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FORMALISM AND ASSUMPTIONS INVOLVED IN PARTIAL WAVE ANALYSIS OF THREE-MESON SYSTEMS

J.D. Hansen*, G.T. Jones**, G. Otter*** and G. Rudolph***

CERN, Geneva

ABSTRACT

The Illinois group has made a partial wave analysis of the reaction $\pi p \to (3\pi)p$. We have modified the Illinois program to study the reaction $Kp \to (K\pi\pi)p$. In this paper a detailed description is given of the partial wave analysis of any reaction of the kind: meson + proton → proton + (3 mesons). It is also shown that, with little modification, the method can be applied to the study of special 4-meson systems like $K\omega$. A discussion is given of the physical assumptions used and of their implications. Two questions are discussed: "Are there ambiguities like the well-known ambiguities encountered in dimeson studies?" and "how is it possible to measure imaginary parts of density matrix elements?"

(*) Now at Niels Bohr Inst. Copenhagen (Denmark)

(**) Now at the University of Birmingham, P.O. Box 363, Birmingham (England)

(***) III. Physikalisches Institut der Techn. Hochschule, Aachen (Germany)

1. INTRODUCTION

Recently, a powerful method of performing a partial wave analysis of the $(\pi^{\pm}\pi^{\pm}\pi^{\mp})$-system was introduced by the Illinois group and applied, by that group, to the reaction $\pi^- p \to (3\pi)^- p$ at various energies [1,2]. The authors have, more recently, been involved in applying a modified version of the Illinois program to the $(K^-\pi^-\pi^+)$-system [3].

The program divides naturally into two parts. The first part calculates amplitudes and special normalization integrals of these amplitudes over relevant regions of phase space, and prepares all the necessary quantities needed by the second part of the program. This latter part is a very fast fitting program designed by G. Ascoli for fitting density matrix elements $\bar{\rho}$ and certain "density parameters" $\bar{C}$, imposing the constraint that the density matrix must be positive definite.

There is no easily available, detailed, discussion of the analysis. Although most of the relevant formulae are in D.V. Brockway's Ph.D. thesis [4], there is nowhere an adequate discussion of the assumptions made or of the method of analysis.

Since it appears that there will be a great deal of time spent on the study of three-particle systems with this program, it is felt that it is worthwhile presenting the formalism in more detail than is customary [5-10] with the following aims in mind:

(1) to state clearly what assumptions are made and to discuss their implication and validity;

(2) to try to give a presentation which will be of use to future users of the Illinois program.

No attempt is made to describe the detailed workings of the program.

It is stressed that this presentation is intended for experimentalists.

2. PLAN

The presentation of the formalism is given in sections 3, 4 and 5. Section 3 gives a derivation of the formula for the cross-section for a reaction of the type a + b → 1 + 2 + 3 + 4. Section 4 presents and discusses the assumptions made to reduce the number of parameters to a workable number. Section 5 describes the method of analysis. Section 6 extends the formalism to reactions other than $K^-p \to K^-\pi^-\pi^+p$ which is the reaction in terms of which the formalism is presented. In Appendix A there is a detailed discussion of what can be learnt about the $(K^-\pi^-\pi^+)$ system from a study of one-dimensional angular distributions alone. Appendix B contains the definition of all angles and reference systems used. Appendix C discusses the question of interferences between different states. Finally, in Appendix D, two important questions are discussed: firstly, "are the solutions to the Illinois partial wave analysis of 3-particle systems unique ?" and, secondly, "how is it possible to measure imaginary parts of density matrix elements ?"

3. THE FORMALISM

The state of a free particle of mass M and spin j can be completely specified by $|\vec{p}\lambda[Mj]>$ where $\vec{p}$ is its 3-momentum and λ its helicity. The quantities M and j (invariants of the Poincaré group) are put in brackets because they are often omitted.

Now consider the reaction $K^- p \to K^- \pi^- \pi^+ p$ to be a special case of the general reaction ab → 1 + 2 + 3 + 4 where particles a, 1, 2 and 3 are spinless while b and 4 are fermions. In the helicity representation just described, the amplitude for this reaction can be written as

$$f_{\lambda_b \lambda_4} \equiv \langle \vec{p}_1 \vec{p}_2 \vec{p}_3 ; \vec{p}_4 \lambda_4 | U | \vec{p}_a ; \vec{p}_b \lambda_b \rangle \qquad 3.1$$

where the momenta and helicities are taken in the overall centre of mass system. The spinless particles in the final state are separated off because this analysis is a study of the $(K^- \pi^- \pi^+)$-system.

The aim now is to derive equation 3.19 for the cross-section in terms of variables in which the analysis of the $(K^- \pi^- \pi^+)$-system can be described conveniently. This procedure involves five changes of basis or of variables. At each step, an attempt is made not only to present the necessary formulae, but also, to motivate and describe in a simple way what physics is in that particular step.

From the experimental point of view the basis $|\vec{p}_1 \vec{p}_2 \vec{p}_3>$ is very natural because the momenta are what one normally measures. However, theoretical results are usually more clearly presented in terms of an angular momentum basis.

Before introducing such a basis, it is convenient to define an intermediate one. The set of variables $\{\vec{p}_1 \vec{p}_2 \vec{p}_3 [M_1 M_2 M_3]\}$ is completely equivalent to the set $\{\vec{P}_{123}\, M_{123}\, \Phi\, \Theta\, M_{12}\, \psi\, \chi [M_1 M_2 M_3]\}$ where

$\vec{P}_{123} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3$ = momentum of (123)-system in overall centre of mass;

M_{123} = invariant (effective) mass of particles 1, 2 and 3;

Φ and Θ are the azimuthal and polar angles of the (13)-system in the (123) rest frame; either the helicity system $(x^{(s)}, y^{(s)}, z^{(s)})$ or the Gottfried-Jackson system $(x^{(t)}, y^{(t)}, z^{(t)})$ can be used, (see Appendix B);

M_{13} = invariant mass of particles 1 and 3;

ψ and χ are the azimuthal and polar angles of particle 3 in the (13)-rest frame, see Appendix B.

Instead of coupling particles 1 and 3, one could just as well have coupled 1 and 2 or 2 and 3, in which case ψ and χ would have been angles defined in the (12)- or (23)-rest frames, while Φ and Θ would have described the position of the (12)- and (23)-systems, respectively, in the (123)-rest frame.

It is stressed that these couplings give three different but completely general descriptions of the (123)-system. It is not assumed that the (123)-system decays via a two-step process. "Direct decay" into three bodies is included in each of the descriptions.

In terms of this new basis equation 3.1 can be written

$$f_{\lambda_b \lambda_4} = \langle \vec{P}_{123}\, M_{123}\, \Phi\, \Theta\, M_{13}\, \psi\, \chi; \vec{p}_4 \lambda_4 | U | \vec{p}_a ; p_b \lambda_b \rangle \qquad 3.2$$

Now, in an attempt to provide some "feel" for the link between the $\{|\vec{P}_{123}\, M_{123}\, \Phi\, \Theta\, M_{13}\, \psi\, \chi\rangle\}$ -basis and an equivalent angular momentum basis, one is reminded of the fact that the state of a quantum-mechanical rigid rotator (two spinless particles "going round each other") can be expressed in terms of $|\theta\phi\rangle$-states or $|\ell m\rangle$-states and also, that

$$\langle \theta\phi | \ell m \rangle = Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\, D^{*(\ell)}_{m0}(\phi,\theta,0) \qquad 3.3.$$

The 3-particle state $|\vec{P}_{123}\, M_{123}\, \Phi\Theta\, M_{13}\, \psi\chi\rangle$ can be pictured loosely as two such systems - with particles 1 and 3 going round each other as well as (13) and 2 going round each other. It seems not unreasonable, then, and is rigorously demonstrated in ref. [9], that the variables Φ, Θ ψ and χ can be replaced, by J, Λ, j and λ where

J = spin of (123)-system,

Λ = helicity of (123)-system in overall centre of mass,

j = spin of (13)-system,

λ = helicity of (13)-system in the (123)-rest system.

Notice that, if the z axis of the (123)-coordinate system is chosen to be the direction of motion of the (123)-system in the overall centre of mass, then Λ is the same as M, the z-component of angular momentum in the (123)-coordinate system.

The link between $|\vec{P}_{123}\, M_{123}\, \Phi\Theta\, M_{13}\, \psi\chi\rangle$ and $|\vec{P}_{123}\, \Lambda[J M_{123}]\, M_{13}\, j\, \lambda\rangle$ is

$$\langle \vec{P}_{123}\, M_{123}\, \Phi\Theta M_{13}\psi\chi \,|\, \vec{P}_{123}\, \Lambda[J M_{123}]\, M_{13} j\, \lambda\rangle = \sqrt{\frac{2J+1}{4\pi}}\sqrt{\frac{2j+1}{4\pi}}\, D^{(J)*}_{\Lambda\lambda}(\Phi,\Theta,0)\, D^{(j)*}_{\lambda 0}(\psi,\chi,0) \qquad 3.4$$

Expanding in terms of these 3-particle states of definite angular momentum, equation 3.2 becomes

$$f_{\lambda_b\lambda_4} = \sum_{J\Lambda j\lambda} \sqrt{\frac{2J+1}{4\pi}}\sqrt{\frac{2j+1}{4\pi}}\, D^{*(J)}_{\Lambda\lambda}(\Phi,\Theta,0)\, D^{*(j)}_{\lambda 0}(\psi,\chi,0)\, \langle \vec{P}_{123}\, \Lambda[J M_{123}] M_{13}\, j\lambda;\, \vec{P}_4\lambda_4 | U | \vec{P}_a;\vec{P}_b\lambda_b\rangle$$

$$\equiv \sum_{J\Lambda j\lambda} \sqrt{\frac{2J+1}{4\pi}}\sqrt{\frac{2j+1}{4\pi}}\, D^{(J)*}_{\Lambda\lambda}(\Phi,\Theta,\Psi)\, d^{(j)}_{\lambda 0}(\chi)\, g^{Jj}_{\Lambda\lambda\lambda_b\lambda_4}(s,t,M_{123},M_{13}) \qquad 3.5$$

In terms of the reaction $K^-p \to K^-\pi^-\pi^+p$, the function g is the amplitude for producing, from an initial K^-p state, a $K^-\pi^-\pi^+p$ state whose $(K^-\pi^-\pi^+)$-system has spin J, mass M_{123} and helicity Λ in the overall centre of mass system while, at the same time, particles 1 and 3 have an invariant mass M_{13}, spin j and helicity λ in the $(K^-\pi^-\pi^+)$-system; the function g also depends on s, t, λ_b and λ_4.

The product $d^{j}_{\lambda b}(\chi)\, g^{Jj}_{\lambda\lambda_b\lambda_4}$ is the Dalitz plot amplitude, which, in general, is also a function of s and t. Notice that, for fixed M_{123} and M_{13}, $\cos\chi$ is linearly related to M^2_{23} or M^2_{12}.

