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CHROMATICITY AND LOW β INSERTIONS IN PARTICLE COLLIDERS

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ABSTRACT

Besides its energy range, the main characteristic of a particle collider is its luminosity. A very important piece of equipment to achieve high luminosities is a low ß insertion, a device which focuses the beam down to a tiny spot size. The focusing limits are not only technological, they are also associated with the aberrations of such systems. This paper deals with the chromatic aberrations. In a first part, the treatment of the focusing errors is reviewed and a formalism using symbolic computation is set up so that high order perturbations can be evaluated although results are limited to the second order betatron tune shift and to its interpretation. In a second part, a completely analytical theory of the final doublet of a low ß insertion is given for a special model of flat beam focusing.

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■ 1 Introduction

A very basic parameter in a particle accelerator is the betatron tune. This tune depends on the particle oscillation amplitude and on the particle momentum. The momentum dependence is called the *chromaticity* and is the object of this paper. The chromaticity is one aspect of the more general *chromatic aberrations* which affect the shape of the beam envelope and we shall show that a detailed insight into the chromaticity problems cannot be disjoint from the full study of the chromatic aberrations. The most important reason why the chromaticity correction has been so widely studied is related to the other class of phenomena, the *geometric aberrations*, which are characteristic of the betatron oscillation amplitude: the tune spread generated by the chromaticity forces indeed that particles which have a tune resonant with some external field error structure to blow up in amplitude with all the unwanted consequences one can imagine such as loss of intensity, radiation background, poor life-time, etc.

The mathematical object which is the core of particle optics studies is the transfer matrix over one period of the machine. The matrix formalism implies a linear treatment of the betatron oscillations and the geometric aberrations are obviously left aside. In the general case, the matrix connects 4-vectors between the input to and the output from the period. These 4-vectors are made of the horizontal and vertical components of the particle position and momentum. If there is no coupling between the horizontal and vertical oscillations, the matrix can be split into two 2*2 sub-matrices, one per plane of oscillation. The betatron tune is deduced from the trace of the matrix and the element (1,2) gives the dimension of the beam envelope through the so-called ß-function.

For a monochromatic beam, the tune has a well defined value. In a real beam where each particle has its own momentum distinct from the momentum of the other particles, the focussing is not uniform for all the beam particles. In most modern machines the tune is a complicated function of the momentum and it must be calculated at least in the frame of a second order perturbation theory. It turns out that the second-order tune-shift is a simple function of the first-order tune-shift and of the first-order ß-distortion. Whereas the betatron tune is a scalar quantity, the ß- distortion is vector-like because it concerns not only the ß-function itself but also its derivative with respect to the longitudinal coordinate of the motion. Hand calculations involved in perturbation theories are usually laborious, I shall therefore resort to symbolic computation to maintain the emphasis on physics without bypassing the necessary algebra.

■ 2 Basic tools

We assume that the theory of the strong focusing in its phase-amplitude formalism is known [1]. For a given momentum, there exists in a circular machine a special trajectory which closes itself after one turn: the *closed orbit* or, for brevity, the *orbit* determined by the configuration of the bending fields. A particle performs about its orbit a *betatron oscillation* under the effect of the quadrupole restoring forces. The betatron oscillation is defined at a point of curvilinear coordinate s along the orbit by the amplitude function ß, its longitudinal derivative $d\beta/ds = -2\alpha$ and the betatron phase μ. These quantities are different for each plane. ß and α satisfy periodic boundary conditions. The betatron phase advance over one turn μ_0 is 2π times the *betatron tune* Q. The phases are defined with respect to an arbitrary origin and the observation point is taken as the boundary for a turn of machine.

