Long-range forces between nonidentical black holes with non-BPS extremal limits

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Motivated by recent studies of long-range forces between identical black holes, we extend these considerations by investigating the forces between two nonidentical black holes. We focus on classes of theories where charged black holes can have extremal limits that are not BPS. These theories, which live in arbitrary spacetime dimension, comprise gravity coupled to N 2-form field strengths and $(N-1)$ scalar fields. In the solutions we consider, each field strength carries an electric charge. The black hole solutions are governed by the $SL(N+1,R)$ Toda equations. In four dimensions, the black hole solutions in the $SL(3,R)$ example are equivalent to the "Kaluza-Klein dyons." We find that any pair of such extremal black holes that are not identical (up to overall scaling) will repel one another. We also show that there can exist pairs of non-extremal, nonidentical black holes which obey a zero-force condition. Finally, we find indications of similar results in the higher examples, such as $SL(4,R)$.

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I. INTRODUCTION

There has been a considerable interest recently in finding detailed ways to quantify the idea that gravity is the weakest force. Such attempts have led to a number of conjectures, with the weak gravity conjecture (WGC) [1] thus far on the strongest footing (see Ref. [2] for a comprehensive review). A natural way to examine the strength of gravity in a given theory is to study the force between well-separated particles and ask whether the gravitational attraction is indeed overwhelmed by the repulsive interactions in the theory. This idea has been made more precise by the repulsive force conjecture (RFC), which was originally stated in Ref. [1] and describes the simple notion that long-range repulsive gauge forces (between identical charged particles) should be at least as strong as all long-range attractive forces. The RFC was reemphasized more recently in Ref. [3], and stronger versions were put forth by Ref. [4]. Moreover, in Ref. [5] it was shown that the RFC can also be understood using the timelike reduction formalism of Ref. [6].

In its weakest form, the RFC argues that effective field theories (EFTs) consistent with quantum gravity must have a state which is "self-repulsive," whether the state is a fundamental particle or a black hole. When multiple charges are present, this requirement must then hold along each direction in charge space.[1] While the WGC and the RFC are clearly related to each other, they are distinct in the presence of scalars [4]. Moreover, studies of black holes in theories with higher derivatives make the difference even more manifest [7] and suggest that the RFC *cannot* be

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[1]The strongest form of the RFC [4] states that in every direction in charge space, there should be a *strongly self-repulsive multiparticle state*—i.e., a multiparticle state where each constituent state repels every other state, including itself.

satisfied by the black hole spectrum alone—unlike the WGC, which can [8]. Taking these considerations into account, it is valuable to better understand what else we can extract from the behavior of long-range forces, in order to eventually clarify to what extent the RFC is a useful criterion for constraining EFTs.

Within the context of the RFC, the interest thus far has been focused on long-range forces between two *identical* copies of the same object. Two identical black holes at rest and asymptotically far from each other will generically attract, except at extremality, when the net force between them will vanish [9]. This is true even in theories with moduli, which mediate new long-range interactions and affect the balance of forces between the black holes.[2] In the simple case of a black hole of mass M and electric charge Q in theories in four dimensions with a single light scalar, the force between two identical copies is[3]

$$F(r) = -\frac{1}{4}\frac{M^2}{r^2} + \frac{Q^2}{r^2} - \frac{\Sigma^2}{r^2} + \cdots, \qquad (1.1)$$

where Σ denotes the *scalar charge* (to be defined precisely later). The contribution of the scalar field precisely offsets the gravitational and electrostatic forces between the black holes, ensuring that all interactions cancel.

Such a no-force condition, which here applies to any pair of identical, extremal black holes, is also well known from studies of supersymmetric Bogomol'nyi-Prasad-Sommerfield (BPS) black holes in supergravity, where there exist multi-black-hole solutions where the individual black holes sit in static equilibrium with zero force between them. However, in the case of extremal BPS black holes in supergravity, the no-force condition holds regardless of whether the black holes are identical or they instead carry different charges. This simple observation motivates us to examine more generally the behavior of long-range forces between two black holes that carry *different* charges and are therefore not identical, working with configurations which are not BPS solutions. We are particularly interested in identifying any generic features of such interactions and under which conditions the force can vanish. As we shall see, for nonidentical black holes, the behavior of the long-range force is quite rich and can still lead to zero-force conditions, albeit in a different manner from the BPS cases arising in supergravity.

A well-known example of a configuration which is not BPS is provided by the so-called "Kaluza-Klein dyon," which is a solution of four-dimensional ungauged $\mathcal{N} = 8$ supergravity, in which a single gauge field carries both electric and magnetic charge. The solution can be described by a consistent truncation of the full $\mathcal{N} = 8$ theory that may instead be obtained as the Kaluza-Klein reduction of pure five-dimensional Einstein gravity; hence the nomenclature "Kaluza-Klein dyon." This theory is described by the four-dimensional Lagrangian

$$\mathcal{L}_4 = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{\sqrt{3}\phi}F^2. \qquad (1.2)$$

For our purposes, it will be more convenient to consider a somewhat different theory comprising gravity, the dilaton, and two distinct gauge fields rather than just a single gauge field. Among other things, this has the advantage that we can describe charged black holes with the feature of having an extremal but not BPS limit in any arbitrary dimension d. The d-dimensional Lagrangian is given by [10]

$$\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{a\phi}F_1^2 - \frac{1}{4}e^{-a\phi}F_2^2, \qquad (1.3)$$

where

$$a = \sqrt{\frac{6(d-3)}{d-2}}. \qquad (1.4)$$

The black hole solutions we shall consider have two independent electric charges, one carried by each of the two field strengths. In the special case of $d = 4$, the solutions will be completely equivalent[4] to the dyonic solutions of the theory described by Eq. (1.2).

For the extremal but non-BPS black holes in the theory described by Eq. (1.3), we find that if the charges (Q_1, Q_2) and $(\tilde{Q}_1, \tilde{Q}_2)$ of two such black holes are unequal,[5] then the force between the two is always repulsive.[6] (See Ref. [11] for some related discussion of black holes that repel one another.) Combining this with the fact that the force between identical black holes is attractive if they are non-extremal (i.e., subextremal), we conclude, by continuity, that that there should exist non-extremal black holes with unequal charges for which the parameters can be tuned so that a zero-force condition holds. It does not appear to be possible to obtain explicit analytic formulas for the general parameter choices that achieve such a zero-force condition.

[2] In theories with higher-derivative corrections, this is no longer true. The long-range force between extremal black holes receives corrections and does not generically vanish [7].

[3] We are neglecting $\mathcal{O}(\frac{1}{r^3})$ terms, as well as velocity-dependent forces.

[4] Essentially, the field strength F_2 in the $d = 4$ specialization of the theory [Eq. (1.3)] is acting like the dualization double of the field strength F_1.

[5] It will always be understood that we are restricting attention to cases where the *signs* of the charges will be the same for the two black holes. Obviously, there is little of interest to discuss if the charges of the two black holes are opposite, since then there would be an attractive electrostatic force that would reinforce the attractive gravitational force, and there could never be any question of achieving a zero-force balance. We shall always assume, without losing generality for the cases that are interesting to discuss, that the charges are all positive.

[6] To be more precise, the force between them is repulsive provided that $(\tilde{Q}_1, \tilde{Q}_2)$ is not a multiple of (Q_1, Q_2)—i.e., that $(\tilde{Q}_1, \tilde{Q}_2) \neq (kQ_1, kQ_2)$ for any constant k.

However, we are able to solve numerically to find examples of nonidentical, non-extremal black holes which have no long-range force between them. We also find an explicit analytical formula for the zero-force condition for a restricted subfamily of these black holes. This balancing of forces is interesting and may hint at a deeper structure, something which we would like to better understand. We shall come back to this point in the Conclusions.

The equations for black hole solutions in the theory described by the Lagrangian (1.3) can be recast as $SL(3,R)$ Toda equations.[7] In Ref. [10], extensions of the theory that give rise to the $SL(n,R)$ Toda equations were also considered, and the black hole solutions were constructed. For the $SL(n,R)$ case, there are now $(n-1)$ field strengths, each carrying an electric charge, and $(n-2)$ dilatonic scalar fields. In the present work, we study the black holes in the $SL(4,R)$ Toda theory, and show that analogous results to those we found for the $SL(3,R)$ Toda black holes arise here also.