The angles Φ, Θ and ψ describe the orientation of the plane of the (123)-system.

At this point, a further change of basis is made - to states of definite relative orbital angular momentum ℓ of the (13)-system and particle 2. This is motivated by the fact that, near threshold, only small ℓ-values contribute, a point which will be discussed later. The amplitude g is now replaced by h, whose meaning differs only in that λ is replaced by ℓ. Explicitly

$$g^{Jj}_{\lambda\lambda_b\lambda_4} = \sum_{\ell} \sqrt{\frac{2\ell+1}{2J+1}}\, \langle \ell 0 j\lambda \,|\, J\lambda\rangle \langle \vec{p}_{123}\Lambda[JM_{123}]M_{13} j\ell; \vec{p}_4\lambda_4 | U | \vec{p}_a;\vec{p}_b\lambda_b\rangle$$

$$\equiv \sum \sqrt{\frac{2\ell+1}{2J+1}}\, \langle \ell 0 j\lambda \,|\, J\lambda\rangle\, h^{J\ell j}_{\lambda\lambda_b\lambda_4}(s,t,M_{123},M_{13}) \quad . \qquad 3.6$$

where $\langle \ell 0 j\lambda \,|\, J\lambda\rangle$ is a Clebsch-Gordon coefficient.

It is worth mentioning that expressing the amplitude in terms of these orbital angular momentum states $|\vec{p}_{123}\Lambda[JM_{123}]M_{13}\, j\,\ell\rangle$ yields a bonus because, in the (123)-rest system, these states are eigenstates of parity P, see ref. [9];

i.e.

$$P\,|\vec{p}_{123}=0\;\Lambda[J^{P}M_{123}]M_{13}\,j\,\ell\rangle = \eta_1\eta_2\eta_3(-1)^{j+\ell}\,|\vec{p}_{123}=0\;\Lambda[J^{P}M_{123}]M_{13}\,j\,\ell\rangle \qquad 3.7$$

where η_i is the intrinsic parity of the i'th particle. The product $P \equiv \eta_1\eta_2\eta_3(-1)^{j+\ell}$ is then the parity of the (123)-system. From now on the quantity P is included in the state vector.

That $|\vec{p}_{123}=0\Lambda[JM_{123}]M_{13}\,j\,\ell\rangle$ should be an eigenstate of parity is not surprising in view of the fact that if a state of spin J is decomposed

into two parts of definite spin and parity, then a particular ℓ-value for their angular momentum corresponds to a definite value for the parity of the original system.

For reasons of symmetry which will become apparent later when the (3π)-system is discussed, the coordinate system is changed. The old angles Θ, Φ and ψ are so defined that the situation $\Theta = \Phi = \psi = 0$ corresponds to the (13)-system going along $z^{(s)}$ with the direction of particle 3 being in the $x^{(s)}z^{(s)}$-plane. For this situation, define θ_{23} to be the angle between the directions of particles 2 and 3 in the (123)-rest frame, (see Appendix B).

The new angles ϕ and θ are the azimuthal and polar angles of particle 3 (the π^+) in the $(x^{(s)}, y^{(s)}, z^{(s)})$-system (the first two Euler angles), while the third Euler angle γ is defined by the direction of the projection of particle 1 (the K^-) onto the plane perpendicular to the direction of particle 3. The configuration $\phi = \theta = \gamma = 0$ corresponds to particle 3 going along the $z^{(s)}$-axis, particle 1 being in the $x^{(s)}z^{(s)}$-plane.

To go from the situation $\Theta = \Phi = \psi = 0$ to the situation $\theta = \phi = \gamma = 0$ it is necessary to rotate by $(\pi-\theta_{23})$ about $y^{(s)}$ and then by π about the direction in which the (13)-system is moving after the rotation by $(\pi-\theta_{23})$.

To change the variables in equation 3.5 from Φ, Θ and ψ to ϕ, θ and γ one can make use of the observation that a state of definite Φ, Θ and ψ can be obtained from a state $\Phi = \Theta = \psi = 0$ in two different ways: either a rotation $R(\Phi, \Theta, \psi)$ or two successive rotations $R(0, \pi-\theta_{23}, \pi)$ and $R(\phi, \theta, \gamma)$. In terms of the irreducible representations of the rotation group. - $D^J_{\Lambda\lambda}$ - one obtains

$$D^{(J)*}_{\Lambda\lambda_{13}}(\Phi,\Theta,\psi) = \sum_{\nu} D^{(J)*}_{\Lambda\nu}(\phi,\theta,\gamma)\; D^{(J)*}_{\nu\lambda_{13}}(0,\pi-\theta_{23},\pi) \qquad 3.8.1$$

The product $d^{j}_{\lambda 0}(\chi)\, g^{Jj}_{\lambda\lambda_b\lambda_4}$ is the Dalitz plot amplitude, which, in general, is also a function of s and t. Notice that, for fixed M_{123} and M_{13}, $\cos\chi$ is linearly related to M^2_{23} or M^2_{12}.

The angles Φ, Θ and ψ describe the orientation of the plane of the (123)-system.

At this point, a further change of basis is made - to states of definite relative orbital angular momentum ℓ of the (13)-system and particle 2. This is motivated by the fact that, near threshold, only small ℓ-values contribute, a point which will be discussed later. The amplitude g is now replaced by h, whose meaning differs only in that λ is replaced by ℓ. Explicitly

$$g^{Jj}_{\lambda\lambda_b\lambda_4} = \sum_{\ell} \sqrt{\frac{2\ell+1}{2J+1}}\, \langle \ell 0 j\lambda \,|\, J\lambda\rangle \langle \vec{p}_{123}\Lambda[JM_{123}]M_{13}j\ell;\vec{p}_4\lambda_4|u|\vec{p}_a;\vec{p}_b\lambda_b\rangle$$

$$\equiv \sum \sqrt{\frac{2\ell+1}{2J+1}}\, \langle \ell 0 j\lambda \,|\, J\lambda\rangle\, h^{J\ell j}_{\lambda\lambda_b\lambda_4}(s,t,M_{123},M_{13}) \quad . \qquad 3.6$$

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$$D^{(J)*}_{\Lambda\lambda_{13}}(\Phi,\Theta,\psi) = \sum_{\nu} D^{(J)*}_{\Lambda\nu}(\phi,\theta,\gamma)\; D^{(J)*}_{\nu\lambda_{13}}(0,\pi-\theta_{23},\pi) \qquad 3.8.1$$

where ν is the helicity of the (12)-system, and where, for clarity, λ is temporarily replaced by λ_{13}.

If particles 2 and 3 are coupled, rather than 1 and 3, a similar relation holds, specifically:

$$D^{(J)*}_{\Lambda\lambda_{23}}(\Phi',\Theta',\psi') = \sum_{\nu} D^{(J)*}_{\Lambda\nu}(\Phi,\Theta,\gamma)\; D^{(J)*}_{-\nu\lambda_{23}}(\pi,\pi-\theta_{13},\pi)$$

where θ_{13} is the angle between particles 1 and 3 in the (123)-rest system λ_{23} is the helicity of the (23)-system in the (123)-rest system and where Φ', Θ' and ψ are the counterparts for the (23)-coupling of Φ, Θ and ψ in the (13)-coupling.

Combining equations 3.5, 3.6 and 3.8.1, (or 3.8.2 for the (23)-coupling) the amplitude $f_{\lambda_b\lambda_4}$ can now be written

$$f_{\lambda_b\lambda_4} = \sum_{J\nu\Lambda} \sqrt{2J+1}\; D^{(J)*}_{\Lambda\nu}(\Phi,\Theta,\gamma) \sum_{\ell_j} h^{J\ell_j}_{\Lambda\lambda_b\lambda_4}\; G^{J\ell_j}_{\nu}(s_1,s_2,M_{123}) \qquad 3.9.1$$

$$G^{J\ell_j}_{\nu}(13) = \sum_{\lambda_{13}} \frac{\sqrt{2\ell+1}\sqrt{2j+1}}{4\pi\sqrt{2J+1}} \langle \ell\, 0\, j\, \lambda_{13} | J\, \lambda_{13}\rangle\; D^{(J)*}_{\nu\lambda_{13}}(0,\pi-\theta_{23},\pi)\; d^{(j)}_{\lambda_{13}0}(\chi)$$

$$= (-1)^{\nu}\epsilon(-1)^{J}\frac{\sqrt{2\ell+1}\sqrt{2j+1}}{4\pi\sqrt{2J+1}} \sum_{\lambda_{13}\geq 0}(2-\delta_{\lambda_{13},0})\langle \ell 0 j\lambda_{13}|J\lambda_{13}\rangle\, d^{(j)}_{\lambda_{13}0}(\chi)\,\frac{1}{2}\left\{ d^{(J)}_{\nu\lambda_{13}}(\theta_{23}) + \epsilon(-1)^{\lambda_{13}} d^{(J)}_{\nu,-\lambda_{13}}(\theta_{23})\right\} \qquad 3.9.2$$

$$G^{J\ell_j}_{\nu}(23) = \sum_{\lambda_{23}} \frac{\sqrt{2\ell+1}\sqrt{2j+1}}{4\pi\sqrt{2J+1}} \langle \ell\, 0\, j\, \lambda_{23} | J\, \lambda_{23}\rangle\; D^{(J)*}_{-\nu\lambda_{23}}(\pi,\pi-\theta_{13},\pi)\; d^{(j)}_{\lambda_{23}0}(\chi)$$

$$= \epsilon(-1)^{J}\frac{\sqrt{2\ell+1}\sqrt{2j+1}}{4\pi\sqrt{2J+1}} \sum_{\lambda_{23}\geq 0}(1-\delta_{\lambda_{23}0})\langle \ell 0 j\lambda_{23}|J\lambda_{23}\rangle\, d^{(j)}_{\lambda_{23}0}(\chi)\,\frac{1}{2}\left\{ d^{(J)}_{\nu\lambda_{23}}(\theta_{13}) + \epsilon(-1)^{\lambda_{23}} d^{(J)}_{\nu,-\lambda_{23}}(\theta_{13})\right\} \qquad 3.9.3$$

where $\epsilon = (-1)^{J+\ell+j}$.

Notice that χ, θ_{23} and θ_{13} are functions of s_1, s_2 and M_{123}.

In equations 3.9.2 and 3.9.3 the quantities j, ℓ and χ are understood to have suffixes (13) and (23) respectively.

Notice that in the case of the (23)-coupling there is an extra $e^{i\nu\pi} \equiv (-1)^{\nu}$ which arises because the particle, whose γ is used, does not change.

If the (12)-coupling were being used, corresponding to expanding the $(K^-\pi^-\pi^+)$-system in terms of $(K^-\pi^-)$-states of definite angular momentum, the (because the direction of the π^+ analyses is opposite to that of the $(K^-\pi^-)$-system) equation 3.9.2 is particularly simple, explicitly:

$$G_\nu^{J^P \ell j}(12) = (-1)^\nu \epsilon (-1)^J \frac{\sqrt{(2j+1)(2\ell+1)}}{4\pi\sqrt{2J+1}} \langle \ell 0 j \nu | J\nu \rangle d^{(j)}_{\nu 0}(\chi) \qquad 3.9.4$$

Use will be made of this in section 6, where the analysis is extended to systems such as $(\bar{K}^0\pi^0\pi^-)$-system where one expects to require resonances in all three two-particle systems.

