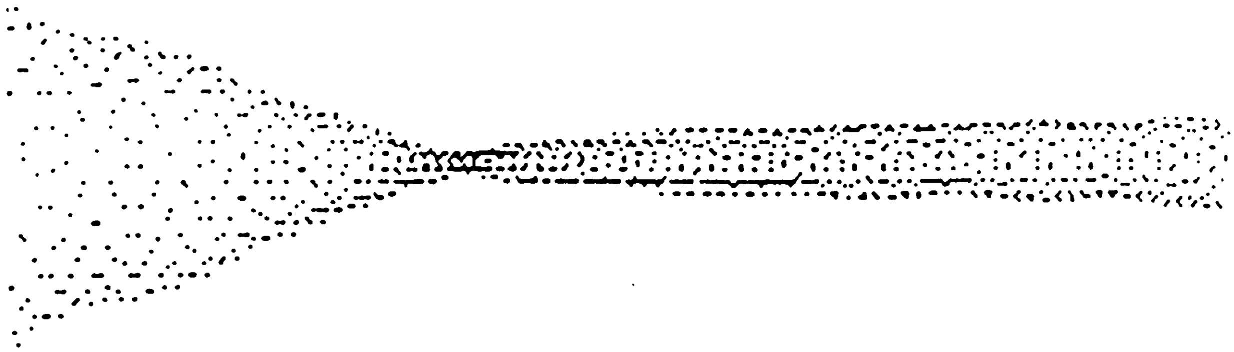
6.2.2 *Advanced Light Source*

The Berkeley Advanced Light Source (ALS)has a 12-fold superperiodicity and each superperiod (Fig. 11) has a mirror symmetry. The computational techniques used for ACOL have been resumed for this machine.

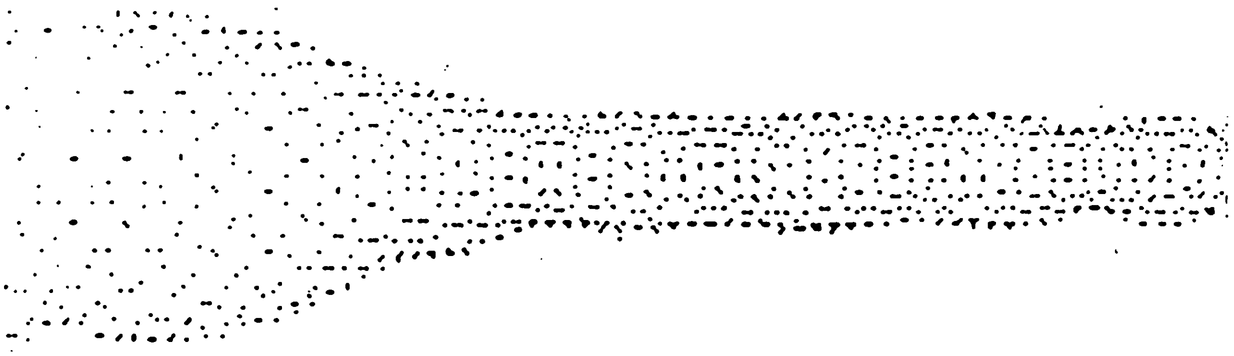
The main difference between the two lattices lies in the *triple bend achromat* which makes tiny orbit dispersion and leads to outstandingly high strengths for the chromaticity sextupoles: $K'l_F = 11.4\ m^{-2}$, $K'l_D = -8.6\ m^{-2}$.