The paper is organized as follows: In Sec. II, we discuss the general non-extremal two-charge black holes of the $SL(3,R)$ Toda theory, and obtain the expression for the force between two well-separated such black holes. Specializing to the case where the two black holes are extremal, we show that the force between them is always non-negative; that is, it is always either repulsive, or else it is zero. Specifically, the force vanishes if and only if the charges (Q_1,Q_2) and $(\tilde{Q}_1,\tilde{Q}_2)$ of the black holes are proportional—i.e., if $(\tilde{Q}_1,\tilde{Q}_2)=(kQ_1,kQ_2)$. The case $k=1$ corresponds to identical black holes. We then turn to the consideration of two non-extremal black holes. If they are identical, we find that the force is negative (i.e., attractive), in line with standard results in the literature. We then show, by means of numerical studies, that for two nonidentical non-extremal black holes, it is possible to choose the mass and charge parameters so that there is zero force between them. We also obtain an explicit analytical formula characterizing the zero-force condition for a special subset of the non-extremal black holes.

In Sec. III, we extend our discussion to black hole solutions of the $SL(4,R)$ Toda theory. We demonstrate that the same general features we found for the $SL(3,R)$ Toda black holes occur in this case also. Namely, the force between two nonidentical extremal black holes is in general repulsive, becoming zero when the black holes are identical and in certain other special cases. Furthermore, we show with numerical examples that one can tune the parameters of two non-extremal black holes such that the force between them vanishes.

In Sec. IV, we present our conclusions and further discussion. Appendix A contains some details of the calculation of the ADM mass and the scalar and electric charges, and it gives the relation between our normalizations for these quantities and the normalizations in Ref. [9]. In Appendix B, we present a calculation of the force between widely separated BPS black holes in a wide class of theories, to illustrate some salient features of the differences between the BPS and the non-BPS extremal black holes.

[7] See Ref. [12] for a study of the link between the timelike formulation of Ref. [6] and Toda equations, in the context of Kaluza-Klein black holes.

II. $SL(3,R)$ TODA BLACK HOLES

The $SL(3,R)$ Toda Lagrangian (1.3) admits two-charge static, asymptotically flat black hole solutions, given by [10]

$$
\begin{aligned}
ds^2 &= -(H_1H_2)^{-\frac{1}{2}} f dt^2 + (H_1H_2)^{\frac{1}{2(d-3)}}(f^{-1}dr^2 + r^2 d\Omega_{d-2}^2),\\
\phi &= \frac{1}{2}\sqrt{\frac{3(d-2)}{2(d-3)}}\log\left(\frac{H_1}{H_2}\right), \qquad f = 1-\frac{m}{r^{d-3}},\\
A_1 &= \sqrt{\frac{d-2}{d-3}}\frac{1-\beta_1 f}{\sqrt{\beta_1\gamma_2}H_1}dt, \qquad A_2\sqrt{\frac{d-2}{d-3}}\frac{1-\beta_2 f}{\sqrt{\beta_2\gamma_1}H_2}dt,\\
H_1 &= \gamma_1^{-1}(1-2\beta_1 f+\beta_1\beta_2 f^2),\\
H_2 &= \gamma_2^{-1}(1-2\beta_2 f+\beta_1\beta_2 f^2),\\
\gamma_1 &= 1-2\beta_1+\beta_1\beta_2, \qquad \gamma_2 = 1-2\beta_2+\beta_1\beta_2, \qquad (2.1)
\end{aligned}
$$

where m, β_1, and β_2 are constants that parametrize the mass and the two electric charges. Adopting convenient normalization, which we define in Appendix A, the mass M, the scalar charge Σ, and the two physical electric charges Q_1 and Q_2 of the black hole solutions are given by

$$
\begin{aligned}
M &= \frac{2(1-\beta_1)(1-\beta_2)(1-\beta_1\beta_2)m}{\gamma_1\gamma_2}\\
&= 2\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)(1-\beta_1\beta_2)m,\\
\Sigma &= \frac{\sqrt{3}(\beta_1-\beta_2)(1-\beta_1\beta_2)m}{\gamma_1\gamma_2},\\
Q_1 &= \frac{\sqrt{2\beta_1\gamma_2}m}{\gamma_1}, \qquad Q_2 = \frac{\sqrt{2\beta_2\gamma_1}m}{\gamma_2}. \qquad (2.2)
\end{aligned}
$$

It will be understood in all that follows that the charges Q_1 and Q_2 are assumed to be non-negative, so that the gauge force between two black holes will always be repulsive.[8]

[8] In this section, we always set the asymptotic (i.e., $r\to\infty$) value of the dilaton to zero in the mass, electric charges, and scalar charge. The reason for this choice is that the Lagrangian in Eq. (1.3) has a shift symmetry under $\phi\to\phi+\bar{\phi}$, with $F_1\to e^{-a\bar{\phi}/2}F_1$ and $F_2\to e^{a\bar{\phi}/2}F_2$. Because of this symmetry, we can scale the physical charges as $Q_1\to e^{a\bar{\phi}/2}Q_1$ and $Q_2\to e^{-a\bar{\phi}/2}Q_2$, and using this scaling, any expression with zero asymptotic value for the dilaton can be dressed with a nonzero asymptotic value. Moreover, since this is a symmetry, it will not change any conclusion we draw for the long-range force.

A different parametrization, which will prove useful for some purposes, is provided by expressing β_1 and β_2 in terms of two new parameters p and q, with

$$\beta_1 = \frac{(p-m)q}{(q+m)p}, \qquad \beta_2 = \frac{(q-m)p}{(p+m)q}. \tag{2.3}$$

In terms of p and q, the mass, scalar charge, and electric charges in Eq. (2.2) become

$$M = p + q, \qquad \Sigma = \frac{\sqrt{3}}{2}(p-q),$$
$$Q_1 = \left[\frac{p(p^2-m^2)}{(p+q)}\right]^{\frac{1}{2}}, \qquad Q_2 = \left[\frac{q(q^2-m^2)}{(p+q)}\right]^{\frac{1}{2}}. \tag{2.4}$$

It is useful also to note that the Hawking temperature of the black hole is given by [10]

$$T = \frac{(d-3)}{4\pi r_+}(\gamma_1\gamma_2)^{\frac{d-2}{4(d-3)}}, \tag{2.5}$$

where r_+ is the radius of the outer horizon, given by $r_+^{d-3} = m$. In terms of the parameters p and q introduced in Eq. (2.3), this becomes

$$T = \frac{(d-3)m}{4\pi}\left[\frac{4(p+q)^2}{pq(p+m)^2(q+m)^2}\right]^{\frac{d-2}{4(d-3)}}. \tag{2.6}$$

As we remarked earlier, in four dimensions, the black holes with two electric charges we are discussing here are equivalent to the KK dyonic black hole solutions of the theory given in Eq. (1.2), with the electric and magnetic charges of the single field strength F in Eq. (1.2) corresponding to the two electric charges Q_1 and Q_2 in Eq. (2.4). In fact, the parametrization using p and q as in Eq. (2.4) is precisely[9] the one used in Ref. [7], where the force between identical Kaluza-Klein dyons was discussed.

The force between two well-separated such black holes is given, up to an overall scale that we suppress for now, by $F = \mathcal{F} r^{-(d-2)}$, where

$$\mathcal{F} = Q_1\tilde{Q}_1 + Q_2\tilde{Q}_2 - \Sigma\tilde{\Sigma} - \frac{1}{4}M\tilde{M}, \tag{2.7}$$

where the untilded and tilded quantities refer to the two black holes, with the untilded quantities being given in terms of parameters (m, β_1, β_2), and the tilded quantities in terms of parameters $(\tilde{m}, \tilde{\beta}_1, \tilde{\beta}_2)$. For the explicit relation of our masses and charges to those defined in Ref. [9] we refer the reader to Appendix A.

[9]Note, however, that our extremality parameter m is twice that used in Ref. [7].

We now examine the nature of the force between the black holes in two cases: first for extremal black holes, and then for non-extremal black holes.

