

Three-point functions in ABJM and Bethe Ansatz

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ABSTRACT: We develop an integrability-based framework to compute structure constants of two sub-determinant operators and a single-trace non-BPS operator in ABJM theory in the planar limit. In this first paper, we study them at weak coupling using a relation to an integrable spin chain. We first develop a nested Bethe ansatz for an alternating $SU(4)$ spin chain that describes single-trace operators made out of scalar fields. We then apply it to the computation of the structure constants and show that they are given by overlaps between a Bethe eigenstate and a matrix product state. We conjecture that the determinant operator corresponds to an integrable matrix product state and present a closed-form expression for the overlap, which resembles the so-called Gaudin determinant. We also provide evidence for the integrability of general sub-determinant operators. The techniques developed in this paper can be applied to other quantities in ABJM theory including three-point functions of single-trace operators.

KEYWORDS: $1/N$ Expansion, AdS-CFT Correspondence, Bethe Ansatz, Integrable Field Theories

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1 Introduction

Over the last fifteen years, there have been breathtaking developments in solving $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) in four dimensions in the large N_c limit. This was mainly thanks to the successful application of *integrability*, a method originally invented to

solve special two-dimensional systems. A key finding which triggered this burst of activity was made by Minahan and Zarembo [1], who recognized that the action of the dilatation operator on a subclass of operators, called single-trace operators, can be mapped to a Hamiltonian of an integrable spin chain. Subsequently the idea was generalized and refined in various directions, and by now the integrability was applied not just to the spectrum of single-trace operators [2–6] but also to their correlation functions [7–23] and even to their $1/N_c$ corrections¹ [25–28]. All these progresses indicate that we are close to a complete solution to an interacting gauge theory in four dimensions; a feat never achieved in the long history of theoretical physics.

More recently the integrability approach was applied to quantities that are far beyond single-trace operators: the paper [29, 30] showed that the correlation function of two determinant operators and one single-trace operator can be computed exactly using the integrability machinery. This came as a surprise since the quantum numbers of determinant operators scale with N_c , and even performing the perturbative computation is a nontrivial task. Nevertheless, by a judicious rewriting one can map these observables to overlaps of states in an integrable two-dimensional system (a spin chain at weak coupling and a string worldsheet at strong coupling), and compute them exactly [29, 30].

In this series of papers [31, 32], we generalize the analysis of [29, 30] to $\mathcal{N} = 6$ supersymmetric Chern-Simons-matter theory in three dimensions, constructed by Aharony, Bergman, Jafferis and Maldacena [33]; *ABJM theory* for short. Soon after the construction of ABJM theory, it was realized in [34–37] that ABJM theory is also integrable in the large N_c limit as far as the spectrum of single-trace operators² is concerned. However, unlike $\mathcal{N} = 4$ SYM, not much is known beyond the spectrum. In particular, we are still lacking an integrability-based framework to compute correlation functions at finite 't Hooft coupling. The goal of this series of papers is to propose and establish the first of such frameworks. In this first paper, we focus on the computation at weak coupling and show that the tree-level structure constants of two determinant operators and one single-trace operator can be computed by overlaps between a matrix product state and a Bethe eigenstate in an integrable spin chain. Furthermore, we present evidence that the relevant matrix product state preserves integrability and conjecture a closed-form expression for the overlaps in terms of Gaudin-like determinants. Similar expressions were found in various different contexts, ranging from the study of quench dynamics [40–48] to the defect one-point functions in $\mathcal{N} = 4$ SYM [49–58].

We should also mention an important difference from $\mathcal{N} = 4$ SYM. In $\mathcal{N} = 4$ SYM, the result in [59] suggests that only the determinant operator corresponds to an integrable boundary state and all the sub-determinant operators are not integrable³. By contrast,

¹See also recent interesting work on non-planar anomalous dimensions [24].

²The full solution to the spectral problem was obtained by Quantum Spectral Curve [38, 39].

³We should however note that the (non-)integrability of the sub-determinant operators is not fully settled even for $\mathcal{N} = 4$ SYM. For instance, there are results on the spectrum of open string attached to the sub-determinant operators which suggest the sub-determinant operators might actually correspond to integrable boundary states [60, 61]. It would be interesting to revisit this question in view of recent discovery of a new integrable boundary condition in $AdS_5 \times S^5$ [62], which was missed in the classification in [63].

the analysis of this paper suggests that the sub-determinant operators are also integrable in ABJM theory, at least at tree level. The possibility of having such a family of integrable boundary states motivates further study of these operators in ABJM theory.

Before describing the contents of the paper, let us explain a couple of more motivations. Two rather obvious reasons were already mentioned; 1. we can test the formalism developed in [29, 30] in other theories, and 2. we establish the first integrability-based framework to compute correlation functions in ABJM theory. Yet another motivation comes from the fact that the structure constants in ABJM theory receive quantum corrections even when all the operators are BPS [64–66]. For the purpose of checking the AdS/CFT correspondence, this is more like a curse than a blessing since it inhibits a direct comparison between the results at weak coupling and the results in supergravity. On the other hand, this feature makes ABJM theory an ideal testing ground for the integrability approach to the correlation functions: in the integrability description, the BPS operator corresponds to the vacuum state of the spin chain and provides the simplest setup for the computation. Unfortunately, this setup was “too simple” for $\mathcal{N} = 4$ SYM since the quantum corrections all vanish [67]. By contrast, in ABJM theory the setup provides a simple yet nontrivial test of the formalism because of the dependence on the coupling constant. In addition, structure constants of BPS single-trace operators in planar ABJM theory can be computed by supersymmetric localization [68–76]. At present, localization computation has not been generalized to sub-determinant operators, but if we succeed in doing so, we will be able to compare two rather different approaches and deepen our understanding on the relation between them.

The rest of the paper is organized as follows. In section 2, we explain the setup to be discussed in this paper: in ABJM theory, even the three-point functions of BPS operators can have several R -symmetry tensor structures and therefore depend on a multitude of structure constants. In order to simplify the analysis, we focus on the *twisted-translated frame*. We show that the structure constant computed in the twisted-translated frame is a particular linear combination of the full structure constants.⁴ In section 3, we first review the basic properties of the $SU(4)$ invariant alternating spin chain, such as the Hamiltonian, the Bethe equations and the relation to the single-trace operator. We then develop the nested coordinate Bethe ansatz and write down wave functions of Bethe states. In section 4, we derive a matrix product state representation of the structure constant of two giant gravitons and a single-trace BPS operator. Such analyses were performed in $\mathcal{N} = 4$ SYM in [29, 30, 59] and in ABJ(M) theory in [77]. The novelty of our analysis is that we derive an explicit expression for the matrix product state for the sub-determinant operators in the twisted-translated frame which can be readily used to evaluate the structure constants. We also evaluate the tree-level structure constants of two non-maximal sub-determinant operators and a single-trace BPS operator. In section 5, we compute the structure constants using the results in the preceding sections and present our conjecture. As was the case with $\mathcal{N} = 4$ SYM, the result for determinant operators exhibits a nontrivial

⁴In principle, we can recover the full structure constants by acting the R -symmetry transformation to each operator, which in the spin-chain language corresponds to adding zero-momentum magnons.

selection rule on the rapidities and is given by a ratio of Gaudin-like determinants. These imply that the determinant operators in ABJM theory correspond to integrable boundary states. One notable difference from $\mathcal{N} = 4$ SYM is that these features persist also for sub-determinant operators, indicating that ABJM theory admits a family of integrable boundary states. Our results also provide the first example of integrable matrix product states in the alternating spin chain. Finally in section 6, we conclude and discuss future directions. Several appendices are included to explain technical details.

2 Setup and Kinematics

2.1 Generalities

Basics. ABJM theory is a $\mathcal{N} = 6$ superconformal Chern-Simons matter theory in three dimensions with a product gauge group $U(N)_k \times U(N)_{-k}$ where k is the Chern-Simons level. It consists of two sets of gauge fields and matter fields in the bi-fundamental representations. See [33, 78] for the explicit form of the Lagrangian. See also [79] for a review of the integrability properties of ABJM theory.

A distinguishing feature of ABJM theory (as compared to $\mathcal{N} = 4$ SYM) is that it admits two different large N limits: the first limit is called the *M-theory limit* and can be defined by $N \rightarrow \infty$ with k fixed. This limit has attracted much attention since it is dual to a M-theory on $AdS_4 \times S^7/Z_k$ and provides one of the most concrete non-perturbative definitions of the M-theory currently available. Unfortunately this limit is difficult to study on the field theory side since it corresponds to a strong coupling limit (unless we focus on the BPS observables and use supersymmetric localization). We therefore study the second limit in this paper, namely the *planar limit*. The planar limit is defined by sending $N \rightarrow \infty$ while keeping the 't Hooft coupling

$$\lambda \equiv \frac{N}{k}, \tag{2.1}$$

fixed. As is the case with the standard planar limit of Yang-Mills theories, the observables in the planar limit can be computed by the planar diagrams— diagrams that can be drawn on a genus 0 Riemann surface. The holographic dual of this limit is given by type-IIA superstring theory in $AdS_4 \times CP^3$.

The main subject of this paper involves gauge invariant local operators in ABJM made out of scalar fields. ABJM theory has two sets of scalar fields; the one that transforms as $(\square, \bar{\square}; \mathbf{4})$ and the other that transforms as $(\bar{\square}, \square; \bar{\mathbf{4}})$ under the $U(N)_k \times U(N)_{-k}$ gauge groups and $SU(4)$ R -symmetry,

$$\begin{aligned} Y^I &: (\square, \bar{\square}; \mathbf{4}), & \Delta^0 &= 1/2, \\ \bar{Y}_I &: (\bar{\square}, \square; \bar{\mathbf{4}}), & \Delta^0 &= 1/2, \end{aligned} \tag{2.2}$$

where $I = 1, \dots, 4$ and Δ^0 is the mass dimension. The simplest gauge invariant operators constructed out of such fields are *single-trace operators*, which in general take the following form:

$$\text{tr} \left(Y^{I_1} \bar{Y}_{J_1} \cdots Y^{I_L} \bar{Y}_{J_L} \right) + \cdots \tag{2.3}$$

As is clear from this expression, it consists of an alternating sequence of Y^I fields and \bar{Y}_I fields. Because of this feature, the spin chain that describes the anomalous dimension of such an operator is an *alternating* spin chain, in which spins living on odd sites are distinct from spins living on even sites. We will give a more detailed description of the spin chain and its Bethe ansatz in section 3.

BPS operators and giant gravitons. General single-trace operators (2.3) do not preserve any supersymmetry. However, for a special choice of the R -symmetry indices, they become BPS and invariant under 1/3 of the supersymmetry transformations. Written explicitly they take the following form,

$$\mathcal{O}_L^\circ(x; n, \bar{n}) \equiv \text{tr} \left[\left((n \cdot Y)(\bar{n} \cdot \bar{Y}) \right)^L \right], \quad (2.4)$$

where the superscript \circ is to indicate that the operator is BPS⁵ and we have

$$n \cdot Y \equiv \sum_{I=1}^4 n_I Y^I, \quad \bar{n} \cdot \bar{Y} \equiv \sum_{I=1}^4 \bar{n}^I \bar{Y}_I. \quad (2.5)$$

Here n and \bar{n} are four-component vectors specifying the R -symmetry polarizations of the operator and they need to satisfy

$$n \cdot \bar{n} = 0, \quad (2.6)$$

in order for the operator to be BPS.

Another class of BPS operators considered in this paper are *giant gravitons* [81–84]. They are defined in terms of sub-determinants as

$$\mathcal{D}_M(x; n, \bar{n}) \equiv \frac{1}{M!} \delta_{[a_1 \dots a_M]}^{[b_1 \dots b_M]} \left[(n \cdot Y)(\bar{n} \cdot \bar{Y}) \right]_{b_1}^{a_1} \dots \left[(n \cdot Y)(\bar{n} \cdot \bar{Y}) \right]_{b_M}^{a_M}, \quad (2.7)$$

with $n \cdot \bar{n} = 0$ and

$$\delta_{[a_1 \dots a_M]}^{[b_1 \dots b_M]} \equiv \sum_{\sigma \in S_M} (-1)^{|\sigma|} \delta_{a_{\sigma_1}}^{b_1} \dots \delta_{b_{\sigma_M}}^{a_M}. \quad (2.8)$$

The operator with a maximal R -charge ($M = N$) is called the *maximal giant gravitons* while others ($M < N$) are called *non-maximal giant gravitons*.

Alternatively, they can be defined in terms of the antisymmetric Schur polynomial [81, 82, 85, 86] as follows:

$$\mathcal{D}_M(x; n, \bar{n}) = \frac{1}{M!} \sum_{\sigma \in S_M} \chi_{A_M}(\sigma) \left[(n \cdot Y)(\bar{n} \cdot \bar{Y}) \right]_{a_1}^{a_{\sigma_1}} \dots \left[(n \cdot Y)(\bar{n} \cdot \bar{Y}) \right]_{a_M}^{a_{\sigma_M}}, \quad (2.9)$$

where χ_{A_M} is a Schur polynomial for the totally antisymmetric representation of size M . See e.g. [77] for explicit definitions. To study the correlation functions of giant gravitons,

⁵Here we are following the notation used in [80].

it is often convenient to consider a generating function $\mathcal{G}(x; n, \bar{n}, t)$ defined by⁶

$$\mathcal{G}(x; n, \bar{n}, t) \equiv \det \begin{pmatrix} \mathbf{1} & -t(\bar{n} \cdot \bar{Y}) \\ t(n \cdot Y) & \mathbf{1} \end{pmatrix} = \det [\mathbf{1} + t^2(n \cdot Y)(\bar{n} \cdot \bar{Y})] . \quad (2.10)$$

To extract the giant gravitons with a fixed charges, we perform the integral of t ;

$$\mathcal{D}_M(x; n, \bar{n}) = \oint \frac{dt}{2\pi i t^{1+2M}} \mathcal{G}(x; n, \bar{n}, t) . \quad (2.11)$$

Holographic dual. Let us also briefly review the dual description of giant gravitons although it is not directly relevant for the analysis performed in this paper. There are two classes of giant gravitons in $AdS_4 \times CP^3$ known in the literature which are conjectured to be dual to 1/3 BPS operators.⁷ The first class is the D2 branes extended in S^2 inside AdS_4 [87–89]. These branes are known to be dual to *symmetric* Schur polynomials and are analogs of the dual giant gravitons in $\mathcal{N} = 4$ SYM. The other class is the D4 branes which are point-like in AdS_4 and extended in the CP^3 direction [83, 87, 90–94]. They are dual to antisymmetric Schur polynomials and are the subject of this paper. In the upcoming paper [31], we will study the correlation functions from these holographic perspectives.

2.2 Structures of two- and three-point functions

We now summarize the structures of two- and three-point functions and the constraints from symmetry, emphasizing the differences from $\mathcal{N} = 4$ SYM.

BPS two-point functions. Let us first consider the two-point functions of 1/3 BPS single-trace operators, $\langle \mathcal{O}_{L_1}^\circ \mathcal{O}_{L_2}^\circ \rangle$. Since n_k 's transform as the anti-fundamentals of $SU(4)$ while \bar{n}_k 's transform as the fundamentals of $SU(4)$, the $SU(4)$ symmetry determines the structure of the two-point function to be

$$\langle \mathcal{O}_{L_1}^\circ \mathcal{O}_{L_2}^\circ \rangle = \delta_{L_1, L_2} \mathcal{N}_{\mathcal{O}_{L_1}^\circ} (d_{12} d_{21})^{L_1} , \quad (2.12)$$

where $\mathcal{N}_{\mathcal{O}_L^\circ}$ is the normalization constant, which at weak coupling reads

$$\mathcal{N}_{\mathcal{O}_L^\circ} = L \lambda^{2L} , \quad (2.13)$$

while d_{ij} 's are defined by

$$d_{ij} \equiv \frac{n_i \cdot \bar{n}_j}{|x_{ij}|} \quad |x_{ij}| \equiv |x_i - x_j| . \quad (2.14)$$

⁶The second equality in (2.10) follows from the invariance of the determinant under the addition and the subtraction of rows or columns,

$$\det \begin{pmatrix} \mathbf{1} & -t(\bar{n} \cdot \bar{Y}) \\ t(n \cdot Y) & \mathbf{1} \end{pmatrix} = \det \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ t(n \cdot Y) & \mathbf{1} + t^2(n \cdot Y)(\bar{n} \cdot \bar{Y}) \end{pmatrix} = \det [\mathbf{1} + t^2(n \cdot Y)(\bar{n} \cdot \bar{Y})] .$$

⁷Some giant gravitons with more or less supersymmetries were also studied in the papers listed below in this paragraph.

Note that, unlike $\mathcal{N} = 4$ SYM, d_{ij} here is not symmetric under the exchange of indices: $d_{ij} \neq d_{ji}$.

The symmetry is powerful enough to determine the two-point functions of (sub)determinant operators as well:

$$\langle \mathcal{D}_M(x_1, n_1, \bar{n}_1) \mathcal{D}_M(x_2, n_2, \bar{n}_2) \rangle = \mathcal{N}_{\mathcal{D}_M} (d_{12} d_{21})^M. \quad (2.15)$$

Here again $\mathcal{N}_{\mathcal{D}_M}$ is the normalization constant.

BPS three-point functions. We then discuss the three-point functions of 1/3-BPS operators. This is where ABJM theory shows significant differences from $\mathcal{N} = 4$ SYM:

- In $\mathcal{N} = 4$ SYM, the R -symmetry structure of the BPS three-point functions is determined completely by the symmetry. On the other hand, the BPS three-point function in ABJM theory admits several different structures.
- The structure constant of 1/2-BPS operators in $\mathcal{N} = 4$ SYM does not depend on the 't Hooft coupling. By contrast, the structure constant of 1/3-BPS operators in ABJM theory is a nontrivial function of the 't Hooft coupling.

To see this explicitly, let us consider the three-point function of BPS single-trace operators. Imposing the $SU(4)$ symmetry, one finds that the three-point function has a multitude of allowed structures labelled by p which is either an integer or a half-integer⁸ [95],

$$\frac{\langle \mathcal{O}_{L_1}^\circ \mathcal{O}_{L_2}^\circ \mathcal{O}_{L_3}^\circ \rangle}{\sqrt{\mathcal{N}_{\mathcal{O}_{L_1}^\circ} \mathcal{N}_{\mathcal{O}_{L_2}^\circ} \mathcal{N}_{\mathcal{O}_{L_3}^\circ}}} = (d_{12} d_{21})^{L_{12|3}} (d_{23} d_{32})^{L_{23|1}} (d_{31} d_{13})^{L_{31|2}} \sum_{p=-\ell_{123}}^{\ell_{123}} C_{123}^{(p)} \xi^p, \quad (2.16)$$

with

$$L_{ij|k} \equiv \frac{L_i + L_j - L_k}{2}, \quad \ell_{123} \equiv \min [L_{12|3}, L_{23|1}, L_{31|2}]. \quad (2.17)$$

Here ξ is the $SU(4)$ cross ratio

$$\xi \equiv \frac{d_{12} d_{23} d_{31}}{d_{21} d_{32} d_{13}} = \frac{(n_1 \cdot \bar{n}_2)(n_2 \cdot \bar{n}_3)(n_3 \cdot \bar{n}_1)}{(n_2 \cdot \bar{n}_1)(n_3 \cdot \bar{n}_2)(n_1 \cdot \bar{n}_3)}, \quad (2.18)$$

and $C_{123}^{(p)}$'s are a set of structure constants. These structure constants can be readily computed at tree level by Wick contractions [64, 84]:

$$C_{123}^{(p)} \stackrel{\lambda \rightarrow 0}{=} \frac{\sqrt{L_1 L_2 L_3}}{N} \times \begin{cases} 2\delta_{p,0} & \sum_{k=1}^3 L_k : \text{ even} \\ \delta_{p, \frac{1}{2}} + \delta_{p, -\frac{1}{2}} & \sum_{k=1}^3 L_k : \text{ odd} \end{cases}. \quad (2.19)$$

On the other hand, [84, 96] performed the computation at strong coupling and the result reads

$$C_{123}^{(p)} \stackrel{\lambda \rightarrow \infty}{=} \frac{1}{N} \sqrt{\frac{h(\lambda)}{\pi}} \frac{\prod_{j=1}^3 \sqrt{1+2L_j} \Gamma[1+L_j]}{\Gamma[1+\frac{L_1+L_2+L_3}{2}]} \prod_{\{i,j,k\}} \frac{\Gamma[1+\frac{L_{ij|k}}{2}]}{\Gamma[1-p+\frac{L_{ij|k}}{2}] \Gamma[1+p+\frac{L_{ij|k}}{2}]}, \quad (2.20)$$

⁸ p takes an integer-value when $\sum_k L_k$ is even while it takes a half-integer value when $\sum_k L_k$ is odd.

where $\prod_{\{i,j,k\}}$ denotes a product over cyclic permutations of $\{1, 2, 3\}$ and $h(\lambda)$ is the so-called *interpolating function* [97] which can be expanded at weak and strong couplings as

$$h(\lambda) = \begin{cases} \lambda - \frac{\pi^2 \lambda^3}{3} + \dots & \lambda \rightarrow 0 \\ \sqrt{\frac{1}{2} \left(\lambda - \frac{1}{24} \right)} - \frac{\log 2}{\pi} + \dots & \lambda \rightarrow \infty. \end{cases} \quad (2.21)$$

The difference between (2.19) and (2.20) provides clear evidence for the coupling dependence of structure constants $C_{123}^{(p)}$.

