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NEW ANALYTICAL CRITERIA FOR LOSS OF LANDAU DAMPING IN LONGITUDINAL PLANE

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Abstract

Landau damping is a very important stabilization mechanism of beams in circular hadron accelerators. In the longitudinal plane, Landau damping is lost when the coherent mode is outside of the incoherent synchrotron frequency spread. In this paper, the threshold for loss of Landau damping (LLD) for constant inductive impedance $\mathrm{Im}Z/k$ is derived using the Lebedev matrix equation (1968). The results are confirmed by direct numerical solutions of the Lebedev equation and using the Oide-Yokoya method (1990). For more realistic impedance models of the ring, new definitions of an effective impedance and the corresponding cutoff frequency are introduced which allow using the same analytic expression for the LLD threshold. We also demonstrate that this threshold is significantly overestimated by the Sacherer formalism based on the previous definition of an effective impedance using the eigenfunctions of the coherent modes.

INTRODUCTION

The loss of Landau damping [1] has been observed in operations of different accelerators (Tevatron [2], RHIC [3], SPS [4] and LHC [5]) and has been studied for many years using different approaches [6–16]. A general way to analyze beam stability is to solve the Vlasov equation linearized for a small perturbation of a stationary particle distribution function. The first self-consistent system of equations suitable for the eigenvalue analysis of longitudinal beam stability was proposed by Lebedev in 1968 [6]. In the recent paper [17], we derived an analytic expression for the LLD threshold in the presence of the constant reactive impedance $\mathrm{Im}Z/k$ using the Lebedev equation. It agrees with the LLD threshold determined numerically by solving the two matrix equations: the Lebedev matrix equation and the Oide-Yokoya equation [18]. In this paper, we present the derivation of the LLD threshold which based on a novel method to compute an effective impedance with an effective cutoff frequency. This allows to evaluate the LLD threshold for complicated impedance models (like the one of the CERN SPS).

MAIN EQUATIONS AND DEFINITIONS

We consider the case of a single RF system, for the sake of simplicity, while the derivations can be adapted to other RF waveforms. The Lebedev equation can be written using variables $(\mathcal{E}, \psi)$, which correspond respectively to the energy and phase of the synchrotron oscillations,

$$\mathcal{E} = \frac{\dot{\phi}^2}{2\omega_{s0}^2} + U_t(\phi), \tag{1}$$

$$\psi = \mathrm{sgn}(\eta \Delta E)\frac{\omega_s(\mathcal{E})}{\sqrt{2}\omega_{s0}}\int_{\phi_{\max}}^{\phi}\frac{d\phi'}{\sqrt{\mathcal{E} - U_t(\phi')}}. \tag{2}$$

Here, ΔE and ϕ are respectively the energy and phase deviations of the particle from the synchronous particle, $\eta = 1/\gamma_{\mathrm{tr}}^2 - 1/\gamma^2$ is the slip factor, γ_{tr} is the Lorentz factor at transition energy, and $f_{s0} = \omega_{s0}/2\pi$ is the frequency of small-amplitude synchrotron oscillations in a bare RF potential. The total potential can be obtained from the sum voltage V_t

$$U_t(\phi) = \frac{1}{V_0 \cos\phi_{s0}}\int_{\Delta\phi_s}^{\phi}\left[V_t(\phi') - V_0\sin\phi_{s0}\right]d\phi', \tag{3}$$

where V_0 is the RF voltage amplitude, and ϕ_{s0} is the synchronous phase. The synchronous phase shift due to intensity effects $\Delta\phi_s$ satisfies the relation $V_0\sin\phi_{s0} = V_0\sin(\phi_{s0} + \Delta\phi_s) + V_{\mathrm{ind}}(\Delta\phi_s)$, where V_{ind} is the induced voltage. In practice, U_t can be obtained for an arbitrary impedance model thanks to an iterative procedure [15]. Then it can be used to compute the synchrotron frequency as a function of the energy of synchrotron oscillations $\omega_s(\mathcal{E}) = 2\pi/T_s(\mathcal{E})$ in Eq. (2) from the period of oscillations including intensity effects.