A. Extremal $SL(3,R)$ black holes

One way of taking the extremal limit of the $SL(3,R)$ black hole solutions is by setting [10]

$$\beta_1 = 1 - q_1^{-\frac{2}{3}} q_2^{-\frac{2}{3}} (q_1^{\frac{2}{3}} + q_2^{\frac{2}{3}})^{\frac{1}{2}} m + q_1^{-\frac{2}{3}} q_2^{-\frac{4}{3}} m^2,$$
$$\beta_2 = 1 - q_1^{-\frac{2}{3}} q_2^{-\frac{2}{3}} (q_1^{\frac{2}{3}} + q_2^{\frac{2}{3}})^{\frac{1}{2}} m + q_1^{-\frac{4}{3}} q_2^{-\frac{2}{3}} m^2, \tag{2.8}$$

in terms of two new charge parameters q_1 and q_2. The extremal limit is then attained by sending $m \to 0$. In this limit, the mass, scalar, and electric charges defined in Eq. (2.2) become

$$M_{\rm ext} = (q_1^{\frac{2}{3}} + q_2^{\frac{2}{3}})^{\frac{3}{2}},$$
$$\Sigma_{\rm ext} = \frac{\sqrt{3}}{2}(q_1^{\frac{2}{3}} - q_2^{\frac{2}{3}})(q_1^{\frac{2}{3}} + q_2^{\frac{2}{3}})^{\frac{1}{2}},$$
$$Q_1^{\rm ext} = q_1, \qquad Q_2^{\rm ext} = q_2. \tag{2.9}$$

In this extremal limit, the (double) horizon is at $r = 0$. Inserting Eq. (2.8) into the expression (2.5) for the Hawking temperature gives

$$T = \frac{(d-3)m}{4\pi}\left[\frac{2}{q_1 q_2}\right]^{\frac{d-2}{2(d-3)}} + \mathcal{O}(m^2), \tag{2.10}$$

which goes to zero, as one would expect, in the extremal limit $m \to 0$.

It is straightforward to verify that if two such extremal black holes have charges that are multiples of one another—i.e., if

$$(\tilde{q}_1, \tilde{q}_2) = (kq_1, kq_2), \tag{2.11}$$

then the force between them, given by substituting Eq. (2.9) into Eq. (2.7), is zero. This includes the special case $k = 1$, corresponding to two identical extremal black holes.

Next, we would like to examine the general case of two nonidentical extremal black holes. The solutions may be reparametrized in terms of (q, θ) and $(\tilde{q}, \tilde{\theta})$, where

$$q_1 = q^3 \cos^3\theta, \qquad q_2 = q^3 \sin^3\theta,$$
$$\tilde{q}_1 = \tilde{q}^3 \cos^3\tilde{\theta}, \qquad \tilde{q}_2 = \tilde{q}^3 \sin^3\tilde{\theta}. \tag{2.12}$$

With the understanding that the charges are all non-negative, we see that we must have

$$0 \le \theta \le \frac{1}{2}\pi, \qquad 0 \le \tilde{\theta} \le \frac{1}{2}\pi. \tag{2.13}$$

Plugging Eq. (2.12) into Eq. (2.9), and then these into the expression (2.7) for the force coefficient, we find

$$\mathcal{F} = q^3\tilde{q}^3 G, \tag{2.14}$$

where

$$G = \cos^3\theta\cos^3\tilde{\theta} + \sin^3\theta\sin^3\tilde{\theta} - \frac{3}{4}\cos 2\theta\cos 2\tilde{\theta} - \frac{1}{4}. \tag{2.15}$$

As we shall now show, G is non-negative when θ and $\tilde{\theta}$ lie anywhere in the square defined by Eq. (2.13).

First, we define $t = \tan\frac{1}{2}\theta$ and $\tilde{t} = \tan\frac{1}{2}\tilde{\theta}$, so we have

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2},$$
$$\sin\tilde{\theta} = \frac{2\tilde{t}}{1+\tilde{t}^2}, \qquad \cos\tilde{\theta} = \frac{1-\tilde{t}^2}{1+\tilde{t}^2}. \tag{2.16}$$

In view of Eq. (2.13), we have $0 \le t \le 1$ and $0 \le \tilde{t} \le 1$. Thus, we can parametrize t and $\tilde{t}$ as

$$t = \frac{a}{1+a}, \qquad \tilde{t} = \frac{b}{1+b}, \tag{2.17}$$

with

$$0 \le a \le \infty, \qquad 0 \le b \le \infty. \tag{2.18}$$

Plugging these substitutions into the definition of G in Eq. (2.15), we find

$$G = \frac{2(a-b)^2 P(a,b)}{(1+2a+2a^2)^3(1+2b+2b^2)^3}, \tag{2.19}$$

where

$$\begin{aligned} P(a,b) = {} & 3(a+b)^2 + 6(a+b)(a^2+4ab+b^2) \\ & + 2(a^4+20a^3b+36a^2b^2+20ab^3+b^4) \\ & + 12ab(a+b)(a^2+4ab+b^2) \\ & + 12a^2b^2(a+b)^2. \end{aligned} \tag{2.20}$$

Since all the coefficients in $P(a,b)$ are positive, it follows that $P(a,b) \ge 0$ for all a and b in the range (2.18), and thus we see from Eq. (2.19) that $G \ge 0$.

Thus, we have shown that the force between any two extremal $SL(3,R)$ Toda black holes is always non-negative, and that it is strictly positive [i.e., repulsive], provided that the charges of the two black holes are not proportional [i.e., provided that Eq. (2.11) is not satisfied for any constant k].

It is worth remarking that taking the extremal limit is somewhat more straightforward in the parametrization using p and q as in Eq. (2.3), since one can now simply set $m = 0$ in the expressions in Eq. (2.4) for the mass, scalar charge, and electric charges, without the need for taking a delicate limit. Indeed, at extremality, the electric charges q_1 and q_2 in Eq. (2.8) are given in terms of p and q by

$$q_1 = \frac{p^{\frac{3}{2}}}{(p+q)^{\frac{1}{2}}}, \qquad q_2 = \frac{q^{\frac{3}{2}}}{(p+q)^{\frac{1}{2}}}, \tag{2.21}$$

as can be seen directly from Eq. (2.4). Note that in this parametrization, one can directly see that the Hawking temperature of the black hole becomes zero in the extremal limit by setting $m = 0$ in Eq. (2.6).

B. Non-extremal $SL(3,R)$ black holes

Now consider non-extremal black holes, characterized by the parameters (m, β_1, β_2) and $(\tilde{m}, \tilde{\beta}_1, \tilde{\beta}_2)$, respectively. The general expression for the force between two distinct non-extremal black holes is the following:

$$\begin{aligned} \mathcal{F} = {} & \frac{m\tilde{m}}{\gamma_1\gamma_2\tilde{\gamma}_1\tilde{\gamma}_2}[2(\beta_1\tilde{\beta}_1)^{\frac{1}{2}}(\gamma_2\tilde{\gamma}_2)^{\frac{3}{2}} + 2(\beta_2\tilde{\beta}_2)^{\frac{1}{2}}(\gamma_1\tilde{\gamma}_1)^{\frac{3}{2}} \\ & - 3(\beta_1-\beta_2)(1-\beta_1\beta_2)(\tilde{\beta}_1-\tilde{\beta}_2)(1-\tilde{\beta}_1\tilde{\beta}_2) \\ & - (1-\beta_1)(1-\beta_2)(1-\beta_1\beta_2)(1-\tilde{\beta}_1)(1-\tilde{\beta}_2) \\ & \times (1-\tilde{\beta}_1\tilde{\beta}_2)]. \end{aligned} \tag{2.22}$$

It can be verified easily that if we consider the case where

$$(\tilde{\beta}_1, \tilde{\beta}_2) = (\beta_1, \beta_2), \tag{2.23}$$

which corresponds to the tilded electric charges being an overall multiple of the untilded electric charges [see the expressions in Eq. (2.2) for Q_1 and Q_2], then the force between the non-extremal black holes will be attractive, with $\mathcal{F}$ given simply by

$$\mathcal{F} = -m\tilde{m}. \tag{2.24}$$

The special case of identical non-extremal black holes arises when the further condition $\tilde{m} = m$ is imposed.[10]

[10]It can also be seen that if the black holes obeying Eq. (2.23) are taken to be extremal, by sending m and $\tilde{m}$ to zero, then the force (2.24) between them becomes zero, as already noted for extremal black holes whose charges are proportional. Note that one must be careful when taking the extremal limit in this parametrization, since the β_i parameters must be taken to 1 at the same time, as seen in the limiting procedure in Eq. (2.8). In particular, having imposed the requirement (2.23), one could not take the $m = 0$ extremal limit for the untilded black hole without also taking the $\tilde{m} = 0$ limit for the tilded black hole, since otherwise the tilded charges $\tilde{Q}_i$ would become infinite.