The expression (2.16) can be readily generalized to the three-point function of two sub-determinant operators and one single-trace BPS operator, which is the main subject of this paper. The resulting expression is

$$\begin{aligned} & \frac{\langle \mathcal{D}_M(x_1; n_1, \bar{n}_1) \mathcal{D}_M(x_2; n_2, \bar{n}_2) \mathcal{O}_L^\circ(x_3; n_3, \bar{n}_3) \rangle}{\mathcal{N}_{\mathcal{D}_M} \sqrt{\mathcal{N}_{\mathcal{O}_L^\circ}}} \\ &= (d_{12} d_{21})^M \left(\frac{d_{23} d_{32} d_{31} d_{13}}{d_{12} d_{21}} \right)^{\frac{L}{2}} \sum_{p=-\frac{L}{2}}^{\frac{L}{2}} D_{M|L}^{(p)} \xi^p, \end{aligned} \quad (2.22)$$

where $D_{M|L}^{(p)}$'s are structure constants. See (4.53) for the tree-level result for $D_{M|L}^{(p)}$.

Non-BPS operators. We now make a few comments on non-BPS operators. In the spin-chain approach [34], we start from the vacuum state, which corresponds to the 1/3-BPS operator, and introduce excitations (magnons) in order to describe non-BPS operators. The operators constructed in this way depend on two sets of data;

1. The R-symmetry polarizations (n and \bar{n}) of the vacuum state $\text{tr}[(n \cdot Y)(\bar{n} \cdot \bar{Y})]^L$.
2. A set of rapidities of magnons \mathbf{u} .

In this paper, we define the normalization of non-BPS operators $\mathcal{N}_{\mathcal{O}}$ in terms of the two-point function with a canonical choice of the R-symmetry polarizations $n_0 \equiv (1, 0, 0, 0)$ and $\bar{n}_0 \equiv (0, 0, 0, 1)$:

$$\langle \mathcal{O}(x_1; n_0, \bar{n}_0) [\mathcal{O}(x_2; n_0, \bar{n}_0)]^\dagger \rangle = \frac{\mathcal{N}_{\mathcal{O}}}{x_{12}^{2\Delta_{\mathcal{O}}}}. \quad (2.23)$$

Here \mathcal{O}^\dagger is a Hermitian-conjugate of the operator \mathcal{O} and $\Delta_{\mathcal{O}}$ is the conformal dimension.

We then define the normalized three-point function by

$$\begin{aligned} & \frac{\langle \mathcal{O}_1(x_1; n_1, \bar{n}_1) \mathcal{O}_2(x_2; n_2, \bar{n}_2) \mathcal{O}_3(x_3; n_3, \bar{n}_3) \rangle}{\sqrt{\mathcal{N}_{\mathcal{O}_1} \mathcal{N}_{\mathcal{O}_2} \mathcal{N}_{\mathcal{O}_3}}} \\ &= \frac{F_{123}}{x_{12}^{\Delta_{\mathcal{O}_1} + \Delta_{\mathcal{O}_2} - \Delta_{\mathcal{O}_3}} x_{23}^{\Delta_{\mathcal{O}_2} + \Delta_{\mathcal{O}_3} - \Delta_{\mathcal{O}_1}} x_{31}^{\Delta_{\mathcal{O}_3} + \Delta_{\mathcal{O}_1} - \Delta_{\mathcal{O}_2}}}, \end{aligned} \quad (2.24)$$

where F_{123} is a sum of all possible allowed R -symmetry and Lorentz invariants times the corresponding structure constants. Non-BPS operators typically have more quantum numbers than the BPS operators. Therefore the number of allowed structures for the non-BPS operators is larger than the one for the BPS operators. In addition, the structures highly depend on the Lorentz and the R -symmetry representations of the operator. These features make it difficult to write down a simple universal expression like (2.16). In what follows, we instead focus on special kinematic configurations, called the *twisted-translated frame*.

Phase ambiguity of structure constants. Note that the normalization of non-BPS operators as defined in (2.23) does not fix the overall phase of the operator \mathcal{O} since a multiplication of a phase to \mathcal{O} ($\mathcal{O} \rightarrow e^{i\phi}\mathcal{O}$) does not change $\mathcal{N}_{\mathcal{O}}$. On the other hand, the three-point functions — and therefore structure constants — do change by such a phase multiplication. This means that the overall phase of the structure constants will not be fixed by our analysis⁹ and one needs to impose further conditions in order to determine it. In section 5, we will show that there is one choice of a phase with which the result from integrability takes a simple form. However, this is more like an answer analysis and we do not have a physical explanation on why it simplifies the final expression. It would be important to come up with a field-theory argument on why this choice of a phase is preferred.

2.3 Twisted translation

The twisted-translated frame was introduced originally in $\mathcal{N} = 4$ SYM in four dimensions in [98] and used also in the integrability analysis [7]. A key feature of this frame is that it makes the correlation functions of 1/2 BPS operators position-independent. A similar topological sector in a large class of $\mathcal{N} \geq 4$ SCFTs in three dimensions was studied by localization in [68–71]. For the case of ABJM theory, the full localization results are not yet available¹⁰, but general symmetry properties of the twisted sector were discussed in [95] and the perturbative computation was performed in [99]. Much like in $\mathcal{N} = 4$ SYM, the twisted-translated frame in ABJM theory provides a useful setup for analyzing the non-BPS three-point functions as well. In fact, it was also discussed in an unpublished work [100, 101] which attempted to construct the hexagon formalism for ABJM theory. As we see below, the structure constant computed in this frame is a particular linear combination of the full structure constants. In principle, we can recover the full structure constants by acting the R -symmetry transformation to each operator, which in the spin-chain language corresponds to adding zero-momentum magnons.

BPS correlation functions. For the correlation functions of BPS operators, the twisted-translated frame is defined by placing all the operators along a single line and aligning the R -symmetry polarizations along an $U(1)$ direction inside $SU(4)$. Written ex-

⁹This phase ambiguity was pointed out already in [80].

¹⁰Two important exceptions are ABJM theory for $k = 1$ [71, 72] and the correlation function of an operator corresponding to the mass deformation [73–75].

plicitly, the twisted-translated BPS single-trace operator reads

$$\hat{\mathcal{O}}_L^\circ(a) \equiv \text{tr} \left[(\mathcal{Y}(a)\bar{\mathcal{Y}}(a))^L \right], \quad (2.25)$$

where \mathcal{Y} and $\bar{\mathcal{Y}}$ are given by

$$\begin{aligned} \mathcal{Y}(a) &\equiv \left(Y^1 + \kappa a Y^4 \right) (0, a, 0), \\ \bar{\mathcal{Y}}(a) &\equiv \left(\bar{Y}_4 - \kappa a \bar{Y}_1 \right) (0, a, 0). \end{aligned} \quad (2.26)$$

Here κ is an arbitrary parameter with mass dimension 1. Similarly, the sub-determinant operators in the twisted-translated frame can be defined by

$$\hat{\mathcal{D}}_M(a) \equiv \frac{1}{M!} \delta_{[a_1 \dots a_M]}^{[b_1 \dots b_M]} (\mathcal{Y}(a)\bar{\mathcal{Y}}(a))_{b_1}^{a_1} \dots (\mathcal{Y}(a)\bar{\mathcal{Y}}(a))_{b_M}^{a_M}. \quad (2.27)$$

An alternative way to describe the twisted-translated frame is to use the *twisted translation* generator, which is a linear combination of the translation and the R-symmetry rotation,

$$\mathcal{T} \equiv iP_2 + \kappa R_1^4, \quad (2.28)$$

where P_2 is the translation along the x_2 direction while R_1^4 is the R-symmetry generator which rotates Y^1 and Y^4 . Using \mathcal{T} , we can express \mathcal{Y} and $\bar{\mathcal{Y}}$ as

$$\mathcal{Y}(a) = e^{\mathcal{T}a} Y^1(0) e^{-\mathcal{T}a}, \quad \bar{\mathcal{Y}}(a) = e^{\mathcal{T}a} \bar{Y}_4(0) e^{-\mathcal{T}a}. \quad (2.29)$$

An important property of \mathcal{T} is that it is \mathcal{Q} exact, i.e.

$$\mathcal{T} = \{ \mathcal{Q}, \bullet \}, \quad (2.30)$$

where \mathcal{Q} is a linear combination of the supersymmetry and superconformal generators which is nilpotent and annihilates the 1/3 BPS operator made out of $\mathcal{Y}(0)$ and $\bar{\mathcal{Y}}(0)$. Because of this property, the correlation functions of twisted-translated 1/3 BPS operators (2.25) and (2.26) become independent of the positions a 's and define a topological subsector. See [95, 101] for more detailed explanation.

This position independence can also be seen directly from the general structure of BPS two- and three-point functions (2.12), (2.15), (2.16) and (2.22). For this purpose, we simply need to set

$$n_j = (1, 0, 0, \kappa a_1), \quad \bar{n}_j = (-\kappa a_j, 0, 0, 1), \quad x_j = (0, a_j, 0), \quad (2.31)$$

in those equations. This leads to $d_{ij} = \kappa \text{sgn}(a_i - a_j)$, and we get

$$\begin{aligned} \langle \hat{\mathcal{O}}_{L_1}^\circ(a_1) \hat{\mathcal{O}}_{L_2}^\circ(a_2) \rangle &= \delta_{L_1, L_2} \mathcal{N}_{\mathcal{O}_{L_1}^\circ} (-\kappa^2)^{L_1}, \\ \langle \hat{\mathcal{D}}_M(a_1) \hat{\mathcal{D}}_M(a_2) \rangle &= \mathcal{N}_{\mathcal{D}_M} (-\kappa^2)^M, \end{aligned} \quad (2.32)$$

$$\frac{\langle \hat{\mathcal{O}}_{L_1}^\circ(a_1) \hat{\mathcal{O}}_{L_2}^\circ(a_2) \hat{\mathcal{O}}_{L_3}^\circ(a_3) \rangle}{\sqrt{\mathcal{N}_{\mathcal{O}_{L_1}^\circ} \mathcal{N}_{\mathcal{O}_{L_2}^\circ} \mathcal{N}_{\mathcal{O}_{L_3}^\circ}}} = (-\kappa^2)^{\frac{L_1+L_2+L_3}{2}} \mathfrak{C}_{123}^\circ, \quad (2.33)$$

$$\frac{\langle \hat{\mathcal{D}}_M(a_1) \hat{\mathcal{D}}_M(a_2) \hat{\mathcal{O}}_L^\circ(a_3) \rangle}{\mathcal{N}_{\mathcal{D}_M} \sqrt{\mathcal{N}_{\mathcal{O}_L^\circ}}} = (-\kappa^2)^{M+\frac{L}{2}} \mathfrak{D}_{M|L}^\circ. \quad (2.34)$$

Here $\mathcal{N}_{\mathcal{O}_L^\circ}$ and $\mathcal{N}_{\mathcal{D}_M}$ are normalizations defined in (2.12) and (2.15) while \mathfrak{C}_{123}° and $\mathfrak{D}_{M|L}^\circ$ are the structure constants in the twisted-translated frame, and they are given by linear combinations of the structure constants defined in (2.16) and (2.22):

$$\mathfrak{C}_{123}^\circ = \sum_{p=-\ell_{123}}^{\ell_{123}} (-1)^p C_{123}^{(p)}, \quad \mathfrak{D}_{M|L}^\circ = \sum_{p=-\frac{L}{2}}^{\frac{L}{2}} (-1)^p D_{M|L}^{(p)}. \quad (2.35)$$

Non-BPS operators. To define the twisted-translated frame for non-BPS operators, we first construct a non-BPS operator by adding magnons (denoted in red below) on top of $\text{tr}[(Y^1 \bar{Y}_4)^L]$ at the origin:

$$\mathcal{O}(0) \equiv \text{tr} \left[\cdots (Y^1 \bar{Y}_4) (\mathbf{Y}^2 \bar{Y}_4) (\mathbf{Y}^1 \bar{Y}_3) (Y^1 \bar{Y}_4) \cdots \right] + \cdots |_{x^\mu=0}. \quad (2.36)$$

As we explain in more detail in the next section, these operators are in one-to-one correspondence with states in the alternating SU(4) invariant spin chain,

$$\text{tr} \left[\cdots (Y^1 \bar{Y}_4) (\mathbf{Y}^2 \bar{Y}_4) (\mathbf{Y}^1 \bar{Y}_3) (Y^1 \bar{Y}_4) \cdots \right] \leftrightarrow |1\bar{4}\mathbf{2}\bar{4}1\bar{\mathbf{3}}1\bar{4}\cdots\rangle. \quad (2.37)$$

We then act the twisted translation $e^{\mathcal{T}a}$ to obtain an operator at a shifted position:

$$\hat{\mathcal{O}}(a) \equiv e^{\mathcal{T}a} \mathcal{O}(0) e^{-\mathcal{T}a}. \quad (2.38)$$

Note that the twisted translation leaves $Y^{2,3}$ and $\bar{Y}_{2,3}$ invariant:

$$e^{\mathcal{T}a} Y^{2,3} e^{-\mathcal{T}a} = Y^{2,3}, \quad e^{\mathcal{T}a} \bar{Y}_{2,3} e^{-\mathcal{T}a} = \bar{Y}_{2,3}. \quad (2.39)$$

As is clear from the definition above, the twisted-translation only involves an SU(2) subgroup of the full SU(4) R-symmetry. As demonstrated e.g. in appendix A of [102], the SU(2) Ward identity allows us to express the kinematic dependence of the three-point function in terms of the conformal dimension Δ and the U(1) R-charge J , which assigns $+1/2$ charge to Y^1 and \bar{Y}_4 and $-1/2$ charge to Y^4 and \bar{Y}_1 . The result reads

$$\frac{\langle \hat{\mathcal{O}}_1(a_1) \hat{\mathcal{O}}_2(a_2) \hat{\mathcal{O}}_3(a_3) \rangle}{\sqrt{\mathcal{N}_{\mathcal{O}_1} \mathcal{N}_{\mathcal{O}_2} \mathcal{N}_{\mathcal{O}_3}}} = \frac{(-\kappa^2)^{\frac{J_1+J_2+J_3}{2}} \mathfrak{C}_{123}}{(a_1 - a_2)^{\gamma_{12|3}} (a_2 - a_3)^{\gamma_{23|1}} (a_3 - a_1)^{\gamma_{31|2}}}, \quad (2.40)$$

with

$$\gamma_{ij|k} \equiv (\Delta_i - J_i) + (\Delta_j - J_j) - (\Delta_k - J_k). \quad (2.41)$$

Here $\mathcal{N}_{\mathcal{O}}$ is the normalization defined in (2.23), and \mathfrak{C}_{123} is the structure constant in the twisted-translated frame, which is F_{123} in (2.24) specialized to the twisted-translated kinematics.

These expressions can be generalized to the correlation function of two sub-determinant operators and a single-trace non-BPS operator. The result reads

$$\frac{\langle \hat{\mathcal{D}}_M(a_1) \hat{\mathcal{D}}_M(a_2) \hat{\mathcal{O}}(a_3) \rangle}{\mathcal{N}_{\mathcal{D}_M} \sqrt{\mathcal{N}_{\mathcal{O}}}} = (-\kappa^2)^{M+\frac{J}{2}} \left(\frac{(a_1 - a_2)}{(a_2 - a_3)(a_3 - a_1)} \right)^{\Delta-J} \mathfrak{D}_{M|\mathcal{O}}, \quad (2.42)$$

with $\mathfrak{D}_{M|\mathcal{O}}$ being the structure constant. The main goal of this paper is to compute $\mathfrak{D}_{M|\mathcal{O}}$ at tree level using the spin-chain description.

3 SU(4) invariant alternating spin chain

In this section, we explain the spin-chain description of single-trace operators made out of scalar fields. After reviewing the basic facts of the SU(4) invariant alternating spin chain and its relation to single-trace operators, we discuss its coordinate Bethe ansatz. In particular, we present explicit expressions for the coordinate Bethe ansatz wave functions, which we will use later to evaluate the structure constants (2.42). An alternative way to construct the wave function is to use the algebraic Bethe ansatz which we review in appendix D.

3.1 Hamiltonian and Bethe equations

The dilatation operator in the scalar sector of ABJM theory at two-loop order is described by the SU(4) invariant alternating spin chain [34, 37]. The Hamiltonian of the spin chain is given by

$$\mathbb{H} = \frac{\lambda^2}{2} \sum_{l=1}^{2L} (2 - 2P_{l,l+2} + P_{l,l+2}K_{l,l+1} + K_{l,l+1}P_{l,l+2}), \quad (3.1)$$

where P_{ab} and K_{ab} are permutation and trace operators acting on the a -th and b -th sites. We denote the set of orthonormal basis of the Hilbert space at each site by $|i\rangle$, $i = 1, \dots, 4$. The two operators act as

$$P|i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle, \quad K|i\rangle \otimes |j\rangle = \delta_{ij} \sum_{k=1}^4 |k\rangle \otimes |k\rangle. \quad (3.2)$$

Using these definitions, it is straightforward to show that P_{ab} and K_{ab} obey the following relations

$$P_{ab}P_{ab} = 1, \quad K_{ab}K_{ab} = 4K_{ab}, \quad P_{ab}K_{bc} = K_{ac}P_{ab} = K_{ac}K_{bc}. \quad (3.3)$$

The spin chain under consideration is an alternating spin chain. The odd and even sites sit in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ representation of SU(4) group and correspond to fields Y^A and \bar{Y}_A in ABJM theory respectively. In order to distinguish odd and even sites, we put bars to the odd sites, namely

$$Y^A \mapsto |A\rangle, \quad \bar{Y}_A \mapsto |\bar{A}\rangle. \quad (3.4)$$

In this notation, the single-trace operator $\text{tr}(Y^{A_1}\bar{Y}_{B_1}Y^{A_2}\bar{Y}_{B_2}\dots)$ is mapped to the following spin-chain state;

$$\text{tr}(Y^{A_1}\bar{Y}_{B_1}Y^{A_2}\bar{Y}_{B_2}\dots) \mapsto |A_1\bar{B}_1A_2\bar{B}_2\dots\rangle. \quad (3.5)$$

One special feature of the Hamiltonian ((3.1)) is that the permutation operator $P_{l,l+2}$ only act on the odd- or even-site spin chains while the trace operator $K_{l,l+1}$ only mix the two. We have

$$K|A\rangle \otimes |\bar{B}\rangle = \delta_{AB} \sum_{C=1}^4 |C\rangle \otimes |\bar{C}\rangle. \quad (3.6)$$

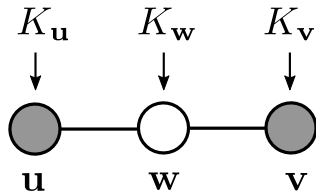


Figure 1. Dynkin diagram of the SU(4) alternating spin chain. The gray circles are the momentum carrying nodes.