Below we will consider particle distributions belonging to a binomial family $g(\mathcal{E}) = (1 - \mathcal{E}/\mathcal{E}_{\max})^{\mu}$. For a finite μ, which is usually the case for proton bunches, a full length τ_{full} is defined as

$$\tau_{\mathrm{full}} = \left[\phi_{\max}(\mathcal{E}_{\max}) - \phi_{\min}(\mathcal{E}_{\max})\right]/\omega_{\mathrm{RF}}. \tag{4}$$

Lebedev Equation

Using the variable and notations described above, an infinite system of equations for harmonics of the line density perturbation $\tilde{\lambda}$ at frequency Ω [6], can be written as

$$\tilde{\lambda}_p(\Omega) = -\frac{\zeta}{h}\sum_{k=-\infty}^{\infty} G_{pk}(\Omega)\,\frac{Z_k(\Omega)/k}{Z_{\mathrm{norm}}}\,\tilde{\lambda}_k(\Omega), \tag{5}$$

which we refer to as the Lebedev equation. Here, ζ is the dimensionless *intensity* parameter,

$$\zeta = -\frac{qN_p h^2\,\omega_0\,Z_{\mathrm{norm}}}{V_0\cos\phi_{s0}}, \tag{6}$$

Z_{norm} is the impedance normalization factor in units of Ohms, which can be arbitrarily chosen (see also Table 1). The

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elements G_{pk}, called beam transfer matrices [19] are

$$G_{pk}(\Omega) = -i\frac{\omega_{s0}}{\pi A_N} \times \sum_{m=1}^{\infty}\int_0^{\mathcal{E}_{\max}} \frac{dg(\mathcal{E})}{d\mathcal{E}}\frac{I_{mk}(\mathcal{E})I^*_{mp}(\mathcal{E})\omega_s(\mathcal{E})}{\Omega^2/m^2-\omega_s^2(\mathcal{E})}d\mathcal{E} \qquad (7)$$

where p and k are the revolution frequency harmonics, m is the azimuthal mode number ($m=1$: dipole mode, $m=2$: quadrupole mode, etc.), the normalization factor for the distribution function is

$$A_N = \omega_{s0}\int_0^{\mathcal{E}_{\max}}\frac{g(\mathcal{E})}{\omega_s(\mathcal{E})}d\mathcal{E}, \qquad (8)$$

and

$$I_{mk}(\mathcal{E}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left[i\frac{k}{h}\phi(\mathcal{E},\psi)-im\psi\right]d\psi, \qquad (9)$$

with h the harmonic number. Note that the elements G_{pk} depend on intensity parameter ζ as they are found after the stationary problem is solved.

The solution of Lebedev equation for particular Ω and ζ exists if the determinant of the following matrix is zero

$$D(\Omega,\zeta) = \det\left|\delta_{pk}+\frac{\zeta}{h}G_{pk}(\Omega)\frac{Z_k(\Omega)/k}{Z_{\text{norm}}}\right| = 0. \qquad (10)$$

Below we will show how this criterion can be used to determine the LLD threshold.

Threshold of Loss of Landau Damping

In this subsection, we will derive the LLD thresholds in a general case for the binomial family of particle distributions and dominating inductive impedance above transition energy $\eta\text{Im}Z/k > 0$ (or space charge below transition).

Analyzing solutions of the dispersion integral obtained from the Vlasov equation for an infinite plasma, N. G. van Kampen [20, 21] found that they have continuous and discrete parts. In application to longitudinal beam dynamics, at low intensities ($\zeta \approx 0$), all van Kampen modes remain within the continuous spectrum, $\Omega = m\omega_s(\mathcal{E})$. Landau damping results then from the phase mixing of these modes which do not represent the collective motion of the particles. Above the threshold the discrete van Kampen modes emerge from the continuous spectrum, implying that Landau damping is lost [9].

