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A SYMBOLIC PACKAGE OF SPECIAL RELATIVITY BASED ON MATHEMATICA

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ABSTRACT

The elements of the Special Relativity theory are collected in a single package which accepts symbolic or numerical input as arguments of a given statement and returns symbolic or graphical and numerical output. In addition, the program can interact with a data base of particle properties.

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A Symbolic Package of Special Relativity based on *Mathematica*

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Abstract

The elements of the Special Relativity theory are collected in a single package which accepts symbolic or numerical input as arguments of a given statement and returns symbolic or graphical and numerical output. In addition, the program can interact with a data base of particle properties.

Introduction

It is well known that a calculation in the four dimension space of Special Relativity becomes rapidly untractable as soon as one ceases to deal with the simple classical configurations. A symbolic program [1] becomes thus helpful to perform lengthy manipulations and acquires its full strength when numerical results and graphical outputs are also available. 4-Vectors and tensors are defined in section 1. The Lorentz transformation for translating trihedra is introduced in section 2 using the rapidity and the spherical angles which define the motion of the moving frame with respect to the observer's frame. Various problems of collisions often encountered in practice are treated in section 3. The theoretical background is drawn from [2]. The symbolic code is not presented, it is considered as an engine that the user has only to call from within a *Mathematica* session using the instruction

<<SpecialRelativity

before executing any of the statements listed in the paper.

1. Vectors and Tensors

A 4-vector A is made of one time-like component A_0 and three space-like components A_1, A_2, A_3. 4-Vector lengths and scalar products are Lorentz invariant. We assume the light velocity equal to 1 so that the particle velocities v are identified with the relativistic parameter ß (ß = v/c).The scalar product of two 4-vectors is given by the statement

LorentzScalar[{A_0, A_1, A_2, A_3}, {B_0, B_1, B_2, B_3}]

The phase of a wave for instance is the scalar product of the pulsation-wave vector (ω, **k**) and of the space-time vector (t, **x**)

```
LorentzScalar[{omega, kx, ky, kz},{t, x, y, z}]
omega t - kx x - ky y - kz z
```

The length of a vector is obtained with

LorentzLength[{ A_0, A_1, A_2, A_3 **}]**

The energy-momentum vector length is the particle mass

LorentzLength[{E, p, 0, 0}]

```
Sqrt[E^2 - p^2]
```

The time and space components of a 4-vector can be extracted using

time[{ A_0, A_1, A_2, A_3 **}]**

space[{ A_0, A_1, A_2, A_3 **}]**

Sometimes, the state of a particle is defined by its length and its space-like vector and it may be necessary to convert this state into a genuine 4-vector using

FourVector[Length, { A_0, A_1, A_2, A_3 **}]**

The only type of tensor which is treated in this paper is the electromagnetic tensor. However, the Lorentz transformations which will be defined later can be applied to any anti-symmetric tensor of rank 2. The electromagnetic tensor of electric field components (E_x, E_y, E_z) and magnetic field components (B_x, B_y, B_z) can be written in the matrix form

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

and is accessible using the statement

EMTensor[$E_x, E_y, E_z, B_x, B_y, B_z$ **]**

The electric field of a charge at rest can be defined in spherical coordinates as

field[e_, phi_, psi_] = EMTensor[e Cos[psi] Sin[phi], e Sin[psi], e Cos[psi] Cos[phi], 0, 0, 0]

2. Lorentz Transformations

By convention, a Lorentz transformation is defined from the particle frame towards an observation system. In the particle frame, the particle is at rest and the only non zero component of a 4-vector is the timelike component which is the proper time for the space - time vector and the mass for the momentum - energy vector. The transformation is completely determined by the momentum vector of the particle in the observation frame.