The Hamiltonian (3.1) is known to be integrable and solvable by the Bethe ansatz. Reflecting the alternating structure of the spin chain, the Bethe state is described by two sets of the momentum carrying Bethe roots which we denote by \mathbf{u} and \mathbf{v} (see figure 1). In addition, there are auxiliary Bethe roots \mathbf{w} . The numbers of rapidities of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are denoted by $K_{\mathbf{u}}, K_{\mathbf{v}}, K_{\mathbf{w}}$ respectively. The rapidities satisfy the Bethe equations

$$\begin{aligned}
 1 = e^{i\phi_{u_j}} &= \left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} \right)^L \prod_{\substack{k=1 \\ k \neq j}}^{K_{\mathbf{u}}} S(u_j, u_k) \prod_{k=1}^{K_{\mathbf{w}}} \tilde{S}(u_j, w_k), \\
 1 = e^{i\phi_{w_j}} &= \prod_{\substack{k=1 \\ k \neq j}}^{K_{\mathbf{w}}} S(w_j, w_k) \prod_{k=1}^{K_{\mathbf{u}}} \tilde{S}(w_j, u_k) \prod_{k=1}^{K_{\mathbf{v}}} \tilde{S}(w_j, v_k), \\
 1 = e^{i\phi_{v_j}} &= \left(\frac{v_j + \frac{i}{2}}{v_j - \frac{i}{2}} \right)^L \prod_{\substack{k=1 \\ k \neq j}}^{K_{\mathbf{v}}} S(v_j, v_k) \prod_{k=1}^{K_{\mathbf{w}}} \tilde{S}(v_j, w_k),
 \end{aligned} \tag{3.7}$$

where the S-matrices $S(u, v)$ and $\tilde{S}(u, v)$ are given by

$$S(u, v) \equiv \frac{u - v - i}{u - v + i}, \quad \tilde{S}(u, v) \equiv \frac{u - v + \frac{i}{2}}{u - v - \frac{i}{2}}. \tag{3.8}$$

The two-loop anomalous dimension of the operator is given by

$$\Delta - L = \lambda^2 \left(\sum_{k=1}^{K_{\mathbf{u}}} \frac{1}{u_k^2 + \frac{1}{4}} + \sum_{k=1}^{K_{\mathbf{v}}} \frac{1}{v_k^2 + \frac{1}{4}} \right). \tag{3.9}$$

The U(1) R-charge J discussed in section 2.3 can be expressed in terms of $K_{\mathbf{u}, \mathbf{v}}$ as

$$J = L - \frac{K_{\mathbf{u}} + K_{\mathbf{v}}}{2}. \tag{3.10}$$

Zero-momentum condition. As is the case with $\mathcal{N} = 4$ SYM, not all solutions to the Bethe equation corresponds to a single-trace operator in ABJM theory. This is because the single-trace operator has an additional *cyclicity* property. In the spin-chain language, this is equivalent to the zero-momentum condition,

$$1 = \prod_{j=1}^{K_{\mathbf{u}}} \frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} \prod_{j=1}^{K_{\mathbf{v}}} \frac{v_j + \frac{i}{2}}{v_j - \frac{i}{2}}. \tag{3.11}$$

3.2 Coordinate Bethe ansatz

The eigenvector of the spin chain (3.1) can be constructed by the nested coordinate Bethe ansatz (CBA). To describe the CBA construction, we map each scalar field to a specific combination of the Bethe roots following the notation in appendix E of [10]. More precisely, we first express odd- and even-sites by bullets \bullet and circles \circ respectively, and place Bethe roots on top of them. Then, the relation between stacks of Bethe roots and the fields in ABJM theory is given by

$$\begin{aligned}
 Y^1 \mapsto |1\rangle &= |\bullet\rangle, & Y^2 \mapsto |2\rangle &= |\bullet^u\rangle, & Y^3 \mapsto |3\rangle &= |\bullet^w\rangle, & Y^4 \mapsto |4\rangle &= |\bullet^{uv}\rangle, \\
 \bar{Y}_1 \mapsto |\bar{1}\rangle &= |\circ^{wv}\rangle, & \bar{Y}_2 \mapsto |\bar{2}\rangle &= |\circ^w\rangle, & \bar{Y}_3 \mapsto |\bar{3}\rangle &= |\circ^v\rangle, & \bar{Y}_4 \mapsto |\bar{4}\rangle &= |\circ\rangle.
 \end{aligned}
 \tag{3.12}$$

We now briefly outline the procedure of constructing eigenvectors. It can be achieved in two steps. In the first step, we distribute the rapidities on different sites of the spin chain and sum over all such possibilities. In the second step, we construct the wave function for each distribution and multiply it to the corresponding distribution. Plugging in the physical solutions of the Bethe roots, we obtain an eigenvector of the Hamiltonian.

Distributing excitations. We consider the spin chain of length $2L$. We first distribute these rapidities on different sites by the following procedure

1. Firstly we distribute the momentum carrying rapidities $\{u_1, \dots, u_{K_u}\}$ and $\{v_1, \dots, v_{K_v}\}$ on top of the ground state $|\bullet_1, \circ_1, \dots, \bullet_L, \circ_L\rangle$. Each odd site can support either a single u -type rapidity, or two rapidities one of uv -type. Likewise, each even site can support a single v -type rapidity, or two rapidities of uv -type.
2. Now we view the physical rapidities $\{u_1, u_2, \dots, u_{K_u}\}$ and $\{v_1, v_2, \dots, v_{K_v}\}$ as inhomogeneities of an *emergent spin chain* of length $K_u + K_v$ and distribute the rapidities $\{w_1, w_2, \dots, w_{K_w}\}$ on the emergent chain. Each w_j can be distributed on top of either u_i or v_i .
3. In the previous two steps, we also generate string configurations without field theory correspondence in (3.12). For example, \bullet^{uv} does not corresponds to any of the fields Y^A or \bar{Y}_A . We set all the states which contain such string configurations to zero.

We give an example of the procedure described above in figure 2.

In what follows, we denote the stack of Bethe roots at site n by \mathfrak{s}_n . The possible configurations are

$$\mathfrak{s}_{2n-1} = \left\{ \bullet_n, \bullet_n^u, \bullet_n^{w_a}, \bullet_n^{w_a u_k v_j} \right\}, \quad \mathfrak{s}_{2n} = \left\{ \circ_n, \circ_n^v, \circ_n^{w_a}, \circ_n^{w_a u_k v_j} \right\}.
 \tag{3.13}$$

A collection of all \mathfrak{s}_n 's for a given distribution will be denoted by $\vec{\mathfrak{s}}$

$$\vec{\mathfrak{s}} \equiv \{ \mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_{2L-1}, \mathfrak{s}_{2L} \}.
 \tag{3.14}$$

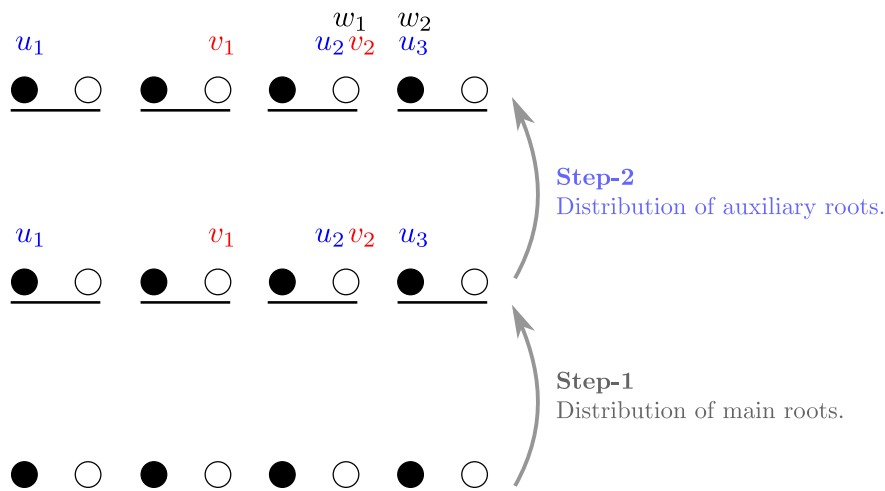


Figure 2. Construction of (one of) the Bethe string configuration corresponding to the ket state $\text{tr}(Y^2 \bar{Y}_4)(Y^1 \bar{Y}_3)(Y^1 \bar{Y}_1)(Y^3 \bar{Y}_4) = |2, \bar{4}, 1, \bar{3}, 1, \bar{1}, 3, \bar{4}\rangle$.

The wave functions: ordered configurations. The next step is to write down a wave function for each distribution of rapidities. We first discuss the distribution in which the rapidities are in the canonical order (namely in the same order as \mathbf{u} , \mathbf{v} and \mathbf{w}). We call such distributions *ordered distributions*. The wave function for a ordered distribution of Bethe roots \vec{s} is given by a product of “single-site wave functions” $\Phi(\mathbf{s})$;

$$\Psi_{\vec{s}}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \prod_{n=1}^{2L} \Phi(\mathbf{s}_n). \tag{3.15}$$

The single-site wave functions with at most a single rapidity are

$$\Phi(\bullet_n) = \Phi(\circ_n) = 1, \quad \Phi(\bullet_n^{u_k}) = \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^n, \quad \Phi(\circ_n^{v_k}) = \left(\frac{v_k + \frac{i}{2}}{v_k - \frac{i}{2}} \right)^n. \tag{3.16}$$

It is slightly more complicated to write a single-site wave function when there is w because the expression depends on the Bethe roots on other sites; or more precisely on the momentum carrying rapidities to the left of the site. For instance, $\Phi(\mathbf{s})$ ’s corresponding to $\{\bullet_n^{\overset{w}{u}}, \circ_n^{\overset{w}{v}}\}$ are given by

$$\begin{aligned} \Phi(\bullet_n^{\overset{w}{u}}) &= \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^n \times \psi(w_a | \mathbf{z}^<) \times \frac{-1}{w_a - u_k - \frac{i}{2}}, \\ \Phi(\circ_n^{\overset{w}{v}}) &= \left(\frac{v_k + \frac{i}{2}}{v_k - \frac{i}{2}} \right)^n \times \psi(w_a | \mathbf{z}^<) \times \frac{1}{w_a - v_k - \frac{i}{2}}, \end{aligned} \tag{3.17}$$

where $\mathbf{z}^<$ denotes all the momentum carrying rapidities that are to the left of the site which supports w . The explicit form is given by

$$\psi(w | \mathbf{z}^<) = \prod_j \frac{w - z_j^< + \frac{i}{2}}{w - z_j^< - \frac{i}{2}}. \tag{3.18}$$

Similarly, the single-site wave functions for the configurations $\{\bullet_n^w, \circ_n^u\}$ are given by

$$\begin{aligned}\Phi(\bullet_n^w) &= \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^n \left(\frac{v_j + \frac{i}{2}}{v_j - \frac{i}{2}}\right)^n \times \psi(w_a | \mathbf{z}^<) \times \frac{-(v_j - \frac{i}{2})}{(w_a - u_k - \frac{i}{2})(w_a - v_j - \frac{i}{2})}, \\ \Phi(\circ_n^u) &= \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^n \left(\frac{v_k + \frac{i}{2}}{v_k - \frac{i}{2}}\right)^n \times \psi(w_a | \mathbf{z}^<) \times \frac{+(u_k + \frac{i}{2})}{(w_a - u_k - \frac{i}{2})(w_a - v_j - \frac{i}{2})}.\end{aligned}\quad (3.19)$$

The wave functions: general configurations. More general configurations of the rapidities can be obtained from the ordered configurations by performing permutations of the Bethe roots. For such configurations, the wave function consists of two factors. The first factor is a product of single-site wave functions Φ ,

$$\prod_{n=1}^{2L} \Phi(\mathbf{s}_n). \quad (3.20)$$

The second factor is a product of S-matrices, which are needed to bring the rapidities in the distribution into the canonical order. For instance, for the distribution in which u_2 is to the left of u_1 , we need to multiply

$$S(u_1, u_2) = \frac{u_1 - u_2 - i}{u_1 - u_2 + i}. \quad (3.21)$$

We multiply similar factors ($S(v_j, v_k)$ and $S(w_j, w_k)$) also for v 's and w 's. Therefore, the wave function for a general distribution is given by

$$\Psi_{\vec{\mathbf{s}}}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbb{S} \times \prod_{n=1}^{2L} \Phi(\mathbf{s}_n), \quad (3.22)$$

where \mathbb{S} is a product of S-matrices described above.

The Bethe eigenstate is given by a summation of all possible distributions; namely

$$|\Psi_{\mathbf{u}, \mathbf{v}, \mathbf{w}}\rangle = \sum_{\vec{\mathbf{s}} \in \text{all possible distributions}} \Psi_{\vec{\mathbf{s}}}(\mathbf{u}, \mathbf{v}, \mathbf{w}) |\vec{\mathbf{s}}\rangle. \quad (3.23)$$

Examples. As explicit examples of the procedure described above, we give three simple states for $L = 2$. The vacuum state is given by

$$|\Omega\rangle = |1, \bar{4}, 1, \bar{4}\rangle. \quad (3.24)$$

First, we consider the state with $K_{\mathbf{u}} = 1$, $K_{\mathbf{v}} = 1$, $K_{\mathbf{w}} = 0$. The state is given by a sum of four terms, and below we list them along with the wave function for each distribution:

$$\begin{aligned}|2, \bar{3}, 1, \bar{4}\rangle &\equiv |\bullet_1^u, \circ_1^v, \bullet_2, \circ_2\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}}\right) \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}}\right), \\ |2, \bar{4}, 1, \bar{3}\rangle &\equiv |\bullet_1^u, \circ_1, \bullet_2, \circ_2^v\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}}\right) \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}}\right)^2, \\ |1, \bar{3}, 2, \bar{4}\rangle &\equiv |\bullet_1, \circ_1^v, \bullet_2^u, \circ_2\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}}\right)^2 \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}}\right), \\ |1, \bar{4}, 2, \bar{3}\rangle &\equiv |\bullet_1, \circ_1, \bullet_2^u, \circ_2^v\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}}\right)^2 \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}}\right)^2.\end{aligned}$$

Second, we consider the state with $K_{\mathbf{u}} = 2$, $K_{\mathbf{v}} = K_{\mathbf{w}} = 0$. It consists of the following two terms

$$\begin{aligned}
 |2, \bar{4}, 2, \bar{4}\rangle_1 &\equiv |\bullet_1^{\mathbf{u}_1}, \circ_1, \bullet_2^{\mathbf{u}_2}, \circ_2\rangle \left(\frac{u_1 + \frac{i}{2}}{u_1 - \frac{i}{2}} \right) \left(\frac{u_2 + \frac{i}{2}}{u_2 - \frac{i}{2}} \right)^2, \\
 |2, \bar{4}, 2, \bar{4}\rangle_2 &\equiv |\bullet_1^{\mathbf{u}_2}, \circ_1, \bullet_2^{\mathbf{u}_1}, \circ_2\rangle \frac{u_1 - u_2 - i}{u_1 - u_2 + i} \left(\frac{u_2 + \frac{i}{2}}{u_2 - \frac{i}{2}} \right) \left(\frac{u_1 + \frac{i}{2}}{u_1 - \frac{i}{2}} \right)^2.
 \end{aligned}$$

The first term is an ordered distribution while the second term is not. This is why the second term comes with an extra factor of the S-matrix $S(u_1, u_2) = (u_1 - u_2 - i)/(u_1 - u_2 + i)$.

As the third example, we consider the state with $K_{\mathbf{u}} = 1$, $K_{\mathbf{v}} = 1$, $K_{\mathbf{w}} = 1$. In this case, there are 12 terms, which we list below

$$\begin{aligned}
 |2, \bar{2}, 1, \bar{4}\rangle &\equiv |\bullet_1^{\mathbf{u}}, \circ_1, \bullet_2^{\mathbf{w}}, \circ_2\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}} \right) \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}} \right) \frac{w - u + \frac{i}{2}}{w - u - \frac{i}{2}} \frac{1}{w - v - \frac{i}{2}}, \\
 |2, \bar{4}, 1, \bar{2}\rangle &\equiv |\bullet_1^{\mathbf{u}}, \circ_1, \bullet_2^{\mathbf{w}}, \circ_2\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}} \right) \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}} \right)^2 \frac{w - u + \frac{i}{2}}{w - u - \frac{i}{2}} \frac{1}{w - v - \frac{i}{2}}, \\
 |1, \bar{2}, 2, \bar{4}\rangle &\equiv |\bullet_1^{\mathbf{w}}, \circ_1, \bullet_2^{\mathbf{u}}, \circ_2\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}} \right)^2 \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}} \right) \frac{1}{w - v - \frac{i}{2}}, \\
 |1, \bar{4}, 2, \bar{2}\rangle &\equiv |\bullet_1^{\mathbf{u}}, \circ_1, \bullet_2^{\mathbf{w}}, \circ_2\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}} \right)^2 \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}} \right)^2 \frac{w - u + \frac{i}{2}}{w - u - \frac{i}{2}} \frac{1}{w - v - \frac{i}{2}}, \\
 |3, \bar{3}, 1, \bar{4}\rangle &\equiv |\bullet_1^{\mathbf{u}}, \circ_1, \bullet_2^{\mathbf{v}}, \circ_2\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}} \right) \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}} \right) \frac{-1}{w - u - \frac{i}{2}}, \\
 |3, \bar{4}, 1, \bar{3}\rangle &\equiv |\bullet_1^{\mathbf{w}}, \circ_1, \bullet_2^{\mathbf{v}}, \circ_2\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}} \right) \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}} \right)^2 \frac{-1}{w - u - \frac{i}{2}}, \\
 |1, \bar{4}, 3, \bar{3}\rangle &\equiv |\bullet_1^{\mathbf{u}}, \circ_1, \bullet_2^{\mathbf{w}}, \circ_2\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}} \right)^2 \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}} \right)^2 \frac{-1}{w - u - \frac{i}{2}}, \\
 |1, \bar{3}, 3, \bar{4}\rangle &\equiv |\bullet_1^{\mathbf{v}}, \circ_1, \bullet_2^{\mathbf{u}}, \circ_2\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}} \right)^2 \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}} \right) \frac{w - v + \frac{i}{2}}{w - v - \frac{i}{2}} \frac{-1}{w - u - \frac{i}{2}}, \\
 |4, \bar{4}, 1, \bar{4}\rangle &\equiv |\bullet_1^{\mathbf{uv}}, \circ_1, \bullet_2, \circ_2\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}} \right) \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}} \right) \frac{-(v - \frac{i}{2})}{(w - v - \frac{i}{2})(w - u - \frac{i}{2})}, \\
 |1, \bar{1}, 1, \bar{4}\rangle &\equiv |\bullet_1^{\mathbf{uv}}, \circ_1, \bullet_2, \circ_2\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}} \right) \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}} \right) \frac{+(u + \frac{i}{2})}{(w - v - \frac{i}{2})(w - u - \frac{i}{2})}, \\
 |1, \bar{4}, 4, \bar{1}\rangle &\equiv |\bullet_1^{\mathbf{uv}}, \circ_1, \bullet_2, \circ_2\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}} \right)^2 \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}} \right)^2 \frac{-(v - \frac{i}{2})}{(w - v - \frac{i}{2})(w - u - \frac{i}{2})}, \\
 |1, \bar{4}, 1, \bar{1}\rangle &\equiv |\bullet_1^{\mathbf{uv}}, \circ_1, \bullet_2, \circ_2\rangle \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}} \right)^2 \left(\frac{v + \frac{i}{2}}{v - \frac{i}{2}} \right)^2 \frac{+(u + \frac{i}{2})}{(w - v - \frac{i}{2})(w - u - \frac{i}{2})}.
 \end{aligned}$$

4 Matrix product states from giant gravitons

To compute the structure constant (2.42) using integrability, we first need to map it to a quantity in the spin chain. For $\mathcal{N} = 4$ SYM, this was done¹¹ in [29, 30, 59], which showed that the structure constant (2.42) corresponds to an overlap between a matrix product state (MPS) and a Bethe eigenstate. This analysis was later generalized to ABJ(M) theory in [77]. In this section, we review and extend the results in [77]. Two main new results are

1. We compute tree-level structure constants of two non-maximal giant gravitons and a single-trace BPS operator.
2. We derive an explicit expression for MPS of two non-maximal giant gravitons in the twisted-translated frame which can be readily used in later sections.

