

On-Shell Covariance of Quantum Field Theory AmplitudesTimothy Cohen,¹ Nathaniel Craig,² Xiaochuan Lu¹ and Dave Sutherland^{3,4}¹*Institute for Fundamental Science, University of Oregon, Eugene, Oregon 97403, USA*²*Department of Physics, University of California, Santa Barbara, California 93106, USA*³*INFN Sezione di Trieste, via Bonomea 265, 34136 Trieste TS, Italy*⁴*Theoretical Physics Department, CERN, 1211 Geneva 23, Switzerland* (Received 3 March 2022; revised 22 November 2022; accepted 4 January 2023; published 27 January 2023)

Scattering amplitudes in quantum field theory are independent of the field parametrization, which has a natural geometric interpretation as a form of “coordinate invariance.” Amplitudes can be expressed in terms of Riemannian curvature tensors, which makes the covariance of amplitudes under nonderivative field redefinitions manifest. We present a generalized geometric framework that extends this manifest covariance to *all* allowed field redefinitions. Amplitudes satisfy a recursion relation to all orders in perturbation theory that closely resembles the application of covariant derivatives to increase the rank of a tensor. This allows us to argue that tree-level amplitudes possess a notion of “on-shell covariance,” in that they transform as a tensor under any allowed field redefinition up to a set of terms that vanish when the equations of motion and on-shell momentum constraints are imposed. We highlight a variety of immediate applications to effective field theories.

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Introduction.—It is well known that scattering amplitudes in quantum field theory are invariant under field redefinitions. The original demonstration of this statement dates back to the early 1960s, where the invariance of the S -matrix was shown for the case of “local” (no derivatives) and “almost local” (a finite number of derivatives) field redefinitions [1–4]. The path integral provides an intuitive framing of this fact. Since amplitudes are computed by performing a weighted integral over all possible field configurations, it is appropriate to think of the field that appears in the Lagrangian as an integration variable. This implies that amplitudes should be independent of the way we choose to parametrize the field, see, e.g., Ref. [5].

This invariance is often obscure at the Lagrangian level, but may be illuminated by geometry: reformulating field theories in terms of geometrically covariant objects can make the invariance of observables under field redefinitions manifest. This so-called “(constant) field space geometry” has indeed been utilized in a variety of scenarios, as we will summarize below. However, this approach faces a major limitation in that it only accommodates field redefinitions without derivatives.

Our goal in this Letter is to generalize the notion of field space geometry to incorporate the full set of allowed field

redefinitions. The key insight is to work with suitably defined off-shell quantities and to compute them in terms of functional derivatives. The resulting “functional geometry” framework allows us to identify objects that are manifestly covariant under all allowed field redefinitions once the on-shell conditions are enforced. In particular, suitably defined tree-level S -matrix elements are “on-shell covariant” in this sense.

Historically, there are three important implications of invariance under field redefinitions that played an essential role in the development of quantum field theory. The first is the notion of gauge invariance, which is a special kind of field redefinition that leaves the Lagrangian explicitly invariant. Indeed, arguing that observables are independent of the gauge choice for non-Abelian gauge theories was one of the main motivations of Ref. [3]. Field redefinitions are also extremely useful for the study of effective field theory (EFT). For example, the primary goal of Ref. [6] was to demonstrate that the linear and nonlinear representations of the Goldstone fields were equivalent, which they achieved using field space geometry. This led to the CCWZ formulation of EFT Lagrangians for the Goldstone boson sector of theories with spontaneously broken global symmetries [7]. Field redefinitions were also central to the development of more general EFTs. One is free to use the equations of motion to write the EFT Lagrangian with the fewest number of derivative operators, which typically leads to a simpler form for the interactions [8]. The search for efficient ways to accommodate basis redundancies in EFTs continues to this day.