Recall that for the case of two *distinct* extremal black holes, we have shown that the long-range force is always positive. Thus, since the force must presumably be a continuous function of the parameters, we conclude that for a general pair of non-extremal black holes, there must exist some choices of the m and β_i parameters which yield a zero-force condition. Indeed, here is a numerical example:

We write β_1 and β_2 as in Eq. (2.8), but we do *not* send m to zero.[11] We do likewise for $\tilde{\beta}_1$ and $\tilde{\beta}_2$ (taking, for convenience, $\tilde{m} = m$). Making a specific choice for the untilded and tilded q_i parameters, namely

$$q_1 = 1, \qquad q_2 = 2, \qquad \tilde{q}_1 = 3, \qquad \tilde{q}_2 = 7, \tag{2.25}$$

we then look at the numerical value of the force coefficient $\mathcal{F}$, as a function of m. We find that, approximately,

$$\begin{aligned} m > 0.066 &\quad \text{implies} \quad \mathcal{F} < 0, \\ m < 0.066 &\quad \text{implies} \quad \mathcal{F} > 0. \end{aligned} \tag{2.26}$$

In other words, for this example of two black holes whose charges are not simply an overall multiple of one another, the force between them is repulsive when they are sufficiently close to being extremal, but it becomes attractive when they are taken to be sufficiently far from extremality. The zero-force condition arises when the non-extremality parameter m is roughly equal to 0.066000632.

As another example, if we take

$$q_1 = 1, \qquad q_2 = 2, \qquad \tilde{q}_1 = 1, \qquad \tilde{q}_2 = 3, \tag{2.27}$$

then the crossover between repulsion and attraction occurs when the non-extremality parameter m is approximately

$$m = 0.2073984664. \tag{2.28}$$

Note that it is necessary to check that the constants $\beta_i, \tilde{\beta}_i, \gamma_i$, and $\tilde{\gamma}_i$ are all non-negative, in order to ensure that the black holes are regular from the horizon to asymptotic infinity. In the examples above, these conditions are indeed satisfied.

We may also give an explicit construction of a special family of nonidentical, non-extremal black holes that obey the zero-force condition. Using the parametrization in terms of p and q as in Eq. (2.3), the expression (2.22) for the force between two non-extremal black holes becomes

[11] At this stage, therefore, we just have a reparametrization of non-extremal black holes in terms of m, q_1, and q_2 rather than m, β_1, and β_2. It is a convenient reparametrization to adopt here since it allows us explore the situation where the black hole is becoming close to extremality, by taking m to be fairly small (in comparison to q_1 and/or q_2). Note that q_1 and q_2 are *not* simply multiples of the physical charges Q_1 and Q_2, except in the actual extremal limit where $m \to 0$.

$$\begin{aligned} \mathcal{F} = &-\frac{1}{4}[(p+q)(\tilde{p}+\tilde{q}) + 3(p-q)(\tilde{p}-\tilde{q})] \\ &+ \left[\frac{p\tilde{p}(p^2-m^2)(\tilde{p}^2-\tilde{m}^2)}{(p+q)(\tilde{p}+\tilde{q})}\right]^{\frac{1}{2}} \\ &+ \left[\frac{q\tilde{q}(q^2-m^2)(\tilde{q}^2-\tilde{m}^2)}{(p+q)(\tilde{p}+\tilde{q})}\right]^{\frac{1}{2}}. \end{aligned} \tag{2.29}$$

As we saw previously, for two identical non-extremal black holes, the force becomes $\mathcal{F} = -m^2$. Note that in order to avoid imaginary charges and negative mass, we should restrict p and q such that $p \geq m$ and $q \geq m$.

Consider now the following specialization: Take

$$\tilde{m} = m, \qquad \tilde{p} = q, \qquad \tilde{q} = p, \tag{2.30}$$

for which the expression (2.29) becomes

$$\mathcal{F} = \frac{2\sqrt{pq}\sqrt{(p^2-m^2)(q^2-m^2)}}{p+q} + \frac{1}{2}(p^2+q^2-4pq). \tag{2.31}$$

For $p = q$, we find $\mathcal{F} = -m^2$ (as is to be expected for two identical non-extremal black holes); the second term is negative, and it outweighs the first term. If instead $p \neq q$, then under certain circumstances the second term in Eq. (2.31) will be positive, and so $\mathcal{F}$ will be positive. Thus, within this considerably simplified class of solutions, we can find explicit expressions for intermediate cases that achieve a zero-force condition:

Writing

$$p = xq, \tag{2.32}$$

we can solve for the ratio q^2/m^2 that makes $\mathcal{F}$ in Eq. (2.31) vanish. This gives

$$\frac{q^2}{m^2} = -\frac{4\left[2x(1+x^2) \pm \sqrt{x}(1+x)\sqrt{1-4x+10x^2-4x^3+x^4}\right]}{(x-1)^2(1-4x-6x^2-4x^3+x^4)}. \tag{2.33}$$

We must choose x so that we have $q^2 - m^2 > 0$, and also $p^2 - m^2 > 0$ (i.e., $q^2x^2 - m^2 > 0$). This implies that we should make the upper sign choice in Eq. (2.33), and so, defining

$$W(x) \equiv -\frac{4\left[2x(1+x^2) + \sqrt{x}(1+x)\sqrt{1-4x+10x^2-4x^3+x^4}\right]}{(x-1)^2(1-4x-6x^2-4x^3+x^4)}, \tag{2.34}$$

we shall have

$$\frac{q^2}{m^2} = W(x), \qquad \frac{p^2}{m^2} = x^2 W(x) = W\left(\frac{1}{x}\right). \tag{2.35}$$

Since sending $x \to x^{-1}$ exchanges the roles of p and q, which corresponds to the symmetry of the $SL(3,R)$ theory under the reflection of the $SL(3,R)$ Dynkin diagram, we can, without loss of generality, restrict attention to taking x, which must be positive, to lie in the interval $1 \leq x \leq \infty$.

In the interval $1 \leq x \leq \infty$, it is evident from Eq. (2.35) that the conditions $p^2 \geq m^2$ and $q^2 \geq m^2$ will be satisfied if $W(x) \geq 1$, and one can see from Eq. (2.34) that this will hold if

$$1 < x < x_+, \quad \text{where } x_+ \approx 5.27451, \tag{2.36}$$

with x_+ being the larger of the two real roots of $1 - 4x - 6x^2 - 4x^3 + x^4 = 0$. The function $W(x)$ is strictly greater than 1 for all x in the interval (2.36), except when $x = 2 + \sqrt{3} \approx 3.73205$, for which W becomes equal to 1.

As a concrete example, if we take $x = 3$, then

$$q = m\sqrt{\frac{15 + 2\sqrt{39}}{23}}, \qquad p = 3m\sqrt{\frac{15 + 2\sqrt{39}}{23}}. \tag{2.37}$$

This satisfies all the necessary constraints, and indeed gives $\mathcal{F} = 0$. For this particular example, the temperatures of the two non-extremal black hole solutions are the same [since, as one can see from Eq. (2.6), if we denote the temperature by $T(p,q,m)$, then we have $T(p,q,m) = T(q,p,m)$]. By contrast, in each of the previous numerical examples we presented, the temperatures were unequal for the two non-extremal black holes for which a no-force condtion held. While the equality of the temperatures in the example in Eq. (2.30) is due to the symmetrical parameter choices of these particular solutions, it may also hint at the existence of a new multicharge black hole. We will return to this point in the Conclusions. Finally, we anticipate that there should not be any obstruction to finding a vanishing force for parameter choices corresponding to properly quantized physical charges, once one is more careful with normalizations and properly reinstates units.