4.1 Derivation of matrix product states

Although the main goal of this paper is to compute the structure constant in the twisted-translated frame (2.42), in this subsection we consider a slightly more general setup; the correlation function of m giant gravitons and a single-trace operator in a general kinematics. More precisely we consider the following correlation function

$$G_m = \left\langle \left(\prod_{j=1}^m \mathcal{G}_j \right) \mathcal{O}(y) \right\rangle, \quad (4.1)$$

where \mathcal{G}_j is a generating function for giant gravitons (cf. (2.10)):

$$\mathcal{G}_j \equiv \det \begin{pmatrix} \mathbf{1} & -t_j(\bar{n}_j \cdot \bar{Y}) \\ t_j(n_j \cdot Y) & \mathbf{1} \end{pmatrix} (x_j). \quad (4.2)$$

To extract the correlation functions of individual giant gravitons, we simply need to perform integrals of t_j 's,

$$\left\langle \left(\prod_{j=1}^m \mathcal{D}_{M_j} \right) \mathcal{O}(y) \right\rangle = \left(\prod_{j=1}^m \oint \frac{dt_j}{t_j^{1+2M_j}} \right) G_m. \quad (4.3)$$

Since giant gravitons carry large R-charges $M_j \sim \mathcal{O}(N)$, most of these integrals in the large N limit can be computed by the saddle-point approximation. We will later see how this works in practice. For now we focus on computing the correlation functions of generating functions.

Let us now compute G_m at tree level. This is given by the following path integral

$$G_m = \frac{1}{Z_Y} \int DY^I D\bar{Y}_I \left(\prod_{j=1}^m \mathcal{G}_j \right) \mathcal{O} \exp \left[-\frac{k}{4\pi} \int d^3x \operatorname{tr}(\partial_\mu \bar{Y}_I \partial^\mu Y^I) \right], \quad (4.4)$$

¹¹The analysis was recently generalized to the four-point function of determinant operators in $\mathcal{N} = 4$ SYM [103].

with

$$Z_Y = \int DY^I D\bar{Y}_I \exp \left[-\frac{k}{4\pi} \int d^3x \operatorname{tr}(\partial_\mu \bar{Y}_I \partial^\mu Y^I) \right], \quad (4.5)$$

The generating functions can be written in terms of path integral of fermions

$$\mathcal{G}_j = \int d\eta_j d\bar{\eta}_j d\chi_j d\bar{\chi}_j \exp \left[(\bar{\chi}_j \chi_j) + (\bar{\eta}_j \eta_j) + t_j \bar{\chi}_j (n_j \cdot Y) \eta_j - t_j \bar{\eta}_j (\bar{n}_j \cdot \bar{Y}) \chi_j \right]. \quad (4.6)$$

Here the fermions χ_i and η_i are in the fundamental representation of the gauge group $U(N)_k$ and $U(N)_{-k}$ respectively while $\bar{\chi}_i, \bar{\eta}_i$ are in the antifundamental representation. The indices are contracted as

$$\bar{\chi}(n \cdot Y) \eta = \bar{\chi}^a (n \cdot Y)_a \bar{b} \eta_b, \quad \bar{\eta}(\bar{n} \cdot \bar{Y}) \chi = \bar{\eta}^{\bar{a}} (\bar{n} \cdot \bar{Y})_{\bar{a}}^b \chi_b. \quad (4.7)$$

We can then rewrite the correlation function G_m as

$$G_m = \frac{1}{Z_Y} \int DY^I D\bar{Y}_I \prod_{i=1}^m d\bar{\chi}_i d\eta_i d\bar{\eta}_i d\chi_i \mathcal{O} \exp(-S_{\chi, \eta}), \quad (4.8)$$

where

$$S_{\chi, \eta} = -\sum_{i=1}^m ((\bar{\eta}_i \eta_i) + (\bar{\chi}_i \chi_i)) + \frac{k}{4\pi} \int d^3x \left[\operatorname{tr}(\partial_\mu \bar{Y}_I \partial^\mu Y^I) - \frac{4\pi}{k} \sum_{i=1}^m t_i \bar{\chi}_i (n_i \cdot Y) \eta_i \delta^3(x - x_i) + \frac{4\pi}{k} \sum_{i=1}^m t_i \bar{\eta}_i (\bar{n}_i \cdot \bar{Y}) \chi_i \delta^3(x - x_i) \right]. \quad (4.9)$$

Since this action is quadratic in Y^I, \bar{Y}_I , we can integrate them out. In practice, this is equivalent to solving the equations of motion for these fields,

$$\begin{aligned} \square (Y^I)_a \bar{b}(x) &= -\frac{4\pi}{k} \sum_{i=1}^m t_i \bar{n}_i^I \chi_{i,a} \bar{\eta}_i^{\bar{b}} \delta^3(x - x_i), \\ \square (Y_I^\dagger)_{\bar{a}}^b(x) &= \frac{4\pi}{k} \sum_{i=1}^m t_i n_{i,I} \eta_{i,\bar{a}} \bar{\chi}_i^b \delta^3(x - x_i). \end{aligned} \quad (4.10)$$

These equations can be solved by Green's function. Using the fact that

$$\square \frac{1}{|x - y|} = -4\pi \delta^3(x - y), \quad (4.11)$$

the solution to (4.10) is given by

$$Y^I(x) = S^I(x) \equiv \frac{1}{k} \sum_{i=1}^m \frac{t_i \bar{n}_i^I \chi_i \bar{\eta}_i}{|x - x_i|}, \quad \bar{Y}_I(x) = \bar{S}_I(x) \equiv -\frac{1}{k} \sum_{i=1}^m \frac{t_i n_{i,I} \eta_i \bar{\chi}_i}{|x - x_i|}. \quad (4.12)$$

Plugging the solution (4.12) back to (4.8), we obtain

$$G_m = \int \prod_{i=1}^m d\eta_i d\bar{\eta}_i d\chi_i d\bar{\chi}_i \mathcal{O}^S \exp(-\tilde{S}_{\chi, \eta}), \quad (4.13)$$

where \mathcal{O}^S is obtained by replacing all Y^I, \bar{Y}_I by their classical solution S^I, \bar{S}_I in the single trace operator. The effective action $\tilde{S}_{\chi,\eta}$ is given by

$$\tilde{S}_{\chi,\eta} = - \sum_i ((\bar{\eta}_i \eta_i) + (\bar{\chi}_i \chi_i)) - \frac{\lambda}{N} \sum_{i,j} t_i t_j \underbrace{\frac{n_i \cdot \bar{n}_j}{x_{ij}}}_{=d_{ij}} (\bar{\chi}_i \chi_j) (\bar{\eta}_j \eta_i). \quad (4.14)$$

For the next step, we integrate in the “bilocal” fields ρ_{ij} and $\bar{\rho}_{ij}$

$$G_m = \frac{1}{Z_{\rho,\bar{\rho}}} \int d\rho d\bar{\rho} d\eta d\bar{\eta} d\chi d\bar{\chi} \mathcal{O}^S \exp \left[- \frac{N}{\lambda} \sum_{\substack{i,j \\ i \neq j}} \rho_{ij} \bar{\rho}_{ji} - \tilde{S}_{\chi,\eta} \right], \quad (4.15)$$

where

$$Z_{\rho,\bar{\rho}} = \int d\rho d\bar{\rho} \exp \left[- \frac{N}{\lambda} \sum_{\substack{i,j \\ i \neq j}} \rho_{ij} \bar{\rho}_{ji} \right]. \quad (4.16)$$

Shifting the bilocal fields $\rho, \bar{\rho}$ by

$$\rho_{ij} \mapsto \rho_{ij} - \frac{\lambda}{N} \sqrt{t_i t_j d_{ij}} (\bar{\chi}_i \chi_j), \quad \bar{\rho}_{ji} \mapsto \bar{\rho}_{ji} - \frac{\lambda}{N} \sqrt{t_i t_j d_{ij}} (\bar{\eta}_j \eta_i), \quad (4.17)$$

the action of (4.15) can be written as

$$\begin{aligned} S_{\rho,\chi,\eta} = & \frac{N}{\lambda} \sum_{\substack{i,j \\ i \neq j}} \rho_{ij} \bar{\rho}_{ji} - \sum_{i,j} \sqrt{t_i t_j d_{ij}} (\rho_{ij} (\bar{\eta}_j \eta_i) + (\bar{\chi}_i \chi_j) \bar{\rho}_{ji}) \\ & - \sum_i ((\bar{\eta}_i \eta_i) + (\bar{\chi}_i \chi_i)). \end{aligned} \quad (4.18)$$

Finally we integrate out the fermions and obtain the expression

$$G_m = \frac{1}{Z_{\rho,\bar{\rho}}} \int d\rho d\bar{\rho} \langle \mathcal{O}^S \rangle_{\chi,\eta} \exp(N S_{\text{eff}}). \quad (4.19)$$

Let us explain the definitions of each quantity in the formula: first the effective action is given by

$$S_{\text{eff}} = - \frac{1}{\lambda} \text{tr}_m(\rho \bar{\rho}) + \text{tr}_m \log \Sigma_\eta + \text{tr}_m \log \Sigma_\chi, \quad (4.20)$$

where tr_k means a trace of matrices of size k . Second $\Sigma_{\eta,\chi}$ are $m \times m$ matrices defined by

$$\Sigma_\eta \equiv \mathbf{1}_m + \hat{\rho}, \quad \Sigma_\chi \equiv \mathbf{1}_m + \hat{\bar{\rho}}, \quad (4.21)$$

where $\mathbf{1}_m$ is the identity matrix of size m and $\hat{\rho}$ and $\hat{\bar{\rho}}$ are defined by

$$\hat{\rho}_{ij} = \sqrt{t_i d_{ij} t_j} \rho_{ij}, \quad \hat{\bar{\rho}}_{ij} = \sqrt{t_j d_{ji} t_i} \bar{\rho}_{ij}. \quad (4.22)$$

Third the expectation value $\langle \mathcal{O}^S \rangle_{\chi, \eta}$ is given by

$$\langle \mathcal{O}^S \rangle_{\chi, \eta} = \frac{\int d\eta d\bar{\eta} d\chi d\bar{\chi} \mathcal{O}^S \exp(\bar{\eta} \Sigma_{\eta}^T \eta + \bar{\chi} \Sigma_{\chi}^T \chi)}{\int d\eta d\bar{\eta} d\chi d\bar{\chi} \exp(\bar{\eta} \Sigma_{\eta}^T \eta + \bar{\chi} \Sigma_{\chi}^T \chi)}, \quad (4.23)$$

where $\Sigma_{\eta, \chi}^T$ means a transposition of the matrix $\Sigma_{\eta, \chi}$.

Let us now explain how to evaluate (4.23) in practice using the following single-trace operator as an example:

$$\mathcal{O}(y) = \text{tr}(Y^{A_1} \bar{Y}_{B_1} \dots Y^{A_L} \bar{Y}_{B_L})(y). \quad (4.24)$$

We first replace the scalar fields in the trace by their classical solutions (4.12)

$$\mathcal{O}(y) \mapsto \mathcal{O}^S(y) = \frac{(-1)^L}{k^{2L}} \sum_{\{i, j\}} \prod_{s=1}^L \frac{t_{i_s} t_{j_s} \bar{n}_{i_s}^{A_s} n_{j_s, B_s}}{|y - x_{i_s}| |y - x_{j_s}|} \prod_{s=1}^L \chi_{i_s} \bar{\eta}_{i_s} \eta_{j_s} \bar{\chi}_{j_s}, \quad (4.25)$$

where $\sum_{\{i, j\}}$ denotes a summation of i_s 's and j_s 's from 1 to m . Then we take the expectation value

$$\langle \mathcal{O}^S \rangle_{\chi, \eta} = \frac{(-1)^L}{k^{2L}} \sum_{\{i, j\}} \prod_{s=1}^L \frac{t_{i_s} t_{j_s} \bar{n}_{i_s}^{A_s} n_{j_s, B_s}}{|y - x_{i_s}| |y - x_{j_s}|} \times \langle \chi_{i_1} \bar{\eta}_{i_1} \eta_{j_1} \bar{\chi}_{j_1} \chi_{i_2} \bar{\eta}_{i_2} \eta_{j_2} \bar{\chi}_{j_2} \dots \rangle_{\chi, \eta}. \quad (4.26)$$

The expectation value of χ 's and η 's can be computed by the Wick contractions, which are given by

$$\langle \bar{\eta}_i^a \eta_{j, b} \rangle = \delta_b^a \left(\Sigma_{\eta}^{-1} \right)_{ij}, \quad \langle \bar{\chi}_i^a \chi_{j, b} \rangle = \delta_b^a \left(\Sigma_{\chi}^{-1} \right)_{ij}, \quad (4.27)$$

where a and b are gauge indices while i and j run from 1 to m (= the number of giant gravitons).

Large N limit. To proceed, we consider the large N limit. There are two simplifications in the large N limit. Firstly, the integrations over ρ and $\bar{\rho}$ are dominated by the saddle points of the action S_{eff} in (4.19). We denote the saddle point solutions by ρ^* and $\bar{\rho}^*$ and the propagators (4.27) can be evaluated at the saddle point. Secondly, the Wick contraction in (4.26) is dominated by the nearest neighboring contraction (see [29] for more detailed explanation). For instance, if we neglect the prefactors in (4.26), the Wick contraction gives

$$\begin{aligned} \langle \chi_{i_1} \bar{\eta}_{i_1} \eta_{j_1} \bar{\chi}_{j_1} \chi_{i_2} \bar{\eta}_{i_2} \eta_{j_2} \bar{\chi}_{j_2} \dots \rangle &\stackrel{N \rightarrow \infty}{=} - \langle \bar{\eta}_{i_1} \eta_{j_1} \rangle \langle \bar{\chi}_{j_1} \chi_{i_2} \rangle \dots \langle \bar{\eta}_{i_L} \eta_{j_L} \rangle \langle \bar{\chi}_{j_L} \chi_{i_1} \rangle \\ &= - N^{2L} \text{tr}_m [\Sigma_{\eta}^{-1} \Sigma_{\chi}^{-1} \dots \Sigma_{\eta}^{-1} \Sigma_{\chi}^{-1}]. \end{aligned} \quad (4.28)$$

Reinstating the prefactors, (4.26) can be written as

$$\langle \mathcal{O}^S \rangle_{\chi, \eta} = -\text{tr}_m [T^{A_1} \bar{T}_{B_1} \dots T^{A_L} \bar{T}_{B_L}], \quad (4.29)$$

where

$$\begin{aligned} T^A &= -\text{diag} \left(\frac{\lambda t_1 \bar{n}_1^A}{|y - x_1|}, \frac{\lambda t_2 \bar{n}_2^A}{|y - x_2|}, \dots, \frac{\lambda t_m \bar{n}_m^A}{|y - x_m|} \right) \cdot \Sigma_{\eta}^{-1}, \\ \bar{T}_B &= \text{diag} \left(\frac{\lambda t_1 n_{1, B}}{|y - x_1|}, \frac{\lambda t_2 n_{2, B}}{|y - x_2|}, \dots, \frac{\lambda t_m n_{m, B}}{|y - x_m|} \right) \cdot \Sigma_{\chi}^{-1}, \end{aligned} \quad (4.30)$$

are $m \times m$ matrices. Alternatively, we can express this as an overlap between a spin-chain state $|A_1 \bar{B}_1 \cdots A_L \bar{B}_L\rangle$ and a matrix product state $|\text{MPS}\rangle$ defined by

$$\langle \mathcal{O}^S \rangle_{\chi, \eta} = \langle \text{MPS} | A_1 \bar{B}_1 \cdots A_L \bar{B}_L \rangle, \quad (4.31)$$

with

$$\langle \text{MPS} | \equiv - \sum_{\{A_s, B_s\}} \langle A_1 \bar{B}_1 \cdots A_L \bar{B}_L | \text{tr}_m [T^{A_1} \bar{T}_{B_1} \cdots T^{A_L} \bar{T}_{B_L}]. \quad (4.32)$$

In deriving these expressions, we implicitly assumed that there is no flat direction around the saddle point. However, this is not always the case as we see below for $m = 2$. See section 4.2 for discussions on this point.

Physically the saddle-point value of $e^{N S_{\text{eff}}}$ gives the m -point function of giant gravitons while the expectation value $\langle \mathcal{O}^S \rangle_{\chi, \eta}$ gives a ratio of the $(m+1)$ -point function and the m -point function,

$$\langle \mathcal{O}^S \rangle_{\chi, \eta} = \frac{\langle (\prod_{j=1}^m \mathcal{D}_{M_j}) \mathcal{O}(y) \rangle}{\langle \prod_{j=1}^m \mathcal{D}_{M_j} \rangle}. \quad (4.33)$$

4.2 Two-point function: saddle point and MPS

To find the solution of saddle point equations, we focus on the case of interest with two giant gravitons, namely we take $m = 2$. We furthermore set the charges of the two giant gravitons to be identical $M_1 = M_2 = M$ and express the ratio M/N by

$$\omega \equiv \frac{M}{N}. \quad (4.34)$$

Note that ω takes values from 0 to 1 and the case with $\omega = 1$ corresponds to the maximal giant graviton.

Setting $m = 2$ in (4.20), we find that the effective action is given by

$$S_{\text{eff}} = -\frac{1}{\lambda} (\rho_{12} \bar{\rho}_{21} + \rho_{21} \bar{\rho}_{12}) + \log \left[(1 - t_1 t_2 \sqrt{d_{12} d_{21}} \rho_{12} \rho_{21}) (1 - t_1 t_2 \sqrt{d_{12} d_{21}} \bar{\rho}_{12} \bar{\rho}_{21}) \right]. \quad (4.35)$$

Taking variation with respect to ρ 's and $\bar{\rho}$'s and imposing $\frac{\delta S_{\text{eff}}}{\delta \rho_{i,j}} = \frac{\delta S_{\text{eff}}}{\delta \bar{\rho}_{i,j}} = 0$, we obtain

$$\rho_{12}^* \bar{\rho}_{21}^* = \rho_{21}^* \bar{\rho}_{12}^* = \lambda \frac{t_1 t_2 \sqrt{d_{12} d_{21}} \rho_{12}^* \rho_{21}^*}{t_1 t_2 \sqrt{d_{12} d_{21}} \rho_{12}^* \rho_{21}^* - 1} = \lambda \frac{t_1 t_2 \sqrt{d_{12} d_{21}} \bar{\rho}_{12}^* \bar{\rho}_{21}^*}{t_1 t_2 \sqrt{d_{12} d_{21}} \bar{\rho}_{12}^* \bar{\rho}_{21}^* - 1}. \quad (4.36)$$

Solving these equations, we find¹²

$$\rho_{ij}^* = \bar{\rho}_{ij}^*, \quad \rho_{12}^* \rho_{21}^* = \bar{\rho}_{12}^* \bar{\rho}_{21}^* = \lambda + \frac{1}{t_1 t_2 \sqrt{d_{12} d_{21}}}. \quad (4.37)$$

¹²In fact there is another solution to the saddle point equations,

$$\rho_{ij}^* = -\bar{\rho}_{ij}^*, \quad \rho_{12}^* \rho_{21}^* = \bar{\rho}_{12}^* \bar{\rho}_{21}^* = -\lambda + \frac{1}{t_1 t_2 \sqrt{d_{12} d_{21}}}.$$

However it will eventually give the same saddle point action and the same MPS (after the integration over θ which will be defined soon), and therefore do not modify our result for the ratio of correlators. Then we will focus only on the first solution in the rest of the paper.