The modulus of the momentum vector is

$$p = \beta \gamma m c$$

c is the light velocity, m the mass of the particle, ß the ratio of the particle speed to c and γ the ratio of the particle energy E to its rest energy m c^2. ß and γ are related by

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

It is convenient to consider the product $\beta\gamma$ as the independent parameter so that

$$\beta = \frac{\beta\gamma}{\sqrt{1+(\beta\gamma)^2}}$$

$$\gamma = \sqrt{1+(\beta\gamma)^2}$$

These expressions are at the origin of the definition of the rapidity ζ

$$\zeta = \sinh^{-1} \beta\gamma$$

which, together with the unit vector $\mathbf{u} = \mathbf{p} / p$ (Figure 1),

$$u_x = \cos\psi \sin\phi$$

$$u_y = \sin\psi$$

$$u_z = \cos\psi \cos\phi$$

enter the matrix of a Lorentz transformation

$$\begin{pmatrix} \cosh\zeta & \sinh\zeta\, u_x & \sinh\zeta\, u_y & \sinh\zeta\, u_z \\ \sinh\zeta\, u_x & 1+(\cosh\zeta-1)\,u_x^2 & (\cosh\zeta-1)\,u_x u_y & (\cosh\zeta-1)\,u_x u_z \\ \sinh\zeta\, u_y & (\cosh\zeta-1)\,u_x u_y & 1+(\cosh\zeta-1)\,u_y^2 & (\cosh\zeta-1)\,u_y u_z \\ \sinh\zeta\, u_z & (\cosh\zeta-1)\,u_x u_z & (\cosh\zeta-1)\,u_y u_z & 1+(\cosh\zeta-1)\,u_z^2 \end{pmatrix}$$

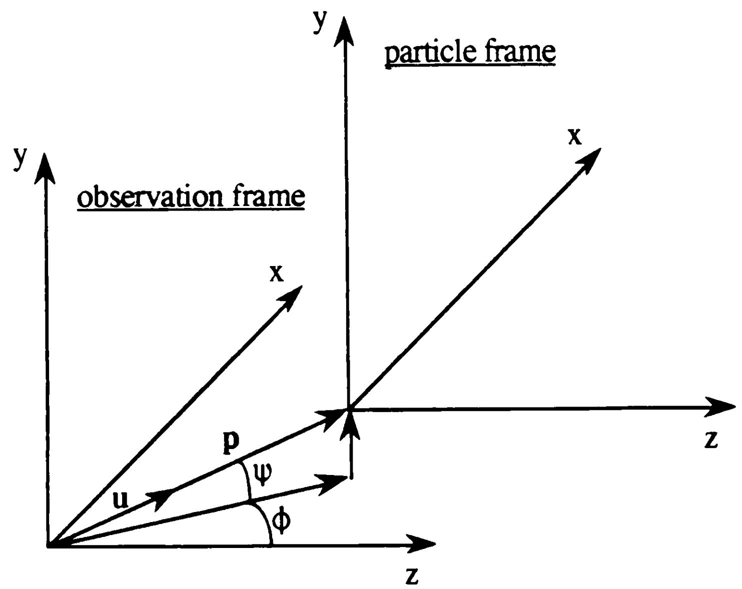


Figure 1. Definition of the reference systems in a Lorentz transformation from the particle frame towards the observation frame.

One notes that a Lorentz transformation is completely defined by the three parameters: $\beta\gamma$, ϕ and ψ, it is called with the statement

Lorentz [$\beta\gamma, \phi, \psi$]

A 4-vector can be submitted to a cascade of n transformations specified by the parameters

$$(\beta\gamma)_i \, , \; \phi_i \, , \; \psi_i \quad (i = 1, n)$$

The associated statement is then

LorentzVector[{{ $\beta\gamma, \phi, \psi$ }$_1$, . . . , { $\beta\gamma, \phi, \psi$ }$_n$ }}, { A_0, A_1, A_2, A_3 }]