It is pointed out in ref. [9] that the constraints imposed by parity conservation in the production process reduce the number of independent amplitudes h. The explicit formula is

$$h^{J\ell j}_{\lambda\lambda_b\lambda_4} = P(-1)^{J+\lambda+\lambda_b+\lambda_4} h^{J\ell j}_{-\lambda-\lambda_b-\lambda_4} \qquad 3.10$$

where P is defined in equation 3.7.

Before presenting a final formula for the amplitude $f_{\lambda_b\lambda_4}$ one further change of basis is made. The new representation efficiently incorporates into the formalism the fact that parity is conserved in the production process. The states $|\vec{p}_{123}=0\ \lambda[J^P M_{123}]\ M_{13} j \ell\rangle$ are replaced by the states $|\vec{p}_{123}=0\ \lambda[J^P M_{123}]\ M_{13} j \ell \eta\rangle$ where $\eta(=\pm 1)$ is the eigenvalue of the reflection operator in the (xz)-(production) plane, $Y = \exp(-i\pi J_y)P$; see ref. [7].

It is shown in ref. [9] that

$$Y|\vec{p}_{123}=0\ \lambda[J^P M_{123}]\ M_{13} j \ell\rangle = P(-1)^{J-\lambda}|\vec{p}_{123}=0\ -\lambda[J^P M_{123}]\ M_{13}\ j \ell\rangle \qquad 3.11$$

Defining

$$|\vec{p}_{123}=0\ \lambda[J^P M_{123}]\ M_{13}\ j \ell \eta\rangle = c_\lambda\{|\vec{p}_{123}=0\ \lambda[J^P M_{123}]\ M_{13} j \ell\rangle + \eta(-1)^J|\vec{p}_{123}=0\ -\lambda[J^P M_{123}]\ M_{13}\ j \ell\rangle \qquad 3.12$$

where $\varepsilon = P(-1)^{J+1}, = (-)^{J+j+\ell}$, $c_\Lambda = \frac{1}{\sqrt{2}}$ for $\Lambda \neq 0$ and $c_\Lambda = \frac{1}{2}$ for $\Lambda = 0$ equations 3.11 and 3.7 show that

$$Y\left|\vec{P}_{123}=0 \wedge [J^P M_{123}] M_{13}\, j\, \ell\, \eta\right\rangle = -\eta \left|\vec{P}_{123}=0 \wedge [J^P M_{123}] M_{13}\, j\, \ell\, \eta\right\rangle \quad 3.13$$

$$P\left|\vec{P}_{123}=0 \wedge [J^P M_{123}] M_{13}\, j\, \ell\, \eta\right\rangle = \eta_1\eta_2\eta_3(-1)^{j+\ell}\left|\vec{P}_{123}=0 \wedge [J^P M_{23}] M_{13} j \ell \eta\right\rangle \quad 3.13a$$

In other words, the new states $\left|\vec{P}_{123}=0 \wedge [J^P M_{123}] M_{13}\, j\, \ell\, \eta\right\rangle$ are eigenstates of Y and of the parity operator P. Incidentally, ε is minus the quantity which is often referred to as "naturality".

For future reference, equation 3.12 is written in an abbreviated form:

$$\left|J^P M^{(s)} \eta\right\rangle = c_{M^{(s)}}\left[\left|J^P M^{(s)}\right\rangle + \eta\varepsilon(-1)^{M^{(s)}}\left|J^P -M^{(s)}\right\rangle\right] \quad 3.12a$$

where Λ = helicity of (123)-system in overall centre of mass

$= M^{(s)} = z^{(s)}$ component of angular momentum of the (123)-system in the $x^{(s)}y^{(s)}z^{(s)}$-coordinate system, which is the so-called s-channel system.

To give an example, equation 3.12 says that the basic $|J^P M^{(s)}\rangle$ states $|1^+1\rangle$, $|1^+0\rangle$ and $|1^+-1\rangle$ are replaced by the equivalent set of $|J^P M^{(s)}\eta\rangle$ states

$$|1^+1+\rangle = \frac{1}{\sqrt{2}}\{|1^+1\rangle - |1^+-1\rangle\}$$
$$|1^+0+\rangle = |1^+0\rangle$$
$$|1^+1-\rangle = \frac{1}{\sqrt{2}}\{|1^+1\rangle + |1^+-1\rangle\}$$

Notice that $|J^P M^{(s)}\eta\rangle$ states with $M^{(s)} \geq 0$ and $\eta = \pm 1$ completely span the space given by $|J^P M^{(s)}\rangle$ where $-J \leq M^{(s)} \leq J$.

This new representation turns out to have two advantages. Firstly, as will be discussed later, the density matrix of the (123)-system is diagonal with respect to η. And, secondly, for $M^{(s)} = 0$, $\eta = +1$ corresponds to a (123)-state being formed by natural parity exchange while $\eta = -1$ corresponds to formation by un-natural parity exchange [11]. For $M^{(s)} \neq 0$, this condition is true to leading order in $(\frac{1}{s})$.

In terms of these $|J^P M^{(s)}$ (or $\Lambda)\eta>$ states, the amplitudes $h^{J^P \ell_j}_{\Lambda\lambda_b\lambda_4}$ are replaced by $h^{J^P \ell_j \eta}_{\Lambda\lambda_b\lambda_4}$ where

$$h^{J^P \ell_j \eta}_{\Lambda\lambda_b\lambda_4} = c_\Lambda \left\{ h^{J^P \ell_j}_{\Lambda\lambda_b\lambda_4} + \eta\epsilon(-1)^\Lambda \, h^{J^P \ell_j}_{-\Lambda\lambda_b\lambda_4} \right\} \qquad 3.14$$

where $\Lambda \geq 0$.

Now, using the relation

$$G^{J^P \ell_j}_{\nu} = \epsilon(-1)^\nu \, G^{J^P \ell_j}_{-\nu} \qquad 3.15$$

which can be obtained from equation 3.9.2 (or 3.9.3 or 3.9.4), the symmetry relation

$$D^{*(J)}_{-\Lambda-\nu}(\phi,\theta,\gamma) = (-1)^{\Lambda-\nu} D^{(J)}_{\Lambda\nu}(\phi,\theta,\gamma) \,, \qquad 3.16$$

and equation 3.14 it is straightforward to show that

$$f_{\lambda_b\lambda_4} = \sum_{J\Lambda(\geq 0)\nu\eta} \sqrt{2J+1} \left\{ D^{(J)*}_{\Lambda\nu}(\phi,\theta,\gamma) + \eta D^{(J)}_{\Lambda\nu}(\phi,\theta,\gamma) \right\} \sum_{\ell_j} c_\Lambda G^{J^P \ell_j}_{\nu} \, h^{J^P \ell_j \eta}_{\Lambda\lambda_b\lambda_4} \qquad 3.17$$

which, in turn, can be re-written as

$$f_{\lambda_b\lambda_4} = 2 \sum_{J\Lambda(\geq 0)\nu} \sqrt{2J+1} \cos(\Lambda\phi+\nu\gamma) \, d^{(J)}_{\Lambda\nu}(\theta) \sum_{\ell_j} c_\Lambda G^{J^P \ell_j}_{\nu} \, h^{J^P \ell_j \, \eta(=+1)}_{\Lambda\lambda_b\lambda_4}$$

$$+ 2i \sum_{J\Lambda(\geq 0)\nu} \sqrt{2J+1} \sin(\Lambda\phi+\nu\gamma) \, d^{(J)}_{\Lambda\nu}(\theta) \sum_{\ell_j} c_\Lambda G^{J^P \ell_j}_{\nu} \, h^{J^P \ell_j \eta(=-1)}_{\Lambda\lambda_b\lambda_4} \qquad 3.18$$

Equation 3.17 can be used to write the cross-section which is proportional to

$$\omega \equiv \sum_{\lambda_b \lambda_4} |f_{\lambda_b \lambda_4}|^2$$

$$= \sum_{\substack{J^P \Lambda(>0)\eta \\ J'^{P'} \Lambda'(>0)\eta'}} \sqrt{(2J+1)(2J'+1)}\, \rho^{(\ell\ell' jj')}_{J^P\Lambda\eta, J'^{P'}\Lambda'\eta}\, c_\Lambda c_{\Lambda'} \left(D^{(J)}_{\Lambda\nu} + \eta D^{(J)*}_{\Lambda-\nu}\right)\left(D^{(J')*}_{\Lambda'\nu'} + \eta D^{(J')}_{\Lambda'-\nu'}\right)\left(G^{J^P \ell j}_{\nu}\right)^* G^{J'^{P'}\ell' j'}_{\nu'} \qquad 3.19$$

where the density matrix ρ is given by

$$\rho^{(\ell\ell' jj')}_{J^P\Lambda\eta, J'^{P'}\Lambda'\eta}(s,t,M_{123},s_2) = \sum_{\lambda_b \lambda_4} \left(h^{J^P \ell j \eta}_{\Lambda \lambda_b \lambda_4}\right)^* h^{J'^{P'}\ell' j'\eta}_{\Lambda' \lambda_b \lambda_4} \qquad 3.20$$

It should be noticed that ρ is diagonal with respect to η. The sum over λ_b and λ_4 expresses the fact that the target is unpolarised and that one does not measure the polarization of the recoiling proton.

Notice that the density matrix ρ depends, in general, on ℓ, ℓ', j, j', s, t, M_{123} and s_2. The s_2-dependence implies that the density matrix elements can vary over the Dalitz plot.

The relationship between this ρ and the one in which the more conventional J^P states are used is

$$\rho_{J^P\Lambda\eta, J'^{P'}\Lambda'\eta} = 2c_\Lambda c_{\Lambda'}\left\{\rho_{J^P\Lambda, J'^{P'}\Lambda'} + \eta\epsilon'(-1)^{\Lambda'}\rho_{J^P-\Lambda, J'^{P'}-\Lambda'}\right\}.$$

Formula 3.19 has been derived using the "s-channel" coordinate system $(x^{(s)}, y^{(s)}, z^{(s)})$ where $z^{(s)}$ is minus the direction of the recoil proton in the $(K\pi\pi)^-$ centre of mass system. Exactly the same formula can be derived by using t-channel axes $(x^{(t)}, y^{(t)}, z^{(t)})$ where $z^{(t)}$ is the direction of the incident K^- in the $(K\pi\pi)^-$ centre of mass. In this latter system, Λ gives the third component of angular momentum of the $(K\pi\pi)^-$-system where the axis of quantisation is now $z^{(t)}$. The density matrix elements in the two systems are related by a rotation, through the crossing angle, in the production plane.