The effective action at this saddle point is given by

$$S_{\text{eff}}^* = -2 - \frac{2}{\lambda t_1 t_2 \sqrt{d_{12} d_{21}}} + \log \left[\lambda^2 t_1^2 t_2^2 d_{12} d_{21} \right]. \quad (4.38)$$

To compute the correlation function of giant gravitons with fixed charges, we need to perform integrals of t 's (4.3). Since M is $\mathcal{O}(N)$, some of those integrals can be evaluated using the saddle-point approximation. Solving the saddle-point equations

$$2\omega \frac{\partial \log t_{1,2}}{\partial t_{1,2}} - \frac{\partial S_{\text{eff}}^*}{\partial t_{1,2}} = 0, \quad (4.39)$$

we get

$$t_1^* t_2^* = \frac{1}{\lambda(\omega - 1)\sqrt{d_{12} d_{21}}}. \quad (4.40)$$

As is clear from this expression, the saddle-point equation only determines the product $t_1 t_2$. Therefore, we still need to perform an integral of the ratio

$$e^{i\theta} \equiv \sqrt{\frac{t_1}{t_2}}. \quad (4.41)$$

as we see below.

MPS for two giant gravitons. Let us place the single trace operator at $y = x_3$,

$$\mathcal{O}(y) = \text{tr}(Y^{A_1} \bar{Y}_{B_1} \dots Y^{A_L} \bar{Y}_{B_L})(x_3), \quad (4.42)$$

and compute the three-point function of two giant gravitons and \mathcal{O} .

Evaluating the matrices T^A and \bar{T}_B at the saddle point (4.36) and (4.40), we get

$$T^A = -\lambda \text{diag} \left(\frac{t_1^* \bar{n}_1^A}{|x_{13}|}, \frac{t_2^* \bar{n}_2^A}{|x_{23}|} \right) \cdot \begin{pmatrix} 1 - \omega & -i\rho_{12}^* \sqrt{\frac{1-\omega}{\lambda}} \left(\frac{d_{12}}{d_{21}} \right)^{\frac{1}{4}} \\ -i\rho_{21}^* \sqrt{\frac{1-\omega}{\lambda}} \left(\frac{d_{21}}{d_{12}} \right)^{\frac{1}{4}} & 1 - \omega \end{pmatrix}, \quad (4.43)$$

$$\bar{T}_B = \lambda \text{diag} \left(\frac{t_1^* n_{1,B}}{|x_{13}|}, \frac{t_2^* n_{2,B}}{|x_{23}|} \right) \cdot \begin{pmatrix} 1 - \omega & -i\rho_{12}^* \sqrt{\frac{1-\omega}{\lambda}} \left(\frac{d_{21}}{d_{12}} \right)^{\frac{1}{4}} \\ -i\rho_{21}^* \sqrt{\frac{1-\omega}{\lambda}} \left(\frac{d_{12}}{d_{21}} \right)^{\frac{1}{4}} & 1 - \omega \end{pmatrix}.$$

To proceed, we make use of the fact that the MPS (4.32) is invariant under the transformation $T^A \mapsto UT^A U^{-1}$ and $\bar{T}_B \mapsto U\bar{T}_B U^{-1}$. By choosing an appropriate U , we can replace ρ_{12}^* and ρ_{21}^* in (4.43) with $\sqrt{\rho_{12}^* \rho_{21}^*}$. As a result, we obtain the following matrices

$$T_\theta^A = \frac{\sqrt{\lambda}}{(d_{12} d_{21})^{\frac{1}{4}}} \begin{pmatrix} -\frac{\bar{n}_1^A e^{i\theta}}{|x_{13}|} \sqrt{1-\omega} & i\frac{\bar{n}_1^A e^{i\theta}}{|x_{13}|} \sqrt{\omega} \left(\frac{d_{12}}{d_{21}} \right)^{\frac{1}{4}} \\ i\frac{\bar{n}_2^A e^{-i\theta}}{|x_{23}|} \sqrt{\omega} \left(\frac{d_{21}}{d_{12}} \right)^{\frac{1}{4}} & -\frac{\bar{n}_2^A e^{-i\theta}}{|x_{23}|} \sqrt{1-\omega} \end{pmatrix}, \quad (4.44)$$

$$\bar{T}_{B,\theta} = \frac{\sqrt{\lambda}}{(d_{12} d_{21})^{\frac{1}{4}}} \begin{pmatrix} \frac{n_{1,B} e^{i\theta}}{|x_{13}|} \sqrt{1-\omega} & -i\frac{n_{1,B} e^{i\theta}}{|x_{13}|} \sqrt{\omega} \left(\frac{d_{21}}{d_{12}} \right)^{\frac{1}{4}} \\ -i\frac{n_{2,B} e^{-i\theta}}{|x_{23}|} \sqrt{\omega} \left(\frac{d_{12}}{d_{21}} \right)^{\frac{1}{4}} & \frac{n_{2,B} e^{-i\theta}}{|x_{23}|} \sqrt{1-\omega} \end{pmatrix}.$$

Here $e^{i\theta}$ is the ratio of t 's defined in (4.41). As is clear from this expression, the MPS depends on θ which parametrizes the flat direction around the saddle point. Therefore, the correct result would be given by an integral over this direction;

$$\langle \mathcal{O}^S \rangle_{\chi, \eta} = \int_0^{2\pi} \frac{d\theta}{2\pi} \langle \text{MPS}_\theta | A_1 \bar{B}_1 \cdots A_L \bar{B}_L \rangle, \quad (4.45)$$

with

$$\langle \text{MPS}_\theta | \equiv - \sum_{\{A_s, B_s\}} \langle A_1 \bar{B}_1 \cdots A_L \bar{B}_L | \text{tr}_m [T_\theta^{A_1} \bar{T}_{B_1, \theta} \cdots T_\theta^{A_L} \bar{T}_{B_L, \theta}]. \quad (4.46)$$

4.3 Structure constants of BPS single-trace operators

Let us now use the results above to derive an expression for the structure constant of the single-trace operator.

Integral representation. We first consider the BPS single-trace operator, given by

$$\mathcal{O}_L^\circ(x_3) = \text{tr}[(n_3 \cdot Y \bar{n}_3 \cdot \bar{Y})^L](x_3). \quad (4.47)$$

Using the results above, we find

$$\frac{\langle \mathcal{D}_M \mathcal{D}_M \mathcal{O}_L^\circ(x_3) \rangle}{\langle \mathcal{D}_M \mathcal{D}_M \rangle} = \int_0^{2\pi} \frac{d\theta}{2\pi} \text{tr}_2 [(\mathcal{T}_\theta \bar{\mathcal{T}}_\theta)^L], \quad (4.48)$$

with

$$\begin{aligned} \mathcal{T}_\theta &\equiv \sum_A n_{3,A} T_\theta^A = \frac{\sqrt{\lambda}}{(d_{12} d_{21})^{\frac{1}{4}}} \begin{pmatrix} -d_{31} e^{i\theta} \sqrt{1-\omega} & id_{31} e^{i\theta} \sqrt{\omega} \left(\frac{d_{12}}{d_{21}}\right)^{\frac{1}{4}} \\ id_{32} e^{-i\theta} \sqrt{\omega} \left(\frac{d_{21}}{d_{12}}\right)^{\frac{1}{4}} & -d_{32} e^{-i\theta} \sqrt{1-\omega} \end{pmatrix}, \\ \bar{\mathcal{T}}_\theta &\equiv \sum_B \bar{n}_3^B \bar{T}_{\theta,B} = \frac{\sqrt{\lambda}}{(d_{12} d_{21})^{\frac{1}{4}}} \begin{pmatrix} d_{13} e^{i\theta} \sqrt{1-\omega} & -id_{13} e^{i\theta} \sqrt{\omega} \left(\frac{d_{21}}{d_{12}}\right)^{\frac{1}{4}} \\ -id_{23} e^{-i\theta} \sqrt{\omega} \left(\frac{d_{12}}{d_{21}}\right)^{\frac{1}{4}} & d_{23} e^{-i\theta} \sqrt{1-\omega} \end{pmatrix}. \end{aligned} \quad (4.49)$$

To evaluate (4.48) it is convenient to consider a *generating function of structure constants*,

$$\sum_{L=0}^{\infty} s^L \frac{\langle \mathcal{D}_M \mathcal{D}_M \mathcal{O}_L^\circ(x_3) \rangle}{\langle \mathcal{D}_M \mathcal{D}_M \rangle} = - \int_0^{2\pi} \frac{d\theta}{2\pi} \text{tr}_2 \left[\frac{1}{1 - s \mathcal{T}_\theta \bar{\mathcal{T}}_\theta} \right]. \quad (4.50)$$

To proceed, we evaluate the right hand side by diagonalizing the matrices $\mathcal{T}_\theta \bar{\mathcal{T}}_\theta$, perform the integral, and read off the coefficient in front of s^L .¹³ The result reads

$$\frac{\langle \mathcal{D}_M \mathcal{D}_M \mathcal{O}_L^\circ(x_3) \rangle}{\langle \mathcal{D}_M \mathcal{D}_M \rangle} = -\lambda^L \left(\frac{d_{23} d_{32} d_{31} d_{13}}{d_{12} d_{21}} \right)^{\frac{L}{2}} \xi^{-\frac{L}{2}} \oint \frac{ds}{2\pi i s^{1+L}} \frac{1 - \xi s^2}{\sqrt{\mathcal{P}(s)}}, \quad (4.51)$$

where $\mathcal{P}(s)$ is given by

$$\mathcal{P}(s) \equiv 1 - 2\omega(1 + \xi)s + \left[-2\xi + 8\xi\omega + \omega^2(1 - \xi)^2 \right] s^2 - 2\omega\xi(1 + \xi)s^3 + \xi^2 s^4. \quad (4.52)$$

¹³Here it is important to keep s being small to make the power series convergent in (4.50). We will give the details of the computations in appendix B.

Dividing this by the normalization factor $\mathcal{N}_{\mathcal{O}_L^c}$ and stripping off the kinematic factors, we arrive at the following expression for the structure constants;

$$\sum_{p=-\frac{L}{2}}^{\frac{L}{2}} \xi^p D_{M|L}^{(p)} = -\frac{\xi^{-\frac{L}{2}}}{\sqrt{L}} \oint \frac{ds}{2\pi i s^{1+L}} \frac{1-\xi s^2}{\sqrt{\mathcal{P}(s)}}. \quad (4.53)$$

For a given L , the integral (4.53) can be readily evaluated. For instance, for small values of L , we obtain

$$\begin{aligned} L=1: \quad & \sum_{p=-\frac{1}{2}}^{\frac{1}{2}} \xi^p D_{M|1}^{(p)} = -\omega \left(\xi^{-\frac{1}{2}} + \xi^{\frac{1}{2}} \right), \\ L=2: \quad & \sum_{p=-1}^1 \xi^p D_{M|2}^{(p)} = -\frac{\omega^2}{\sqrt{2}} (\xi^{-1} + 4(1-\omega^{-1}) + \xi), \\ L=3: \quad & \sum_{p=-\frac{3}{2}}^{\frac{3}{2}} \xi^p D_{M|3}^{(p)} = -\frac{\omega^3}{\sqrt{3}} \left[\xi^{-\frac{3}{2}} + \frac{3(1-\omega)(1-3\omega)}{\omega^2} \left(\xi^{-\frac{1}{2}} + \xi^{\frac{1}{2}} \right) + \xi^{\frac{3}{2}} \right]. \end{aligned} \quad (4.54)$$

Closed-form expressions for special cases. It is however difficult to write down a closed-form expression for general L , ξ and ω . The exceptions are for the maximal giant graviton ($\omega = 1$) and for the twisted-translated kinematics ($\xi = -1$). The former is given by

$$\sum_{p=-\frac{L}{2}}^{\frac{L}{2}} \xi^p D_{N|L}^{(p)} = -\frac{1}{\sqrt{L}} \left(\xi^{-\frac{L}{2}} + \xi^{\frac{L}{2}} \right), \quad (4.55)$$

which is equivalent to

$$D_{N|L}^{(p)} = -\frac{1}{\sqrt{L}} \left(\delta_{p, \frac{L}{2}} + \delta_{p, -\frac{L}{2}} \right). \quad (4.56)$$

On the other hand, in the twisted-translated kinematics ($\xi = -1$), we have

$$\begin{aligned} \mathfrak{D}_{M|L} &= \sum_{p=-\frac{L}{2}}^{\frac{L}{2}} (-1)^p D_{M|L}^{(p)} \\ &= \begin{cases} -\frac{(-1)^{\frac{L}{2}}}{\sqrt{L}} \left[P_{\frac{L}{2}}(-1+4\omega-2\omega^2) + P_{\frac{L}{2}-1}(-1+4\omega-2\omega^2) \right] & L : \text{ even} \\ 0 & L : \text{ odd} \end{cases}, \end{aligned} \quad (4.57)$$

where $P_{\frac{L}{2}}$ is the Legendre polynomial.

4.4 Structure constants of non-BPS single-trace operators

We next study the structure constant of the non-BPS single-trace operator. As mentioned in section 2, in this paper we focus on the twisted-translated frame, which amounts to setting

$$n_{1,2} = (1, 0, 0, \kappa a_{1,2}), \quad \bar{n}_{1,2} = (-\kappa a_{1,2}, 0, 0, 1), \quad x_{1,2} = (0, a_{1,2}, 0). \quad (4.58)$$

Result for the structure constant. To further simplify the analysis, we consider a symmetric configuration in which the single-trace operator is at the origin and the giant gravitons are placed symmetrically around it; namely we set $a_1 = -a_2 = +1$ and $a_3 = 0$ in (2.42). Then the matrices that enter in the MPS representation, (4.44), simplify to the following,

$$\begin{aligned}
 T_\theta^1 &= \sqrt{\kappa\lambda} e^{-i\frac{\pi}{4}} \begin{pmatrix} e^{i\theta}\sqrt{1-\omega} & -ie^{i(\theta-\frac{\pi}{4})}\sqrt{\omega} \\ ie^{-i(\theta-\frac{\pi}{4})}\sqrt{\omega} & -e^{-i\theta}\sqrt{1-\omega} \end{pmatrix}, \\
 \bar{T}_{4,\theta} &= \sqrt{\kappa\lambda} e^{-i\frac{\pi}{4}} \begin{pmatrix} e^{i\theta}\sqrt{1-\omega} & -ie^{i(\theta+\frac{\pi}{4})}\sqrt{\omega} \\ ie^{-i(\theta+\frac{\pi}{4})}\sqrt{\omega} & -e^{-i\theta}\sqrt{1-\omega} \end{pmatrix}, \\
 T_\theta^4 &= \sqrt{\frac{\lambda}{\kappa}} e^{-i\frac{\pi}{4}} \begin{pmatrix} -e^{i\theta}\sqrt{1-\omega} & ie^{i(\theta-\frac{\pi}{4})}\sqrt{\omega} \\ ie^{-i(\theta-\frac{\pi}{4})}\sqrt{\omega} & -e^{-i\theta}\sqrt{1-\omega} \end{pmatrix}, \\
 \bar{T}_{1,\theta} &= \sqrt{\frac{\lambda}{\kappa}} e^{-i\frac{\pi}{4}} \begin{pmatrix} e^{i\theta}\sqrt{1-\omega} & -ie^{i(\theta+\frac{\pi}{4})}\sqrt{\omega} \\ -ie^{-i(\theta+\frac{\pi}{4})}\sqrt{\omega} & e^{-i\theta}\sqrt{1-\omega} \end{pmatrix},
 \end{aligned} \tag{4.59}$$

and all the other $T_\theta^{A'}$'s and $\bar{T}_{B,\theta}$ are zero.

To compute the structure constant $\mathfrak{D}_{M|\mathcal{O}}$, we then compute the matrix trace in (4.46), perform the θ integral in (4.45) and divide the result by the normalization $\mathcal{N}_{\mathcal{O}}$. As discussed in [80], $\mathcal{N}_{\mathcal{O}}$ can be expressed in terms of the norm of the spin-chain state $\langle \mathcal{O}|\mathcal{O} \rangle$ as follows

$$\mathcal{N}_{\mathcal{O}} = L\lambda^{2L} \langle \mathcal{O}|\mathcal{O} \rangle, \tag{4.60}$$

where the prefactor L is the number of different Wick contractions related by cyclic permutations (see [80] for details) and λ^{2L} comes from the normalization of propagators. Finally, factoring out the kinematic factor in (2.42), we arrive at the expression,

$$\mathfrak{D}_{M|\mathcal{O}} = \frac{(-1)^{J+1}}{2^{\Delta-J} \sqrt{L} \langle \mathcal{O}|\mathcal{O} \rangle} \int_0^{2\pi} \frac{d\theta}{2\pi} \langle B_\theta|\mathcal{O} \rangle, \tag{4.61}$$

where the matrix product state $\langle B_\theta|$ is defined by

$$\langle B_\theta| \equiv \sum_{A_s, B_s=1,4} \langle A_1 \bar{B}_1 \cdots A_L \bar{B}_L | \text{tr}_2 \left[\mathbf{t}^{A_1} \bar{\mathbf{t}}_{B_1} \cdots \mathbf{t}^{A_L} \bar{\mathbf{t}}_{B_L} \right], \tag{4.62}$$

with¹⁴

$$\begin{aligned}
 \mathbf{t}^1 &= \begin{pmatrix} e^{i\theta}\sqrt{1-\omega} & -i\sqrt{\omega} \\ i\sqrt{\omega} & -e^{-i\theta}\sqrt{1-\omega} \end{pmatrix}, & \bar{\mathbf{t}}_4 &= \begin{pmatrix} e^{i\theta}\sqrt{1-\omega} & \sqrt{\omega} \\ \sqrt{\omega} & -e^{-i\theta}\sqrt{1-\omega} \end{pmatrix}, \\
 \mathbf{t}^4 &= \begin{pmatrix} -e^{i\theta}\sqrt{1-\omega} & i\sqrt{\omega} \\ i\sqrt{\omega} & -e^{-i\theta}\sqrt{1-\omega} \end{pmatrix}, & \bar{\mathbf{t}}_1 &= \begin{pmatrix} e^{i\theta}\sqrt{1-\omega} & \sqrt{\omega} \\ -\sqrt{\omega} & e^{-i\theta}\sqrt{1-\omega} \end{pmatrix}.
 \end{aligned} \tag{4.63}$$

¹⁴Recall that ω is related to the charge of the giant graviton M by $\omega \equiv M/N$.