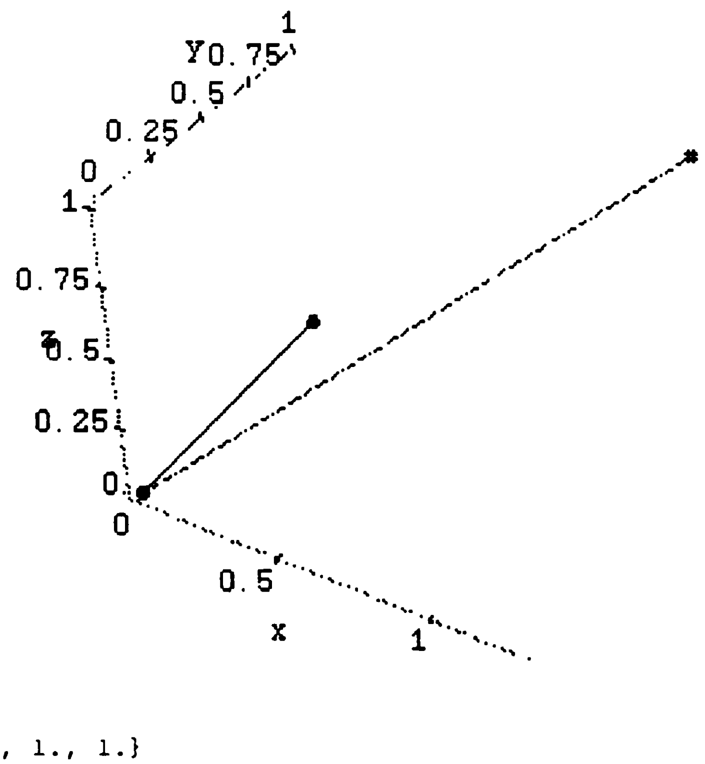
The Doppler shift is derived by applying a Lorentz transformation to the (ω, **k**) 4-vector. In the following example, the source moves along the z-axis and radiates towards the observer along the same axis:

LorentzVector[{{bg, 0, 0}},{omega, 0, 0, -k}]

```
{ - (bg k) + Sqrt[1 + bg^2] omega,0,0, - Sqrt[1 + bg^2] k + bg omega }
```

When the input is numeric, the transformed 4-vector and the graph of the initial and final spacelike vectors are returned in blue and red respectively (unfortunately, the high quality of the plots is altered by the black and white reproduction). In the following example, two transformations are cascaded.

LorentzVector[{{1, 0, 0},{.9, Pi/2, 0}},{1, 0, 1, 0}] // N




```
{1.90263, 1.27279, 1., 1.}
```

A second rank tensor written in matrix form F becomes

$$F' = L \, F \, L^T$$

after the Lorentz transformation L. For multiple Lorentz transformations, the statement

LorentzTensor[{{ βγ, φ, ψ }$_1$,..., { βγ, φ, ψ }$_n$ }}, tensor]

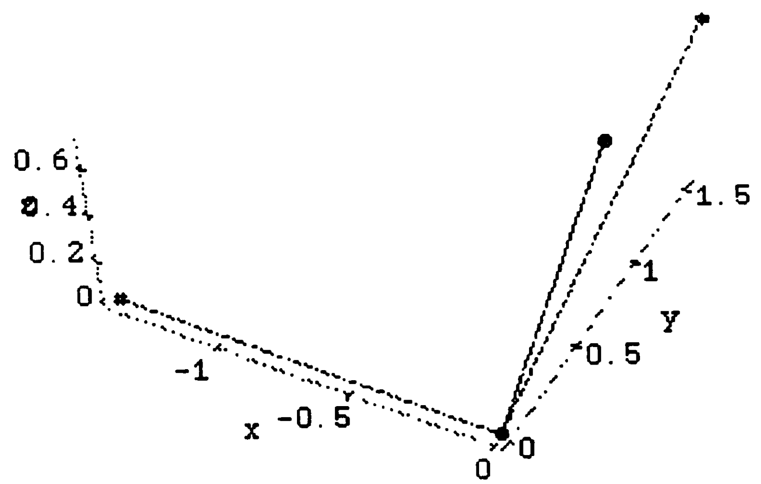
returns the transformed tensor and, for numerical input, the initial and final configurations of the component vectors. In the next example, we give the special form of the transformed tensor when the boost is along the z-axis and make it familiar by re-introducing β and γ separately instead of βγ.

```
LorentzTensor[{{bg, 0, 0}}, EMTensor[ex, ey, ez, bx, by, bz]] /.
        Sqrt[1+bg^2]->g /. bg->ß*g  // Factor] // MatrixForm
```

```
0                  -(g (ex + by ß))   -(g (ey - bx ß))   -ez
g (ex + by ß)      0                  -bz                g (by + ex ß)
g (ey - bx ß)      bz                 0                  -(g (bx - ey ß))
ez                 -(g (by + ex ß))   g (bx - ey ß)      0
```