Usually the analysis is done in the t-channel because the dominance of certain states is more marked in it than in the s-channel.

Appendix B defines all reference systems and angles used in the analysis of the (3π)-system [1,2] and the $(K^-\pi^-\pi^+)$-system [3].

Before going on to discuss the assumptions in the next section, an important property of the density matrix is discussed.

From equations 3.10 and 3.13a it can be shown in a straightforward manner that

$$h^{J^P e j \eta}_{\Lambda \lambda_b \lambda_4} = \eta (-1)^{l+\lambda_b+\lambda_4} \; h^{J^P e j \eta}_{\Lambda\, -\lambda_b\, -\lambda_4} \qquad 3.21$$

Using equations 3.20 and 3.21 it can further be shown that

$$\rho^{(ll'jj')}_{J^P\Lambda\eta, J'^{P'}\Lambda'\eta} = 2\left\{ \left(h^{J^P e j \eta}_{\Lambda \uparrow \uparrow}\right)^* h^{J'^{P'} e' j' \eta}_{\Lambda' \uparrow \uparrow} + \left(h^{J^P e j \eta}_{\Lambda \uparrow \downarrow}\right)^* h^{J'^{P'} e' j' \eta}_{\Lambda \uparrow \downarrow} \right\} \qquad 3.22$$

where $\uparrow(\downarrow)$ corresponds to helicity $\frac{1}{2}(-\frac{1}{2})$.

Bearing in mind the mathematical fact that a matrix A_{ik} formed from a vector a_i according to the prescription $A_{ik} = a_i a_k^*$ can only have one non-zero eigenvalue, it can be seen that ρ can have two eigenvalues for each η-value, corresponding to helicity flip and helicity non-flip at the proton vertex.

At this point attention is drawn to Appendix A where there is a discussion of what can be learnt from studying the one-dimensional angular distributions alone.

4. ASSUMPTIONS

(a) From the 1-dimensional distributions it appears that the data can be fitted without introducing J-values above a certain J_{max}. This has previously been found to be the case in the study of the (3π)-system produced in πp interactions .

Assumption 1: assume $J \leq J_{max}$, where J_{max} is estimated from a Fourier analysis of the 1-dimensional angular distributions.

(b) To emphasize the fact that the amplitude $f_{\lambda_b \lambda_4}$ can be written in terms of $(K^-\pi^+)$, $(\pi^-\pi^+)$ or $(K^-\pi^-)$-couplings, let it be written symbolically as:

$$f_{\lambda_b\lambda_4} = \sum_{j(K^-\pi^+)=0}^{\infty} \equiv \sum_{j(\pi^-\pi^+)=0}^{\infty} \equiv \sum_{j(K^-\pi^-)=0}^{\infty} \qquad 4.1$$

where $j(K^-\pi^+)$, $j(\pi^-\pi^+)$ and $j(K^-\pi^-)$ are the spins of the $(K^-\pi^+)$-, $(\pi^-\pi^+)$- and $(K^-\pi^-)$-systems respectively. Another description which will be useful for discussing assumption 2, is given by

$$f_{\lambda_b\lambda_4} = \frac{1}{3}\left\{ \sum_{j(K^-\pi^+)=0}^{\infty} + \sum_{j(\pi^-\pi^+)=0}^{\infty} + \sum_{j(K^-\pi^-)=0}^{\infty} \right\} \qquad 4.2$$

Assumption 2: assume that $j(K^-\pi^+)$, $j(\pi^-\pi^+)$ and $j(\pi^-K^-)$ are restricted to be less than or equal to $j_{max}(K^-\pi^+)$, $j_{max}(\pi^-\pi^+)$ and $j_{max}(K^-\pi^-)$ respectively, and that one can write

$$f_{\lambda_b\lambda_4} = \sum_{j(K^-\pi^+)=0}^{j_{max}(K^-\pi^+)} + \sum_{j(\pi^-\pi^+)=0}^{j_{max}(\pi^-\pi^+)} + \sum_{j(K^-\pi^-)=0}^{j_{max}(K^-\pi^-)} \qquad 4.3$$

The fact that the 1/3 is missing in equation 4.3 reflects the assumption that the high partial waves in any one particular coupling are covered by the low partial waves in the other two couplings and the hope that there is not very much overlap between the truncated contributions.

In the $(K^-\pi^-\pi^+)$-case [3] it is assumed that $j_{max}(K^-\pi^+) = 2$, $j_{max}(\pi^-\pi^+) = 2$ and that the contribution of the third coupling can be ignored altogether. There can still, in principle, be overlaps between some or all of the $j(K^-\pi^+) = 0,1$ and 2 contributions and the $j(\pi^-\pi^+) = 0,1$ or 2 contributions, resulting in "double counting". (To express things in terms which should not be used until after assumption 3, overlap between, for example, $j(K^-\pi^+) = 1$ and $j(\pi^-\pi^+) = 1$ contributions would correspond to interference between $K^*(890)\pi$ and $K\rho$ decay modes of the $(K\pi\pi)$-system).

This assumption is motivated by the observation that the $(K^-\pi^+)$- and $(\pi^-\pi^+)$-effective mass distributions show $K^*(890)$, $K^*(1420)$, ρ and f signals and also, phase shift analyses [13,14] find that S-, P- and D-waves are enough to describe $(K^-\pi^+)$- and $(\pi^-\pi^+)$-systems for masses up to those encountered in this analysis. The I = 3/2 phase shifts for the $(K^-\pi^-)$-system are small and are neglected.

(c) The amplitude h is seen to depend on the masses M_{13} and M_{23} for the (13)- and (23)-couplings respectively. If one invokes the idea of a strong Watson final state 2-particle interaction [17], it is reasonable to factorize off this dependence.

Assumption 3: assume

$$h^{J^P e_{ij\eta}}_{\lambda\lambda_b\lambda_4}(s,t,M_{123},M_n) = \bar{h}^{J^P e_{ij\eta}}_{\lambda\lambda_b\lambda_4}(s,t,M_{123}) \cdot BW^{(j,n)}(M_n)\, q^j(M_n)\, p^l(M_{123},M_n) \qquad 4.4$$

where the index n, introduced here, is 1, 2 or 3 depending on whether the (23)-, (13)- or (12)-coupling is being used and where the indices ℓ and j refer to the coupling in question.

The product $p^{\ell}.BW.q^{j}$ is a parametrization based on the picture of the $(K\pi\pi)$-system breaking up by a two-step decay. The barrier factors at the first and second decay vertices are p^{ℓ} and q^{j} respectively, where p is the momentum of the di-meson system of interest in the $(K\pi\pi)$-rest frame, ℓ is its orbital angular momentum relative to the lone meson, and q is the momentum of one of the decay products of the di-meson system in the di-meson rest frame. The function BW parametrises the mass dependence of the di-meson system either by a Breit-Wigner or by phase shifts [17].

(d) Looking at the indices on $R^{J^P \ell j n \eta}_{\Lambda\lambda_b\lambda_4}$ it is clear that the density matrix defined in equation 3.20 has many elements. For example, consider the (123)-system to decay only by the modes shown in table 4.1. This would correspond to a (5 x 5)-density matrix which is described by 25 real parameters.

In an attempt to reduce the number of parameters a further assumption is made. (Appendix D discusses further implications of assumptions 3 and 4).

Assumption 4: assume

$$\bar{R}^{J^P \ell j n \eta}_{\Lambda\lambda_b\lambda_4}(s,t,M_{123}) = T^{J^P \eta}_{\Lambda\lambda_b\lambda_4}(s,t,M_{123})\; C^{J^P \ell j n}(s,t,M_{123}) \qquad 4.5$$

Equation 4.5 assumes that C is independent of the (discrete) variables λ_b, λ_4, Λ and η and that T is independent of the discrete variables ℓ, j and n. Notice that, if in a given J^P state there were but one resonance, this assumption would be automatically satisfied. Furthermore, C would be independent of s and t,

making the amplitude split into production and decay parts. In this paper T and C will be loosely referred to as the production and decay parts of the amplitude despite the fact that C depends on s and t.

It is now possible to define new "reduced matrix elements" by

$$\rho_{J M \eta, J' P' M' \eta}(s,t,M_{123}) = \sum_{\lambda_b \lambda_4} \left(T^{J^P \eta}_{M \lambda_b \lambda_4}\right)^* T^{J' P' \eta}_{M' \lambda_b \lambda_4} \qquad 4.6$$

In accordance with custom, Λ will henceforth be replaced by M. The program tries to determine these density matrix elements and the $C^{J^P j n}$. Notice that if one again considers only the states given in table 4.1, the number of parameters is, as a result of assumption 4, reduced to nine density matrix parameters plus four complex parameters $C^{J^P j n}$. In sect. 5 it is shown that, by means of a suitable normalization convention, it is possible to put one $C^{J^P j n}$ equal to one for every J^P state considered; in which case, only two of the $C^{J^P j n}$ remain in the example being considered. Thus, assumption 4 has reduced the number of parameters from 25 to 13.

(e) With sufficient statistics one would determine ρ and C as functions of s, t and M_{123} by fitting in small intervals of these variables; then the rest of this paragraph (e) would be redundant.

In reality, the intervals are quite wide and one has to average over each interval. One writes

$$T^{J^P\eta}_{M\lambda_b\lambda_4}(s,t,M_{123})\,C^{J^P\rho_{jn}}(s,t,M_{123}) = \bar{T}^{J^P\eta}_{M\lambda_b\lambda_4}\,\bar{C}^{J^P\rho_{jn}}\,R^{J^PM\eta\rho_{jn}}(s,t,M_{123}) \quad 4.7$$

and fits in each interval for the constants $\bar{\rho}$ and $\bar{C}$, where

$$\bar{\rho}_{J^PM\eta,\,J'^{P'}M'\eta} = \sum_{\lambda_b\lambda_4}\left(\bar{T}^{J^P\eta}_{M\lambda_b\lambda_4}\right)^* \bar{T}^{J'^{P'}\eta}_{M'\lambda_b\lambda_4} \quad 4.8$$

The "shape function" R is given by

$$R^{J^PM\eta\rho_{jn}}(s,t,M_{123}) = FT^{J^PM\eta\rho_{jn}}(t)\;F^{J^PM\eta\rho_{jn}}(M_{123}) \quad 4.9$$

and it contains the best knowledge available on the dependence of the amplitude on t and M_{123} (averaged over s if experiments of different energies are being combined). The function FT can be different in different M_{123}-intervals (in accordance with well-known experimental results) and the function F can be different in different t-intervals.