Simplification for the maximal giant graviton. For the maximal giant gravitons $\omega = 1$, the matrices in (4.63) all become off-diagonal, and the products of \mathbf{t} and $\bar{\mathbf{t}}$ take the following simple form;

$$\begin{aligned} \mathbf{t}_1 \bar{\mathbf{t}}_4 &= \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, & \mathbf{t}_1 \bar{\mathbf{t}}_1 &= \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \\ \mathbf{t}_4 \bar{\mathbf{t}}_4 &= \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, & \mathbf{t}_1 \bar{\mathbf{t}}_1 &= \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \end{aligned} \tag{4.64}$$

Since these matrices do not depend on θ , the integral of θ can be trivially performed. As a result, we can replace (4.61) with the following expression

$$\mathfrak{D}_{N|\mathcal{O}} = -\frac{(-i)^J}{2^{\Delta-J}} \frac{\langle \mathcal{B}|\mathcal{O} \rangle}{\sqrt{L} \langle \mathcal{O}|\mathcal{O} \rangle}, \tag{4.65}$$

where the state $\langle \mathcal{B}|$ is defined by

$$\langle \mathcal{B}| \equiv \sum_{A_s, B_s=1,4} \langle A_1 \bar{B}_1 \cdots A_L \bar{B}_L | (1 + (-1)^J). \tag{4.66}$$

Here J is the U(1) R-charge, which counts

$$J = L - (\text{number of } 4 \text{ on odd sites}) - (\text{number of } \bar{1} \text{ on even sites}). \tag{4.67}$$

We can also check that the result (4.65) matches the one computed from a different approach explained in appendix A.

In the rest of this paper, we use (4.61) and (4.65) to evaluate the structure constants of non-BPS single-trace operators. For this purpose, we need to compute two quantities in the spin chain; the overlap $\langle \mathcal{B}|\mathcal{O} \rangle$ and the norm $\langle \mathcal{O}|\mathcal{O} \rangle$. In the next section, we present conjectures for these observables based on the coordinate Bethe ansatz we developed in section 3.

5 Main results

In this section, we use the coordinate Bethe ansatz in section 3 and evaluate the overlap $\langle \mathcal{B}|\mathcal{O} \rangle$ and the norm $\langle \mathcal{O}|\mathcal{O} \rangle$. For reader's convenience, we first present a summary of results and later explain the details. The numerical data used for checking our formula, such as the solutions to the Bethe equations and the results for the overlap, is summarized in appendix C.

In what follows, we label a Bethe state in terms of its Bethe roots; namely we express $|\mathcal{O} \rangle$ as $|\mathbf{u}, \mathbf{w}, \mathbf{v} \rangle$ where \mathbf{u} , \mathbf{w} and \mathbf{v} are sets of Bethe roots for each Dynkin node (see figure 1).

5.1 Summary of results

Selection rules for the overlap. As a result of the numerical experiments, we found a set of selection rules in order for the overlap to be non-zero. The selection rules apply both to the maximal giant graviton $\langle \mathcal{B}|\mathcal{O} \rangle$ and the non-maximal giant gravitons $\int \frac{d\theta}{2\pi} \langle B_\theta|\mathcal{O} \rangle$.

The first and the most obvious selection rule is

0. The numbers of Bethe roots for each node need to satisfy

$$K_{\mathbf{u}} = K_{\mathbf{v}} = K_{\mathbf{w}}. \quad (5.1)$$

This simply follows from the fact that $\langle B_\theta |$ only contains fields Y^1, Y^4, \bar{Y}_1 and \bar{Y}_4 : in order for $|\mathbf{u}, \mathbf{w}, \mathbf{v}\rangle$ to contain kets involving $Y^1, Y^4, \bar{Y}_1, \bar{Y}_4$ only, we need to set $K_{\mathbf{u}} = K_{\mathbf{v}} = K_{\mathbf{w}}$. Note that this is just a consequence of the global symmetry, not the integrability of the matrix product state.

In addition to this constraint, the overlap also obeys the following ‘‘parity conditions’’:

1. The U(1) R-charge of the state, $J = L - \frac{K_{\mathbf{u}} + K_{\mathbf{v}}}{2}$, must be even.
2. The rapidities of the right node must be (-1) times the rapidities of the left node:

$$\mathbf{v} = -\mathbf{u}. \quad (5.2)$$

3. The rapidities of the middle node should be parity-symmetric, namely

$$\mathbf{w} = \begin{cases} (w_1, -w_1, w_2, -w_2, \dots) & K_{\mathbf{w}} : \text{even}, \\ (w_1, -w_1, w_2, -w_2, \dots, 0) & K_{\mathbf{w}} : \text{odd}. \end{cases} \quad (5.3)$$

These selection rules are almost identical to the selection rules found for the structure constant of two giant gravitons and a single-trace operator in the SO(6) sector of $\mathcal{N} = 4$ SYM [29, 30]. The only difference is the roles of the rapidities: in ABJM, the left and the right nodes are momentum carrying nodes and the middle node is auxiliary while in $\mathcal{N} = 4$ SYM, the middle node is momentum carrying and the left and right nodes are auxiliary.

As discussed in [29, 43, 46, 104], the existence of these selection rules indicate that the matrix product states $\langle \mathcal{B} |$ and $\int \frac{d\theta}{2\pi} \langle B_\theta |$ are integrable boundary states. In particular, we want to emphasize that the non-maximal giant gravitons also satisfy the selection rule, implying that they correspond to integrable boundary states as well.

Determinant formula for the maximal giant graviton. For the maximal giant graviton ($\omega = 1$), we found a closed-form expression for the structure constant. Before writing down the result, let us first clarify our convention and ordering of the rapidities satisfying the selection rules;

$$\begin{aligned} \mathbf{u} &= (u_1, u_2, \dots, u_{K_{\mathbf{u}}}), \\ \mathbf{v} &= -\mathbf{u} = (-u_1, -u_2, \dots, -u_{K_{\mathbf{u}}}), \\ \mathbf{w} &= \begin{cases} (w_1, -w_1, w_2, -w_2, \dots, w_{\frac{K_{\mathbf{w}}}{2}}, -w_{\frac{K_{\mathbf{w}}}{2}}) & K_{\mathbf{w}} : \text{even} \\ (w_1, -w_1, w_2, -w_2, \dots, w_{\lceil \frac{K_{\mathbf{w}}}{2} \rceil}, -w_{\lceil \frac{K_{\mathbf{w}}}{2} \rceil}, 0) & K_{\mathbf{w}} : \text{odd} \end{cases}, \end{aligned} \quad (5.4)$$

where $\lceil \frac{K_{\mathbf{w}}}{2} \rceil$ means the largest integer not larger than $\frac{K_{\mathbf{w}}}{2}$.

The result for the normalized overlap reads

$$\frac{\langle \mathcal{B} | \mathbf{u}, \mathbf{w}, -\mathbf{u} \rangle}{\sqrt{\langle \mathbf{u}, \mathbf{w}, -\mathbf{u} | \mathbf{u}, \mathbf{w}, -\mathbf{u} \rangle}} = -2 \sqrt{\prod_{j=1}^{K_{\mathbf{u}}} \left(u_j^2 + \frac{1}{4}\right) \prod_{k=1}^{\lceil \frac{K_{\mathbf{w}}}{2} \rceil} \frac{1}{w_k^2(w_k^2 + \frac{1}{4})} \frac{\det G_+}{\det G_-}}. \quad (5.5)$$

Here $\det G_{\pm}$ are the Gaudin-like determinants which we will explain in the next subsection. The result leads to the following expression for the structure constant:

$$\mathfrak{D}_{N|\mathcal{O}} = \frac{i^J + (-i)^J}{2^{\Delta-J} \sqrt{L}} \sqrt{\prod_{j=1}^{K_{\mathbf{u}}} \left(u_j^2 + \frac{1}{4}\right) \prod_{k=1}^{\lceil \frac{K_{\mathbf{w}}}{2} \rceil} \frac{1}{w_k^2(w_k^2 + \frac{1}{4})} \frac{\det G_+}{\det G_-}}. \quad (5.6)$$

In (5.5) and (5.6), we neglected an overall phase factor which is ambiguous (see the discussion in section 2.2). These are the main results of this paper. Below we explain the details of the formula and the derivation.

For the non-maximal giant gravitons, we have not been able to find a closed-form expression analogous to (5.6). The results for a sample of Bethe states are summarized in appendix C. We leave it for future investigations to find a determinant formula for the non-maximal giant gravitons.

5.2 Norms and Gaudin determinant

Norms for the coordinate Bethe states. The norms of the Bethe states are often given by the so-called Gaudin determinant, which are given by the logarithmic derivative of the Bethe equations. In the case at hand, the relevant Gaudin matrix is given by

$$G = \begin{pmatrix} \partial_{u_i} \phi_{u_j} & \partial_{u_i} \phi_{w_j} & \partial_{u_i} \phi_{v_j} \\ \partial_{w_i} \phi_{u_j} & \partial_{w_i} \phi_{w_j} & \partial_{w_i} \phi_{v_j} \\ \partial_{v_i} \phi_{u_j} & \partial_{v_i} \phi_{w_j} & \partial_{v_i} \phi_{v_j} \end{pmatrix}. \quad (5.7)$$

where ϕ 's was defined through the Bethe ansatz equations (3.7).

We conjecture that the norms of the Bethe states in the SU(4)-invariant alternating spin chain are given by

$$\begin{aligned} \langle \mathbf{u}, \mathbf{w}, \mathbf{v} | \mathbf{u}, \mathbf{w}, \mathbf{v} \rangle &= \left(\prod_{i < j} \frac{S(u_i, u_j)}{S(u_i^*, u_j^*)} \right)^{\frac{1}{2}} \left(\prod_{i < j} \frac{S(v_i, v_j)}{S(v_i^*, v_j^*)} \right)^{\frac{1}{2}} \left(\prod_{i < j} \frac{S(w_i, w_j)}{S(w_i^*, w_j^*)} \right)^{\frac{1}{2}} \\ &\times \left(\prod_j \frac{1}{\partial_u p(u_j)} \right) \left(\prod_k \frac{1}{\partial_v p(v_k)} \right) \det G, \end{aligned} \quad (5.8)$$

where the states are normalized using the coordinate Bethe ansatz described in section 3 and $\langle \mathbf{u}, \mathbf{w}, \mathbf{v} | \equiv [|\mathbf{u}, \mathbf{w}, \mathbf{v}\rangle]^{\dagger}$. $S(u, v)$ is the S-matrix

$$S(u, v) \equiv \frac{u - v - i}{u - v + i}, \quad (5.9)$$

while $p(u)$ is the momentum of a magnon with rapidity u

$$p(u) \equiv \frac{1}{i} \log \frac{u + \frac{i}{2}}{u - \frac{i}{2}} \quad \Rightarrow \quad \partial_u p(u) = \frac{-1}{u^2 + \frac{1}{4}}. \quad (5.10)$$

The conjecture (5.8) passes several nontrivial tests: it correctly reproduces the results for the $SU(2) \times SU(2)$ sector (see for instance [105]) and it is satisfied by all the Bethe states listed in appendix C.

Simplifications for the parity-symmetric states. For the states satisfying the parity conditions $|\mathbf{u}, \mathbf{w}, -\mathbf{u}\rangle$, the formula for the norm simplifies further.

First the prefactor in (5.8) simplifies to

$$\langle \mathbf{u}, \mathbf{w}, -\mathbf{u} | \mathbf{u}, \mathbf{w}, -\mathbf{u} \rangle = \prod_{k=1}^{\lceil \frac{K_{\mathbf{w}}}{2} \rceil} \frac{(w_k - \frac{i}{2})(w_k^* + \frac{i}{2})}{(w_k + \frac{i}{2})(w_k^* - \frac{i}{2})} \prod_{j=1}^{K_{\mathbf{u}}} \left(u_j^2 + \frac{1}{4} \right)^2 \det G. \quad (5.11)$$

Note that we are using the convention and the ordering given in (5.4).

Second the Gaudin determinant for the parity-symmetric states factorizes into a determinants of submatrices. This factorization was already observed in $\mathcal{N} = 4$ SYM, and we will show below that a similar factorization holds also for ABJM. Since the derivations for the even $K_{\mathbf{w}}$ cases is similar to the one for the odd $K_{\mathbf{w}}$ cases, we first discuss the odd $K_{\mathbf{w}}$ cases.

For this purpose, we divide the rapidities $\mathbf{w} = (w_1, -w_1, w_2, -w_2, \dots, 0)$ into three parts, $\mathbf{w}^{(+)} = (w_1, w_2, \dots)$ denotes the ‘‘positive’’ part of Bethe roots, $\mathbf{w}^{(-)} = (-w_1, -w_2, \dots)$ denotes the ‘‘negative’’ part, and $\mathbf{w}^{(0)} = (0)$. As the first step, We reorder the rows and columns of Gaudin determinant to express it as

$$\det G = \det \begin{pmatrix} U_u & U_+ & U_0 & U_v & U_- \\ W_{+u} & W_{++} & W_{+0} & W_{+v} & W_{+-} \\ W_{0u} & W_{0+} & W_{00} & W_{0v} & W_{0-} \\ V_u & V_+ & V_0 & V_v & V_- \\ W_{-u} & W_{-+} & W_{-0} & W_{-v} & W_{--} \end{pmatrix}, \quad (5.12)$$

with

$$\begin{aligned} [U_u]_{ij} &\equiv \partial_{u_i} \phi_{u_j}, & [U_{\pm}]_{ij} &\equiv \partial_{u_i} \phi_{w_{\pm j}}, & [U_0]_{ij} &\equiv \partial_{u_i} \phi_{w^{(0)}}, & [U_v]_{ij} &\equiv \partial_{u_i} \phi_{v_j}, \\ [W_{\pm u}]_{ij} &\equiv \partial_{w_i^{(\pm)}} \phi_{u_j}, & [W_{\pm\pm}]_{ij} &\equiv \partial_{w_i^{(\pm)}} \phi_{w_j^{(\pm)}}, & [W_{\pm 0}]_{ij} &\equiv \partial_{w_i^{(\pm)}} \phi_{w^{(0)}}, \\ [W_{\pm\mp}]_{ij} &\equiv \partial_{w_i^{(\pm)}} \phi_{w_j^{(\mp)}}, & [W_{\pm v}]_{ij} &\equiv \partial_{w_i^{(\pm)}} \phi_{v_j}, & [W_{0u}]_{ij} &\equiv \partial_{w^{(0)}} \phi_{u_j}, \\ [W_{0\pm}]_{ij} &\equiv \partial_{w^{(0)}} \phi_{w_j^{(\pm)}}, & [W_{00}]_{ij} &\equiv \partial_{w^{(0)}} \phi_{w^{(0)}}, & [W_{0v}]_{ij} &\equiv \partial_{w^{(0)}} \phi_{v_j}, \\ [V_u]_{ij} &\equiv \partial_{v_i} \phi_{u_j}, & [V_{\pm}]_{ij} &\equiv \partial_{v_i} \phi_{w^{(\pm)}_j}, & [V_0]_{ij} &\equiv \partial_{v_i} \phi_{w^{(0)}}, & [V_v]_{ij} &\equiv \partial_{v_i} \phi_{v_j}. \end{aligned} \quad (5.13)$$

To proceed, we use the following relations that are valid when the selection rules are satisfied,

$$\begin{aligned} U_u &= V_v, & U_v &= V_u, & W_{\pm u} &= W_{\mp v}, & U_{\pm} &= V_{\mp}, & U_0 &= V_0, \\ W_{+\pm} &= W_{-\pm}, & W_{0u} &= W_{0v}, & W_{0+} &= W_{0-}, & W_{+0} &= W_{-0}. \end{aligned} \quad (5.14)$$

Then we can rewrite the determinant by adding the rows and subtracting the columns.

$$\det G = \det \begin{pmatrix} U_u & U_+ & U_0 & U_v & U_- \\ W_{+u} & W_{++} & W_{+0} & W_{+v} & W_{+-} \\ W_{0u} & W_{0+} & W_{00} & W_{0v} & W_{0+} \\ U_v & U_- & U_0 & U_u & U_+ \\ W_{+v} & W_{+-} & W_{+0} & W_{+u} & W_{++} \end{pmatrix} \quad (5.15)$$

$$= \det \begin{pmatrix} U_u + U_v & U_+ + U_- & 2U_0 & 0 & 0 \\ W_{+u} + W_{+v} & W_{++} + W_{+-} & 2W_{+0} & 0 & 0 \\ W_{0u} & W_{0+} & W_{00} & 0 & 0 \\ U_v & U_- & U_0 & U_u - U_v & U_+ - U_- \\ W_{+v} & W_{+-} & W_{+0} & W_{+u} - W_{+v} & W_{++} - W_{+-} \end{pmatrix}. \quad (5.16)$$

We thus obtain the factorization formula for the Gaudin determinant of the parity-symmetric states $\det G = \det G_+ \det G_-$ with

$$\begin{aligned} \text{Odd } K_{\mathbf{w}} : \quad G_+ &= \begin{pmatrix} U_u + U_v & U_+ + U_- & 2U_0 \\ W_{+u} + W_{+v} & W_{++} + W_{+-} & 2W_{+0} \\ W_{0u} & W_{0+} & W_{00} \end{pmatrix}, \\ G_- &= \begin{pmatrix} U_u - U_v & U_+ - U_- \\ W_{+u} - W_{+v} & W_{++} - W_{+-} \end{pmatrix}. \end{aligned} \quad (5.17)$$

For the even $K_{\mathbf{w}}$ cases, we can repeat the derivation above simply by omitting the row and the column involving \mathbf{w}_0 . As a result we obtain the following factorization formula $\det G = \det G_+ \det G_-$;

$$\begin{aligned} \text{Even } K_{\mathbf{w}} : \quad G_+ &= \begin{pmatrix} U_u + U_v & U_+ + U_- \\ W_{+u} + W_{+v} & W_{++} + W_{+-} \end{pmatrix}, \\ G_- &= \begin{pmatrix} U_u - U_v & U_+ - U_- \\ W_{+u} - W_{+v} & W_{++} - W_{+-} \end{pmatrix}. \end{aligned} \quad (5.18)$$

The same submatrices G_{\pm} appear in the expression for the overlap (5.5).

To summarize, the norm of the parity-symmetric state is given by

$$\langle \mathbf{u}, \mathbf{w}, -\mathbf{u} | \mathbf{u}, \mathbf{w}, -\mathbf{u} \rangle = \prod_{k=1}^{\lceil \frac{K_{\mathbf{w}}}{2} \rceil} \frac{(w_k - \frac{i}{2})(w_k^* + \frac{i}{2})}{(w_k + \frac{i}{2})(w_k^* - \frac{i}{2})} \prod_{j=1}^{K_{\mathbf{u}}} \left(u_j^2 + \frac{1}{4} \right)^2 \det G_+ \det G_-. \quad (5.19)$$

5.3 Some details on the overlap

We now explain the details of how we arrive at the formula (5.5). For this purpose, let us first present our conjecture for the overlap $\langle \mathcal{B} | \mathbf{u}, \mathbf{w}, -\mathbf{u} \rangle$ itself:

$$\langle \mathcal{B} | \mathbf{u}, \mathbf{w}, -\mathbf{u} \rangle = 2(-1)^L \prod_{j=1}^{K_{\mathbf{u}}} \left(u_j^2 + \frac{1}{4} \right) \left(u_j + \frac{i}{2} \right) \prod_{k=1}^{\lceil \frac{K_{\mathbf{w}}}{2} \rceil} \frac{1}{w_k(w_k + \frac{i}{2})} \det G_+. \quad (5.20)$$

We stress that the state $|\mathbf{u}, \mathbf{w}, -\mathbf{u}\rangle$ was constructed using the nested coordinate Bethe ansatz, with the order of the magnon rapidities given by (5.4).