III. $SL(4,R)$ TODA BLACK HOLES

$SL(n,R)$ Toda black holes are discussed in Ref. [10], and additional explicit details are given for the $SL(4,R)$ case. The Lagrangian for the $SL(n,R)$ case is given by

$$\mathcal{L} = R - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{4}\sum_{i=1}^{n-1} e^{\vec{a}_i\cdot\vec{\phi}} F_i^2, \tag{3.1}$$

where the $(n-1)$ dilaton vectors $\vec{a}_i$ satisfy

$$\begin{aligned}
\vec{a}_i\cdot\vec{a}_i &= \frac{1}{3}(n-2)(n^2+2n+3)\frac{d-3}{d-2}, \quad (\text{no sum on } i),\\
\vec{a}_i\cdot\vec{a}_{i+1} &= -\frac{1}{6}(n^3-n+12)\frac{d-3}{d-2},\\
\vec{a}_i\cdot\vec{a}_j &= -\frac{2(d-3)}{d-2}, \qquad i \neq j-1, j, j+1.
\end{aligned} \tag{3.2}$$

There are $n-2$ dilatonic scalars, so the dilaton vectors are $(n-2)$-component vectors. For the case of $SL(4,R)$, we can satisfy the conditions in Eq. (3.2) by choosing

$$\begin{aligned}
\vec{a}_1 &= (\sqrt{8\nu}, \sqrt{10\nu}), \qquad \vec{a}_2 = (-\sqrt{18\nu}, 0),\\
\vec{a}_3 &= (\sqrt{8\nu}, -\sqrt{10\nu}), \quad \text{where } \nu \equiv \frac{d-3}{d-2}.
\end{aligned} \tag{3.3}$$

The $SL(4,R)$ Toda black hole solutions, involving three field strengths, each carrying an electric charge, and two dilatonic scalar fields, were constructed in Ref. [10]. They are given by

$$\begin{aligned}
ds^2 &= -(H_1H_2H_3)^{-\frac{1}{5}} f\,dt^2\\
&\quad + (H_1H_2H_3)^{\frac{1}{5(d-3)}}(f^{-1}dr^2 + r^2 d\Omega_{d-2}^2),\\
\vec{\phi} &= \frac{1}{10\nu}\sum_i \vec{a}_i \log H_i,\\
A_1 &= \sqrt{\frac{3}{5\nu}}\frac{1-2\beta_1 f + \beta_1\beta_2 f^2}{\sqrt{\beta_1\gamma_2}H_1}dt,\\
A_3 &= \sqrt{\frac{3}{5\nu}}\frac{1-2\beta_1 f + \beta_3\beta_2 f^2}{\sqrt{\beta_3\gamma_2}H_3}dt,\\
A_2 &= \sqrt{\frac{4}{5\nu}}\frac{1-3\beta_2 f + \frac{3}{2}\beta_2(\beta_1+\beta_3)f^2 - \beta_1\beta_2\beta_3 f^3}{\sqrt{\beta_2\gamma_1\gamma_3}H_2}dt,
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
H_1 &= \gamma_1^{-1}(1 - 3\beta_1 f + 3\beta_1\beta_2 f^2 - \beta_1\beta_2\beta_3 f^3),\\
H_2 &= \gamma_2^{-1}(1 - 4\beta_2 f + 3\beta_2(\beta_1+\beta_3)f^2\\
&\qquad - 4\beta_1\beta_2\beta_3 f^3 + \beta_1\beta_2^2\beta_3 f^4),\\
H_3 &= \gamma_3^{-1}(1 - 3\beta_3 f + 3\beta_2\beta_3 f^2 - \beta_1\beta_2\beta_3 f^3),\\
f &= 1 - \frac{m}{r^{d-3}},
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\gamma_1 &= 1 - 3\beta_1 + 3\beta_1\beta_2 - \beta_1\beta_2\beta_3,\\
\gamma_2 &= 1 - 4\beta_2 + 3\beta_2(\beta_1+\beta_3) - 4\beta_1\beta_2\beta_3 + \beta_1\beta_2^2\beta_3,\\
\gamma_3 &= 1 - 3\beta_3 + 3\beta_2\beta_3 - \beta_1\beta_2\beta_3.
\end{aligned} \tag{3.6}$$

The ADM mass of the $SL(4,R)$ Toda black hole is given in Ref. [10], and it can be written (in our choice of overall normalization) as.[12]

$$M = \left(\frac{3k_1}{\gamma_1} + \frac{4k_2}{\gamma_2} + \frac{3k_3}{\gamma_3}\right)\frac{m}{5}, \tag{3.7}$$

where

$$\begin{aligned} k_1 &= 1 - \beta_1 - \beta_1\beta_2 + \beta_1\beta_2\beta_3, \\ k_2 &= 1 - 2\beta_2 + 2\beta_1\beta_2\beta_3 - \beta_1\beta_2^2\beta_3, \\ k_3 &= 1 - \beta_3 - \beta_2\beta_3 + \beta_1\beta_2\beta_3. \end{aligned} \tag{3.8}$$

With the overall normalization we are adopting, the three electric charges are given by [10]

$$Q_1 = \frac{\sqrt{6\beta_1\gamma_2}m}{\sqrt{5}\gamma_1}, \quad Q_2 = \frac{\sqrt{8\beta_2\gamma_1\gamma_3}m}{\sqrt{5}\gamma_2}, \quad Q_3 = \frac{\sqrt{6\beta_3\gamma_2}m}{\sqrt{5}\gamma_3}. \tag{3.9}$$

The two scalar charges, read off from the coefficients of the r^{-d+3} terms in the asymptotic expressions for the dilaton fields, are given, in our normalization, by the 2-vector

$$\vec{\Sigma} = \frac{1}{5\sqrt{2\nu}}\sum_i \vec{a}_i \ell_i, \tag{3.10}$$

and hence

$$\Sigma_1 = \frac{2\ell_1 - 3\ell_2 + 2\ell_3}{5}, \qquad \Sigma_2 = \frac{\ell_1 - \ell_3}{\sqrt{5}}, \tag{3.11}$$

where ℓ_i are the coefficients of the r^{-d+3} terms in the expressions for $H_i = 1 + \frac{\ell_i}{r^{d-3}} + \cdots$, and are given by

$$\begin{aligned} \ell_1 &= \frac{3\beta_1 m}{\gamma_1}(1 - 2\beta_2 + \beta_2\beta_3), \quad \ell_3 = \frac{3\beta_3 m}{\gamma_3}(1 - 2\beta_2 + \beta_1\beta_2), \\ \ell_2 &= \frac{4\beta_2 m}{\gamma_2}\left(1 - \frac{3}{2}(\beta_1 + \beta_3) + 3\beta_1\beta_3 - \beta_1\beta_2\beta_3\right). \end{aligned} \tag{3.12}$$

The force between two black holes, characterized by untilded and tilded parameters $(m, \beta_1, \beta_2, \beta_3)$ and $(\tilde{m}, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3)$, is given by

$$\mathcal{F} = \sum_{i=1}^{3} Q_i\tilde{Q}_i - \vec{\Sigma}\cdot\vec{\tilde{\Sigma}} - \frac{1}{4}M\tilde{M}. \tag{3.13}$$

If the two black holes are identical, this gives

$$\mathcal{F} = -m^2. \tag{3.14}$$

As expected, this vanishes in the extremal case ($m = 0$) and is negative—implying an attractive force—in the subextremal case.

As we also saw in the case of the $SL(3,R)$ Toda system, studying the force between two nonidentical black holes is a lot more involved. However, the logic here is similar to that of the previous section. As we shall discuss in more detail below, in the case of unequal extremal black holes the force can be positive, and although we do not have a general proof in this case, we can expect that it will always be positive, just as we saw for the $SL(3,R)$ examples.

By continuity, given that the force between identical non-extremal black holes is negative, we can again expect, just as for $SL(3,R)$ black holes, that there should exist nonidentical pairs of non-extremal $SL(4,R)$ black holes for which a zero-force condition holds. Here, we present one explicit numerical example. We consider two non-extremal $SL(4,R)$ black holes with parameters

$$\begin{aligned} m = \tilde{m}, \quad (\beta_1, \beta_2, \beta_3) &= \left(\frac{1}{5}, \frac{1}{3}, \frac{1}{4}\right), \\ (\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3) &= \left(\frac{1}{5} + w, \frac{1}{3} + w, \frac{1}{4} + w\right). \end{aligned} \tag{3.15}$$

When $w = 0$, the force is just given by $\mathcal{F} = -m^2$, as mentioned previously. We find that the force becomes zero if the parameter w is tuned to

$$w \approx 0.03742503. \tag{3.16}$$

In the case of extremal $SL(4,R)$ Toda black holes, we can follow the procedure described in Ref. [10] for taking the extremal limit, by writing[13] the β_i in terms of new parameters a, b, c, as follows:

$$\beta_1 = 1 - am + a^2bm^2, \quad \beta_2 = 1 - am + \frac{1}{2}a^3(c-1)m^3, \quad \beta_3 = 1 - am - a^2bm^2, \tag{3.17}$$

and then sending m to zero. Before taking this limit, the temperature is given by [10]

$$T = \frac{(d-3)}{4\pi r_+}(\gamma_1\gamma_2\gamma_3)^{\frac{d-2}{10(d-3)}}, \tag{3.18}$$

which implies, using the parametrization in Eq. (3.17), that

[12]Again, we choose to set the asymptotic value of the dilaton to zero for reasons mentioned in footnote 8.