As an example of a tensor transform, we consider the Coulomb field of a particle at rest defined in the previous section with the function **field** and look for its transform in the the observation frame:

```
LorentzTensor[{{2, 0, 0}}, field[1, 0, Pi/4]] // N // MatrixForm
```




```
0.         0.  -1.58114  -0.707107
0.         0.  0.        0.
1.58114    0.  0.        1.41421
0.707107   0.  -1.41421  0.
```

The electric field is dilated in the y-direction and the magnetic field lies along the x axis.

3. Collisions

In the collision process, the particle trajectories are in the same plane before and after the collision.The origin is at the collision point and the two axes (z, x) are the bissectrices of the momentum vectors of the colliding particles. The angle between the incident momentum vector and the z-axis is θ. The center of mass moves in the plane which has just been defined and that we call horizontal. New particles may be created in reference frames defined by the angles ϕ and ψ; the y-axis is vertical (Figure 2).

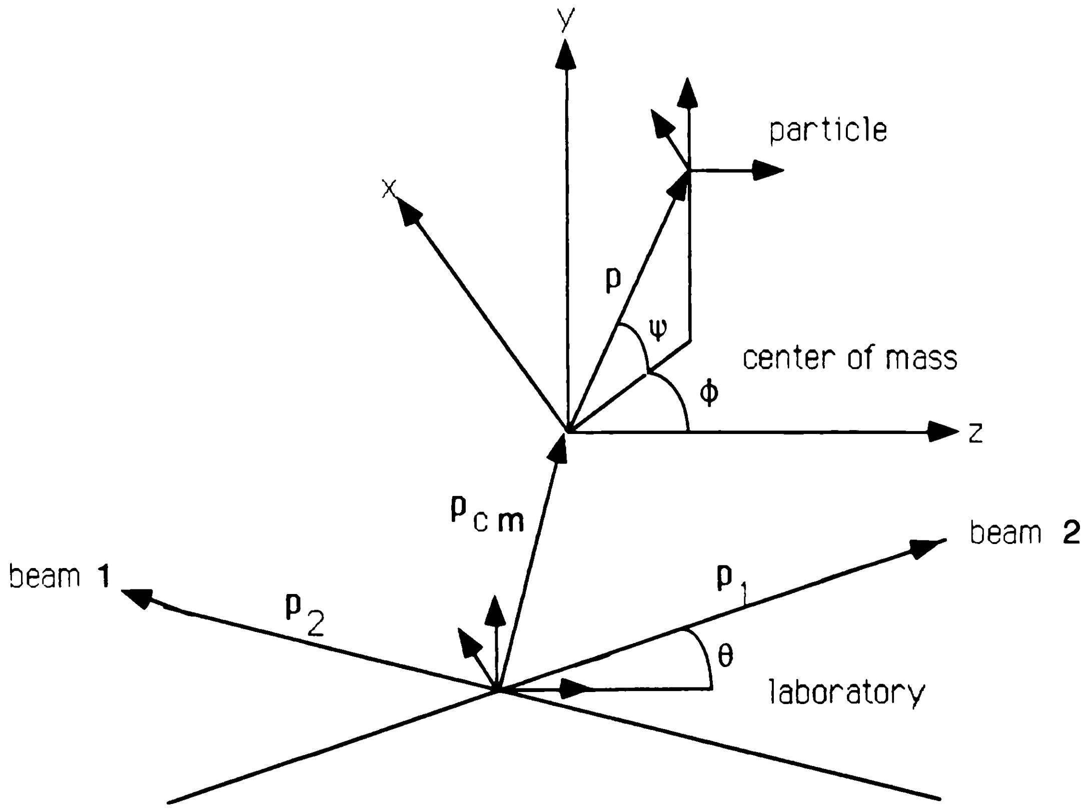


Figure 2. Reference frames

The center of mass energy-momentum 4-vector is the sum of the energy-momenta of the incident particles. Its length is usually denoted $\sqrt{s}$. The statement

CMEnergyMomentum[mass1, momentum1, mass2, momentum2, θ]

returns the expression of the center of mass energy-momentum vector

CMEnergyMomentum[m1,p1,m2,p2,t] // Factor

```
{ Sqrt[m1^2 + p1^2] + Sqrt[m2^2 + p2^2], (p1 - p2) Cos[t], (p1 + p2) Sin[t],0 }
```

and, for numerical input, a numerical output and the graph of the momentum vectors of the incident particles which are added to give the center of mass momentum.

A classical problem consists of determining the momentum of the incident particles to aim at a resonance of mass $\sqrt{s}$. It is solved using

Incidentmomentum[mass1, mass2, momentum2, θ, √s]

in the general case where the incident particles have different momenta; the subscript 1 is referred to beam 1 of unknown momentum and the subscript 2 to beam2 whose characteristics are given.

Incidentmomentum[0, 0, p2, t, ss]

```
      ss^2
-------------
4 p2 Cos[t]^2
```

If the two incident particles have the same, but unknown, momentum, the above statement has to set the optional argument *EqualMomenta* to *True*:

Incidentmomentum[mass1, mass2, momentum, θ, √s, EqualMomenta->True]

The previous example becomes in that case:

Incidentmomentum[0, 0, p, t, ss, EqualMomenta->True]

```
   ss
---------
2 Cos[t]
```

The purpose of the following statements is to establish expressions for the momentum of the particles created in the collision so that the various Lorentz transformations which may be required could be determined.

The center of mass momentum is given by

CMmomentum[momentum1, momentum2, θ]

CMmomentum[p1, p2, t]

```
Sqrt[p1^2 + p2^2 - 2 p1 p2 Cos[2t]]
```

In a 2-body decay, the total energy is equally shared by the two particles and equal to √s / 2 or, in other words, to half the mass M of the initial particle which is desintegrated. The momentum of the created particle of mass m is given by

Particlemomentum[M, m]

Particlemomentum[M, m]

```
      M^2
Sqrt[----- - m^2]
       4
```

4. Particle Properties Data Base

A calculation has almost always to be concluded by numerical applications. It is therefore necessary to have a data base where the particle properties are stored. Here, the data base has a limited number of information but it can be extended at will. The statement

Particle

returns the list of particles:

Particle

```
{Bmeson, Photon, Upsilon}
```

and

Data

the list of properties:

Data

```
{mass, lifetime}
```

Information is retrieved by typing

property [*particle*]

For instance, the mass of the Upsilon is given in electron-volt by

mass[Upsilon]

```
1.058 10^10
```

Conclusion

An application relevant to Special Relativity theory and based on the functionality and the programing language of a general symbolic program (*Mathematica*) has been described. A natural extension would include rotating frames. What has been said for the complexity of the calculations in the 4-dimension space of Special Relativity is still truer in the field of General Relativity and of tensor manipulation. Here too, applications have been written [3,4]. The technique of transmitting knowledge and information through an application where text and interactive statements are merged belongs to the realm of electronic textbooks which will play a more and more important role in the future.

ACKNOWLEDGMENTS

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