The function F is given by

$$F^{(b)} = \left\{M^{-1}_{123}\int \mathcal{N}_a\,ds_1 ds_2\right\}^{-\frac{1}{2}} BWA^{(b)}(M_{123}) \qquad \text{where } a \equiv \{J^P\rho_{jn}\},\; b \equiv \{J^PM\eta\rho_{jn}\} \quad 4.10$$

Here

$$\mathcal{N}_a = \sum_\nu \left| G^{J^P\rho_{jn}}_\nu(s_1,s_2,M_{123})\; p^\ell\; BW^{(j,n)}\; q^\ell \right|^2$$

If BWA is kept constant the quantity $F^{(b)}$ in equation 4.10 forces $d\sigma/dM_{123}$ for a given state to be independent of M_{123} in the interval considered. However, the contribution to $d\sigma/dM_{123}$ from interference between different states can still be functions of M_{123} (see Appendix C); explicit calculation shows that this remaining dependence is weak.

The function BWA describes the mass-dependence. In the program there is a routine BWA which enables one to put in explicit dependence on M_{123} when the dependence is believed to be strong; for example, for large M_{123} intervals near thresholds.

The functions FT and BWA (and hence R) are normalised to one in each interval considered. The reason for doing this will be discussed in sect. 5.

5. METHOD OF ANALYSIS

(1) The likelihood function

The partial wave analysis of the (Kππ)-system is done by using the extended maximum likelihood method [15] to determine the quantities $\bar{C}^{J^P \varrho_j n}$ and $\bar{\rho}_{J^P M\eta, J'^{P'} M'\eta}$. The function that is maximised is

$$\mathcal{L} = \sum_{i=1}^{N_{INPUT}} \ln \omega(\tau_i) - \int \omega(\tau) A(\tau) d\tau + \sum_{i=1}^{N_{INPUT}} \ln A(\tau_i, x_i) \qquad 5.1$$

where N_{INPUT} = number of experimental events used;

$$\omega = \sum_{\substack{J^P M\eta \\ J'^{P'} M'}} \left(\sum_{\varrho j n} \bar{C}^{J^P \varrho_j n} \mathcal{M}^{J^P}_{M\eta \varrho_j n} \right)^* \bar{\rho}_{J^P M\eta, J'^{P'} M'\eta} \left(\sum_{\varrho' j' n'} \bar{C}^{J'^{P'} \varrho' j' n'} \mathcal{M}^{J'^{P'}}_{M'\eta \varrho' j' n'} \right) \qquad 5.2$$

τ represents the seven phase space variables t, M_{123}, s_1, s_2, ϕ, θ, γ;

x represents the three spatial variables (x, y, z);

$A(\tau_i, X_i)$ is the acceptance of the ith event (every real event in the laboratory is characterised by τ and x);

$A(\tau) \equiv \int A(\tau, x) dx$.

The product $\omega(\tau_i) A(\tau_i, x_i)$ is the probability of finding an event at the "point" (τ_i, x_i). The quantity $\ln A(\tau_i, x_i)$ is a constant and is ignored in the fit. Notice then that to perform a fit it is only necessary to know the acceptance integrated over x, $A(\tau)$.

The "matrix element" $\mathcal{M}^{J^P}_{M_\eta \rho_{j\eta}}$ has the explicit form (see equations 3.19, 4.4, to 4.9)

$$\mathcal{M}^{J^P}_{M_\eta \rho_{j\eta}} = \sum_\nu c_m \left(\mathcal{D}^{(J)*}_{M\nu}(\phi,\theta,\gamma) + \eta \mathcal{D}^{(J)}_{M-\nu}(\phi,\theta,\gamma) \right) H_\nu^{(J^P M_\eta \rho_{j\eta})} \qquad 5.3.1$$

where

$$H_\nu^{(J^P M_\eta \rho_{j\eta})} = G_\nu^{(J^P \rho_{j\eta})} p^\ell B w^{(j\eta)} q^j F_T^{(J^P M_\eta \rho_{j\eta})} F^{(J^P M_\eta \rho_{j\eta})} \qquad 5.3.2$$

To facilitate the discussion of normalization which comes later, it is convenient to write

$$\int \omega A \, d\tau = \sum \bar{c}^K \bar{c} \, \bar{f} \int \mathcal{M}^* \mathcal{M} \, A \, d\tau \qquad 5.4$$

where the indices which have been left out are as in equation 5.2

By

$$\int d\tau \equiv \sum_s N_s / N_{INPUT} \int d\phi \, d(\cos\theta) \, d\gamma \, ds_1 ds_2 \frac{dM_{123}}{M_{123}} \, dt$$

is meant integration over that region of phase space in which the fit is being formed (defined by the interval in M_{123} and t). The sum over s is needed if events from more than one energy s are used. Then, N_s is the number of events of energy s and $N_{INPUT} = \sum N_s$.

One can, for convenience distinguish between two kinds of conditions when $A \neq 1$ over the whole of phase space.

(a) It might be decided to reject some events because they are felt to be unsatisfactorily measured; for example, if the reaction $K^- p \to K^- \pi^- \pi^+ p$ is studied at 10 GeV/c in a bubble chamber, events with a fast K^- and a fast π^- often give fits to two reactions, corresponding to interchange of the K^- and the π^-; in ref. [3],

such events are removed from the analysis - at the expense of increasing statistical errors - and are corrected for as if they had not been detected in the first place.

When events are removed in this way, A is either 0 or 1, depending on the position in phase space. The region of phase space over which A = 1 is referred to as "cut phase space" and one could write

$$\int_{\text{uncut}} \omega A \, d\tau = \int_{\text{cut}} \omega \, d\tau$$

where "uncut phase space" is the whole of phase space over which a fit is being performed. It is for this reason that $\int_{\text{uncut}} \omega A \, d\tau$ is referred to as "cut integral" in the Illinois program, even when A varies over phase space. For the cases where A = 0 or 1 there are routines called CHRM 10 and CHRM 11 where the acceptance is applied to the experimental and Monte Carlo events (used for producing integrals) respectively.

(b) The detection efficiency of one's apparatus may not be 100% in which case A can vary over phase space. In this case there is a routine BOBK for applying to the Monte Carlo events that acceptance which is believed to exist for the experimental events.

Now, since ωA is the probability of finding an event

$$\int \omega A \, d\tau = 1 \qquad \text{or} \qquad N_{INPUT} \int \omega A \, d\tau = N_{INPUT} \qquad 5.5$$

This condition is maintained throughout the fitting procedure by replacing $\bar{\rho}_{J^P M_\eta, J'^{P'} M'_{\eta'}}$ at every step by $\bar{\rho}_{J^P M_\eta, J'^{P'} M'_{\eta'}} \Big/ \int \omega A \, d\tau$. All this normalization condition says is that, if an event is being used in the fit, the probability of finding it somewhere must be one.

(2) Interpretation of density matrix elements and C's

The normalization discussed at the end of sect. 4 ensures that

$$\int \mathcal{M}_a^* \mathcal{M}_a \, d\tau = 1 \qquad a \equiv \{J^P M \eta \ell j n\} \qquad 5.6$$

This condition will be used later in this section.

Now the fit actually determines the products $(\bar{C}^{J^P \ell j n})^* \bar{C}^{J'^{P'} \ell' j' n'} \bar{\rho}_{J^P M \eta, J'^{P'} M' \eta}$ (see equation 5.4) which are here called "full density matrix elements". If there are more than one decay mode of a particular J^P, one $\bar{C}^{J^P \ell j n}$ can be fixed to an arbitrary complex number. For simplicity this is chosed to be one.

At this stage one defines the "predicted number of events" to be the number of events in the whole of phase space for a detection efficiency A = 1; it is given by

$$N_{PREDICTED} = N_{INPUT} \int \omega \, d\tau$$

$$= \sum_{\substack{J^P M \eta \ell j n \\ J'^{P'} M' \eta' \ell'}} N_{INPUT} (\bar{C}^{J^P \ell j n})^* \bar{C}^{J'^{P'} \ell' j' n'} \rho_{J^P M \eta, J'^{P'} M' \eta} \int (\mathcal{M}^{J^P}_{M \eta \ell j n})^* \mathcal{M}^{J'^{P'}}_{M' \eta' \ell' j' n'} \, d\tau \qquad 5.7$$

Using equations 5.6 and 5.7 it is now possible to interpret the quantities $\bar{\rho}$ and $\bar{C}$.

For the case when $\{J^P M \eta \ell j n\} \equiv \{J'^{P'} M' \eta' \ell' j' n'\}$, equations 5.6 and 5.7 allow one to interpret $N_{INPUT} |\bar{C}^{J^P \ell j n}|^2 \rho_{J^P M \eta, J^P M \eta}$ as the predicted number of events of the type $\{J^P M \eta \ell j n\}$.

From equation 5.7 alone the quantity

$$N_{INPUT}\left\{(\bar{C}^{J^P\ell jn})^*\bar{C}^{J'^{P'}\ell'j'n'}\bar{\rho}_{J^PM\eta,J'^{P'}M'\eta'}\int(\mathcal{M}^{\pi^-P}_{M\eta\ell jn})^*\mathcal{M}^{J'P'}_{M'\eta'\ell'j'n'}\,d\tau\right\}+\text{same with } J^PM\eta\ell jn \text{ replaced by } J'^{P'}M'\eta'\ell'j'n'$$

is interpreted as the predicted number of events which exhibit interference between combinations $\{J^PM\eta\ell jn\}$ and $\{J'^{P'}M'\eta'\ell'j'n'\}$.

(i) This interference, when integrated over the whole of phase space, can only occur when $\{J^PM\eta\} \equiv \{J'^{P'}M'\eta'\}$ and when $n \neq n'$; see Appendix C. Amongst other things, this means that it is not possible to have interference between $K^*(890)\pi$ and $K^*(1420)\pi$ decay modes of the same $\{J^PM\eta\}$; and also, that s- and d-wave decays of a given $\{J^PM\eta\}$ into a given mode do not interfere. It should be noted that the conditions derived here for the $K^-\pi^-\pi^+$ system do not all carry over to the situation where one has identical particles.

(ii) Imaginary parts of density matrix elements can only be determined by measuring interference terms between states of different n or j, see Appendix D.

To illustrate the interpretations of $\bar{C}$ and $\bar{\rho}$ one considers an example. Assume that the only $\{J^PM\eta\ell jn\}$ combinations used in the fits are $\{1^+0+0\ 11\}$ and $\{1^+0+0\ 12\}$, corresponding to the dominant s-wave decays of the Q via $K^*(890)\pi$ and $K\rho$ respectively. The fit determines

$$N_{INPUT}|\bar{C}^{1^+011}|^2\bar{\rho}_{1^+01,1^+01} = \text{predicted number of } (K^*\pi)\text{-events} \equiv N(K^*\pi)$$

$$N_{INPUT}|\bar{C}^{1^+012}|^2\bar{\rho}_{1^+01,1^+01} = \text{predicted number of } (K\rho)\text{-events} \equiv N(K\rho)$$

$$N_{INPUT}\,2\,Re(\bar{C}^{1^+011})^*\bar{C}^{1^+012}\bar{\rho}_{1^+01,1^+01}\int(\mathcal{M}^{1^+}_{0^+011})^*\mathcal{M}^{1^+}_{0^+012}\,d\tau \equiv N(\text{INTERFERENCE})$$

If the last number were zero then the ratio $|\bar{C}^{1^+011}|^2/|\bar{C}^{1^+012}|^2$ would be the $K^*\pi/K\rho$ branching ratio. The quantity $(\bar{C}^{1^+011})^*\bar{C}^{1^+012}$ can be written $R\exp(i\Delta\phi)$ where $\Delta\phi$ is the relative phase of the $K^*\pi$ and $K\rho$ decay modes.