[13]A sign error in Ref. [10] in the expression for β_2 is corrected here.

$$T = \frac{(d-3)m}{4\pi}[a^{10}(c^2-9b^2)(3+3b^2-2c)]^{\frac{d-2}{10(d-3)}} + \mathcal{O}(m^2), \tag{3.19}$$

which indeed vanishes when $m = 0$.

Using Eq. (3.17) and then sending m to zero gives charges and mass as follows:

$$Q_1 = \frac{\sqrt{6(3+3b^2-2c)}}{\sqrt{5}a(c-3b)}, \qquad Q_2 = \frac{2\sqrt{2}\sqrt{c^2-9b^2}}{\sqrt{5}a(3+3b^2-2c)},$$
$$Q_3 = \frac{\sqrt{6(3+3b^2-2c)}}{\sqrt{5}a(c+3b)},$$
$$\Sigma_1 = \frac{6(9b^2-18b^4+6c+9b^2c-7c^2+c^3)}{5a(3+3b^2-2c)(c^2-9b^2)},$$
$$\Sigma_2 = \frac{6b(3-c)}{\sqrt{5}a(c^2-9b^2)},$$
$$M = \frac{4(36b^2c+9c-54b^2-27b^4-3c^2-c^3)}{5a(3+3b^2-2c)(c^2-9b^2)}. \tag{3.20}$$

The parameters b and c are constrained by the requirements that

$$0 \le b \le 1, \qquad 3b \le c \le \frac{3}{2}(1+b^2), \tag{3.21}$$

where we have, without loss of generality, required b to be non-negative [there is a symmetry of the solution, corresponding to reflecting the $SL(4,R)$ Dynkin diagram, under sending $b \to -b$ and exchanging the "1" and "3" labels on the electric charges]. It will be convenient to parametrize the allowed ranges of values for b and c in terms of two parameters x and y that can independently range over $0 \le x \le \infty$ and $0 \le y \le \infty$ by writing

$$c = 3b + \frac{3(1-b)^2 y}{2(1+y)}, \qquad b = \frac{x}{1+x}. \tag{3.22}$$

From looking at numerous numerical examples, it would seem that the force between any two unequal-charge extremal $SL(4,R)$ Toda black holes is positive (i.e., replulsive). Since we were able to prove this explicitly for the analogous $SL(3,R)$ case, it would seem likely that the force will be always repulsive in this case too. Although we have not constructed an explicit proof in general, we are able to show that when the parameters of the two extremal black holes are close to one another, the force is always repulsive. Specifically, we may consider the situation where

$$\tilde{a} = a, \qquad \tilde{b} = b + \epsilon_1, \qquad \tilde{c} = c + \epsilon_2, \tag{3.23}$$

and then look at the expression for the long-range force at leading order in the small quantities ϵ_1 and ϵ_2. (Since the parameters a and $\tilde{a}$ appear only as overall scaling factors in the expressions for masses and charges, there is no loss of generality, from the point of view of establishing a positivity result, in simply taking $\tilde{a}$ to be equal to a.) The leading-order terms in the expression for the force arise at the quadratic order in the ϵ parameters, so $\mathcal{F}$ takes the form

$$\mathcal{F} = h_{ij}\epsilon_i\epsilon_j + \mathcal{O}(\epsilon^3). \tag{3.24}$$

We find that

$$\det(h_{ij}) = \frac{64(1+x)^{10}(1+y)^5(1+4x+2x^2+2y+4xy+2x^2y)}{225a^4y^2(4x+4x^2+y+4xy+4x^2y)^2}, \tag{3.25}$$

which is non-negative, since $0 \le x \le \infty$ and $0 \le y \le \infty$. We also find that $\sum_i h_{ii}$ is manifestly non-negative. Thus, it must be that the two eigenvalues of h_{ij} are non-negative, and so at least to quadratic order in perturbations around the case of identical extremal black holes, the force is always repulsive, as anticipated.

IV. DISCUSSION AND CONCLUSIONS

Motivated quite naturally by studies of the WGC, there has been recent interest in exploring the relation between black hole extremality bounds and zero-force conditions. In particular, the RFC encodes the idea that gravity is the weakest force by requiring (in its simplest form) the existence of self-repulsive states. However, while the WGC has been examined extensively—and partial proofs have appeared in a variety of contexts—the RFC is thus far less studied and much less understood. While the two conjectures are identical in two-derivative theories that contain only gravity and electromagnetic forces (where a charged state with $Q > M$ will clearly repel itself), they are known to be distinct in the presence of scalar fields, which can mediate new long-range interactions. In this context, the fact that long-range forces between identical extremal black holes vanish independently of the complexity of the matter sector (which may include a variety of scalar and gauge fields) is quite nontrivial.

The situation is more complicated in the presence of higher-derivative corrections, where there are examples of theories in which forces between extremal black holes do not have definite signs, and of directions in charge space with no self-repulsive states [7]. Indeed, the analysis of Ref. [7] suggests that at least the simplest versions of the RFC may be violated by the black hole spectrum—in stark contrast with the WGC, which could in principle be satisfied entirely by black holes. While this does not rule out the RFC—it simply requires the existence of self-repulsive fundamental particles—it does highlight the fact that the conjecture is fundamentally different from the

WGC, and raises the question of to what extent long-range forces can be used to constrain low-energy EFTs and how much nontrivial information they encode. Higher-derivative corrections to long-range forces were also studied in Ref. [13], which computed α' corrections to families of heterotic multicenter black hole solutions. Interestingly, in the cases studied in Ref. [13], the zero-force condition between extremal black holes was preserved even in the presence of higher derivatives, for both supersymmetric and nonsupersymmetric solutions. Once again, this raises the question of how to interpret more generally the implications of the balancing of forces.

Motivated by these issues, in this paper we have extended the studies of long-range forces to black holes that are *not* identical. We were primarily interested in whether one could identify any generic features in the behavior of the corresponding forces, and perhaps shed light on what properties of black hole solutions can be captured by treating them as widely separated point particles, working essentially in a nonrelativistic, weak-field limit. As we have seen, the behavior of the forces is quite rich and leads to a number of novel results. In particular, any pair of (non-BPS) extremal black holes that are not identical (up to overall scaling) repel one another. We constructed a proof for the case of $SL(3,R)$ Toda black holes, and similar considerations seemingly apply to the more complicated example of $SL(4,R)$ Toda black holes. Since we have restricted our attention to somewhat simple classes of black holes which carry only two (electric) charges [in the $SL(3,R)$ Toda case], a natural next step is to extend our analysis to the most general class of STU black holes in four dimensions, to see whether the features we have identified here persist or not for broader classes of solutions, including magnetic charges. Work in this direction is in progress [14].

We have also identified pairs of non-extremal, non-identical black holes which obey a zero-force condition. While such balancing of forces is well known in the context of supersymmetric BPS black holes in supergravity, where it holds even for black holes that carry different charges, it is unexpected here. Moreover, for multicenter BPS black holes, the cancellation of the forces holds for arbitrary relative positions of the centers. On the other hand, in the theories we have examined, the zero-force condition applies only to well-separated black holes (i.e., in the long-range limit) and for specific choices of parameters. Nonetheless, we wonder whether it may indicate the existence of new multicenter black hole solutions (perhaps at nearby points in phase space), especially for cases where the two black holes have the same temperature. Indeed, it would be valuable to better understand to what extent no-force conditions such as the ones we have found in this paper can be used as a diagnostic tool for identifying and generating new solutions, aided by suitable constraints on, e.g., the temperature, regularity properties, and so on. It would also be interesting to understand this logic in light of the results of Ref. [13], where both supersymmetric and nonsupersymmetric black holes, corrected by higher derivatives, satisfy a no-force condition.