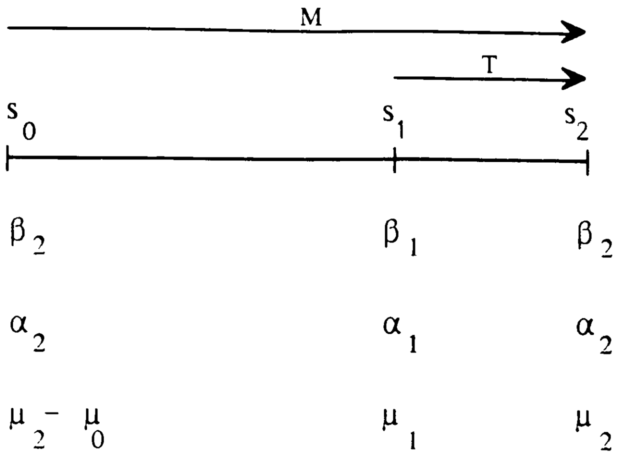

Fig.1 Unfolded machine with an error of infinitesimal length at s_1

The transfer matrix M over one turn and the transfer matrix T between two points are coded symbolically [2] and the notation is self-explanatory at the exception of the variable "a" which stands for "α".

```
                    (*    Ground state matrices *)

(* Transfer matrix between s1 and s2 *)

T1={{Sqrt[ß2/ß1]               , 0                 },
    {(a1-a2)/Sqrt[ß2*ß1]       ,Sqrt[ß1/ß2]        }}
T2={{a1 Sqrt[ß2/ß1]            ,Sqrt[ß2*ß1]        },
    {-(1+a1*a2)/Sqrt[ß2*ß1],-a2 Sqrt[ß1/ß2]}}
T =T1 Cos[μ2-μ1]+T2 Sin[μ2-μ1]

(* Transfer matrix over the period ending at s2 *)

M1={{1,0},
    {0,1}}
M2={{a2           , ß2},
    {-(1+a2^2)/ß2,-a2}}
M =M1 Cos[μ0]    +M2 Sin[μ0];
```

■ 3 First order tune and ß perturbation

The perturbation of the machine ground state is due to focusing errors that we suppose here to have a chromatic origin but the treatment is equally valid for any type of error. A quadrupole is characterized by a focusing strength

$$k = \frac{e}{p} G$$

where e is the particle charge, p the particle momentum and G the gradient of the quadrupole field. For another particle with a relative momentum error Δp/p, the change in focusing strength is

$$dk = - k \frac{dp}{p}$$

The matrix Δ represents the perturbation of a thin focusing quadrupole of length l due to a momentum error. The chromatic corrections are performed by sextupoles located in regions where the orbit position depends on the momentum via the *orbit dispersion* D so that the linear focusing of the sextupoles can compensate the linear chromatic errors of the quadrupoles; in that case, the error is

$$dk = \frac{e}{p} G' D \frac{dp}{p}$$

where G' is the transverse derivative of the sextupole gradient. The coding is restricted to the first type of error.The perturbed matrix ΔM over one turn where the transfer from s0 to s1 is defined by T^-1.M gives the tune-shift ΔQ, the relative ß error Δß/ß (dß) and the α error Δα (da2) via the trace (Tr), the element ΔM_{12} (ΔM[[1,2]]) and the difference of the diagonal elements respectively. The formalism

```
argument//function
```

is equivalent to

```
function[argument]
```

but is easier to manipulate when several operations on the same argument are to be concatenated.

```
            (*    First order perturbation matrices *)

(*  Infinitesimal perturbation matrix at s1 *)

Δ ={{0       ,0},
   {k dp 1,0}}

(*  Inverse transfer matrix from s1 to s2 *)

Tinv=Inverse[T1] Cos[μ2-μ1]+Inverse[T2] Sin[μ2-μ1]

(*  Perturbed period matrix *)

ΔM=T.Δ.Tinv.M;


             (*    Package of trigonometry rules *)

<<Trigonometry.m
```

```
            (*    First order tune-shift *)

(* Perturbed trace of the period matrix *)

Tr=Sum[ΔM[[i,i]],{i,2}]//Expand//Factor//TrigCanonical

(* Tune-shift *)

ΔQ=-Tr/(4 Pi Sin[μ0])

-(dp k 1 ß1)
------------
    4 Pi

            (*    First order ß perturbation *)

dß2=(ΔM[[1,2]]/ß2-2 Pi ΔQ Cos[μ0])/Sin[μ0]//Factor

(* Rules needed to linearize trigonometric functions *)

EIFactor1=
    p_Rational*E^(Complex[0,m_] a_+Complex[0,n_] b_)->
    p E^(I (m a+ n b))
EIFactor2=
    Complex[0,p_Rational]*
               E^(Complex[0,l_] a_+Complex[0,m_] b_)->
    (I p) E^(I (l a+m b))
TrigLin[e_]:=(e//TrigToComplex//Expand)/.EIFactor2/.
            EIFactor1//ComplexToTrig//Expand

(* Linearization of the trigonometric functions in dß/ß *)

dß2[[7]]=TrigLin[dß2[[7]]]

(* Final expression of dß/ß *)

dß2

dp k 1 ß1 Cos[μ0 + 2 (μ1 - μ2)]
-------------------------------
           2 Sin[μ0]
```

```
(*    First order α perturbation *)

da2=
  (ΔM[[1,1]]-ΔM[[2,2]]-a2 Cos[μ0] 4 Pi ΔQ)/(2 Sin[μ0])//
  Factor

(* Linearization of the trigonometric functions in dß/ß *)

da2[[7]]=TrigLin[da2[[7]]]

(* Final expression of dα *)

da2

-(dp k 1 ß1 (-(a2 Cos[μ0 + 2 (μ1 - μ2)]) +

      Sin[μ0 + 2 (μ1 - μ2)])) / (2 Sin[μ0])
```