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Geometry has been used in many guises to capture the effects of nonderivative field redefinitions; see, e.g., Refs. [9–12]. It has seen something of a revival in recent years, largely due to Refs. [10,11], which introduced a geometric picture of the EFT for the scalar sector of the standard model of particle physics. This so-called Higgs EFT (HEFT) provides a generic low energy approximation to extensions of the standard model that is expressed directly in the electroweak broken phase of the theory. Reference [10] further showed how to express scattering amplitudes in terms of curvature tensors constructed from the field space metric of the HEFT Lagrangian. This picture has found various important applications, such as robustly classifying which HEFTs admit linear realizations of electroweak symmetry [13], determining the scale of unitarity violation [14,15] for the standard model scalar sector, providing results for low-point observables in the standard model effective field theory to all orders in the fields [16], and explaining the soft theorems for generalized Goldstone bosons from a new point of view [17]. There has also been recent progress developing related ideas in the mathematical physics literature [18–21].

Yet the power of field space geometry to describe EFTs is limited by its inability to accommodate derivative field redefinitions. In this Letter, we solve this problem by developing a generalized notion of field space geometry incorporating the full space of allowed field redefinitions. In particular, we construct a generalization of the field space metric and Christoffel symbol, which will be the essential building blocks for a novel expression of an off-shell recursion relation for computing amplitudes, which holds to all orders in perturbation theory. (The connection to the famous Berends-Giele off-shell recursion [22], as well as the Vilkovisky-DeWitt effective action [23,24], will be made below.) This will allow us to show that amputated position-space correlators \mathcal{M} at tree level are on-shell covariant in the sense that they transform under a general field redefinition $\phi(x) \rightarrow \tilde{\phi}(x)$ as

$$\begin{aligned} \tilde{\mathcal{M}}(x_1 \cdots x_n) &= \frac{\delta\phi(y_1)}{\delta\tilde{\phi}(x_1)} \cdots \frac{\delta\phi(y_n)}{\delta\tilde{\phi}(x_n)} \mathcal{M}(y_1 \cdots y_n) \\ &+ \text{evanescent}, \end{aligned} \quad (1)$$

where “evanescent” denotes terms that vanish on-shell. The derivations for the main results presented here, along with some first applications, will appear in a forthcoming companion paper [25].

Covariant amplitudes.—We begin by defining scattering amplitudes in a way that will be useful for making their on-shell covariance manifest. The objects of interest are the amplitudes \mathcal{A} that have been stripped of their LSZ residue factors:

$$\bar{\mathcal{M}}(p_1, \dots, p_n) \equiv -Z_\eta^{-n/2} \mathcal{A}(p_1, \dots, p_n). \quad (2)$$

These objects are known to be covariant under field redefinitions *without* derivatives, see, e.g., Refs. [10,11,14]. In this section, we will review the textbook methods for computing them directly from the path integral using so-called functional methods. This will allow us to show that such (tree-level) amplitudes are covariant under general field redefinitions, up to terms that vanish on shell, see Sec. IV.

We begin by defining the generating functional $W[J]$ for a generic theory of commuting fields $\eta(x)$, whose indices (such as those of flavor and spin) we suppress. In terms of a path integral,

$$e^{iW[J]} \equiv \int \mathcal{D}\eta e^{i(S[\eta] + \int d^4x J(x)\eta(x))}, \quad (3)$$

where $\mathcal{D}\eta$ is the functional integration measure, $S[\eta]$ is the action defined as a functional of η , and $J(x)$ is an external source. One can extract connected correlation functions of η from this object by taking functional derivatives with respect to J . For example, the vacuum expectation value is given by

$$\phi^x \equiv \langle \eta(x) \rangle_J = \frac{\delta W}{\delta J_x}, \quad (4)$$

where we have introduced the shorthand $J_x \equiv J(x)$ and $\phi^x \equiv \phi(x)$. The propagator is

$$iD^{xy} \equiv i\langle \eta(x)\eta(y) \rangle_{J,\text{conn}} = \frac{\delta^2 W}{\delta J_x \delta J_y}, \quad (5)$$

where $\langle \dots \rangle_{J,\text{conn}}$ denotes that this is defined for general J , and only includes the connected contributions.