Remembering that one of the $\bar{C}$'s can be put equal to one; in which case $\rho_{J^P M\eta, J^P M\eta}$ gives the number of predicted events decaying via that particular mode whose $\bar{C}$ is put equal to one.

3. How the positivity constraint is applied.

Next, the incorporation into the fit of the constraint that the density matrix of the (Kππ)-system must be positive definite will be discussed. In this analysis, it is the "reduced" density matrix $\bar{\rho}$ which is kept positive definite. If $\bar{\rho}$ is positive then so is the "full" density matrix $\bar{C}^* \bar{C} \bar{\rho}$

A hermitian matrix (in this case the density matrix) can be described by its N eigenvalues λ_I and N normalized eigenvectors $v^{(I)}$ according to the relation

$$\rho_{ij} \equiv \lambda_1 v_i^{(1)*} v_j^{(1)} + \lambda_2 v_i^{(2)*} v_j^{(2)} + \cdots \lambda_N v_i^{(N)*} v_j^{(N)} \qquad 5.8.1$$

$$\equiv V_i^{(1)*} V_j^{(1)} + V_i^{(2)*} V_j^{(2)} + \cdots V_i^{(N)*} V_j^{(N)} \qquad 5.8.2$$

where $V_i^{(I)} \equiv \lambda_I^{\frac{1}{2}} v_i^{(I)}$ and the suffices i and j refer to $J^P M\eta$-values. This relation is just an unfamiliar way of expressing the fact that a hermitian matrix can be diagonalized by the unitary matrix formed from its eigenvectors.

(For simplicity of presentation only, assume that only states with $\eta = +1$ are present. Exactly the same holds for the $\eta = -1$ part of the density matrix).

By transforming from "density-matrix-element-space" to "eigenvalue- and eigenvector-space" the non-linear positivity conditions on the density matrix elements become very simple, namely that $\lambda_I \geq 0$. These are imposed at every step in the fits. In passing, the normalization of the eigenvectors is taken into account by using the angles which specify them.

4. Implications of having a recoiling spin 1/2 particle.

Now that the "eigenvalue-and eigenvector space" has been defined it is possible to discuss some very important implications of having a recoiling spin 1/2 proton (as opposed to a recoiling Δ or anything).

Equation 3.22 or 4.8 shows that the existence of a recoiling proton means that an (NXN)-density matrix is defined by two N-vectors (4N real numbers) and has only two non-zero eigenvalues. (In the "reduced" density matrix $\bar{\rho}$ being determined, some deviation from this ideal situation is expected because approximations have been made. One possible example is that equation 4.7 assumes the same dependence on s, t and M_{123} for helicity-flip and helicity-non-flip amplitudes in the interval being fitted.)

The fitting program determines a density matrix which has N^2 real parameters in it.

It would then seem, at first sight, to be possible (for $N \geq 4$) to determine the 2N complex amplitudes of equation 4.8, i.e. to perform an amplitude analysis. This is not so and the reason for this will now be discussed.

The density matrix being measured is of the form

$$\bar{\rho}_{ij} = f^{(i)*}_{\uparrow\uparrow} f^{(j)}_{\uparrow\uparrow} + f^{(i)*}_{\uparrow\downarrow} f^{(j)}_{\uparrow\downarrow} \qquad 5.9$$

The question is: "is it possible to determine uniquely the amplitudes f by measuring ρ_{ij}?". The fitted density matrix in the program is given by

$$\rho_{ij}(\text{fitted}) = V_i^{(1)*} V_j^{(1)} + V_i^{(2)*} V_j^{(2)} ,$$

see equation 5.8.2. It is tempting to identify these V with the amplitudes f. However, this cannot be done because if $\bar{\rho}$ can be formed from the vectors $V_i^{(1)}$ and $V_i^{(2)}$ it can also be formed from the vectors $W_i^{(1)}$ and $W_i^{(2)}$ (see ref. [16]), where

$$\begin{pmatrix} W_i^{(1)} \\ W_i^{(2)} \end{pmatrix} = \begin{pmatrix} A e^{i\alpha} & \sqrt{1-A^2}\, e^{i\beta} \\ \sqrt{1-A^2}\, e^{i\gamma} & A e^{i(\pi+\beta+\gamma-\alpha)} \end{pmatrix} \begin{pmatrix} V_i^{(1)} \\ V_i^{(2)} \end{pmatrix} \qquad 5.10$$

The matrix in equation 5.10 is the most general unitary matrix, with four free parameters ($0 \leq A < 1$ and α, β and γ can take any values).

All that equation 5.10 says is that a rotation in the spin space of the proton does not change the density matrix, and it is therefore impossible to do an amplitude analysis if one sums over the spin projections of the recoiling proton. However, if, for example, one knew one particular $T^{J^P M\eta}_{\uparrow\uparrow}$ and $T^{J^P M\eta}_{\uparrow\downarrow}$ then the parameters A, α, β and γ could be determined and hence an amplitude analysis performed.

However, bearing all this in mind one might still ask "Is it still possible to learn something from the eigenvalues of the density matrix?". It is. If the density matrix found in the fitting program has two non-zero eigenvalues then both spin-flip and spin-non-flip amplitudes are present. If, on the other hand, only one non-zero eigenvalue is found, it is not possible to say that only spin-flip OR spin-non-flip is present. This can be seen by considering the case when the ratio of the spin-flip amplitude to the spin-non-flip amplitude is the same for all $J^P M\eta$ states. This situation - normally called "spin coherence" - is described by

$$\bar{T}^{J^P\eta}_{M\uparrow\downarrow} = a\, \bar{T}^{J^P\eta}_{M\uparrow\uparrow} \qquad 5.11$$

where a is an arbitrary complex number, independent of $J^P M$ and where $\bar{T}$ is defined in equation 4.8. In this case

$$\bar{\rho}_{J^P M\eta, J'^{P'} M'\eta} = (1+|a|^2)\, \bar{T}^{J^P\eta}_{M\uparrow\uparrow}\, \bar{T}^{J'^{P'}\eta}_{M'\uparrow\uparrow} \qquad 5.12$$

which can only have one non-zero eigenvalue. Incidentally, a special case of this would be the complete absence of either helicity-flip or helicity-non-flip amplitudes. This situation implies maximal interference between all states.

5. Brief Discussion of errors.

Before going on to make a few remarks about how the errors are calculated it is worth saying that as a result of the density matrix having rank 2, the number of independent parameters in the problem is (for $N \geq 4$) reduced to $4N - 2$. This makes the errors on the N^2 (inter-dependent) parameters calculated by the fit smaller than might have been expected had they been independent.

A few remarks about the error calculation will be made. Let the log-likelihood function be written in the form of a Taylor expansion:

$$\mathcal{L} = \mathcal{L}^0 + \sum_\alpha \left(\frac{\partial \mathcal{L}}{\partial p_\alpha}\right)_{p_\alpha = p_\alpha^0} (p_\alpha - p_\alpha^0) + \frac{1}{2}\sum_{\alpha\beta} (p_\alpha - p_\alpha^0)\left(\frac{\partial^2 \mathcal{L}}{\partial p_\alpha \partial p_\beta}\right)_{\substack{p_\alpha = p_\alpha^0 \\ p_\beta = p_\beta^0}} (p_\beta - p_\beta^0) + \cdots$$

$$\equiv \mathcal{L}^0 + \sum_\alpha V_\alpha \delta p_\alpha - \frac{1}{2}\sum_{\alpha\beta} \delta p_\alpha G_{\alpha\beta} \delta p_\beta + \cdots$$

where p_α are the parameters to be fitted and the superfix "0" corresponds to the starting value of any particular iteration.

Then, in the absence of constraints, the errors on the parameters p_α are given [15] by

$$\delta p_\alpha \, \delta p_\beta = (G^{-1})_{\alpha\beta} \qquad 5.13$$

where the $(G^{-1})_{\alpha\beta}$ are evaluated at the final set of parameter values i.e. the values of p_α which give the maximum likelihood.

The same formula for calculating the errors is used in this analysis, BUT:

(i) even if the maximum value of the likelihood function is found to be in the physical region (i.e. satisfy positivity of the denisty matrix) there is a chance that a positivity-violating set of parameters might be within the symmetric errors given by equation 5.13;

(ii) if the real maximum value of the likelihood function happens to correspond to one or some of the eigenvalues being negative, the values of the parameters used to calculate $(G^{-1})_{\alpha\beta}$ are those which correspond to the situation obtained after the constraints are applied. These are "off-maximum-likelihood" values and so $V_\alpha \neq 0$ which implies that equation 5.13 is not strictly true;

(iii) the error calculation does not take into account the constraints of equation 5.5 and of the positivity of the eigenvalues of the density matrix.

Despite, these difficulties, there is reason to believe that the errors calculated according to the method just described are not without meaning. Starting from the results of a particular fit on experimental data, several sets of "theoretical experimental data" were created by Monte Carlo methods. These were then fitted like real experimental data samples. The results were in agreement within the errors.

6. APPLICATIONS TO OTHER SYSTEMS

The formalism derived in section 3 for the reaction $K^- p \to (K^- \pi^- \pi^+) p$ is, apart from small modifications, applicable to the study of three-meson systems produced in other reactions. Examples are

$$\left.\begin{aligned} &\pi^{\pm} p \to (\pi^{\pm} \pi^{\pm} \pi^{+}) p \\ &K^{\pm} p \to (K^{\pm} K^{\pm} K^{+}) p \\ &\pi^{+} p \to (\pi^{+} \pi^{-} \pi^{o}) \Delta^{++} \\ &\pi^{-} p \to (\pi^{+} \pi^{-} \pi^{o}) n \\ &K^{-} p \to (\pi^{-} \pi^{o} K^{o}) p \\ \text{and} \quad &K^{+} p \to (\pi^{+} \pi^{o} K^{o}) p . \end{aligned}\right\} \qquad 6.1$$

Furthermore, there are special cases where the analysis can be applied to a 4-meson system, for example

$$K^{\pm} p \to (K^{\pm} \pi^{+} \pi^{-} \pi^{o}) p \qquad 6.2$$

where the $(\pi^+ \pi^- \pi^o)$-system forms an ω.

This section is in two parts. Sect. 6.1 describes the modifications needed to make the analysis applicable to various three-meson systems while sect. 6.2 concerns itself with the 4-meson case.

6.1 Three-meson systems.

The only difference between most of the reactions of type 6.1 and the reaction $K^- p \to K^- \pi^- \pi^+ p$ used in sect. 3 is that the amplitudes have to satisfy certain restrictions arising from the identity of particles.