One notes that the beam envelope perturbation is fully described by $(\Delta\beta/\beta, \Delta\alpha)$ or a couple of linear combinations of these variables. The combination $(\Delta\beta/\beta, \Delta\alpha - \alpha\,\Delta\beta/\beta)$ is especially interesting since its norm is constant and proportional to the tune-shift:

$$\left(\frac{\Delta\beta}{\beta}\right)^2 + \left(\Delta\alpha - \alpha\,\frac{\Delta\beta}{\beta}\right)^2 = \left(\frac{2\pi\,\Delta Q}{\sin\mu_0}\right)^2$$

The vector $(\Delta\beta/\beta, \Delta\alpha - \alpha\,\Delta\beta/\beta)$ was first introduced [3] in the context of the design of special devices called *low-β insertions* and is sometimes called W. A low-ß insertion focuses a particle beam to a tiny spot and must be matched to the cells of the lattice; more precisely, the (β, α) vector should be the same at the cell-insertion interface whether it is viewed from the cell or from the insertion. If this is not the case, an error propagates itself through the whole lattice and the beam is blown-up. The norm which has just been defined is a good measure of this error. As a matter of fact, even if the matching is exact for a given orbit, it cannot be so for all the orbits precisely because of the finite beam momentum spread to which unavoidable chromatic errors are associated; then we are back to the evaluation of these errors using the same formalism. Until now, we have treated linear (or first order) perturbations and the total error on the chromaticity or on the β-perturbation in a machine of length L is just the superposition of all the individual errors.

$$\Delta Q = \frac{1}{4\pi}\int_0^L k\,\beta\,ds$$

$$\frac{\Delta\beta_2}{\beta_2} = \frac{\Delta p/p}{2\sin\mu_0}\int_{s_2}^{s_2+L} k(s_1)\,\beta(s_1)\cos\left(\mu_0 + 2\,(\mu(s_1) - \mu(s_2))\right) ds_1$$

$$\Delta\alpha_2 - \alpha_2\,\frac{\Delta\beta_2}{\beta_2} = \frac{-\,\Delta p/p}{2\sin\mu_0}\int_{s_2}^{s_2+L} k(s_1)\,\beta(s_1)\cos\left(\mu_0 + 2\,(\mu(s_1) - \mu(s_2))\right) ds_1$$

Numerical simulations of machines equipped with low-ß insertions show very non linear dependences of Q or ß on the momentum. For this reason, we shall push the perturbation treatment to the second order.

■ 4 Second order tune-shift

The simplest contibution to the second order tune-shift comes from the expansion of the denominator of the focusing strength and gives -(dp/p) ΔQ. Another simple contribution is due to octupoles which play at the second order the role played by sextupoles at the first order, their effect is given by

$$dk = \frac{1}{2}\frac{e}{p}G'' \left(D\frac{\Delta p}{p}\right)^2$$

where G" is the second derivative of the field gradient.

More complicated are the terms associated with the correlations between first order errors. For this purpose, a second source of perturbation has to be explicitly introduced (Fig. 2). The setting-up of the calculation is the same as in the first order case, the only variant lies in the procedure of simplification of the trigonometric functions. The present treatment is limited to the tune-shift but it could be extended to ß and α perturbtions if they were needed since the one turn perturbed matrix is evaluated.