Combining this with the result for the norm of the parity-symmetric state (5.18), we obtain

$$\begin{aligned} & \frac{\langle \mathcal{B} | \mathbf{u}, \mathbf{w}, -\mathbf{u} \rangle}{\sqrt{\langle \mathbf{u}, \mathbf{w}, -\mathbf{u} | \mathbf{u}, \mathbf{w}, -\mathbf{u} \rangle}} \\ &= 2(-1)^L \prod_{j=1}^{K_{\mathbf{u}}} \left(u_j + \frac{i}{2} \right) \prod_{k=1}^{\lceil \frac{K_{\mathbf{w}}}{2} \rceil} \frac{1}{w_k(w_k + \frac{i}{2})} \left[\prod_{k=1}^{\lceil \frac{K_{\mathbf{w}}}{2} \rceil} \frac{(w_k + \frac{i}{2})(w_k^* - \frac{i}{2})}{(w_k - \frac{i}{2})(w_k^* + \frac{i}{2})} \right]^{\frac{1}{4}} \sqrt{\frac{\det G_+}{\det G_-}}. \end{aligned} \tag{5.21}$$

To arrive at the formula (5.5), we then utilize the phase ambiguity discussed in section 2.2. For instance, multiplying the following phase¹⁵

$$\begin{aligned} e^{i\varphi} &\equiv (-1)^{L+1} \left(\prod_{k=1}^{\lceil \frac{K_{\mathbf{w}}}{2} \rceil} \frac{(w_k + \frac{i}{2})(w_k^* + \frac{i}{2})}{(w_k - \frac{i}{2})(w_k^* - \frac{i}{2})} \right)^{\frac{1}{4}} \prod_{j=1}^{K_{\mathbf{u}}} \left(\frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}} \right)^{\frac{1}{2}} \\ &= (-1)^{L+1} \left(\prod_{k=1}^{\lceil \frac{K_{\mathbf{w}}}{2} \rceil} \frac{(w_k + \frac{i}{2})(w_k^* + \frac{i}{2})}{(w_k - \frac{i}{2})(w_k^* - \frac{i}{2})} \right)^{\frac{1}{4}} \prod_{j=1}^{K_{\mathbf{u}}} \left(\frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}} \right)^{\frac{1}{2}}, \end{aligned} \tag{5.22}$$

we get

$$\frac{e^{i\varphi} \langle \mathcal{B} | \mathbf{u}, \mathbf{w}, -\mathbf{u} \rangle}{\sqrt{\langle \mathbf{u}, \mathbf{w}, -\mathbf{u} | \mathbf{u}, \mathbf{w}, -\mathbf{u} \rangle}} = -2 \sqrt{\prod_{j=1}^{K_{\mathbf{u}}} \left(u_j^2 + \frac{1}{4} \right) \prod_{k=1}^{\lceil \frac{K_{\mathbf{w}}}{2} \rceil} \frac{1}{w_k^2(w_k^2 + \frac{1}{4})} \frac{\det G_+}{\det G_-}}, \tag{5.23}$$

which coincides with (5.5).

6 Conclusion

In this paper, we studied the tree-level structure constants of a single-trace non-BPS operator and two sub-determinant operators (also known as giant gravitons) in ABJM theory in the planar limit. Much like in $\mathcal{N} = 4$ SYM, these structure constants can be computed by overlaps between a matrix product state and a Bethe eigenstate. To evaluate them explicitly, we developed the coordinate Bethe ansatz for the alternating $SU(4)$ spin chain. As a result of the computation, we found that the overlap obeys a selection rule similar to the ones found in defect one-point functions in $\mathcal{N} = 4$ SYM. We also found a closed-form expression for the overlaps for the maximal giant gravitons.

This paper is the first installment of our studies of the structure constants of giant gravitons in ABJM theory. In the second paper [31], we will analyze these quantities at

¹⁵Here we used the fact that \mathbf{u} as a set must be invariant under the complex conjugation (otherwise, the energy and the higher conserved charges generically will take complex values). Similar results for XXX and XXZ spin chains were proved in [106].

strong coupling using a holographic description. The results of this paper and the next will be used in the third paper [32] to test and verify the nonperturbative approach based on the integrable bootstrap.

There are several future directions worth pursuing. First it would be desirable to prove the determinant formula for the overlap conjectured in this paper. One possible strategy is to map it to a partition function of a lattice model (a vertex model) as was done for $\mathcal{N} = 4$ SYM [42]. Another possible strategy is to use the algebraic Bethe ansatz and Separation of Variables¹⁶ following the recent work [109]. Second the coordinate Bethe ansatz developed in this paper will be useful for computing other quantities in ABJM theory, most notably the three-point functions of single-trace operators¹⁷. Such computations will provide valuable data for developing the hexagon formalism for ABJM theory, which is yet to be established. Works in this direction are in progress. Third it is important to perform the computation at a loop level in order to see if the structure found in this paper persists, and to have more data to test the bootstrap approach discussed in the third paper. Finally the results in this paper suggest that non-maximal giant gravitons in ABJM theory might also correspond to integrable boundary states. It would be interesting to explore this further¹⁸, e.g. by analyzing the boundary condition of the string worldsheet at strong coupling [62, 63] or by computing the Hamiltonian of the open spin chain attached to the non-maximal giant gravitons (see [113–115] for the results on the maximal giant graviton). In addition, it would be desirable to derive a closed-form expression for the overlap for the non-maximal giant gravitons. Another interesting question is to generalize the analysis in this paper to the dual giant gravitons and see whether they lead to integrable boundary conditions.

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A Partially contracted giant graviton

In this appendix, we study the structure constants of two giant graviton and one single-trace operator of length $2L$ using the partially-contracted-giant-graviton (PCGG) approach [29] in order to cross-check the results in the main text. For simplicity, we only discuss the maximal giant gravitons but the analysis can be readily extended to non-maximal giant gravitons.

¹⁶The relation between overlaps of integrable boundary states and Separation of Variables was studied also in recent papers [107, 108] from different perspectives.

¹⁷See [110] for results in the $SU(2) \times SU(2)$ sector.

¹⁸It would also be useful to revisit the analysis in [77] in order to understand the difference between $\mathcal{N} = 4$ SYM and ABJM. This is important since, even for $\mathcal{N} = 4$ SYM, there are contradicting claims on the (non-)integrability of the non-maximal giant gravitons in the literature [60, 61] (see also [111, 112]).

Two-point function. The main focus of this appendix is the three-point function

$$G_2 = \langle \mathcal{D}_1(x_1) \mathcal{D}_2(x_2) \mathcal{O}(x_3) \rangle, \quad (\text{A.1})$$

where

$$\mathcal{D}_i(x_i) = \det \left((n_i \cdot Y)(\bar{n}_i \cdot \bar{Y}) \right) = \det(n_i \cdot Y) \det(\bar{n}_i \cdot \bar{Y}). \quad (\text{A.2})$$

However, it is useful to first analyze the two-point function

$$\langle \mathcal{D}_1(x_1) \mathcal{D}_2(x_2) \rangle = \langle \det(n_1 \cdot Y) \det(\bar{n}_2 \cdot \bar{Y}) \rangle \langle \det(n_2 \cdot Y) \det(\bar{n}_1 \cdot \bar{Y}) \rangle. \quad (\text{A.3})$$

At the leading order, we can compute it by Wick contraction. The propagator is given by

$$\langle (n_1 \cdot Y(x))_a^{\bar{b}} (\bar{n}_2 \cdot \bar{Y}(y))_{\bar{c}}^d \rangle = \frac{1}{k} \frac{n_1 \cdot \bar{n}_2}{|x - y|} \delta_a^d \delta_{\bar{c}}^{\bar{b}}. \quad (\text{A.4})$$

The determinant can be written as

$$\det X = \frac{1}{N!} \epsilon_{a_1 \dots a_N} \epsilon^{b_1 \dots b_N} X_{b_1}^{a_1} \dots X_{b_N}^{a_N}. \quad (\text{A.5})$$

Therefore, the two-point function of a single determinant is given by

$$\begin{aligned} \langle \det(n_1 \cdot Y)(x_1) \det(\bar{n}_2 \cdot \bar{Y})(x_2) \rangle &= \frac{1}{k^N (N!)^2} \epsilon_{a_1 \dots a_N} \epsilon^{b_1 \dots b_N} \epsilon_{c_1 \dots c_N} \epsilon^{d_1 \dots d_N} \\ &\quad \times \frac{(n_1 \cdot \bar{n}_2)^N}{|x_{12}|^N} \delta_{a_1}^{d_1} \dots \delta_{a_N}^{d_N} \delta_{b_1}^{c_1} \dots \delta_{b_N}^{c_N} \cdot N! \\ &= \frac{(n_1 \cdot \bar{n}_2)^N N!}{k^N |x_{12}|^N} = \frac{d_{12}^N}{k^N} N!, \end{aligned} \quad (\text{A.6})$$

where we have used $\epsilon_{a_1 \dots a_N} \epsilon^{a_1 \dots a_N} = N!$. Then, we get the following result for the two-point function of giant gravitons

$$\langle \mathcal{D}_1(x_1) \mathcal{D}_2(x_2) \rangle = \frac{1}{k^{2N}} (d_{12} d_{21})^N (N!)^2. \quad (\text{A.7})$$

Partially contracted giant graviton. To compute the three-point function involving a single-trace operator of length $2L$ (A.1), we first perform the Wick contractions between the two giant gravitons *partially* so that it leaves precisely $2L$ uncontracted scalar fields. We then perform the Wick contraction between these scalar fields with the single trace operator. We see that, in ABJM theory, the giant graviton factorizes into the product of two determinants (A.2). In the large N limit, the partially contracted giant graviton at the leading order is given by the sum of the following two contributions:

- Contract completely $\det(n_1 \cdot Y)$ and $\det(\bar{n}_2 \cdot \bar{Y})$ and take the leading term of the partially contracted $\det(n_2 \cdot Y)$ and $\det(\bar{n}_1 \cdot \bar{Y})$.
- Contract completely $\det(n_2 \cdot Y)$ and $\det(\bar{n}_1 \cdot \bar{Y})$ and take the leading term of the partially contracted $\det(n_1 \cdot Y)$ and $\det(\bar{n}_2 \cdot \bar{Y})$.

The leading term of the partially contracted determinant operators have been worked out in [29] in the study of giant gravitons in $\mathcal{N} = 4$ SYM, and we can simply recycle the result. For instance, the PCGG obtained from the partial contraction of $\det(n_2 \cdot Y)$ and $\det(\bar{n}_1 \cdot \bar{Y})$ is a non-local single trace operator

$$(N - L)! \frac{(d_{21})^{N-L} (-1)^{L+1}}{k^{N-L}} \frac{1}{L} \text{tr} \left[\left((n_2 \cdot Y)(x_2) (\bar{n}_1 \cdot \bar{Y})(x_1) \right)^L \right]. \quad (\text{A.8})$$

A similar expression can be derived for the partial contraction of $\det(n_1 \cdot Y)$ and $\det(\bar{n}_2 \cdot \bar{Y})$. Combining the two contributions, we find that the full PCGG is given by

$$G_L(x_1, x_2) = N!(N - L)! \frac{(d_{21}d_{12})^N (-1)^{L+1}}{k^{2N-L}} \frac{1}{L} \left(\frac{\text{tr} \left[\left((n_2 \cdot Y)(x_2) (\bar{n}_1 \cdot \bar{Y})(x_1) \right)^L \right]}{(d_{21})^L} + \frac{\text{tr} \left[\left((n_1 \cdot Y)(x_1) (\bar{n}_2 \cdot \bar{Y})(x_2) \right)^L \right]}{(d_{12})^L} \right).$$

In order to take the large N limit, it is useful to divide it by the two-point function of giant gravitons (A.7). We then get

$$\frac{G_L(x_1, x_2)}{\langle \mathcal{D}_1(x_1) \mathcal{D}_2(x_2) \rangle} \stackrel{N \rightarrow \infty}{=} \frac{(-1)^{L+1}}{L} \left(\frac{1}{\lambda} \right)^L \left(\frac{\text{tr} \left[\left((n_2 \cdot Y)(x_2) (\bar{n}_1 \cdot \bar{Y})(x_1) \right)^L \right]}{(d_{21})^L} + \frac{\text{tr} \left[\left((n_1 \cdot Y)(x_1) (\bar{n}_2 \cdot \bar{Y})(x_2) \right)^L \right]}{(d_{12})^L} \right).$$

Twisted-translated frame and MPS. To compare this with the results in the main text, we consider the twisted-translated kinematics and set $a_1 = -a_2 = 1$ and $a_3 = 0$. Computing the Wick contractions between the PCGG (A.9) and the single-trace operator and dividing the result by the normalization $\mathcal{N}_{\mathcal{O}}$, we find that the structure constant is given by

$$\mathfrak{D}_{N|\mathcal{O}} = -\frac{(-i)^J}{2^{\Delta-J}} \frac{\langle \mathcal{B}|\mathcal{O} \rangle}{\sqrt{L} \langle \mathcal{O}|\mathcal{O} \rangle}, \quad (\text{A.9})$$

which precisely matches with the result in the main text (4.65).

B Derivation of BPS three-point functions

Here we explain how to derive the expression for the three-point function of two (non-)maximal giant gravitons and a single-trace BPS operator given in (4.51).

The starting point of the analysis is the generating function of structure constants (4.50). In order to extract the structure constant for a single-trace operator with a fixed charge, we perform the integration over s ; namely

$$\frac{\langle \mathcal{D}_M \mathcal{D}_M \mathcal{O}_L^\circ \rangle}{\langle \mathcal{D}_M \mathcal{D}_M \rangle} = - \oint_{|s|=\epsilon \ll 1} \frac{ds}{2\pi i s^{1+L}} \oint_{|z|=1} \frac{dz}{2\pi i z} \text{tr}_2 \left[\frac{1}{1 - s \mathcal{T}_\theta \bar{\mathcal{T}}_\theta} \right]. \quad (\text{B.1})$$

Here we replaced the integration variable θ with $z \equiv e^{i\theta}$. Note also that the integration of s is performed in a region near the origin $|s| \ll 1$ since the generating function $1/(1 - s \mathcal{T}_\theta \bar{\mathcal{T}}_\theta)$ is expected to have a finite radius of convergence when expanded in s and therefore one has to take $|s|$ to be sufficiently small in order to use the formula.

As the next step, we diagonalize the matrix $1/(1 - s \mathcal{T}_\theta \bar{\mathcal{T}}_\theta)$. We also change the integration variables as

$$\tilde{s} = \lambda \frac{d_{13} d_{32}}{d_{12}} s, \quad \tilde{z} = \left(\frac{d_{13} d_{31}}{d_{23} d_{32}} \right)^{\frac{1}{4}} z, \quad (\text{B.2})$$

in order to simplify the expression. As a result we get

$$\begin{aligned} \frac{\langle \mathcal{D}_M \mathcal{D}_M \mathcal{O}_L^\circ \rangle}{\langle \mathcal{D}_M \mathcal{D}_M \rangle} &= - \left(\lambda \frac{d_{13} d_{32}}{d_{12}} \right)^L \oint_{|\tilde{s}|=\epsilon \ll 1} \frac{d\tilde{s}}{2\pi i \tilde{s}^{1+L}} \oint_{|z|=1} \frac{d\tilde{z}}{2\pi i \tilde{z}} \\ &\times \left[1 + \frac{\tilde{z}^2 (1 - \xi \tilde{s}^2)}{\sqrt{\xi} \tilde{s} (\omega - 1) (1 + \tilde{z}^4) + \tilde{z}^2 ((\xi + 1) \tilde{s} \omega - 2)} \right]^{-1}. \end{aligned} \quad (\text{B.3})$$

We then perform the integration of \tilde{z} by closing the contour and picking up contributions from poles inside the contour. When $|\tilde{s}| \ll 1$, we find that the integrand has three poles inside the contour $|z| = 1$:

$$\tilde{z} = 0, \quad \pm \frac{1}{\xi^{1/4}} \sqrt{\frac{1 + \tilde{s}^2 \xi - (1 + \xi) \tilde{s} \omega - \sqrt{(1 + \tilde{s}^2 \xi - \tilde{s} (1 + \xi) \omega)^2 - 4 \tilde{s}^2 \xi (\omega - 1)^2}}{2 \tilde{s} (\omega - 1)}}. \quad (\text{B.4})$$

Computing the residues from these poles (and redefining \tilde{s} as s), we obtain (4.51).

C Data and numerics

In this appendix, we explain our method to numerically solve the Bethe equation and provide some data for the overlap with the non-maximal giant graviton.

C.1 Numerical solutions to the Bethe equations

Solving the Bethe equation. To solve the Bethe equations (3.7) numerically, we follow the following strategy: first we divide the auxiliary roots \mathbf{w} into two subsets $\mathbf{w} = \mathbf{w}^{(1)} \cup \mathbf{w}^{(2)}$,

introduce a parameter ϵ to modify the Bethe equations into the following:

$$\begin{aligned}
 \left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} \right)^L &= \prod_{\substack{k=1 \\ k \neq j}}^{K_{\mathbf{u}}} \frac{u_j - u_k + i}{u_j - u_k - i} \prod_{k=1}^{K_{\mathbf{w}}^{(1)}} \frac{u_j - w_k^{(1)} - \frac{i}{2}}{u_j - w_k^{(1)} + \frac{i}{2}} \prod_{k=1}^{K_{\mathbf{w}}^{(2)}} \frac{u_j - w_k^{(2)} - \frac{i}{2}}{u_j - w_k^{(2)} + \frac{i}{2}} \epsilon, \\
 \left(\frac{v_j + \frac{i}{2}}{v_j - \frac{i}{2}} \right)^L &= \prod_{\substack{k=1 \\ k \neq j}}^{K_{\mathbf{v}}} \frac{v_j - v_k + i}{v_j - v_k - i} \prod_{k=1}^{K_{\mathbf{w}}^{(1)}} \frac{v_j - w_k^{(1)} - \frac{i}{2}}{v_j - w_k^{(1)} + \frac{i}{2}} \prod_{k=1}^{K_{\mathbf{w}}^{(2)}} \frac{v_j - w_k^{(2)} - \frac{i}{2}}{v_j - w_k^{(2)} + \frac{i}{2}} \epsilon, \\
 1 &= \prod_{k=1}^{K_{\mathbf{u}}} \frac{w_j^{(1)} - u_k - \frac{i}{2}}{w_j^{(1)} - u_k + \frac{i}{2}} \prod_{k=1}^{K_{\mathbf{v}}} \frac{w_j^{(1)} - v_k - \frac{i}{2}}{w_j^{(1)} - v_k + \frac{i}{2}} \\
 &\quad \times \prod_{\substack{k=1 \\ k \neq j}}^{K_{\mathbf{w}}^{(1)}} \frac{w_j^{(1)} - w_k^{(1)} + i}{w_j^{(1)} - w_k^{(1)} - i} \prod_{k=1}^{K_{\mathbf{w}}^{(2)}} \frac{w_j^{(1)} - w_k^{(2)} + i}{w_j^{(1)} - w_k^{(2)} - i} \epsilon, \\
 1 &= \prod_{k=1}^M \frac{w_j^{(2)} - u_k - \frac{i}{2}}{w_j^{(2)} - u_k + \frac{i}{2}} \prod_{k=1}^N \frac{w_j^{(2)} - v_k - \frac{i}{2}}{w_j^{(2)} - v_k + \frac{i}{2}} \\
 &\quad \times \prod_{\substack{k=1 \\ k \neq j}}^{K_{\mathbf{w}}^{(2)}} \frac{w_j^{(2)} - w_k^{(2)} + i}{w_j^{(2)} - w_k^{(2)} - i} \prod_{k=1}^{K_{\mathbf{w}}^{(1)}} \frac{w_j^{(2)} - w_k^{(1)} + i}{w_j^{(2)} - w_k^{(1)} - i} \epsilon.
 \end{aligned}$$

The modified Bethe equations (C.1) go back to the original equations when $\epsilon = 1$ while they become two decoupled SU(3) Bethe equations when $\epsilon = 0$.

Second we solve the SU(3) Bethe equations using the rational Q-system¹⁹ by Marboe and Volin [117]. We then use these solutions as “seed” solutions for $\epsilon = 0$ and use `FindRoots` in *Mathematica* to generate solutions for $\epsilon = 1$.