For the dipole mode ($m = 1$), the LLD threshold ζ_{th} is reached when there is a coherent mode whose frequency Ω equals the maximum incoherent frequency $\hat{\omega}_s = \max[\omega_s(\mathcal{E})]$, i.e., $\Omega = \hat{\omega}_s$. This implies that for any infinitesimally small increase of intensity this mode will move outside the incoherent frequency band. At low intensities, the synchrotron frequency distribution in a single RF system is a monotonic function of the energy of synchrotron oscillations $\mathcal{E}$. Assuming that still holds at the LLD threshold for a dipole mode $m = 1$, we can search for a value of the parameter ζ, at which $\Omega = \hat{\omega}_s = \omega_s(0)$ is a solution of Eq. (5). Since, as follows from definition (9), $I_{mk}(0) = 0$, the integral (7) defining the elements G_{pk} converges for all p and k.

The solution of the Lebedev equation can be found thanks to the following property of the matrix

$$\det\left[\exp(\varepsilon X)\right] = \exp\left[\varepsilon\ \text{tr}(X)\right], \qquad (11)$$

where $\text{tr}(X)$ is the trace of an arbitrary square matrix X, and ε is the small parameter $\varepsilon \ll 1$, which will be defined later. Consider that X also depends on ε, $X(\varepsilon) = X(0) + \varepsilon(dX/d\varepsilon)(0) + ...$, expansion up to the first order of ε yields,

$$\det[I+\varepsilon X(\varepsilon)] = \det(\exp\{\ln[I+\varepsilon X(\varepsilon)]\}) = \exp(\text{tr}\{\ln[I+\varepsilon X(\varepsilon)]\}) \approx 1+\varepsilon\ \text{tr}[X(0)], \qquad (12)$$

with the identity matrix I. Thus, we get a general expression for the LLD threshold from Eq. (10)

$$\zeta_{\text{th}} = -h\left[\sum_{k=-\infty}^{\infty}G_{kk}(\Omega)\frac{Z_k(\Omega)/k}{\text{Im}Z/k}\right]^{-1}. \qquad (13)$$

Naturally, the parameter $\varepsilon \propto \zeta$. Its dependence on the bunch length will be deduced below. Thus, G_{kk} needs to be evaluated at zero intensity, as inclusion of potential well distortion will already result in keeping a higher-order term of the parameter ε.

In the present work, the elements G_{kk} are calculated analytically for short bunches in a single RF system, while similar derivations for the double RF system can be found in [22]. Keeping only one element of the sum over azimuthal harmonics ($m = 1$), from Eq. (7) we can obtain G_{kk} at the LLD threshold ($\Omega = \omega_{s0}$)

$$G_{kk} \approx i\frac{16\mu(\mu+1)}{\pi\phi_{\max}^4}\left[1-{}_1F_2\left(\frac{1}{2};2,\mu;-y^2\right)\right], \qquad (14)$$

where ${}_pF_q(a_1,..,a_p;b_1,...,b_q;z)$ is the generalized Hypergeometric function with $y = k\phi_{\max}/h$, and $\phi_{\max}$ corresponds to a half bunch length expressed in radians. These matrix elements can be presented as a combination of Bessel functions for the particular values of μ. For example, in the case of $\mu = 1/2$, $\mu = 1$, and $\mu = 2$, one obtains $G_{kk} \propto \left[1-J_1(2y)/y\right]$, $G_{kk} \propto \left[1-J_0^2(y)-J_1^2(y)\right]$, and $G_{kk} \propto \left[1/2-J_0^2(y)-J_1^2(y)+J_0(y)J_1(y)/y\right]$, respectively. As $G_{kk} \propto 1/\phi_{\max}^4$, we can now define the small parameter

$$\varepsilon = \zeta/\phi_{\max}^4$$

and check the validity of expansion (12) later by comparison with exact semi-analytic calculations.