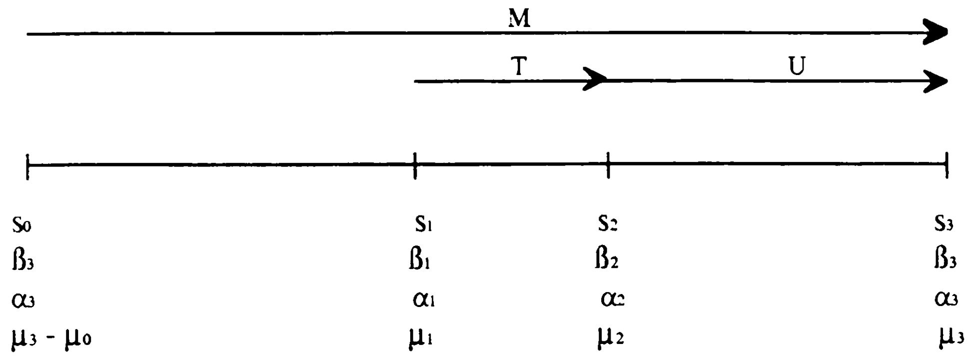

Fig.2 Unfolded machine with two errors of infinitesimal length at s1 and s2

```
            (*    Second order perturbation matrices *)

(*  Infinitesimal perturbation matrix at s1 *)

Δ1 ={{0         ,0},
     {k1 dp 11,0}}

(*  Infinitesimal perturbation matrix at s2 *)

Δ2 ={{0         ,0},
     {k2 dp 12,0}}

(*  Direct and inverse transfer matrices from s2 to s3 *)

U1={{Sqrt[ß3/ß2]              , 0              },
    {(a2-a3)/Sqrt[ß3*ß2]     ,Sqrt[ß2/ß3]     }}
U2={{a2 Sqrt[ß3/ß2]          ,Sqrt[ß3*ß2]     },
    {-(1+a2*a3)/Sqrt[ß3*ß2],-a3 Sqrt[ß2/ß3]}}
U =U1 Cos[μ3-μ2]+U2 Sin[μ3-μ2]
Uinv=Inverse[U1] Cos[μ3-μ2]+Inverse[U2] Sin[μ3-μ2]

(*  Perturbed period matrix *)

M1={{1,0},
    {0,1}}
M2={{a3              , ß3},
    {-(1+a3^2)/ß3,-a3}}
M =M1 Cos[μ0]    +M2 Sin[μ0]
ΔΔM=U.Δ2.T.Δ1.Tinv.Uinv.M;


                 (*    Second order tune-shift *)

(*  Perturbed trace of the period matrix *)

Tr=Sum[ΔΔM[[i,i]],{i,2}]//Together//Factor//TrigCanonical
Tr[[9]]=Tr[[9]]//TrigLin
Tr=Tr//TrigLin//Simplify//TrigCanonical//Factor

(*  Tune-shift *)

ΔΔQ=-Tr/(4 Pi Sin[μ0])

  2
dp  k1 k2 11 12 ß1 ß2 (Cos[μ0] - Cos[μ0 + 2 μ1 - 2 μ2])
--------------------------------------------------------
                     8 Pi Sin[μ0]
```

The tune-shift is made of two terms: one is purely scalar, the other one depends on the phase difference between the two errors. The second term has a simple and very useful interpretation, it is just proportional to the ß perturbation at s_2 created by the error located at s_1. The expression of the global second order tune shift contains a double integral which describes the correlation between the individual errors:

$$\Delta_2 Q = -\frac{dp}{p}\Delta Q + \frac{2\pi}{\tan\mu_0}(\Delta Q)^2 - \left(\frac{dp}{p}\right)^2 \int_0^L dQ(s_2) \int_{s_2}^{s_2+L} \frac{d\beta_2}{\beta_2}(s_1)$$

In practice, the double integral is the dominant term in the particle colliders where the interaction region is equipped with a *low-β insertion* that we have already met in section 3 and that we shall discuss further in the next section.

■ 5 Low ß insertion

The luminosity of a particle collider is inversely proportional to the cross section area of the beam at the ineraction point (IP). To get high luminosities, the first measure consists of removing the contribution of the momentum spread to the beam size by suppressing the orbit dispersion. All the chromaticity which is generated in this region has therefore to be corrected elsewhere in the lattice and the sextupoles have then naturally large focusing strengths which are the sources of harmful non linearities. For this reason, all the ingenuity deployed in the various methods of chromaticity corrections [4] is dedicated to reducing the sextupole correlations as much as possible, this is true for chromatic and geometric aberrations as well.

Another aspect of this problem is a careful design of the low ß insertions. As surprising as it may seem, there is little analytical theory for these types of structures, their calculation results from the designer's expertise and from numerical optimisations which do not take chromatic and, a fortiori, amplitude dependent errors into account. A breakthrough has been accomplished with the use of *telescopes* [5,6] which give a rational approach to the problem of focusing round beams (*symmetric* focusing). Here, I shall try to sketch the elements of a theory of low-ß insertion for a flat beam (*asymmetric* focusing).

The simplest low-ß structure is shown in Fig.3 where the quadrupoles are thin lenses. The purpose of such a structure is to match the two sets of known ß-values at the ends of the insertion. A parameter that is also known is the distance l_1 because it is determined by the room needed for the detection of the particles produced at the interaction point. The problem consists of finding the strengths f_1 and f_2 of the lenses and the lengths l_1 and l_2 of the straight sections. We shall impose α to be zero at the IP, a usual condition which simplifies the calculation without altering the generality of the problem.