We will find it more convenient to work with the one-particle irreducible (1PI) effective action $\Gamma[\phi]$, which is the Legendre transform of $W[J]$:

$$-\Gamma \equiv J_x \phi^x - W, \quad (6)$$

and hence satisfies

$$\frac{\delta(-\Gamma)}{\delta\phi^x} = J_x, \quad (7a)$$

$$\frac{\delta^2(-\Gamma)}{\delta\phi^x \delta\phi^y} = \left[\frac{\delta^2 W}{\delta J_x \delta J_y} \right]^{-1} \equiv -iD_{xy}^{-1}, \quad (7b)$$

where D^{xy} is the all-orders propagator. Note the summation convention $J_x \phi^x = \int d^4x J(x)\phi(x)$.

To generate our objects of interest, we simply compute

$$\mathcal{M}_{x_1 \cdots x_n} \equiv -(-iD_{x_1 y_1}^{-1}) \cdots (-iD_{x_n y_n}^{-1}) \frac{\delta^n W}{\delta J_{y_1} \cdots \delta J_{y_n}}, \quad (8)$$

where $\mathcal{M}_{x_1 \dots x_n}$ are precisely the LSZ-residue-factor-stripped amplitudes in position space

$$(2\pi)^4 \delta^4(p_1 + \dots + p_n) \bar{\mathcal{M}}(p_1, \dots, p_n) = \int \left[\prod_{i=1}^n d^4 x_i e^{i p_i x_i} \right] \mathcal{M}_{x_1 \dots x_n} |_{J=0}, \quad (9)$$

where we have taken all external momenta to be outgoing. Note that the quantities $\mathcal{M}_{x_1 \dots x_n}$ defined in Eq. (8) are functionals of J , and so we must impose an on-shell condition $J = 0$ when going from $\mathcal{M} \rightarrow \mathcal{A}$ (taking $J = 0$ is equivalent to taking the origin defined in [6]), which is equivalent to enforcing that ϕ satisfies the η equations of motion through Eq. (7a). Furthermore, the physical amplitudes $\mathcal{A}(p_1, \dots, p_n)$ only correspond to observables for on-shell momenta p_i , while $\mathcal{M}_{x_1 \dots x_n}$ as defined in Eq. (8) are off shell. This ambiguity is automatically removed via Eq. (9) by contracting with the one-particle wave functions $e^{i p_i x_i}$, which are zero eigenfunctions of $-iD_{x_i y}^{-1} |_{J=0}$ when the momenta p_i are on shell. This eliminates any terms proportional to an inverse propagator that acts on a leg in \mathcal{M} . In summary, as far as the amplitudes \mathcal{A} are concerned, only the “physical” pieces of $\mathcal{M}_{x_1 \dots x_n}$ contribute, which are obtained by enforcing the following two conditions

$$\frac{\delta(-\Gamma)}{\delta \phi^z} = J_z = 0 \quad (\text{equations of motion}), \quad (10a)$$

$$\frac{\delta^2(-\Gamma)}{\delta \phi^{x_i} \delta \phi^{y_j}} |_{J=0} = -iD_{x_i y_j}^{-1} |_{J=0} = 0 \quad (\text{on-shell legs}). \quad (10b)$$

This defines what we mean by “on shell” in this Letter.

Off-shell recursion.—The definitions presented in the previous section will naturally lead to a geometric interpretation. To see this explicitly, we will use the fact that the $\mathcal{M}_{x_1 \dots x_n}$ satisfy the following tensorlike recursion relation (derived below):

$$\begin{aligned} \mathcal{M}_{x_1 \dots x_n x} &= \frac{\delta}{\delta \phi^x} \mathcal{M}_{x_1 \dots x_n} - \sum_{i=1}^n G_{xx_i}^y \mathcal{M}_{x_1 \dots \hat{x}_i y \dots x_n} \\ &\equiv \nabla_x \mathcal{M}_{x_1 \dots x_n}, \end{aligned} \quad (11)$$

where the notation $x_1 \dots \hat{x}_i y \dots x_n$ denotes the string $x_1 \dots x_n$ with x_i replaced by y . As the second line in Eq. (11) implies, the right-hand side can be viewed as an analog of a covariant derivative on the field configuration space manifold, where

$$G_{x_1 x_2}^y \equiv iD^{yz} \mathcal{M}_{z x_1 x_2} = iD^{yz} \frac{\delta^3(-\Gamma)}{\delta \phi^z \delta \phi^{x_1} \delta \phi^{x_2}}. \quad (12)$$

We can interpret the covariant derivative in Eq. (11) as generating parallel transport on the field space manifold.