For the case of the inductive impedance $Z_k = ikZ_{\text{norm}}$, the sum in Eq. (13) can also be analytically evaluated by approximating it with an integral

$$\frac{1}{h}\sum_{k=-\infty}^{\infty}G_{kk}(\Omega)\frac{Z_k(\Omega)/k}{Z_{\text{norm}}} \approx \frac{i}{h}\int_{-\infty}^{\infty}G_{kk}(\Omega)\,dk \to \infty,$$

which diverges for $\mu > 0$. This can be easily seen from the asymptotic behavior of the elements G_{kk} as they saturate at

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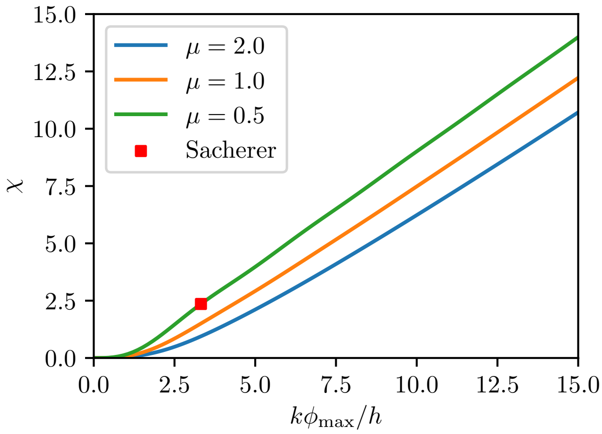

Figure 1: Examples of the function $\chi(k\phi_{\max}/h, \mu)$ defined by Eq. (17) for three different distribution functions from the binomial family ($\mu = 0.5, 1,$ and 2). The values $k\phi_{\max}/h$, for which the LLD thresholds corresponds to that used in Sacherer [7] criterion, is shown with a square.

a constant value for larger k. Once we have truncated the sum at arbitrary $k_{\max}$, the LLD threshold becomes

$$\zeta_{\rm th} = \frac{\pi \phi_{\max}^5}{32\mu(\mu+1)\chi(k_{\max}\phi_{\max}/h, \mu)}, \tag{15}$$

or in terms of intensity

$$N_{p,\rm th} = -\frac{\pi V_0 \cos\phi_{s0}\phi_{\max}^5}{32qh^2\omega_0\mu(\mu+1)\chi(k_{\max}\phi_{\max}/h, \mu)Z_{\rm norm}}, \tag{16}$$

where we introduced the function

$$\chi(y,\mu) = y\left[1 - {}_2F_3\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}, 2, \mu; -y^2\right)\right]. \tag{17}$$

Examples of this function for different μ values are shown in Fig. 1.

For $\mu = 1/2$, the Sacherer formalism proposes the LLD threshold which can be written in our notations [23]

$$\zeta_{\rm th,S} = \phi_{\max}^5/18. \tag{18}$$

It can be recovered from Eq. (15) for $y_{\max} \approx 3.32$ (see also Fig. 1). This means that for a given bunch length the commonly used criteria Eq. (18) is accurate only for the special choice of the cutoff frequency $f_c \approx 1/\tau_{\rm full}$. Based on this, for space charge below transition energy with a rather high cutoff frequency, the LLD threshold will be significantly overestimated for bunches with $\tau_{\rm full} \gg 1/f_c$. Instead, our criteria Eq. (15) gives a simplified expression

$$\zeta_{\rm th} \approx \frac{\pi\phi_{\max}^4 h}{32\mu(\mu+1)k_{\max}}, \tag{19}$$

since the generalized Hypergeometric function ${}_2F_3$ approaches zero for $y \to \infty$. One can see that the threshold is inversely proportional to the cutoff frequency, and the fifth power in the dependence on the bunch length is replaced by the fourth.