For example, consider the reaction

$$K^- p \to (\pi^- \pi^o K^o) p \ .$$

If the $(K\pi\pi)$-system is assumed to have I = 1/2 the $(\pi^-\pi^o)$-system must have I = 1. The total amplitude must be symmetric with respect to interchange of the two pions and so the space part of the amplitude must be antisymmetric with respect to this interchange.

The symmetry properties of the (3π)-system are more complicated and have been extensively treated in the literature (e.g. ref. [18]) and will not be discussed here.

Nevertheless, one presents here the correctly symmetrised amplitudes needed if the program is used to study the reactions like those in equation 6.1.

The symmetry constraints force certain $C^{J^P \rho_{jn}}$ for different couplings (n-values) to be equal and so, instead of the $H_\nu^{J^P M_\eta \rho_{jn}}$ of equation 5.3.2 one uses linear combinations of them. The indices $J^P M_\eta \rho_j$, all but n, are omitted.

Table 6.1 gives examples of various amplitudes.

6.2 Four-meson systems.

Consider reactions of the type

$$K^- p \to \pi^+ \pi^- \pi^o K^- p$$

with the $(\pi^+\pi^-\pi^o)$-system forming an ω. Thinking of this as a b → 1 + 2 + 3 + 4 + 5, the equation corresponding to equation 3.1 is

$$f_{\lambda_b \lambda_4} = \langle \vec{p}_1 \vec{p}_2 \vec{p}_3 \vec{p}_4 ; \vec{p}_5 \lambda_5 | U | \vec{p}_a ; p_b \lambda_b \rangle \qquad 6.3$$

The set of twelve variables $\{\vec{p}_1 \vec{p}_2 \vec{p}_3 \vec{p}_4\}$ can be replaced by the equivalent set $\{\vec{P}_{1234}\ M_{1234}\ \Phi\ \Theta\ M_{123}\ \alpha\ \beta\ \gamma\ s_{12}\ s_{23}\}$ where

$\vec{P}_{1234} = \sum_{i=1}^{4} \vec{p}_i$ = momentum of (1234)-system in overall CM system;

M_{1234} = invariant mass of the (1234)-system;

Φ, Θ are the azimuthal and polar angles of the (123)-system in the (1234)-rest system;

M_{123} = invariant mass of the (123)-system, i.e. the ω-mass;

The quantities $\alpha,\beta,\gamma,s_{12},s_{23}$ are the 5 variables which describe the (3π)-system:

α,β are the azimuthal and polar angles of the normal to the (3π)-plane in the (123)-rest system (they are two Euler angles);

γ is the third Euler angle, specifying the direction of one of the pions;

s_{12}, s_{23} are the effective masses of two out of the three di-pion systems (Dalitz plot variables).

Changing to an angular momentum basis (of equation 3.5) gives

$$f_{\lambda_b\lambda} = \sum_{J\Lambda j\lambda} \sqrt{\frac{2J+1}{4\pi}}\sqrt{\frac{2j+1}{4\pi}}\, D^{(J)*}_{\Lambda\lambda}(\Phi,\Theta,0)\, D^{(j)*}_{\lambda K}(\alpha,\beta,\gamma)\; g^{J_j}_{\Lambda\lambda\lambda_b\lambda_c K}(s,t,M_{1234},M_{123},s_{12},s_{23}) \qquad 6.4$$

where

J = spin of the (1234)-system;

Λ = helicity of (1234)-system in overall centre of mass;

j = spin of (123)-system;

λ = helicity of (123)-system in (1234)-system;

K = projection of spin j along the normal to the (3π)-plane; see ref [7].

For the special case where the spin-parity of the (3π)-system is 1^- (the ω-meson)

$$g^{J\,j(=1)}_{\Lambda\lambda\lambda_b\lambda_c\,K(=\pm1)} \equiv 0 \qquad 6.5.1$$

Taking this special case, the index K can be dropped and equation 6.4 reduces to

$$f_{\lambda_b\lambda_c} = \sum_{J\Lambda\lambda} \sqrt{\frac{2J+1}{4\pi}}\sqrt{\frac{3}{4\pi}}\, D^{(J)*}_{\Lambda\lambda}(\phi,\theta,\alpha)\, d^{j(=1)}_{\lambda 0}(\beta)\; g^{J\,j(=1)}_{\Lambda\lambda\lambda_b\lambda_c} \qquad 6.5$$

Ignoring the s_{12} and s_{23} dependence of g (e.g. by integrating over s_{12} and s_{23}) it is then seen that equation 6.5 has the same form as equation 3.5 because the ω is a $J^P = 1^-$ particle.

This shows that the reaction $K^- p \to (K^- \omega)p$ and similar reactions can be investigated in terms of the formalism that has been developed to study the 3-meson system.

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APPENDIX A

By integrating the decay distribution over the Dalitz plot variables and also over two of the three Euler angles, the 1-dimensional distribution of the third angle can be calculated. Although, in practice, these 1-dimensional angular distributions are appreciably affected by cuts on the data, it is nevertheless felt thatit is worthwhile discussing them in some detail because they provide clues as to which states should be included in the fit. Remember that the axis of quantization is the direction of the incident K^- in the $(K\pi\pi)$-rest system; that is the so-called t-channel is being used.

Using equations 3.19 and 3.20 the full decay distribution can be written (the same is true after the assumptions of chapter 4 have been made) as

$$W(\theta,\phi,\gamma) = N(\theta,\phi,\gamma) + U(\theta,\phi,\gamma) \qquad \text{A.1}$$

where the natural parity exchange ($\eta = 1$) part $N(\theta,\phi,\gamma)$ is given by

$$N(\theta,\phi,\gamma) = \sum_{\substack{J^P J'^{P'} \ell\ell' jj' \\ \nu\nu' \Lambda(\gamma,\theta)\Lambda'(\gamma,0)}} \cos(\Lambda\phi+\nu\gamma)\cos(\Lambda'\phi+\nu'\gamma) \sum_{J^P\Lambda\eta=1,\, J'^{P'}\Lambda'\eta=1}^{(\ell\ell' jj')} d^{J}_{\Lambda\nu}(\theta)\, d^{J'}_{\Lambda'\nu'}(\theta)\, C_\Lambda C_{\Lambda'} \left(G^{J^P\ell j}_{\nu}\right)^{*} G^{J'^{P'}\ell' j'}_{\nu'} \qquad \text{A.1a}$$

and the un-natural parity exchange ($\eta = -1$) part $U(\theta,\phi,\gamma)$is given by

$$U(\theta,\phi,\gamma) = -\sum_{\substack{J^P J'^{P'} \ell\ell' jj' \\ \nu\nu' \Lambda(\gamma,\theta)\Lambda'(\gamma,0)}} \sin(\Lambda\phi+\nu\gamma)\sin(\Lambda'\phi+\nu'\gamma) \sum_{J^P\Lambda\eta=-1,\, J'^{P'}\Lambda'\eta=-1}^{(\ell\ell' jj')} d^{J}_{\Lambda\nu}(\theta)\, d^{J'}_{\Lambda'\nu'}(\theta)\, C_\Lambda C_{\Lambda'} \left(G^{J^P\ell j}_{\nu}\right)^{*} G^{J'^{P'}\ell' j'}_{\nu'} \qquad \text{A.1b}$$

This distribution W has two useful symmetry properties. The first is that it is unchanged when one replaces (ϕ θ γ) by ($-\phi,\theta, -\gamma$). This is due to parity conservation in the reaction a + b → 1 + 2 + 3 + 4. The parity operation reverses all particle directions (including those which are used to define the coordinate frame). This symmetry condition means that it is only necessary to look at ϕ and γ in the range (0, π).

The second symmetry relation is that, if all the J^P states have the same parity P, then $W(\phi,\theta,\gamma)$ is unchanged when one replaces (ϕ,θ,γ) by $(\pi + \phi, \pi - \theta, \pi - \gamma)$. To prove this one needs equations A.1, 3.19 and some properties of the d-functions.

These two symmetry conditions imply that, if only states of one parity are present, each of the one-dimensional angular distributions are symmetric about $\pi/2$. This applies separately to the natural and un-natural parity exchange parts because they do not interfere.

For completeness a summary will now be given of what can be learnt from the one-dimensional distributions.

The ϕ-distribution

1. A Fourier analysis of the ϕ-distribution can give a lower limit to the maximum value of J required.
2. If only states with $\Lambda = \Lambda' = 0$ are present, the ϕ-distribution has to be flat. (Such a situation is known as t-channel or s-channel helicity conservation depending on whether one is working in the Gottfried-Jackson or the helicity frame).
3. If the distribution is not symmetric about $\pi/2$ then there are contributions from states of different parity and Λ's differing by an odd number.

The γ-and θ-distributions

Again a Fourier analysis can give a lower limit to the maximum value of J required and asymmetry about $\pi/2$ can be taken as evidence of the existence of states of different parity.

It is stressed again that experimental biasses can affect the one-dimensional distributions and so care must be taken.

APPENDIX B

Definition of angles and reference systems

In the (123)-rest system, two sets of coordinate axes can be defined - the so-called "s-channel" (helicity) and "t-channel" axes (Gottfried-Jackson).

The s-channel system is given by:

$$\hat{z}^{(s)} = -\hat{p}_4$$

$$\hat{y}^{(s)} = \hat{p}_a \wedge \hat{z}^{(s)}$$

$$\hat{x}^{(s)} = \hat{y}^{(s)} \wedge \hat{z}^{(s)} .$$

The t-channel system is given by:

$$\hat{z}^{(t)} = \hat{p}_a$$

$$\hat{y}^{(t)} = \hat{p}_4 \wedge \hat{z}^{(t)}$$

$$\hat{x}^{(t)} = \hat{y}^{(t)} \wedge \hat{z}^{(t)} .$$

All the vectors in the above definitions are in the (123)-rest system.

Notice that $\hat{y}^{(s)} = \hat{y}^{(t)}$ = normal to production plane.

The two angles, θ and ϕ, will now be defined with respect to a set of axes $(\hat{x}, \hat{y}, \hat{z})$ which is the same as $(\hat{x}^{(s)}, \hat{y}^{(s)}, \hat{z}^{(s)})$ or $(\hat{x}^{(t)}, \hat{y}^{(t)}, \hat{z}^{(t)})$.

$\cos\theta = \hat{p}_3 . \hat{z}$.

ϕ = azimuthal angle of particle 3 in $(\hat{x}, \hat{y}, \hat{z})$.

To define the angle γ it is necessary to define a new set of axes $(\hat{x}', \hat{y}', \hat{z}')$ in the (12)-rest frame:

$$\hat{z}' = \hat{p}_3$$

$$\hat{y}' = \hat{z} \wedge \hat{z}'$$

$$\hat{x}' = \hat{y}' \wedge \hat{z}' \ ;$$

γ is the azimuthal angle of particle 1 in this system.

The angles θ_{23} and θ_{13} are defined by:

$$\cos \theta_{23} = \hat{p}_2 \cdot \hat{p}_3$$

$$\cos \theta_{13} = \hat{p}_3 \cdot \hat{p}_1$$

They are independent of whether s-channel or t-channel systems are being used.