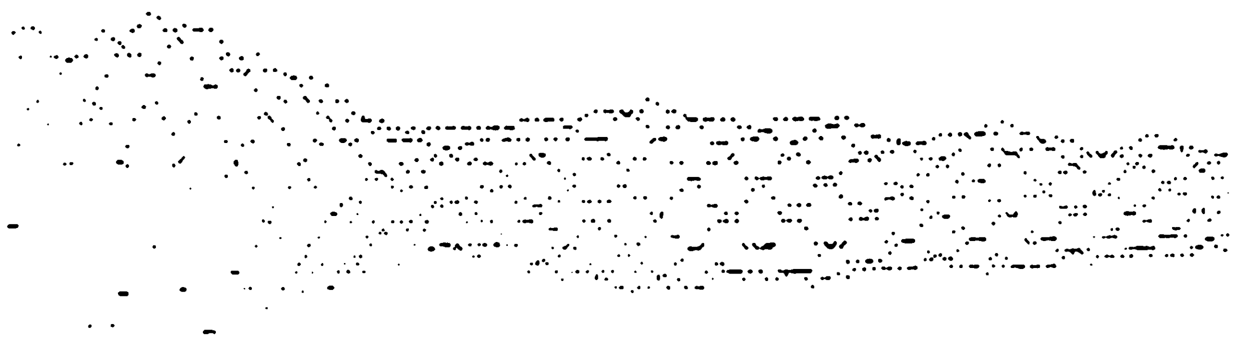
- a - horizontal motion before correction



- b - vertical motion before correction



- c - horizontal motion after correction



- d - vertical motion after correction

Fig. 10 Motion of the bunch center of gravity

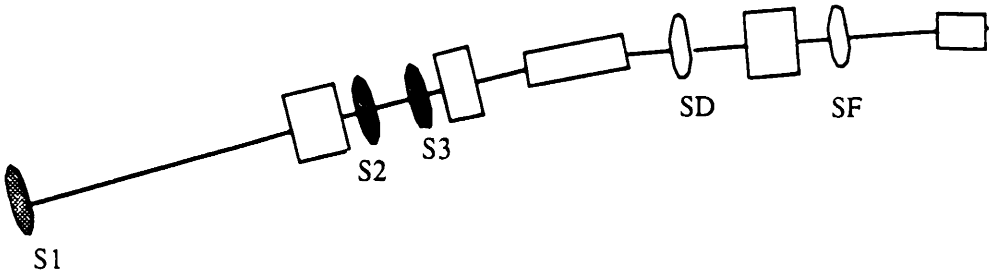


Fig. 11 ALS half superperiod

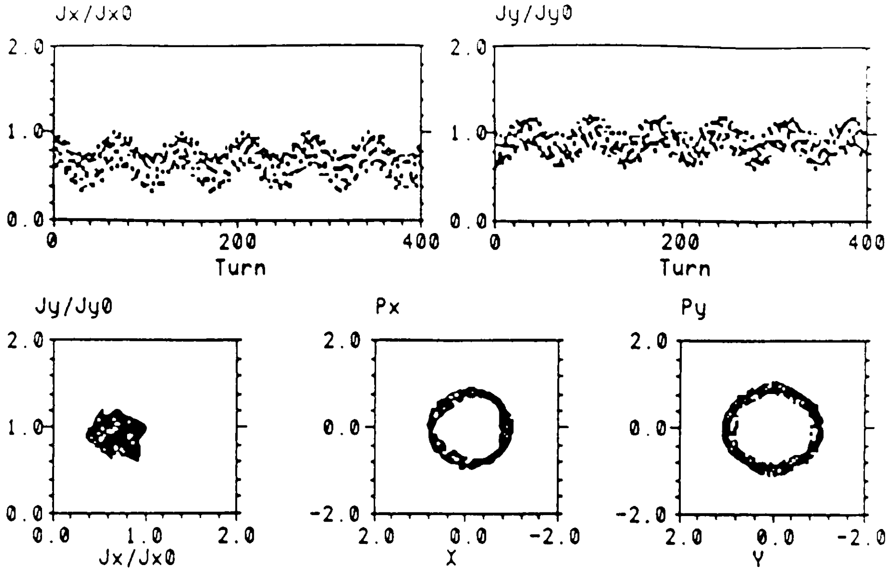
Two schemes were defined: one with a single weak sextupole ($K'l_2 = .87\ m^{-2}$), and another one with three stronger sextupoles ($K'l_1 = -4.49\ m^{-2}$, $K'l_2 = 5.96\ m^{-2}$, $K'l_3 = -13.2\ m^{-2}$). From the inspection of the scatter plots (Fig. 12), it seems that the last scheme (case c) should give the best results. As a matter of fact, the improvement of the dynamic aperture (Fig. 13), if there is any, with respect to case b does not justify the complication of the three sextupole scheme. Each couple (x, y) which defines the dynamic aperture corresponds to the first unstable particle of a sample of 25 particles, all with the same initial amplitude but with different initial phases. To illustrate the importance of that definition, Fig. 14 shows the dynamic aperture which would be deduced from a single particle (scattered points) as compared to the one which results from a multiple particle tracking (connected points). As soon as a particle is near its limit of stability, the first-order perturbation theory is no longer sufficient and future schemes will be based on formulae expanded to the second order.