Table 1: The Machine and RF Parameters of the LHC at Injection Energy and of the SPS at Extraction Energy [25]

Parameter	Units	LHC	SPS
Circumference, C	m	26658.86	6911.55
Harmonic number, h		35640	4620
Transition gamma, $\gamma_{\rm tr}$		55.76	17.95
RF frequency, $f_{\rm RF}$	MHz	400.79	200.39
Beam energy, E_0	TeV	0.45	0.45
RF voltage, V_0	MV	6	7.2
Norm. factor, $Z_{\rm norm}$	Ohm	0.07	1

COMPARISON WITH MELODY CODE

The analytic threshold according to Eq. (15) can be compared with semi-analytical results obtained using code MELODY (Matrix Equations for LOngitudinal beam DYnamics calculations) [24] in applications to the LHC and SPS. The main accelerator parameters are listed in Table 1. In calculations based on the Lebedev equation (5), the determinant $D(\hat{\omega}_s, \zeta)$ is numerically evaluated for different ζ changed iteratively until condition Eq. (10) is satisfied.

In the Oide-Yokoya method [18], the Vlasov equation is converted into a matrix equation and its eigenvalues are calculated as a function of the intensity parameter ζ. To find the threshold, the difference between the maximum eigenfrequency and the maximum incoherent frequency is evaluated. The threshold corresponds to the intensity where the difference vanishes (see details in [17]).

Inductive Impedance

Here we will first show the results for the truncated inductive impedance $Z_k = ikZ_{\rm norm}$ for $|k| < k_{\max}$ and $Z_k = 0$ elsewhere. Figure 2 shows the LLD threshold as a function of the full bunch length calculated for two different cutoff frequencies using analytic equation (15) and code MELODY. One can see that numerical results obtained using the Oide-Yokoya method and the Lebedev equation agree, with the maximum 2% relative error in the covered bunch-length range. They are close to the analytic expression (15), while, as expected, there is some discrepancy for larger bunch lengths since the analytic threshold was derived in short-bunch approximation while still taking the synchrotron frequency spread into account. We also observe that the dependence on the bunch length is even slightly weaker than the fourth power. The dependence on the cutoff frequency can be also seen in Fig. 2 and results, for two values of the unperturbed bunch length with increasing cutoff frequencies, in Fig. 3. It confirms the vanishing of the LLD threshold for $f_c \to \infty$.

Effective Impedance

Above we discussed the results for a truncated inductive impedance, while in reality the impedance could be a much more complicated function of frequency. An example of

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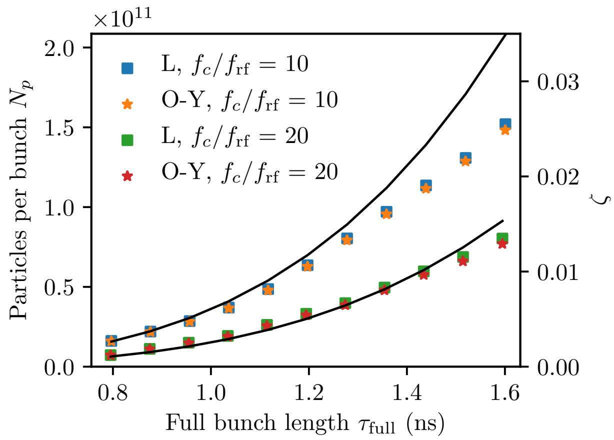

Figure 2: The LLD intensity threshold as a function of the full bunch length τ_{full} calculated using the Lebedev equation (5) and the Oide-Yokoya method for different cutoff frequencies ($f_c = k_{\text{max}} f_0$) of the inductive impedance. The analytic predictions from Eq. (15) are plotted as solid lines. Case of $\eta > 0$ and other parameters are $Z_{\text{norm}} = 0.07$ Ohm, $V_0 = 6$ MV, and $\mu = 2$. The corresponding intensity parameter ζ is shown on the second vertical axis.