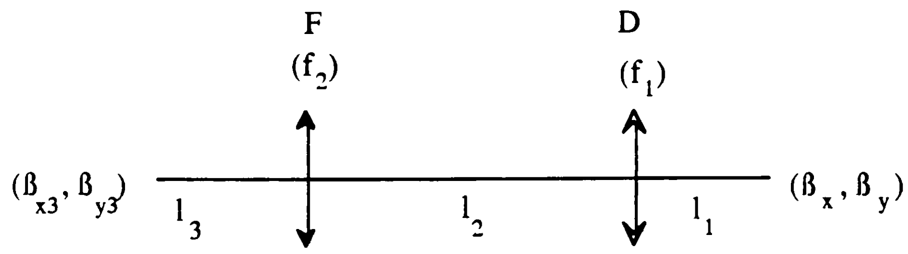

Fig.3 Final doublet of a low ß insertion

■ 5.1 Transfer of the ß function

The betatron functions are gathered in the matrix

$$\sigma = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \frac{1+\alpha^2}{\beta} \end{pmatrix}$$

which is transformed in a magnetic channel of matrix A according to the relation

$$\sigma_2 = A\, \sigma_1\, A$$

The transfer matrices are defined for straight sections (strt) of length l and for focusing (qF) and defocusing (qD) quadrupoles of focal length f.

```
(*  σ matrix *)

sigma[ß_,alpha_]={{ ß      , -alpha        },
                  {-alpha, (1+alpha^2)/ß}}

(*  2*2 transfer matrices *)

qF[f_]    ={{ 1   ,0},     (*      focusing quadrupole *)
            {-1/f,1}}

qD[f_]    ={{ 1   ,0},     (*      defocusing quadrupole *)
            { 1/f,1}}

strt[l_]={{ 1   ,l},     (*      straight section *)
          { 0   ,1}}

(*  transfer of σ-matrix *)

sigmaTransfer[s_,m_]:=m.s.Transpose[m];
```

■ 5.2 Calculation method

A natural pocedure to determine the value of (f_1,l_2,f_2,l_3) would be to trace the horizontal and vertical σ matrices from the interaction point to the insertion input and to equate the ß and α functions after tracing to their values at the entrance to the insertion. Unfortunately, and this is the reason why the theory of low ß insertion does not exist, the system of four equations obtained this way leads to equations of order higher than 4 after elimination of the variables and cannot be solved analytically. We shall therefore impose intermediate conditions so that (f_1,l_2) and (f_2,l_3) will be the solutions of two independent systems of two equations:

$$\beta_{y2} = \beta_{y3}$$

$$\beta_{x2} = \beta_{y1}$$

These conditions are justified by the asymmetry of the final focus and the matching of the insertion to a FODO cell. In addition, β_x is assumed tobe much larger than β_y,which is classical in this type of calculation. As a consequence, the expansion of the solutions will be limited to the term in $\sqrt{\beta_y}$.

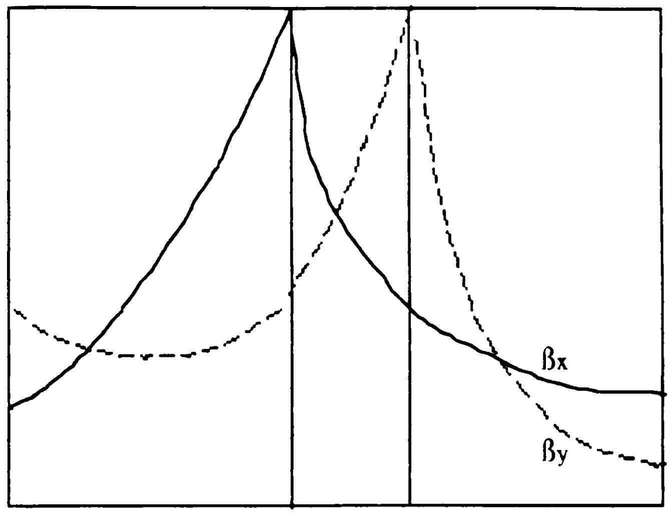

Fig.4 Sketch of ß functions in the final focus

This model is close to the design of a low ß insertion recently proposed for a B-Factory [7].