7. Conclusion

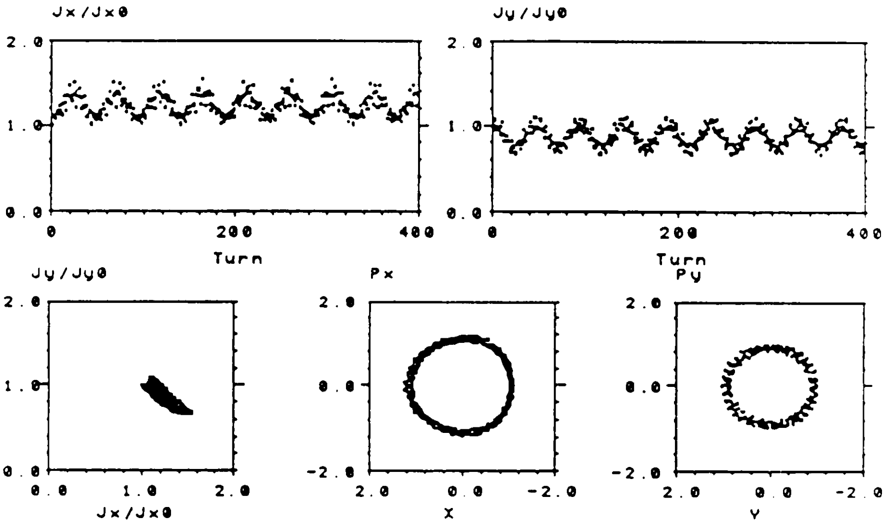
A theory of the non-linear betatron oscillation has been presented based on a solution of the action-angle equations of motion expanded to the second order. Emphasis has been put on the action variable because it is directly related to the beam size. The intricacy of this type of calculation is greatly alleviated by the use of symbolic programs. The insight given by analytical calculations is applied to the design of correction schemes. Until now, these schemes have been derived using the first-order formulae only and they provide substantial improvements of the particle transverse dynamics but, when the criterion of the dynamical aperture is used, it becomes necessary to resort to second-order expressions. A correction scheme has been tested experimentally and it turns out that solutions made of a single family of correctors, and therefore very easy to implement in a real machine, may lead to substantial improvements in the use of the machine aperture.