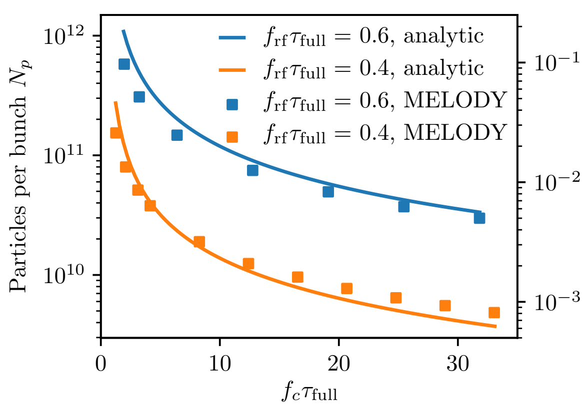

Figure 3: LLD intensity threshold in the logarithmic scale as a function of the cutoff frequency of a reactive impedance $f_c = k_{\text{max}} f_0$ (multiplied by the full bunch length τ_{full}) for two different values of τ_{full}. The analytic predictions from Eq. (15) are shown as solid lines and the results of semi-analytic calculations using MELODY as squares. Parameters as in Fig. 2. The corresponding intensity parameter ζ is shown on the second vertical axis.

the SPS impedance model [26] is shown in Fig. 4. We aim to comprise the frequency dependence of a particular impedance model in just two numbers: $(\text{Im}Z/k)_{\text{eff}}$, the equivalent (effective) value of the inductive impedance and k_{eff}, the corresponding effective cutoff frequency. In Ref. [17] we proposed the following definition of the effective impedance

$$(\text{Im}Z/k)_{\text{eff}} = \sum_{k=-k_{\text{eff}}}^{k_{\text{eff}}} G_{kk} \text{Im}\,(Z_k/k) \Big/ \sum_{k=-k_{\text{eff}}}^{k_{\text{eff}}} G_{kk}, \quad (20)$$

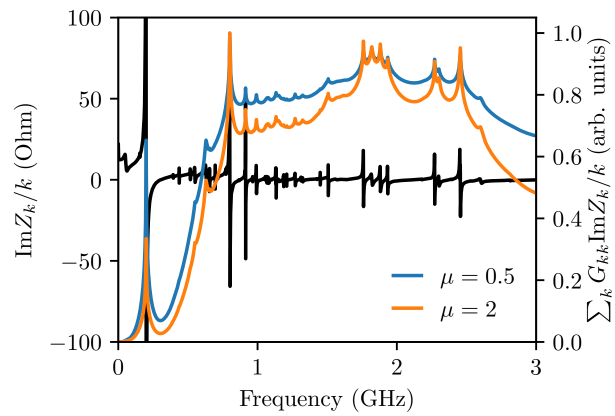

Figure 4: The present SPS reactive impedance model $\text{Im}Z/k$ (black curve) after the impedance reduction campaign during the 2nd Long Shutdown (LS2) 2019-2020. The values of the cumulative sum used for calculation of the effective impedance (20) are shown as blue ($\mu = 0.5$) and orange ($\mu = 2$) curves ($\phi_{\text{max}} = 2$).

while k_{eff} was chosen such that $\text{Im}Z_k/k < 0$ for $k > k_{\text{eff}}$. Then the LLD threshold can be written as

$$\zeta_{\text{th}} = \frac{\pi \phi_{\text{max}}^5}{32\mu\,(\mu+1)\,\chi\,(k_{\text{eff}}\phi_{\text{max}}/h,\mu)} \frac{Z_{\text{norm}}}{(\text{Im}Z/k)_{\text{eff}}}, \quad (21)$$

and in terms of intensity one gets

$$N_{p,\text{th}} = -\frac{\pi V_0 \cos\phi_{s0} \phi_{\text{max}}^5}{32qh^2\omega_0\mu(\mu+1)\,\chi(k_{\text{eff}}\phi_{\text{max}}/h,\mu)\,(\text{Im}Z/k)_{\text{eff}}}. \quad (22)$$

This formula was successfully verified for the case of a single broadband resonator impedance with quality factor $Q = 1$ and different values of the resonant frequency f_r.