This motivates identifying $G_{x_1 x_2}^y$ as the “functional Christoffel symbol.” In addition, the inverse propagator $-iD_{xy}^{-1}$ and the propagator iD^{xy} can be viewed as the metric and inverse metric on this manifold. (As we will discuss in Sec. IV, $G_{x_1 x_2}^y$ and $-iD_{xy}^{-1}$ are indeed generalizations of the Christoffel symbol and metric on the constant field space manifold; see Eq. (25)(26) below. As a metric, $-iD_{xy}^{-1}$ is distinct from existing field space metrics such as, e.g., the metric on the orbit space of Yang-Mills theory [26–28]. Note that Eq. (12) has a factor of 2 relative to the definition of a metric connection.)

We emphasize that the recursion relation Eq. (11) holds off shell, in particular, for general $J \neq 0$, which means that the functional Christoffel symbol $G_{xx_i}^y$ encodes all the interactions in the theory, not just the cubic ones. It can be algebraically derived from Eq. (8) by repeatedly applying

$$\left[\frac{\delta}{\delta J_y}, iD^{y_i z_i} \right] = -(iD^{yw})(iD^{y_i w_i}) G_{ww_i}^{z_i}; \quad (13)$$

see Ref. [25] for details. Since Eq. (8) holds (in principle) to all orders in perturbation theory, the recursion relation Eq. (11) is an all-order result. One can also understand it from a diagrammatic point of view, where an amputated correlator $\mathcal{M}_{x_1 \dots x_n}$ (again to all orders in perturbation) consists of tree diagrams of 1PI vertices that are glued together by full propagators, so any term in $\mathcal{M}_{x_1 \dots x_n}$ is a product of the following two types of ingredients:

$$k\text{-point 1PI vertices: } -i \frac{\delta^k(-\Gamma)}{\delta \phi^{y_1} \dots \delta \phi^{y_k}}, \quad (14a)$$

$$(\text{full}) \text{ propagators: } D^{y_1 y_2}. \quad (14b)$$

Now consider adding an additional leg to an amplitude, labeled by x . There are three ways this can be accomplished. First is to connect this leg to a particular 1PI vertex

$$\begin{aligned} -i \frac{\delta^k(-\Gamma)}{\delta \phi^{y_1} \dots \delta \phi^{y_k}} &\rightarrow -i \frac{\delta^{k+1}(-\Gamma)}{\delta \phi^{y_1} \dots \delta \phi^{y_k} \delta \phi^x} \\ &= \frac{\delta}{\delta \phi^x} \left[-i \frac{\delta^k(-\Gamma)}{\delta \phi^{y_1} \dots \delta \phi^{y_k}} \right]. \end{aligned} \quad (15)$$

Second is to split a propagator into two propagators using an insertion of the three-point 1PI vertex:

$$\begin{aligned} D^{y_1 y_2} &\rightarrow D^{y_1 z_1} \left[-i \frac{\delta^3(-\Gamma)}{\delta \phi^{z_1} \delta \phi^x \delta \phi^{z_2}} \right] D^{z_2 y_2} \\ &= \frac{\delta}{\delta \phi^x} D^{y_1 y_2}. \end{aligned} \quad (16)$$

Putting together all the possible ways of performing both types of these insertions amounts to taking the functional