A more challenging question is to understand how to connect our results to expectations from the RFC. As we have seen, in the theories we have studied, distinct extremal black holes repel. If this observation is a generic feature of top-down theories that support broader classes of black hole solutions, it may help to identify the kinds of repulsive multiparticle states that the strong version of the RFC would call for. We leave these questions to future work.

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APPENDIX A: MASS AND CHARGES FOR STATIC BLACK HOLES

1. Mass

The static d-dimensional black-hole metrics can be written in the form

$$ds^2 = -uf\,dt^2 + u^{-\frac{1}{d-3}}(f^{-1}dr^2 + r^2 d\Omega_{d-2}^2), \quad \text{(A1)}$$

where u and f are functions only of r.

We can expand the metric around Minkowski spacetime by introducing the coordinates

$$x = t, \qquad x_i = rn_i, \quad \text{where } n_i n_i = 1, \quad \text{(A2)}$$

and the unit vector n_i is parametrized in terms of the $(d-2)$ angular coordinates on the unit $(d-2)$-sphere. We shall have $dn_i dn_i = d\Omega_{d-2}^2$. We can now write the metric (A1) as follows:

$$\begin{aligned} ds^2 &= -uf(dx^0)^2 + u^{-\frac{1}{d-3}}(f^{-1}-1)dr^2 \\ &\quad + u^{-\frac{1}{d-3}}(dr^2 + r^2 d\Omega_{d-2}^2), \\ &= -uf(dx^0)^2 + u^{-\frac{1}{d-3}}(f^{-1}-1)\frac{x_i x_j}{r^2}dx^i dx^j \\ &\quad + u^{-\frac{1}{d-3}}dx^i dx^i, \\ &= \eta_{\mu\nu}dx^\mu dx^\nu + h_{\mu\nu}dx^\mu dx^\nu, \end{aligned} \quad \text{(A3)}$$

with

$$h_{00} = 1 - uf, \quad h_{ij} = (u^{-\frac{1}{d-3}} - 1)\delta_{ij} + u^{-\frac{1}{d-3}}(f^{-1} - 1)\frac{x_i x_j}{r^2},$$
$$h_{0i} = 0. \tag{A4}$$

The metric functions f and u have the form

$$f = 1 - m\rho, \quad u = 1 - \sigma\rho + \mathcal{O}(\rho^2), \quad \text{where } \rho = \frac{1}{r^{d-3}}, \tag{A5}$$

and m and σ are constants. If we assume that r is very large, so that the metric is nearly Minkowski, we can focus just on terms up to linear order in ρ. We then have

$$h_{00} = (m + \sigma)\rho + \cdots, \quad h_{ij} = \frac{\sigma}{d-3}\delta_{ij}\rho + m\rho\frac{x_i x_j}{r^2} + \cdots, \tag{A6}$$

which implies

$$\partial_j h_{ij} = (m - \sigma)\frac{x_i}{r^{d-1}} + \cdots. \tag{A7}$$

Applying the ADM mass formula, we find

$$\begin{aligned} M_{\text{ADM}} &= \frac{1}{16\pi}\int_\Sigma d\Sigma_i(\partial_j h_{ij} - \partial_i h_{jj}) \\ &= \frac{1}{16\pi}\int_\Sigma (d-2)(m+\sigma)d\Omega_{d-2} \\ &= (d-2)(m+\sigma)\frac{\Omega_{d-2}}{16\pi}. \end{aligned} \tag{A8}$$

Applied to the $SL(3,R)$ Toda black holes, this gives

$$M_{\text{ADM}} = \frac{(d-2)\Omega_{d-2}}{16\pi}\frac{(1-\beta_1)(1-\beta_2)(1-\beta_1\beta_2)}{\gamma_1\gamma_2}, \tag{A9}$$

in agreement with the expression given in Ref. [10].

In Ref. [9], the black hole mass is calculated using the fact that in De Donder gauge, the trace-reversed linearized metric perturbation $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$, satisfying the gauge condition $\partial^\mu \bar{h}_{\mu\nu} = 0$, obeys $\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$. In particular, in Ref. [9], the mass M_H is read off from the expression (setting $\kappa^2 = \frac{1}{2}$ to accord with our conventions)

$$\bar{h}_{00} = \frac{M_H}{(d-3)\Omega_{d-2}}\frac{1}{r^{d-3}} + \cdots. \tag{A10}$$

The linearized metric $h_{\mu\nu}$ given in Eq. (A4) is in fact not in De Donder gauge: We have

$$h = -h_{00} + h_{ii} = \frac{2\sigma}{d-3}\rho + \cdots, \tag{A11}$$

and since the metric is static, we need only check

$$\partial_j \bar{h}_{ij} = \partial_j h_{ij} - \partial_i h = \frac{m x_i}{r^{d-1}} + \cdots, \tag{A12}$$

which is nonzero at this leading order. We can easily make a coordinate transformation to put $h_{\mu\nu}$ in De Donder gauge at the leading order by considering

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}. \tag{A13}$$

Let us try taking ξ_μ to be given by

$$\xi_0 = 0, \qquad \xi_i = \frac{\alpha x_i}{r^{d-1}}, \tag{A14}$$

where α is a constant to be determined. Thus, we shall have

$$h \to h' = h + \frac{2\alpha}{r^{d-3}}, \tag{A15}$$

and so

$$\partial_j \bar{h}_{ij} \to \partial_j \bar{h}'_{ij} = \partial_j \bar{h}_{ij} - \frac{\alpha x_i}{r^{d-1}} = \frac{[m - (d-3)\alpha]x_i}{r^{d-1}} + \cdots, \tag{A16}$$

implying that $\bar{h}'_{\mu\nu}$ will be in De Donder gauge at leading order if we choose $\alpha = m(d-3)^{-1}$. Finally, note that this gives

$$\bar{h}'_{00} = \bar{h}_{00} + \frac{m}{d-3}\rho + \cdots, \tag{A17}$$

and so

$$\begin{aligned} \bar{h}'_{00} &= \left[m + \frac{(d-2)\sigma}{d-3}\right]\rho + \frac{m}{d-3}\rho + \cdots \\ &= \frac{(d-2)(m+\sigma)}{d-3}\rho + \cdots. \end{aligned} \tag{A18}$$

Thus, we find that the mass M_H in Ref. [9] is given by

$$M_H = (d-2)\Omega_{d-2}(m+\sigma) = 16\pi M_{\text{ADM}}. \tag{A19}$$

In terms of the rescaled mass M that we defined in Eq. (2.2), we therefore have

$$M_H = \frac{1}{2}(d-2)\Omega_{d-2}M. \tag{A20}$$

2. Scalar charge

The role of scalar charge in black hole thermodynamics was described in Ref. [15]. In Eq. (4.16) of Ref. [9], the scalar charge $\mu = M'(\phi_0)$ is read off from the leading falloff term in the large-distance expansion of the scalar field:

$$\phi = \phi_0 - \frac{G^{-1}_{\phi\phi} M'(\phi_0)}{(d-3)\Omega_{d-2}} \frac{1}{r^{d-3}} + \cdots. \tag{A21}$$

If we write the large-distance forms of the two functions H_1 and H_2 in the solution (2.1) as

$$H_i = 1 + \ell_i \rho + \cdots, \tag{A22}$$

then we see from the expression for ϕ in Eq. (2.1) that

$$\phi = \frac{1}{2}\sqrt{\frac{3(d-2)}{2(d-3)}}(\ell_2 - \ell_1)\rho + \cdots, \tag{A23}$$

and therefore the scalar charge is

$$\mu = M'(\phi_0) = \sqrt{\frac{3(d-2)(d-3)}{8}}\Omega_{d-2}(\ell_2 - \ell_1). \tag{A24}$$

Now, for the solution (2.1), we have

$$\ell_1 = \frac{2\beta_1(1-\beta_2)m}{\gamma_1}, \qquad \ell_2 = \frac{2\beta_2(1-\beta_1)m}{\gamma_2}, \tag{A25}$$

and so the scalar charge is given by

$$\mu = M'(\phi_0) = \sqrt{\frac{3(d-2)(d-3)}{2}} \frac{(\beta_2 - \beta_1)(1-\beta_1\beta_2)m}{\gamma_1\gamma_2}. \tag{A26}$$