In the recent study [27], further investigations were done for the impedance defined as a sum of broadband and narrowband resonator impedance models

$$Z_k = \frac{R}{1 + iQ\left(\frac{kf_0}{f_r} - \frac{f_r}{kf_0}\right)} + \frac{R_2}{1 + iQ_2\left(\frac{kf_0}{f_{r,2}} - \frac{f_{r,2}}{kf_0}\right)} \quad (23)$$

with shunt impedances $R = Z_{\text{norm}} Q f_r/f_0$, and $R_2 = Z_{\text{norm}} Q_2 f_{r,2}/f_0$, respectively. To study the impact of the impedance of the resonance structure with the resonant frequency below the cutoff frequency of the main broadband impedance, the systematic analysis of different cases with various $Q_2 \in [1, 1000]$ and $f_{r,2} \in (0, f_r)$ was performed. As the result, a new definition of the effective cutoff frequency was suggested

$$k_{\text{eff}} = \arg\max \sum_{k'=0}^{k} G_{k'k'} \text{Im}(Z_{k'}/k'), \quad (24)$$

which corresponds to a value of k for which the maximum of the cumulative sum is reached. For a single broadband resonator both old and new methods give the same effective cutoff frequency. For the SPS impedance model the cumulative sum of expression Eq. (24) is presented in Fig. 4,

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which reaches the maximum value at $k_{\mathrm{eff}} f_0 \approx 0.8$ GHz (for $\phi_{\max} \approx 1$). In Fig. 5, the analytic prediction (21) for a smooth distribution with $\mu = 2$ is compared with the full semi-analytic calculations using MELODY. The maximum error is about 40% for long bunches which is only slightly larger than for the case of a simple constant inductive impedance $\mathrm{Im}Z/k = Z_{\mathrm{norm}}$ (see Fig. 2).

It is worth to compare this new definition of the effective impedance with the one based on the Sacherer formalism $(\mathrm{Im}Z/k)_{\mathrm{eff,S}}$. This formalism requires to find a set of orthogonal coherent modes for a given distribution. Then, the effective impedance is computed as a sum of their power spectral harmonics multiplied by $\mathrm{Im}Z_k/k$. For parabolic bunches ($\mu = 0.5$) and dipole mode one gets [23]

$$(\mathrm{Im}Z/k)_{\mathrm{eff,S}} = \sum_{k=-\infty}^{\infty} \frac{J^2_{3/2}(k\phi_{\max}/h)}{|k\phi_{\max}/h|} \mathrm{Im}\left(Z_k/k\right) \Bigg/ \sum_{k=-\infty}^{\infty} \frac{J^2_{3/2}(k\phi_{\max}/h)}{|k\phi_{\max}/h|}, \quad (25)$$

so that the LLD threshold is $\zeta_{\mathrm{th,S}} = \phi^5_{\max}/18 \times Z_{\mathrm{norm}}/\left(\mathrm{Im}Z/k\right)_{\mathrm{eff,S}}$. As expected for very short bunches Sacherer criterion agrees with threshold (21) and MELODY results, while for longer bunches it significantly overestimates the LLD threshold (see Fig. 5). However, we also see a non-monotonic behavior of the LLD threshold obtained using MELODY for this particle distribution. This can be understood from the fact that distribution with $\mu = 0.5$ has abrupt tails which affect the synchrotron frequency $\omega_s(\mathcal{E})$ as a function of the synchrotron oscillation energy via potential well distortion and its derivative $d\omega_s(\mathcal{E})/d\mathcal{E}$ can become a non-monotonic function. This was also observed for a simplified impedance model (23), where the discrepancy between MELODY and analytic predictions was due to potential well distortion [27]. If this type of distribution is expected in operation, we suggest firstly to estimate the LLD threshold using Eq. (21) and then to check the presence of non-monotonic behavior of the derivative of the synchrotron frequency as a function of the synchrotron oscillation energy.