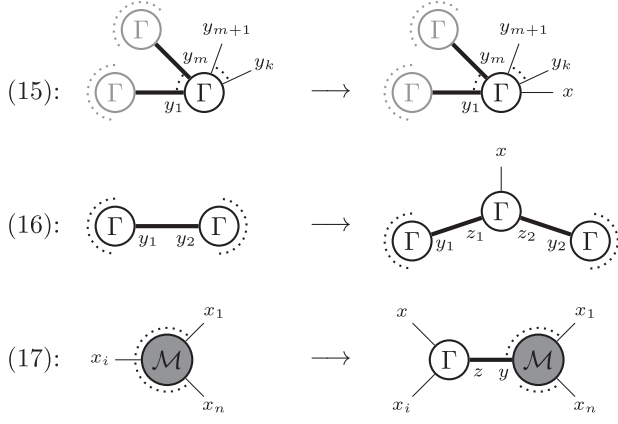


FIG. 1. The n -point amputated correlator \mathcal{M} comprises tree graphs of 1PI vertices (Γ), full propagators (thick lines), and n amputated external legs (thin lines). Diagrammatically, the recursion relation Eq. (11) encodes all ways to add a $(n+1)$ th amputated leg to these tree graphs. The functional derivative $(\delta/\delta\phi^x)$ adds the new leg to all vertices and propagators [cf. Eq. (15)(16), respectively], whereas the connection term combines the new leg with an existing external leg [Eq. (17)].

derivative $(\delta/\delta\phi^x)$ of the amplitude, namely, the first term in Eq. (11).

However, these two operations do not cover all the ways of adding a leg to an amplitude. The third option is to attach a three-point 1PI vertex to a leg, which turns that leg into a propagator that is connected with two new legs:

$$\begin{aligned} \mathcal{M}_{x_1 \dots x_i \dots x_n} &\rightarrow -i \frac{\delta^3(-\Gamma)}{\delta\phi^x \delta\phi^{x_i} \delta\phi^z} D^{zy} \mathcal{M}_{x_1 \dots \hat{x}_i y \dots x_n} \\ &= -G_{xx_i}^y \mathcal{M}_{x_1 \dots \hat{x}_i y \dots x_n}. \end{aligned} \quad (17)$$

The functional Christoffel symbol terms in Eq. (11) are accounting for all possible ways of doing this operation. A more detailed derivation of this recursion relation will be presented in [25]; we summarize our present discussion in Fig. 1.

It is worth noting that our recursion relation Eq. (11) reduces to the Berends-Giele recursion relation [22,29–31] when $J=0$ is enforced. To make this connection more concrete, we extend Eq. (12) to define an analog of the generalized Christoffel symbols

$$G_{x_1 \dots x_n}^y \equiv iD^{yz} \mathcal{M}_{zx_1 \dots x_n}, \quad (18)$$

which, similar to \mathcal{M} , satisfy the following recursion relation for general $J \neq 0$

$$\begin{aligned} G_{x_1 \dots x_n x}^z &= \frac{\delta}{\delta\phi^x} G_{x_1 \dots x_n}^z - \sum_{i=1}^n G_{xx_i}^y G_{x_1 \dots \hat{x}_i y \dots x_n}^z \\ &\equiv \nabla_x G_{x_1 \dots x_n}^z, \end{aligned} \quad (19)$$

as expected for the usual generalized Christoffel symbols.

We see from Eq. (8)(9) that as far as the physical amplitudes $\mathcal{A}(p_1, \dots, p_n)$ are concerned, it is sufficient to study these generalized Christoffel symbols at $J=0$. On the other hand, through Eq. (4)(8)(18), one can derive that the quantities $G_{x_1 \dots x_n}^y|_{J=0}$ play a central role in the relation between ϕ^y and the “raised” source field $\hat{J}^x \equiv (iD^{xy}|_{J=0})J_y$:

$$\phi^y = \hat{J}^y - \sum_{n=2}^{\infty} \frac{1}{n!} (G_{x_1 \dots x_n}^y|_{J=0}) \hat{J}^{x_1} \dots \hat{J}^{x_n}. \quad (20)$$

Therefore, one can derive $G_{x_1 \dots x_n}^y|_{J=0}$ [and hence obtain the physical amplitudes $\mathcal{A}(p_1, \dots, p_n)$] by computing $\phi^y[\hat{J}^y]$ order-by-order in \hat{J}^y . Doing this at the tree-level by iteratively solving the equation of motion in Eq. (7a) about $J=0$ is the Berends-Giele recursion procedure [22,29–31]. This approach efficiently computes the quantities $G_{x_1 \dots x_n}^y|_{J=0}$, and hence the physical amplitudes. Our Eq. (19) [or, equivalently, Eq. (11)] reveals that $G_{x_1 \dots x_n}^y$ possess a more general recursion structure that holds also in the case $J \neq 0$. This is the insight that allows us to define the generalized notion of field space geometry that can accommodate derivative field redefinitions.