In terms of the rescaled scalar charge Σ that we defined in Eq. (2.2), we therefore have

$$\mu = M'(\phi_0) = \sqrt{\frac{(d-2)(d-3)}{2}}\Omega_{d-2}\Sigma. \tag{A27}$$

3. Electric charges

In Eq. (4.14) of Ref. [9], the electric charge Q_H is read off from the asymptotic form of the electromagnetic potential $A = -\Phi dt$, with

$$\Phi = \Phi_0 + \frac{Q_H}{(d-3)\Omega_{d-2}} \frac{1}{r^{d-3}} + \cdots. \tag{A28}$$

(We have set $e = 1$ to accord with our conventions, and also we have allowed for a possible constant term Φ_0 in the potential at infinity.) Writing the gauge potentials A_1 and A_2 in our Eq. (2.1) as $A_i = \Phi_i dt$, we have

$$\begin{aligned}\Phi_1 &= \text{const.} - \frac{[\beta_1 m - (1-\beta_1)\ell_1]}{\sqrt{\beta_1\gamma_2}}\sqrt{\frac{d-2}{d-3}}\rho + \cdots \\ &= \text{const.} + \sqrt{\frac{d-2}{d-3}}\frac{\sqrt{\beta_1\gamma_2}m}{\gamma_1} + \cdots,\end{aligned} \tag{A29}$$

with an equivalent expression for Φ_2 obtained by exchanging the 1 and 2 subscripts. Thus, we have the charges Q_i^H given by

$$\begin{aligned}Q_1^H &= \sqrt{(d-2)(d-3)}\Omega_{d-2}\frac{\sqrt{\beta_1\gamma_2}m}{\gamma_1}, \\ Q_2^H &= \sqrt{(d-2)(d-3)}\Omega_{d-2}\frac{\sqrt{\beta_2\gamma_1}m}{\gamma_2}.\end{aligned} \tag{A30}$$

In terms of the rescaled electric charges Q_1 and Q_2 defined in Eq. (2.2), the charges are therefore given by

$$\begin{aligned}Q_1^H &= \sqrt{\frac{(d-2)(d-3)}{2}}\Omega_{d-2}Q_1, \\ Q_2^H &= \sqrt{\frac{(d-2)(d-3)}{2}}\Omega_{d-2}Q_2.\end{aligned} \tag{A31}$$

4. Force between widely separated black holes

From Eq. (4.17) in Ref. [9], we will have, in our $SL(3,R)$ Toda case, $F_{12}^H = \mathcal{F}_H / r^{d-2}$, with

$$\mathcal{F}_H = \frac{1}{\Omega_{d-2}}\left[Q_1^H\tilde{Q}_1^H + Q_2^H\tilde{Q}_2^H - \mu\tilde{\mu} - \frac{(d-3)}{2(d-2)}M_H\tilde{M}_H\right]. \tag{A32}$$

Expressed in terms of the rescaled masses and charges of our Eq. (2.2), we therefore have

$$\mathcal{F}_H = \frac{1}{2}(d-2)(d-3)\Omega_{d-2}\mathcal{F}, \tag{A33}$$

with the force coefficient given by

$$\mathcal{F} = Q_1\tilde{Q}_1 + Q_2\tilde{Q}_2 - \Sigma\tilde{\Sigma} - \frac{1}{4}M\tilde{M}, \tag{A34}$$

as given earlier in Eq. (2.7).

APPENDIX B: BPS BLACK HOLES

For comparison with the results for the forces between the non-BPS extremal black holes that we have been considering in this paper, it is interesting to look at examples of BPS extremal black holes. A convenient and rather general class of examples can be constructed as solutions for the following theories, which were discussed in Ref. [16]. The d-dimensional Lagrangian is

$$\mathcal{L} = R - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{4}\sum_{\alpha=1}^{N} e^{\vec{c}_\alpha\cdot\vec{\phi}}F_\alpha^2, \tag{B1}$$

where $\vec{\phi}$ denotes an $(N-1)$-vector of dilatonic scalar fields, and the constant vectors $\vec{c}_\alpha$ describing the couplings of these scalars to the N 2-form gauge fields F_α obey the relation

$$\vec{c}_\alpha \cdot \vec{c}_\beta = 4\delta_{\alpha\beta} - \frac{2(d-3)}{d-2}. \tag{B2}$$

In dimensions $d \leq 10$, these Lagrangians arise as consistent truncations of the maximal supergravities obtained by the toroidal reductions of $d = 11$ supergravity. One can also consider theories of the form (B1) more generally, in any arbitrary dimension.

The extremal N-charge black hole solutions of the theory (B1) are given by [16]

$$ds^2 = -\left(\prod_\alpha H_\alpha\right)^{-\frac{d-3}{d-2}} dt^2 + \left(\prod_\alpha H_\alpha\right)^{\frac{1}{d-2}} (dr^2 + r^2 d\Omega_{d-2}^2),$$
$$F_\alpha = dt \wedge dH_\alpha^{-1}, \qquad \vec{\phi} = \frac{1}{2}\sum_\alpha \vec{c}_\alpha \log H_\alpha, \tag{B3}$$

where

$$H_\alpha = 1 + \frac{q_\alpha}{r^{D-3}}. \tag{B4}$$

Calculating the ADM mass as discussed in Appendix A, we find

$$M_{\text{ADM}} = \frac{(d-3)\Omega_{d-2}}{16\pi}\sum_\alpha q_\alpha. \tag{B5}$$

The mass M_H, scalar charges $\vec{\mu}$ and electric charges Q_α^H, calculated as described in Appendix A, are therefore given by

$$M_H = (d-3)\Omega_{d-2}\sum_\alpha q_\alpha,$$
$$\vec{\mu} = \frac{1}{2}(d-3)\Omega_{d-2}\sum_\alpha \vec{c}_\alpha q_\alpha,$$
$$Q_\alpha^H = (d-3)\Omega_{d-2} q_\alpha. \tag{B6}$$

The force between two distant BPS black holes, defined as in the formula (A32), gives

$$\mathcal{F}_H = (d-3)^2\Omega_{d-2}\left[\sum_\alpha q_\alpha \tilde{q}_\beta - \frac{1}{4}\sum_{\alpha,\beta}\vec{c}_\alpha \cdot \vec{c}_\beta q_\alpha \tilde{q}_\beta - \frac{d-3}{2(d-2)}\sum_{\alpha,\beta} q_\alpha \tilde{q}_\beta\right]. \tag{B7}$$

After making use of the relation (B2), we see that the force between any pair of BPS black holes is equal to zero. This should be contrasted with the situation for the $SL(3,R)$ extremal black holes, which are not BPS, where the force is zero only if the electric charges of one black hole are an overall multiple of the electric charges of the other black hole.

It can easily be verified that more generally, one can consider configurations of the form

$$ds^2 = -\left(\prod_\alpha H_\alpha\right)^{-\frac{d-3}{d-2}} dt^2 + \left(\prod_\alpha H_\alpha\right)^{\frac{1}{d-2}} dy^i dy^i,$$
$$F_\alpha = dt \wedge dH_\alpha^{-1}, \qquad \vec{\phi} = \frac{1}{2}\sum_\alpha \vec{c}_\alpha \log H_\alpha, \tag{B8}$$

where y^i are coordinates on the Euclidean $(d-1)$-dimensional transverse space, and the functions H_α depend on the y^i coordinates. The equations of motion following from the Lagrangian (B1) are satisfied if the H_α are arbitrary harmonic functions on the $(d-1)$-dimensional Euclidean space. Multi-black-hole solutions are obtained by taking the H_α to be sums of elementary harmonic functions of the form

$$H_\alpha = \sum_a \frac{q_{\alpha a}}{|\vec{y} - \vec{y}_{\alpha a}|^{d-3}}. \tag{B9}$$

(Global considerations impose certain constraints on the strengths $q_{\alpha a}$ of the elementary functions located at the mass centers $\vec{y}_{\alpha a}$.) It should be emphasized that the ability to construct multicenter solutions of this form is intimately related to the fact that the force between any pair of single-center black holes is zero. This is very different from the situation for the non-BPS extremal black holes we have been considering in this paper.

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