CONCLUSION

Loss of Landau damping (LLD) in the longitudinal plane can be an important performance limitation of existing and future synchrotrons. In the present paper, the analytic expression for the LLD threshold of the dipole oscillations is discussed for the case of a single RF system and a particle distribution of the binomial family.

The new analysis shows that the LLD threshold is zero for a constant inductive impedance $\mathrm{Im}Z/k$ above transition (the LHC case) or capacitive (space charge) below. Once a finite cutoff frequency is introduced, the threshold becomes inversely proportional to the cutoff frequency f_c for $f_c \gg 1/\tau_{\mathrm{full}}$ (τ_{full} is the full bunch length). We have confirmed this dependence by solving the Lebedev matrix equation semi-analytically as well as using the Oide-Yokoya method, also showing that both numerical methods agree extremely well. The commonly used dependence of the LLD threshold on the bunch length to the fifth power is justified only in the low cutoff-frequency limit ($f_c \leq 1/\tau_{\mathrm{full}}$). The LLD threshold obtained by the Sacherer approach can be reproduced only when $f_c \approx 1/\tau_{\mathrm{full}}$. The dependence of the threshold on the bunch length changes to the power of four for the case of a higher cutoff frequency ($f_c \gg 1/\tau_{\mathrm{full}}$).

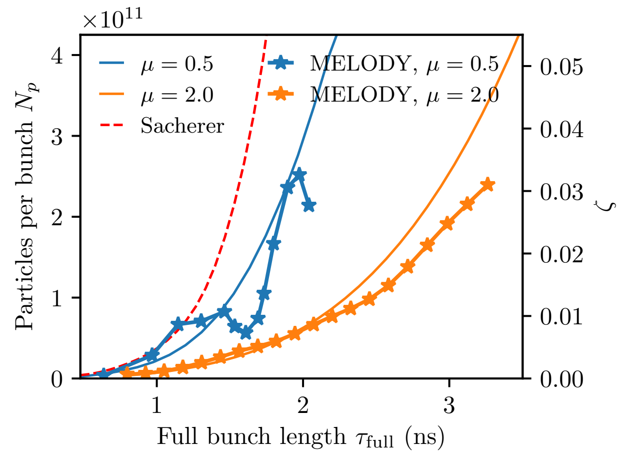

Figure 5: The LLD threshold as a function of bunch length for the SPS impedance model (Fig. 4). The stars connected with lines are calculated by MELODY. The lines are the prediction from the analytical formula (21). The red curve is LLD threshold calculated using Sacherer effective impedance (25). Case of $\eta > 0$ and other parameters are $Z_{\mathrm{norm}} = 1$ Ohm, $V_0 = 7.2$ MV. The corresponding intensity parameter ζ is shown on the second vertical axis.

We introduced a new definition of the effective impedance and the corresponding cutoff frequency to estimate the LLD threshold of more complicated impedance models. It does not require finding a set of the orthogonal modes which is essential for the Sacherer formalism. For the case of a smooth particle distribution function ($\mu = 2$) and the impedance model of the CERN SPS the agreement of analytic expression with full semi-analytic calculations is very good. For parabolic bunches ($\mu = 0.5$), the deviations are larger due to impact of potential well distortion. We also have shown that the Sacherer definition of the effective impedance can significantly overestimate the LLD threshold, especially for the bunch lengths $\tau_{\mathrm{full}} > 1/f_c$.

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