Functional geometry.—In Sec. III, we derived our first main result, Eq. (11), which shows that higher-point correlators can be constructed through the repeated application of a covariant-derivative-like operation. This result holds to all orders in perturbation theory, and suggests the existence of a geometric structure on the field configuration space manifold, which we call “functional geometry.”

In this section we show that the objects $\mathcal{M}_{x_1 \dots x_n}$ indeed transform covariantly under general field redefinitions, up to terms that vanish on shell. For simplicity, we restrict the analysis to tree level. We can parametrize a general field redefinition $\phi(x) \rightarrow \tilde{\phi}(x)$ that could include derivatives using a functional relation $\phi[\tilde{\phi}]$. At tree level, the 1PI effective action transforms as a scalar

$$\tilde{\Gamma}[\tilde{\phi}] = \tilde{S}[\tilde{\phi}] = S[\phi[\tilde{\phi}]] = \Gamma[\phi[\tilde{\phi}]]. \quad (21)$$

This allows us to derive the following transformation law for the amplitudes [25]:

$$\tilde{\mathcal{M}}_{x_1 \dots x_n} = \left(\frac{\delta\phi^{y_1}}{\delta\tilde{\phi}^{x_1}} \dots \frac{\delta\phi^{y_n}}{\delta\tilde{\phi}^{x_n}} \right) \mathcal{M}_{y_1 \dots y_n} + U_{x_1 \dots x_n}, \quad (22)$$

where

$$\begin{aligned} U_{x_1 \dots x_n} &= a_{x_1 \dots x_n y_1} \frac{\delta(-\Gamma)}{\delta\phi^{y_1}} \\ &+ \sum_{i=1}^n b_{x_1 \dots \hat{x}_i \dots x_n y_1} \frac{\delta\phi^{y_2}}{\delta\tilde{\phi}^{x_i}} \frac{\delta^2(-\Gamma)}{\delta\phi^{y_1} \delta\phi^{y_2}}, \end{aligned} \quad (23)$$

for some a and b , are a set of evanescent terms that vanishes if the field redefinition $\phi[\tilde{\phi}]$ preserves the two on-shell conditions in Eq. (10). In other words, on-shell covariance implies that \mathcal{M} transforms like a tensor up to a set of evanescent terms that do not contribute to physical observables. In fact, $G_{x_1\dots x_n}^y$ and $-iD_{x_1x_2}^{-1}$ only transform as a Christoffel symbol and a metric in the same sense, i.e., up to evanescent terms.

Finally, let us show that our functional geometry formalism is a generalization of the constant field space geometric picture explored in Refs. [10–12]. To this end, we consider the generic form of the Lagrangian,

$$\mathcal{L} = -V + \frac{1}{2}g_{ab}(\partial_\mu\phi^a)(\partial^\mu\phi^b), \quad (24)$$

to derive $G_{x_1x_2}^y$ explicitly. Making flavor indices (a, b, \dots) explicit, one can check that it satisfies

$$\begin{aligned} & \lim_{q^2 \rightarrow \infty} \int d^4x_1 d^4x_2 d^4y e^{ip_1x_1 + ip_2x_2} e^{-iqy} \\ & \times [G_{ab}^c(x_1, x_2, y)|_{\partial_\mu\phi_i=0}] \\ & = (2\pi)^4 \delta^4(p_1 + p_2 - q) \\ & \times \frac{1}{2} g^{cd}(g_{da,b} + g_{db,a} - g_{ab,d}). \end{aligned} \quad (25)$$

We see that restricting to constant field configurations and taking the high momentum transfer limit, $G_{x_1x_2}^y$ reduces to the Christoffel symbol in Refs. [10–12], i.e., the second line in Eq. (25). Similarly, the inverse propagator $-iD_{x_1x_2}^{-1}$ reduces to the metric on the constant field space manifold when taking the same limit:

$$\begin{aligned} & \int d^4x_1 d^4x_2 e^{ip_1x_1} e^{-ip_2x_2} [-iD_{ab}^{-1}(x_1, x_2)|_{\partial_\mu\phi_i=0}] \\ & = (2\pi)^4 \delta^4(p_1 - p_2) (-p_2^2 g_{ab} + V_{,ab}). \end{aligned} \quad (26)$$

This shows that we have indeed generalized the building blocks of Riemannian geometry in Refs. [10–12] as promised.