Acknowledgements

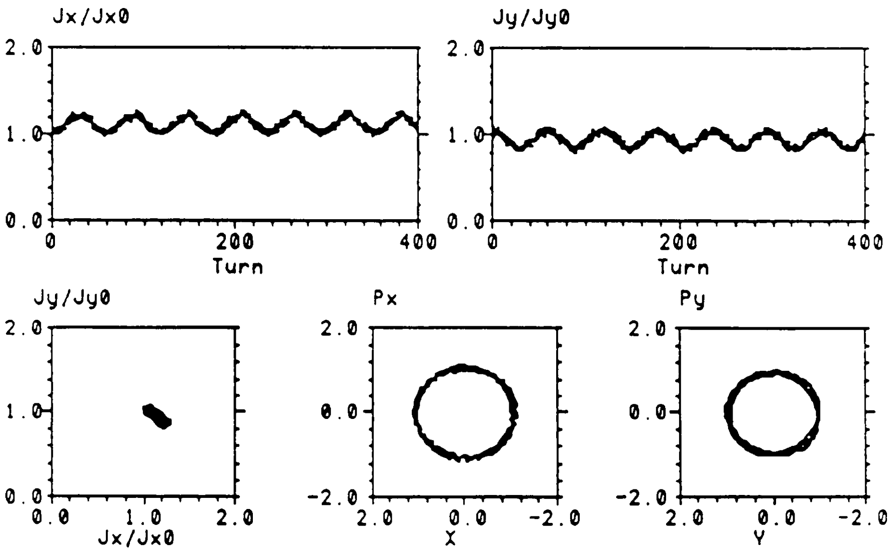
The theory has been elaborated with J. Bengtsson. Numerical tracking for ALS is due to H. Nishimura. H.H. Umstaetter has designed the ACOL sextupolar pole face windings. ACOL experimental results are due to P. Krejcik. Discussions with M. Chanel and J. Bengtsson have been very stimulating thanks to the experience they have developed in the operation of the LEAR machine at CERN.



- a -



- b -



- c -

Fig. 12 Transverse dynamics in ALS for the bare machine (a), a one- (b) and a three- (c) sextupole correction scheme

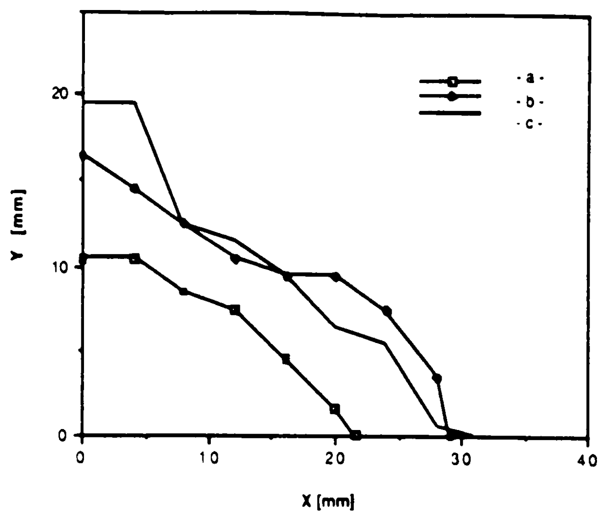


Fig. 13 Dynamic apertures for the bare machine (a) and two correction schemes (b,c).

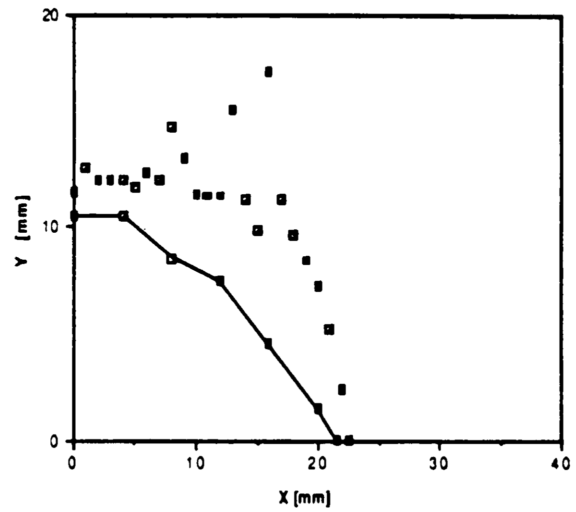


Fig. 14 Comparison of the dynamic apertures deduced from a single (scattered points) and a multiple tracking (solid line)

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