■ 5.3 Downstream parameters

We define the first system of two equations with (f_1,l_2) as unknowns.

```
(* first equation: vertical transfer from IP to F1 input *)

s=sigma[sqßy^2,0]
t=strt[l1]
s1d=Map[Normal[Series[#,{sqßy,0,0}]]&,
        sigmaTransfer[s,t],
        {2}]
t=strt[l2].qD[-f1]
ßv2d=sigmaTransfer[s1d,t][[1,1]]/.l2/f1->u1//Factor

                   2
(-l1 - l2 + l1 u1)
-------------------
            2
        sqßy
```

At F1 input, ßy is a perfect square and the first equation is of first degree in l_2 and the reduced variable l_2/f_1 (u1):

$$l_1 \frac{l_2}{f_1} - l_2 - l_1 - \beta_y \sqrt{m_y} = 0$$

The sign ambiguity has been solved by guessing that f₁ would be small enough to make the first term dominant and the ß magnification mᵥ has been introduced.

```
eq1=Sqrt[ßv2d sqßy^2]-Sqrt[my] sqßy^2;
```

The second equation is obtained by tracing the horizontal ß function and is of second degree with respect to the variables. The solutions are meaningful if they are positive.

```
(* second equation: horizontal transfer from IP to F1 input *)

t=strt[l2].qD[f1].strt[l1]
s=sigma[ßx,0]
ßh2d=sigmaTransfer[s,t][[1,1]]/.l2/f1->u1
ßv1d=s1d[[1,1]]
eq2=Collect[Numerator[Together[ßh2d-ßv1d]],{l2,u1}]
eq2[[7]]=Factor[eq2[[7]]]
eq2[[8]]=Factor[eq2[[8]]]
eq2

  2      2                    2       2       2                    2
l1   sqßy   + 2 l1 l2 sqßy   + l2   sqßy   + 2 l1 l2 sqßy   u1 -

    2              2    2              2          2       2
  l1   ßx + sqßy   ßx   + 2 sqßy   u1 (l1   + ßx ) +

       2    2      2      2
  sqßy   u1   (l1   + ßx )


(* determination of l2 *)

l2=l2/.Solve[{eq1==0,eq2==0},{l2,u1}][[1,1]]//
       Simplify
l2=Normal[Series[l2,{sqßy,0,2}]]
l2=l2-l2[[1]]-l2[[2]]+((l2[[1]]+l2[[2]])//Factor)
l2[[3]]=l2[[3]]//Factor
l2[[4]]=l2[[4]]//Factor
l2

     2          3/2                      2       2
-2 l1   sqßy ßx        2 l1 (2 l1   + ßx )
---------------- - ------------------ -
     2      2 3/2                 2       2
(4 l1   + ßx )              4 l1   + ßx

                     2          2       2                2
  Sqrt[my] sqßy   (2 l1   + ßx )           l1   Sqrt[ßx]
  --------------------------- + ---------------------
                2      2                            2       2
          4 l1   + ßx                sqßy Sqrt[4 l1   + ßx ]
```

```
(* determination of f1 *)

f11=Normal[Series[12/u1/.Solve[eq1==0,u1],{sqßy,0,2}]]
Do[f11[[1,i]]=f11[[1,i]]//Factor,{i,3}]
f1=f11[[1]]

          2                       2      2
     sqßy  ßx    sqßy Sqrt[4 l1    + ßx  ]
l1 - -------- - ---------------------
        l1               Sqrt[ßx]
```

The output of the calculation is given in alphabetic order. let us re-write it as ordered expansions of $\sqrt{\beta_y}$:

$$l_2 = \sqrt{\frac{\beta_x}{4\beta_x^2 + l_1^2}}\,\frac{l_1^2}{\sqrt{\beta_y}} - \frac{2\,l_1\left(\beta_x^2 + 2\,l_1^2\right)}{\beta_x^2 + 4\,l_1^2} - 2\,l_1^2\left(\frac{\beta_x}{\beta_x^2 + 4\,l_1^2}\right)^{3/2}\sqrt{\beta_y} - \frac{\sqrt{m_y}\left(\beta_x^2 + 2\,l_1^2\right)}{\beta_x^2 + 4\,l_1^2}$$

$$f_1 = l_1 - \sqrt{\frac{4\,l_1^2 + \beta_x^2}{\beta_x}}\,\sqrt{\beta_y} - \frac{\beta_x}{l_1}\,\beta_y$$

Since m_y is of order $1/m_y$, l_2 has been expanded up to β_y. The interpretation of f_1 is particularly interesting: f_1 is somewhat smaller than l_1 as expected and does not depend on the magnification of the doublet. In brief, the downstream parameters of the low ß insertion, l_2 and f_1, are almost completely defined by the characteristics of the interaction point. It is also posible at this stage to determine the vertical ß magnification of the doublet.