Conclusions and outlook.—In this Letter, we have presented a generalization of field space geometry that accommodates field redefinitions involving derivatives. The key building blocks are the functional metric and functional Christoffel symbol introduced here. These objects enabled us to write down a covariant derivative on the field space manifold. When acting on an amplitude, the parallel transport generated by this covariant derivative yields a new amplitude with an additional leg. This provides a new type of off-shell recursion relation for computing amplitudes. We then leveraged this recursion relation to demonstrate that tree-level amplitudes have a manifest notion of on-shell covariance, namely, that they transform like tensors up to terms which vanish when the

on-shell conditions are enforced. This gives us a new way of understanding the invariance of amplitudes under field redefinitions reframed in terms of functional geometry.

There are many settings where this generalization can be applied. One immediate application is the generalization of invariant criteria for EFTs to linearly realize a symmetry [10,11,14] to accommodate derivative field redefinitions. We claim that if there exists a fixed point $\phi_0(x)$ on the field configuration space manifold, which corresponds to the point where the symmetry can be linearly realized, then we can solve for it using

$$\left. \frac{\delta^n(-\Gamma)}{\delta\pi_{x_1} \cdots \delta\pi_{x_n}} \right|_{\phi=\phi_0(x)} = 0, \quad (27)$$

where the π 's are the Goldstone directions on the manifold. The existence of the fixed point $\phi_0(x)$ determined using this criterion cannot be obscured using derivative field redefinitions anywhere on the field space manifold. Our framework should similarly generalize our understanding of how Lagrangian terms map onto kinematic structures in amplitudes, and allow the generalization of the analysis of soft-theorems presented in Ref. [17] to accommodate the full set of allowed field redefinitions.

This Letter opens many directions for future research. Given the supporting details and explicit examples in [25], one obvious extension will be to understand how the functional geometry manifests beyond tree level. We know that such a framework must exist, since it is well known that amplitudes are invariant under field redefinitions to any order in perturbation theory. Although our recursion relation Eq. (11) holds to all orders in perturbation theory, the 1PI effective action is no longer invariant beyond tree level, i.e., Eq. (21) does not hold. An understanding of the covariance properties of the loop level 1PI effective action will likely make it clear how to geometrize the loop level amplitudes as well.

Another area that is worth studying is the connection to the Vilkovisky-DeWitt formulation of the path integral [23,24]. Their approach is to make the covariance under field redefinitions (without derivatives) manifest off shell by modifying the coupling between the field and the source that appears in the path integral, see Eq. (3). It would be very interesting to understand how to incorporate the new methods provided by functional geometry into this way of writing the path integral to make covariance under all allowed field redefinitions manifest. Aspects of our construction are also reminiscent of DeWitt's manifestly covariant formulation of quantum field theory [32,33] which we will explore further in [25].

Functional geometry provides a language with which to understand the on-shell covariance of amplitudes in quantum field theory. Given the foundational role played by the freedom to perform field redefinitions when defining our theories, we are optimistic that the new approach proposed here will lead to further insights.

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Note added.—Recently, we were made aware of the exciting related work by Cheung, Helset, and Parra-Martinez [34]. These authors propose the existence of a “geometry-kinematics” duality, which allows them to identify the building blocks of a generalized geometry that accommodates field redefinitions including derivatives. Our constructions differ at the detailed level. For example, as the authors note, the geometric invariants that define their kinematic geometry depend on choices made when specifying the initial metric, which is not the case for the functional geometry developed here. It will be very interesting to understand the relation between these two approaches. We are grateful to the authors for sharing their draft with us.

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