As a result, we obtained the solutions to the original Bethe equations (3.7). In what follows we only consider the solutions with $K_{\mathbf{u}} = K_{\mathbf{v}} = K_{\mathbf{w}}$ that are relevant for the computation of the overlap. Note also that the solutions listed below are by no means exhaustive. It would be desirable to improve the algorithm for solving the Bethe equations and perform more systematic checks. In particular, the generalization of Marboe-Volin algorithm for the alternating spin chain is an important open problem.

Solutions to the Bethe equations. Let us now list the solutions we obtained. Here the Bethe roots denoted in red satisfy both the selection rules and the zero-momentum condition while the Bethe roots denoted in blue only satisfy the zero-momentum condition. All the other Bethe roots do not satisfy the zero-momentum condition and therefore do not correspond to single-trace operators in ABJM theory. For later convenience, we also numbered each solution in red, as [n].

¹⁹However, the Q system there only works for the case when $L - M \geq M - N \geq N$, where L is the length of the SU(3) spin chain and M, N are number of two types of magnons. Sometimes this constraint is too strong to obtain the SU(4) solutions. In some cases, we can fix this problem using the bosonic duality [116].

L	K_w	$[\mathbf{u}, \mathbf{v}, \mathbf{w}]$
1	1	$[\{0\}, \{0\}, \{0\}]$ [1]
2	1	$[\{\frac{1}{2\sqrt{3}}\}, \{-\frac{1}{2\sqrt{3}}\}, \{0\}]$
		$[\{\frac{1}{2}\}, \{\frac{1}{2}\}, \{\frac{1}{2}\}]$
		$[\{-\frac{1}{2}\}, \{-\frac{1}{2}\}, \{-\frac{1}{2}\}]$
3	1	$[\{0\}, \{0\}, \{0\}]$ [2]
		$[\{\frac{1}{2}\}, \{-\frac{1}{2}\}, \{0\}]$ [3]
		$[\{-\frac{\sqrt{3}}{2}\}, \{-\frac{\sqrt{3}}{2}\}, \{-\frac{\sqrt{3}}{2}\}]$
4	1	$[\{\frac{\sqrt{7}-2}{6}\}, \{-\frac{\sqrt{7}+2}{6}\}, \{-\frac{1}{3}\}]$
5	1	$[\{0\}, \{0\}, \{0\}]$ [4]
		$[\{-\frac{1}{\sqrt{12}}\}, \{\frac{1}{\sqrt{12}}\}, \{0\}]$ [5]
6	1	$[\{-0.3987366944412021\}, \{0.3987366944412021\}, \{0\}]$
		$[\{0.13397459621556135\}, \{0.13397459621556138\}, \{0.13397459621556135\}]$
		$[\{0.45440762733328904\}, \{0.15990657179334503\}, \{0.307157099563317\}]$
		$[\{\frac{1}{2}\}, \{\frac{1}{2}\}, \{\frac{1}{2}\}]$
2	2	$[\{\sqrt{\frac{3}{20}}, -\sqrt{\frac{3}{20}}\}, \{-\sqrt{\frac{3}{20}}, \sqrt{\frac{3}{20}}\}, \{\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\}]$ [6]
3	2	$[\{-0.34554024732023516, 0.34554024732023516\},$ $\{0.34554024732023516, -0.34554024732023516\},$ $\{0.51108108452939387i, -0.51108108452939387i\}]$
		$[\{-0.51320279936202029, 0.062678427970585504\},$ $\{0.51320279936202029, -0.062678427970585504\},$ $\{-0.41623324836102756, 0.41623324836102756\}]$
		$[\{-0.67910322332093357, 0.025611245506665032\},$ $\{0.53448799950517913, -0.74702142547534923\},$ $\{-0.76136871888869386, 0.32835601699647453\}]$
		$[\{-0.56944513222254722, 0.56944513222254722\},$ $\{0.56944513222254722, -0.56944513222254722\},$ $\{0.55472575089337952i, -0.55472575089337952i\}]$ [7]
4	2	$[\{-0.66931302374657110, -0.10647062449199778\},$ $\{0.66931302374657110, 0.10647062449199778\},$ $\{-0.48978976193047651, 0.48978976193047651\}]$ [8]
		$[\{-0.028299146374196785, 0.55999384176787242\},$ $\{-0.028299146374196785, 0.55999384176787242\},$ $\{0.26584734769683781 - 0.50624190637875312i,$ $0.26584734769683781 + 0.50624190637875312i\}]$

6	2	$\{0.34675919853122809, 0.99366751548216189\},$ $\{-0.34675919853122809, -0.99366751548216189\},$ $\{0.71060441235307684, -0.71060441235307684\}$ [9]
		$\{-1.0773108398074073, 0.49798319908651183\},$ $\{0.089846608200097142, 0.42002515253515016\},$ $\{-0.59531374844468910, 0.56058580845186502\}$
6	2	$\{0.40946507692652007, 0.10041044458589016\},$ $\{-0.39457524195942078, 0.15869120668880070\},$ $\{0.44210986670775702, -0.30511412358686194\},$
		$\{0.11567004621958770, 0.44116926182762157\},$ $\{-0.40026535781351243, 0.48672949230748789\},$ $\{0.57166303815947364, -0.25001131688888128\}$
		$\{0.37830228050349018, 1.0458410010838975\},$ $\{-0.055390813251937810, -0.35320869561294462\},$ $\{0.78252761450158240, -0.27475572814032976\}$
		$\{0.36681212663617484, 1.0519697940662345\},$ $\{0.20899528319467480, -1.0040495234017774\}$ $\{0.82203932258846606, -0.51017548234081269\}$
		$\{0.13241485029334295, 0.45759251757083131\},$ $\{-0.098732170839668265, 0.48070339312713823\},$ $\{0.59462910995790549, -0.10863981488208338\}$
3	3	$\{-0.61842989257770833, 0, 0.61842989257770833\}$ $\{0.61842989257770833, 0, -0.61842989257770833\},$ $\{0.71410132990930250, -0.71410132990930250, 0\}$ [10]
5	3	$\{-0.90018200552947429, 0.031648482693564728, 0.37680381182936445\}$ $\{0.90018200552947429, -0.031648482693564728, -0.37680381182936445\},$ $\{-0.75676650863660755, 0.75676650863660755, 0\}$ [11]
6	4	$\{\pm 0.53714639382219377 \pm 0.51401134667644172i\}$ $\{\mp 0.53714639382219377 \mp 0.51401134667644172i\},$ $\{\pm 1.6004024229193260i, \pm 0.50622703524559230i\}$ ²⁰ [12]
		$\{-0.098635382965062615, 0.14753207194168752,$ $-1.0179905746843994, 0.47877616601418219\},$ $\{0.098635382965062615, -0.14753207194168752,$ $1.0179905746843994, -0.47877616601418219\},$ $\{-0.89439365253962075, 0.89439365253962075,$ $0.22001470275637868, -0.22001470275637868\}$ [13]

²⁰Here the precise meaning of the expression like “ $\{\pm \text{Re}(u) \pm i\text{Im}(u)\}, \{\mp \text{Re}(u) \mp i\text{Im}(u)\}, \{\pm w_1, \pm w_2\}$ ” is “ $\{\text{Re}(u) + i\text{Im}(u), -\text{Re}(u) + i\text{Im}(u), \text{Re}(u) - i\text{Im}(u), -\text{Re}(u) - i\text{Im}(u)\}, \{-\text{Re}(u) - i\text{Im}(u), \text{Re}(u) - i\text{Im}(u), -\text{Re}(u) + i\text{Im}(u), \text{Re}(u) + i\text{Im}(u)\}, \{w_1, -w_1, w_2, -w_2\}$ ”.

C.2 Overlaps for non-maximal giant gravitons

Here we summarize the results for overlaps for non-maximal giant gravitons $\omega \neq 1$. Below we list the results for the ratio between the structure constant of the maximal giant graviton $\mathfrak{D}_{N|\mathcal{O}}$ and the structure constant of the non-maximal giant graviton $\mathfrak{D}_{M|\mathcal{O}}$; $r \equiv \mathfrak{D}_{M|\mathcal{O}}/\mathfrak{D}_{N|\mathcal{O}}$.

$$\begin{aligned}
 [1] \quad & r = \omega, \\
 [2] \quad & r = -2\omega^3 + 4\omega^2 - \omega, \\
 [3] \quad & r = -2\omega^3 + 4\omega^2 - \omega, \\
 [4] \quad & r = 6\omega^5 - 24\omega^4 + 30\omega^3 - 12\omega^2 + \omega, \\
 [5] \quad & r = 6\omega^5 - 24\omega^4 + 30\omega^3 - 12\omega^2 + \omega, \\
 [6] \quad & r = 2\omega^2 - \omega, \\
 [7] \quad & r = -4\omega^4 + 8.80557589\omega^3 - 3.77669700\omega^2 - 0.028878886\omega, \\
 [8] \quad & r = -4\omega^4 + 8.21090774\omega^3 - 3.39983829\omega^2 + 0.188930549\omega, \\
 [9] \quad & r = 12\omega^6 - 47.8851733\omega^5 + 62.8683251\omega^4 - 30.0180998\omega^3 \\
 & \quad + 3.96919074\omega^2 + 0.065757247\omega, \\
 [10] \quad & r = -2\omega^3 + 4\omega^2 - \omega, \\
 [11] \quad & r = -8\omega^5 + 20.17658812\omega^4 - 12.92497528\omega^3 \\
 & \quad + 1.74975464\omega^2 - 0.001367480\omega, \\
 [12] \quad & r = -16\omega^6 + 42.978665\omega^5 - 33.996976\omega^4 + 8.5586366\omega^3 \\
 & \quad - 0.54003958\omega^2 - 0.00028639193\omega, \\
 [13] \quad & r = -16\omega^6 + 49.281471\omega^5 - 51.190705\omega^4 + 23.285677\omega^3 \\
 & \quad - 5.0615656\omega^2 + 0.68512348\omega.
 \end{aligned} \tag{C.1}$$

Here [n]'s refer to the numbers in the table above.

D Nested algebraic Bethe ansatz

In this appendix, we review the nested algebraic Bethe ansatz approach for the SU(4) alternating spin chain [34, 37]. We begin with the following set of four R -matrices,

$$R_{ab} = u + P_{ab}, \tag{D.1}$$

$$R_{\bar{a}\bar{b}} = u + P_{\bar{a}\bar{b}}, \tag{D.2}$$

$$R_{a\bar{b}} = -(u+2) + K_{a\bar{b}}, \tag{D.3}$$

$$R_{\bar{a}b} = -(u+2) + K_{\bar{a}b}, \tag{D.4}$$

where a (\bar{a}) denotes the spin in the fundamental (anti-fundamental) representation of SU(4). These R -matrices satisfy the following eight Yang-Baxter equations,

$$R_{\mathbf{ab}}(u-v)R_{\mathbf{ac}}(u)R_{\mathbf{bc}}(v) = R_{\mathbf{bc}}(v)R_{\mathbf{ac}}(u)R_{\mathbf{ab}}(u-v), \tag{D.5}$$

where \mathbf{a} can take value of a or \bar{a} independently.

We now define two monodromy matrices using two different auxiliary spaces in $\mathbf{4}$ and $\bar{\mathbf{4}}$ representations,

$$T_a(u) = R_{a\mathbf{1}}(u)R_{a\bar{\mathbf{1}}}(u) \cdots R_{aL}(u)R_{a\bar{L}}(u), \quad (\text{D.6})$$

$$T_{\bar{a}}(u) = R_{\bar{a}\mathbf{1}}(u)R_{\bar{a}\bar{\mathbf{1}}}(u) \cdots R_{\bar{a}L}(u)R_{\bar{a}\bar{L}}(u). \quad (\text{D.7})$$

The corresponding transfer matrices

$$\tau(u) = \text{Tr}_a T_a(u), \quad \bar{\tau}(u) = \text{Tr}_{\bar{a}} T_{\bar{a}}(u), \quad (\text{D.8})$$

commute with each other

$$[\tau(u), \tau(v)] = 0, \quad [\bar{\tau}(u), \bar{\tau}(v)] = 0, \quad [\tau(u), \bar{\tau}(v)] = 0, \quad (\text{D.9})$$

due to the above Yang-Baxter equations. We can generate two Hamiltonians from the transfer matrices,

$$H_{\text{odd}} = (\tau(0))^{-1} \frac{d}{du} \tau(u) \Big|_{u=0}, \quad H_{\text{even}} = (\bar{\tau}(0))^{-1} \frac{d}{du} \bar{\tau}(u) \Big|_{u=0}. \quad (\text{D.10})$$

The true Hamiltonian is given by $H = H_{\text{odd}} + H_{\text{even}}$ which is the same as planar two-loop anomalous dimension matrix in the scalar sector of ABJM theory, up to rescaling and shifting by a constant.

Due to (D.9), there exist u -independent common eigenstates of $\tau(u)$ and $\bar{\tau}(u)$. We now construct eigenstates of $\tau(u)$ using nested algebraic Bethe ansatz [118]. For a nice review, we refer to [119]. We first write the monodromy matrix $T_a(u)$ as 4×4 matrix whose elements are operators acting on the Hilbert space of the spin chain $\mathcal{H} \cong (\mathbf{C}^4)^{\otimes 2L}$,

$$T_a(u) = \begin{pmatrix} A(u) & B_1(u) & B_2(u) & B_3(u) \\ C_1(u) & D_{22}(u) & D_{23}(u) & D_{24}(u) \\ C_2(u) & D_{32}(u) & D_{33}(u) & D_{34}(u) \\ C_3(u) & D_{42}(u) & D_{43}(u) & D_{44}(u) \end{pmatrix}.$$

The RTT relation

$$R_{ab}(u-v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u-v), \quad (\text{D.11})$$

from Yang-Baxter equations, leads to the following commutation relations,

$$A(u)B_i(v) = \frac{u-v-1}{u-v} B_i(v)A(u) + \frac{1}{u-v} B_i(u)A(v), \quad (\text{D.12})$$

$$D_{ij}(u)B_k(v) = \frac{1}{u-v} B_{k'}(v)D_{ij'}(u) \left(R^{\text{SU}(3)}(u-v) \right)_{kj}^{k'j'} - \frac{1}{u-v} B_j(u)D_{ik}(v), \quad (\text{D.13})$$

where i, j, k, \dots take values in $2, 3, 4$ and

$$\left(R^{\text{SU}(3)}(u-v) \right)_{kj}^{k'j'} = (u-v) \delta_k^{k'} \delta_j^{j'} + \delta_j^{k'} \delta_k^{j'} \quad (\text{D.14})$$

is an SU(3) R-matrix. And the red part in the above formulas is the origin of the unwanted terms in the following.

Now we consider the subspace \mathcal{H}_1 of \mathcal{H} spanned by the state

$$|1, \bar{i}_1, 1, \bar{i}_2, \dots, 1, \bar{i}_L\rangle \quad (\text{D.15})$$

with $\bar{i}_k = 2, 3, 4$. Any state $|1\rangle$ in this subspace satisfy the following important property,

$$A(u)|1\rangle = (-u - 2)^L(u + 1)^L|1\rangle, \quad (\text{D.16})$$

$$C_i(u)|1\rangle = 0, \quad (\text{D.17})$$

$$D_{ij}(u)|1\rangle \in \mathcal{H}_1. \quad (\text{D.18})$$

Consider the following state in \mathcal{H} ,

$$|\psi\rangle = B_{i_1}(\mu_1) \cdots B_{i_M}(\mu_M) X^{i_1 \cdots i_M} |1\rangle, \quad (\text{D.19})$$

with $|1\rangle \in \mathcal{H}_1$ and $X^{i_1 \cdots i_M}$ to-be-determined coefficients. The acts of $\tau(u) = A(u) + D_{jj}(u)$ on $|\psi\rangle$ is given by

$$A(u)|\psi\rangle = (-u - 2)^L(u + 1)^L \prod_{i=1}^M \frac{u - \mu_i - 1}{u - \mu_i} |\psi\rangle + \text{unwanted terms}, \quad (\text{D.20})$$

$$\begin{aligned} D_{jj}(u)|\psi\rangle &= \left(\prod_{i=1}^M \frac{1}{u - \mu_i} \right) B_{k_1}(\mu_1) \cdots B_{k_M}(\mu_M) D_{j l_M}(u) R^{\text{SU}(3)}(u - \mu_M)_{i_M l_{M-1}}^{k_M l_M} \\ &\times R^{\text{SU}(3)}(u - \mu_{M-1})_{i_{M-1} l_{M-2}}^{k_{M-1} l_{M-1}} \cdots R^{\text{SU}(3)}(u - \mu_2)_{i_2 l_1}^{k_2 l_2} R^{\text{SU}(3)}(u - \mu_1)_{i_1 j}^{k_1 l_1} \\ &\times X^{i_1 \cdots i_M} |1\rangle + \text{unwanted terms}. \end{aligned} \quad (\text{D.21})$$

Now we need to compute the eigenstate of

$$\begin{aligned} D_{j l_M}(u) R^{\text{SU}(3)}(u - \mu_M)_{i_M l_{M-1}}^{k_M l_M} R^{\text{SU}(3)}(u - \mu_{M-1})_{i_{M-1} l_{M-2}}^{k_{M-1} l_{M-1}} \cdots R^{\text{SU}(3)}(u - \mu_2)_{i_2 l_1}^{k_2 l_2} \\ \times R^{\text{SU}(3)}(u - \mu_1)_{i_1 j}^{k_1 l_1} \end{aligned} \quad (\text{D.22})$$

acting on $(\mathbf{C}^3)^{\otimes M} \otimes \mathcal{H}_L$. It is not hard to see that the problem is reduced to finding eigenvalue of transfer matrix on an integrable SU(3) spin chain with length $M + 2L$. On this chain, μ_1, \dots, μ_M play the roles of inhomogeneity parameters. One can reduce further to an SU(2) spin chain and finally diagonalize the transfer matrix. Collecting all of the wanted terms, we get

$$\begin{aligned} \Lambda(u) &= (-u - 2)^L(u + 1)^L \prod_{i=1}^{K_u} \frac{u - \mu_i - 1}{u - \mu_i} \\ &+ (-u)^L(u + 1)^L \prod_{j=1}^{K_v} \frac{u - \nu_j + 1}{u - \nu_j} \\ &+ (-u)^L(u + 2)^L \prod_{i=1}^{K_u} \frac{u - \mu_i + 1}{u - \mu_i} \prod_{k=1}^{K_w} \frac{u - \lambda_k - 1}{u - \lambda_k} \\ &+ (-u)^L(u + 2)^L \prod_{j=1}^{K_v} \frac{u - \nu_j - 1}{u - \nu_j} \prod_{k=1}^{K_w} \frac{u - \lambda_k + 1}{u - \lambda_k}. \end{aligned} \quad (\text{D.23})$$

Demanding the vanishing of the residue of the spurious pole at $u = \mu_i, \nu_j, \lambda_k$ leads to Bethe equations. After performing the substitution $\mu_i = iu_i - 1/2, \nu_j = iv_j - 3/2, \lambda_k = iw_k - 1$, these Bethe equations becomes exact the Bethe ansatz equations (3.7) in subsection 3. When these Bethe equations are satisfied, the above $\Lambda(u)$ is the eigenvalue of $\tau(u)$, and the corresponding eigenstate can be constructed using the above procedure. We have checked, for small L , that such states are also eigenstates of $\bar{\tau}(u)$. It should be very valuable to prove this for the general case.

The Bethe states from algebraic Bethe ansatz and the ones from coordinate Bethe ansatz are related by

$$|\mathbf{u}, \mathbf{v}, \mathbf{w}\rangle^{al} \propto \prod_{i < j} \frac{u_i - u_j + i}{u_i - u_j} \prod_{i < j} \frac{v_i - v_j + i}{v_i - v_j} \prod_{i < j} \frac{w_i - w_j + i}{w_i - w_j} |\mathbf{u}, \mathbf{v}, \mathbf{w}\rangle^{co}, \quad (\text{D.24})$$

with a proportional factor which is invariant under the permutations of rapidities of the same type.

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