The angles χ_{12}, χ_{23} and χ_{13} (which are also independent of whether s-channel or t-channel systems are being used) are defined in the (12)-, (23)- and (13)-rest systems respectively. Thus:

$\cos \chi_{12} = -\hat{p}_3 \cdot \hat{p}_1$ evaluated in (12)-rest system;

$\cos \chi_{23} = -\hat{p}_1 \cdot \hat{p}_3$ evaluated in (23)-rest system;

$\cos \chi_{13} = -\hat{p}_2 \cdot \hat{p}_3$ evaluated in (13)-rest system.

Notice that they are not cyclically defined; this is so because it is easier to demonstrate the symmetry between particles 1 and 2 when the angles are not cyclically defined. (The program was originally written for the $(\pi^- \pi^- \pi^+)$-system.)

APPENDIX C

In Appendix A the distribution given by equation 3.19 was integrated over all variables except one Euler angle and the 1-dimensional angular distributions thus obtained were discussed. This appendix discusses the interferences which remain when (a) one integrates over the three Euler angles (i.e. what interferes on the Dalitz plot?) and (b) one integrates over ϕ, θ, γ and s_1 and s_2.

(a) Orthogonality of the rotation matrices $D^J_{MJ}(\phi,\theta,\gamma)$ implies that it is impossible for states of different J or M to interfere on the Dalitz plot. Furthermore, equation 3.15 implies that states with different parity cannot interfere on the Dalitz plot.

(b) The only interferences which can occur when one integrates over ϕ, θ, γ, s_1 and s_2 are interferences between decays of a particular $|J^P M\eta>$-state into final states of different n.

A few illustrative examples are given:

(i) The $K^*(890)\pi$ and $K^*(1420)\pi$ decay modes of a particular $|J^P M\eta>$-state can interfere on the Dalitz plot, but there is no interference between them when one integrates over ϕ, θ, γ, s_1 and s_2;

(ii) The same is true for the s- and d-wave decays of the $|J^P M\eta> = |1^+ 0+>$ state into $K^*(890)\pi$;

(iii) The $K^*(890)\pi$ and $K\rho$ decay modes of given $|J^P M\eta>$-state do interfere even after integrating over all five decay variables.

It should be noticed that the conditions derived here for the $(K^-\pi^-\pi^+)$-system do not all carry over automatically to the case where one has identical particles.

APPENDIX D

The aim of this appendix is to discuss two questions. First of all, in the study of di-meson systems it is well known that there are ambiguities; that is, that given one set of parameters which describes the data it is possible to construct another which describes the data in the same way. Is the same true for three-meson systems? The second question is concerned with the measurability of the imaginary parts of off-diagonal density matrix elements. That there must be off diagonal elements follows if more than two $\{J^P M\eta\}$ combinations exist because the presence of a recoiling nucleon implies that the density matrix has rank 2. A rank 2 matrix with more than 2 non-zero diagonal elements must have non-zero off diagonal elements. These are, in general complex. In the special case when all the production amplitudes are relatively real - "phase coherence" - the off-diagonal density matrix elements are real.

It will be shown that the answers (for an alternative treatment see ref [19]) are related to assumptions 3 and 4. For clarity of presentation the discussion is in two parts, the first part treats the situation when assumption 4 is NOT made while the second part deals with the implications of making this assumption.

Assumption 4 not made.

It is important to stress that if assumption 4 is not made then for every decay mode (specified by j, ℓ and n) there exists a different density matrix.

a) Existence of ambiguities.

Consider equation 3.19 together with assumption 3 (equation 4.4).

of a fixed n, $\ell = 0$ and $j = j' = 1$ which might correspond to s-wave decays via $K^*(890)\pi$. In such a situation it is not possible to measure the imaginary parts of the interference the $M = 0$ and $M' = 1$ amplitudes

$$\left(\text{i.e. } \mathrm{Im}\ \rho^{01n,01n}_{1^+0+,1^+1+}\right) .$$

Assumption 4 made.

Assumption 4 replaces the density matrix $\rho^{\ell j n,\ell' j' n'}_{J^P M\eta,J'^{P'}M'\eta'}$ by the "reduced density matrix" $\bar{\rho}_{J^P M\eta,J'^{P'}M'\eta'}$ and complex decay parameters $\bar{C}^{J^P\ell j n}$ corresponding to various decay modes. It can be seen that different decay modes of a $|J^P M\eta\rangle$-state are associated with the same density matrix elements.

a) Removal of ambiguities

The ambiguities which exist when assumption 4 is not made are no longer present because the decay modes are now dependent on each other.

b) Measurement of imaginary parts.

Consider equation 5.2 written in the form

$$\omega = \sum_{bb'} \bar{\rho}_{bb'}\ \mathcal{N}_b^*\ \mathcal{N}_{b'} \qquad \text{D.3}$$

where $$\mathcal{N}_b \equiv \mathcal{N}_{J^P M\eta} \equiv \sum_{\ell j n} \bar{C}^{J^P\ell j n}\ \mathcal{M}^{J^P}_{M\eta\ell j n}$$

Equations 5.3.1 and 5.3.2 can be used to show that $\mathcal{N}_b^*\mathcal{N}_b$ is always complex. Consequently both real and imaginary parts of all density matrix elements after explicity in equation D.3 and are thus measurable.

$$\omega = \sum \sqrt{(2J+1)(2J'+1)}\ \bar{\rho}^{\ell j n, \ell' j' n'}_{J^P M\eta, J'^{P'} M'\eta}\ BW^*(j,n)\ BW(j',n')\ q^{j+j'} p^{\ell+\ell'}\ C_M C_{M'}$$

$$\times \left(D^{(J)}_{M\nu} + \eta\, D^{(J)*}_{-M-\nu}\right)\left(D^{(J')*}_{M'\nu'} + \eta\, D^{(J')}_{M'-\nu'}\right)\left(G^{J^P \ell j}_{\nu}\right)^*\ G^{J'^{P'}\ell' j'}_{\nu'} \qquad \text{D.1}$$

If $j = j'$ and $n = n'$ it is possible for more than one set of density matrix elements to give the same decay distribution ω - ambiguities can exist. The simplest example is when $j = j' = 0$. In this case, for fixed s_2, equation 3.19 describes the decay of systems of various spin-particles J^P into two spinless particles - like a decay into two pions which is known to have ambiguities.

b) Measurement of imaginary parts

For simplicity, equation D.1 is written in the form

$$\omega = \sum_{aa'} \mathcal{M}^*_a\, \rho_{aa'}\, \mathcal{M}_{a'}$$
$$= \sum_{aa'} \left\{ (\mathrm{Re}\,\rho_{aa'}) \cos\phi - (\mathrm{Im}\,\rho_{aa'}) \sin\phi \right\} |\mathcal{M}_a| \cdot |\mathcal{M}_{a'}| \qquad \text{D.2}$$

where $a \equiv \{J^P M \eta\, \ell j n\}$

and $\phi = \arg(\mathcal{M}_{a'}) - \arg(\mathcal{M}_a)$

From equation D.2 it is seen that $\mathrm{Re}\,\rho_{aa'}$ is measurable if $\cos\phi \neq 0$ somewhere on the Dalitz plot, while $\mathrm{Im}\,\rho_{aa'}$ is measurable if $\sin\phi \neq 0$ somewhere on the Dalitz plot. The only imaginary contribution to $\mathcal{M}^*_a \mathcal{M}_{a'}$ comes from the product $BW(j,n)\, BW(j',n')$ which is therefore responsible for the measurability of the imaginary parts of the density matrix elements.

However, even now, not all imaginary parts of density matrix elements can be determined. This is the case when $n = n'$ and $j = j'$ because the product $BW^*(j,n) BW(j',n')$ becomes real and hence only the corresponding real part of the density matrix can be determined. For example, consider the case

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So, when assumption 4 is made (and one has more than one decay mode of a particular state) the ambiguities discussed no longer exist, and it is in principle possible to measure the imaginary parts of all density matrix elements. If there is but one decay mode, the situation concerning both the ambiguities and the measurability of imaginary parts is similar to what it was before assumption 4 was made.

To end, a cautionary remark is made about the situation when there are identical particles. Consider for example, the case of the $(\pi^-\pi^-\pi^+)$-system. Here, assumption 4 only serves to reduce the number of parameters being fitted. The removal of ambiguities and the measurability are already guaranteed because the imposition of Bose' symmetry ensures that different decay modes interfere.

J P ℓ M η	Decay mode
0 - P 0 +	$K^{*}-$
0 - P 0 +	Kρ
1 + S 0 +	$K^{*}\pi$
1 + S 0 +	Kρ
1 + S 1 +	$K^{*}\pi$

Table 4.1

Table 4.1 Example of set of states given to show how assumption 4 reduces the number of parameters.

Table 6.1 Table of amplitudes to be used in Illinois Partial wave Program for various three-meson systems.

PARTICLES 1 2 3	ISOSPIN (if necessary)	AMPLITUDE (spins of dimeson $\leq$ 1 only)
$K^- \pi^- \pi^+$ $K^+ \pi^+ \pi^-$		$H_{13}(K^*\pi) + \bar{C}_1 H_{23}(K\rho) + \bar{C}_2 H_{23}(K\varepsilon) + \bar{C}_3 H_{13}(\pi)$
$\pi^- \pi^- \pi^+$ $\pi^+ \pi^+ \pi^-$		$[H_{13}(\rho\pi) + H_{23}(\rho\pi)] + \bar{C}\,[H_{13}(\varepsilon\pi) + H_{23}(\varepsilon\pi)]$
$\pi^- \pi^0 \bar{K}^0$ $\pi^+ \pi^0 K^0$	$I = \frac{1}{2}$	$[H_{23}(K^*\pi) - H_{13}(K^*\pi)] + \bar{C}\, H_{12}(K\rho)$
$\pi^- K^- K^+$ $\pi^+ K^+ K^-$	$I = 1$	$H_{13}(K^*K) + \bar{C}\, H_{23}(\phi\pi)$
$K^- K^- K^+$ $K^+ K^+ K^-$	$I = \frac{1}{2}$	$\lvert H_{13}(\phi, K) + H_{23}(\phi, K)\rvert$
$\bar{K}^0 \pi^+ \pi^-$ $K^0 \pi^- \pi^+$		$H_{13}(K^*\pi) + \bar{C}\, H_{23}(\rho K)$
$\pi^+ \pi^- \pi^0$	$I = 0$	$H_{13}(\rho\pi) - H_{23}(\rho\pi) + H_{12}(\rho\pi)$ *
	$I = 1$	$[H_{13}(\rho\pi) + H_{23}(\rho\pi)]$ * $+ \bar{C}\, H_{12}(\varepsilon\pi)$

* The sign of $H_{23}(\rho\pi)$ comes from the non-cyclic definition of the angles; see Appendix C.

TABLE 6.1