5.4 Upstream parameters

Here, the variables (f_2,l_3) will be derived mainly from the vertical tracking for which both the boundary condition and an existence criterion give the two required equations in a rather simple way.

```
(* third equation: vertical ß transfer from F1 input to
   insertion input; u2 is the reduced variable l3/f2 *)

t=strt[l3].qF[-f2]
s={{ßy2,-ay2},{-ay2,(1+ay2^2)/ßy2}}
sv3d=sigmaTransfer[s,t]//Expand
eq3=sv3d[[1,1]]-ßy2/.f2->l3/u2
eq3=Collect[eq3,{u2,l3}]

                                                  2
                                  2    1       ay2
-2 ay2 l3 - 2 ay2 l3 u2 + l3   (--- + ----) + 2 u2 ßy2 +
                                  ßy2    ßy2

   2
 u2  ßy2
```

We can consider this equation as an equation in l_3.

```
l3=l3/.Solve[eq3==0,l3][[1]]
l3[[2,2,1]]=l3[[2,2,1]]//Factor
l3=l3//Factor
```

```
                         2                 2
(ay2 + ay2 u2 + Sqrt[ay2  - 2 u2 - u2 ]) ßy2
-----------------------------------------------
                           2
                     1 + ay2
```

The solution is real if the argument of the √ is not negative, which implies for u_2 the restriction:

$$0 < u_2 \leq u_{20} = \sqrt{1 + \alpha_{y2}^2} - 1$$

and for l_3 the limit values

$$\left(\frac{2\, \alpha_{y2}\, \beta_{y2}}{1 + \alpha_{y2}^2}, \frac{\alpha_{y2}\, \beta_{y2}}{\sqrt{1 + \alpha_{y2}^2}} \right)$$

For intermediate values characterized by the coefficient $k = u_2 / u_{20}$

$$l_3 = \frac{\alpha_{y2}\,(1 + k\, u_{20}) + \sqrt{u_{20}\,(1 - k)(u_{20}\,(1 + k) + 2)}}{1 + \alpha_{y2}^2}\, \beta_{y2}$$

5.5 Summary of the low ß parameters in a symbolic code

The four parameters of the final doublet are thus determined. The horizontal plane has been left floating in this treatment but we keep k as a "button" to adjust the horizontal ß value.

```
(* Low ß parameters; the term in ßy is neglected in the
expression of l2; the output is the list
    {f1,l2,f2,l3,
    {ßy1,-αy1},{ßx1,-αx1},{ßy2,-αy2},{ßx2,-αx2},{ßy3,-αy3},{ßx3,-αx3}

 Lowß[ßx_,ßy_,l1_,k_]:=
    Block[{fd,ld,ff,lf,qdy,qdx,qfy,qfx,
           ss1,ss2,ss3,u20,ty,tx,
           s0x,s0y,s1x,s1y,s2x,s2y,s3x,s3y},
 fd=l1-ßy*ßx/l1-Sqrt[ßy*(4*l1^2 + ßx^2)/ßx];
 ld=-2*l1^2*Sqrt[ßy]*(ßx/(4*l1^2 + ßx^2))^(3/2)-
    2*l1*(2*l1^2 + ßx^2)/(4*l1^2 + ßx^2)+
    l1^2*Sqrt[ßx/(ßy*(4*l1^2 + ßx^2))];
 s0y={{ßy,0 },{0    ,1/ßy}};
 s0x={{ßx,0 },{0    ,1/ßx}};
 ss1={{1 ,l1},{0    ,1   }};
 ss2={{1 ,ld},{0    ,1   }};
 qdy={{1 ,0 },{-1/fd,1   }};
 qdx={{1 ,0 },{ 1/fd,1   }};
 s1y=ss1.s0y.Transpose[ss1];
 s1x=ss1.s0x.Transpose[ss1];
 ty =ss2.qdy.ss1;
 tx =ss2.qdx.ss1;
 s2y=ty.s0y.Transpose[ty];
 s2x=tx.s0x.Transpose[tx];
 u20=Sqrt[1+s2y[[1,2]]^2]-1;
 lf=s2y[[1,1]]*
   (-s2y[[1,2]] (1+k u20)+Sqrt[(1-k) u20 ((1+k) u20+2)])/
   (1+s2y[[1,2]]^2);
 ff=lf/(k u20);
 ss3={{1 ,lf},{0    ,1   }};
 qfx={{1 ,0 },{-1/ff,1   }};
 qfy={{1 ,0 },{ 1/ff,1   }};
 ty=ss3.qfy;
 tx=ss3.qfx.ss2.qdx.ss1;
 s3y=ty.s2y.Transpose[ty];
 s3x=tx.s0x.Transpose[tx];
 {fd,ld,ff,lf,
  s1y[[1]],s1x[[1]],
  s2y[[1]],s2x[[1]],
  s3y[[1]],s3x[[1]]}//MatrixForm]
```

■ 5.6 Example of application

In very high luminosity particle colliders such as high energy linear colliders or B-factories, it is important to have a very asymmetric focusing. We shall take as an example

$$\beta_x = 1 \text{ cm} \quad \beta_y = 10 \text{ m} \quad l_l = 1 \text{ m}$$

The coefficient k will be determined graphically so that

$$\beta_{x3} < \beta_{y3} \;\&\; \alpha_{x3}\,\alpha_{y3} < 0$$

```
ßx3= Lowß[10.,.01,1.,k][[1,10,1]]
ax3=-Lowß[10.,.01,1.,k][[1,10,2]]
Show[Plot[{ßx3,ax3},{k,.01,1.},DisplayFunction->Identity],
     Graphics[{Text["ßx3",{.5,950}],Text["ax3",{.5,-720}]}],
     DisplayFunction->$DisplayFunction, AxesLabel->{"k"," "}]
```

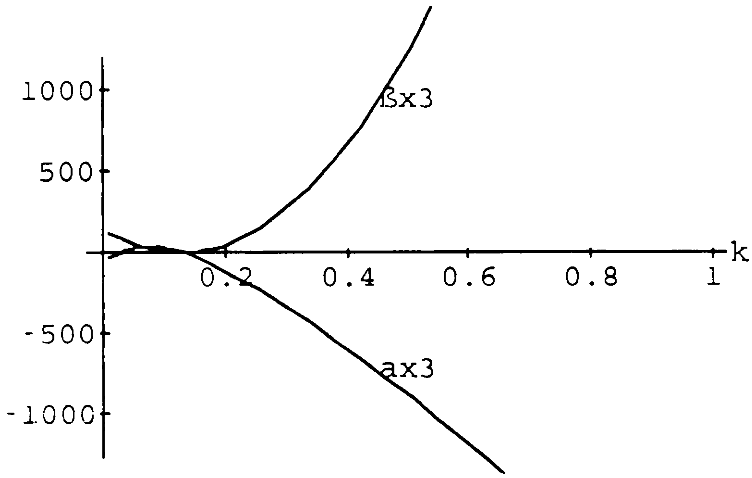

Fig.5 Variations of β_{x3} and α_{x3} as a function of the "existence" parameter k

The special case k=1 was tempting from a calculation stand point because it provided simple expressions for the upstream parameters, it has clearly to be discarded for the present application. By exploring the low range of k, it turns out that .115 fulfills our requirements.

```
LowB[10.,.01,1.,.115]

0.57751

1.13337

0.58243

0.771803

{100.01, 100.}

{10.1, 0.1}

{2.92857, -12.483}

{89.4421, 52.4168}

{2.92857, 7.45478}

{1.45307, -12.8544}
```

We note that the horizontal ß value is a little too small at QF, this is due to the truncation of the expression of l_2 which is not quite valid for very large magnification of $ß_y$ such as the present one (~300). This is in fact of no practical importance.

6 Conclusion

The theory of the focusing perturbations and more especially of the chromaticity has been reviewed. The results can already be found in the classical paper by Courant and Snyder published more than thirty years ago. The progress made in the past fifteen years concerns the interpretation of these results particularly in the implementation of chromaticity correction methods which take the interplay between high order chromaticity and ß distortions into account. The symbolic presentation which was systematically used here paves the way to higher order perturbation theories if a further analytical insight is neccessary. The chromaticity problem is still of great importance today in the context of the extreme luminosities that modern particle colliders have to achieve. A basic device in these machines is the low ß insertion, a source of chromatic aberrations which are very difficult to correct. We have derived a model for the final doublet of the insertion; the quadrupole strengths and the length of the straight sections are expressed in a closed form for the case of a very asymmetric focusing (flat beam). In this model, the chromaticity of the doublet is roughly the same in the horizontal and vertical planes. The theory of low ß insertions is still a field open to research: the definition of the intermediate constraints could be made more flexible, the matching section between the doublet and the regular lattice is not treated, momentum errors should be explicitly introduced. We think that such a theory is within reach, the main limitation was due to untractable calculations which can now be performed using the easy and sophisticated programs of symbolic computing which are now available.

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