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Renormalization of the topological charge density in QCD with dimensional regularization

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To all orders of perturbation theory, the renormalization of the topological charge density in dimensionally regularized QCD is shown to require no more than an additive renormalization proportional to the divergence of the flavour-singlet axial current. The proof is based on the standard BRS analysis of the QCD vertex functional in the background gauge and exploits the special algebraic properties of the charge density through the Stora–Zumino chain of descent equations.

1. Introduction

All known consistent forms of dimensional regularization of QCD break chiral symmetry and the symmetry is then only recovered after renormalization and removal of the regularization. In the flavour-singlet channel, the situation is further complicated by the chiral anomaly, a term proportional to the topological charge density in the axial-current conservation equation, which requires renormalization as do the other terms in that equation.

Parity-odd fields are in general not easy to deal with in dimensional regularization, because the fifth Dirac matrix γ_5 and the Levi–Civita symbol $\epsilon_{\mu\nu\rho\sigma}$ are not naturally defined in dimensions other than four. In QCD this technical difficulty can however be bypassed by representing such fields through totally antisymmetric tensor fields

[1,2]. Using this representation, the renormalization of the axial quark densities, the axial currents and the chiral anomaly has been worked out to high order in the gauge coupling [2–5].

In these computations, a multiplicative renormalization of the topological charge density turned out to be unnecessary, thus suggesting that the density is finite to all orders up to additive renormalizations [5]. The absence of a divergent multiplicative renormalization is perhaps not unexpected in view of the Adler–Bardeen theorem [6] or simply because the topological charge of classical fields assumes integer values. Many years ago, Breitenlohner, Maison and Stelle [7] attempted to trace back the finiteness of the charge density to its algebraic properties, but their argumentation remained incomplete and was partially incorrect to the extent of being inconclusive. To date a rigorous all-order discussion of the situation in dimensionally regularized QCD appears to be missing and the principal goal here is to fill this gap, using the standard BRS analysis of the QCD vertex functional [8–10], the background gauge [11–14] and the Stora–Zumino descent equations [15–17].

After going through some definitions and preliminary material in sect. 2, the rather special algebraic properties of the topological charge density, as expressed through the descent equations, are exposed in sect. 3. The symmetries of the QCD vertex functional in presence of a background gauge field and the sources for the descendants of the charge density are then discussed. These strongly constrain the form of the divergent parts of the vertex functional and, as shown in sect. 5, eventually exclude a multiplicative renormalization of the charge density. The paper ends with some comments on the axial anomaly and a few concluding remarks.

2. Preliminaries

2.1 Background field technique

The theory is set up in Euclidean space in the standard manner with any number $N_{\rm f} \geq 0$ of quarks in the fundamental representation of the gauge group $\mathrm{SU}(N)$ (see appendix A for any unexplained notation).

In the background field formalism, the fundamental gauge potential $A_\mu(x)$ is normalized such that the associated field tensor $F_{\mu\nu}(x)$ and the gauge-covariant derivatives do not involve the gauge coupling. The QCD action in D dimensions is then given by

$$S = \int \mathrm{d}^D x \left\{ -\frac{1}{2g_0^2} \mathrm{tr}\{F_{\mu\nu}F_{\mu\nu}\} + \sum_{r=1}^{N_\mathrm{f}} \overline{\psi}_r (\not{D} + m_{0,r})\psi_r \right\}, \tag{2.1}$$

where the index r of the quark fields $\overline{\psi}_r$ and ψ_r labels the quark flavours, g_0 is the bare coupling and $m_{0,r}$ the bare mass of the quark number r.

The background field technique permits the theory to be probed with greater respect for the gauge symmetry than is the case when probed in conventional ways [11–14]. Let $B_\mu(x)$ be a smooth classical gauge potential and consider the decomposition

$$A_\mu(x) = B_\mu(x) + g_0 q_\mu(x) \tag{2.2}$$

of the fundamental gauge potential A_μ in the background field B_μ and the quantum field q_μ, which is now the field integrated over in the functional integral. A possible choice of the gauge-fixing and associated ghost action is then

$$S_\mathrm{gf} = -\lambda_0 \int \mathrm{d}^D x \, \mathrm{tr}\left\{D_\mu q_\mu D_\nu q_\nu\right\}, \qquad D_\mu = \partial_\mu + \mathrm{Ad}\, B_\mu, \tag{2.3}$$

$$S_\mathrm{gh} = -2 \int \mathrm{d}^D x \, \mathrm{tr}\left\{D_\mu \bar{c}\,(D_\mu + g_0 \mathrm{Ad}\, q_\mu)\, c\right\}, \tag{2.4}$$

c and $\bar{c}$ being the ghost and antighost fields. As is quite clear from these expressions, and further discussed in subsect. 3.2, this way of fixing the gauge preserves a classical gauge symmetry.

The theory with total action

$$S_\mathrm{tot} = S + S_\mathrm{gf} + S_\mathrm{gh} \tag{2.5}$$

has a regular perturbation expansion in Feynman diagrams, if the background field is treated as an additional source field, i.e. if the functional integral is expanded in a powers series in this field. In particular, at vanishing background field, the theory in the standard Lorentz-covariant gauge is recovered. All-important is then the fact that the theory renormalizes in the same way with and without background field, the latter requiring no renormalization.

An introduction to the subject and a proof of the renormalizability of the dimensionally regularized theory in presence of the background field is provided in the first few sections of ref. [18]. Some of the strategies described there will again be used here, but the presentation in the following is intended to be self-contained.

2.2 Tensor fields and the topological charge density

In $D = 4$ dimensions, the topological charge density is given by

$$q_{\rm top}(x) = -\frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}{\rm tr}\{F_{\mu\nu}(x)F_{\rho\sigma}(x)\}. \tag{2.6}$$

Since the Levi–Civita symbol $\epsilon_{\mu\nu\rho\sigma}$ is not a well defined object in dimensions other than four, the use of eq. (2.6) in any dimension would require the first four dimensions to be distinguished from the -2ϵ extra dimensions.

Following refs. [1–5], such a distinction can be avoided by noting that the totally antisymmetric tensor field

$$(FF)_{\mu\nu\rho\sigma}(x) = F^a_{\mu\nu}(x)F^a_{\rho\sigma}(x) + F^a_{\nu\rho}(x)F^a_{\mu\sigma}(x) + F^a_{\nu\sigma}(x)F^a_{\rho\mu}(x) \tag{2.7}$$

is, in four dimensions, proportional to $\epsilon_{\mu\nu\rho\sigma}$ times the charge density. This tensor field is well defined in any dimension and thus provides a possible representation of the charge density in the framework of dimensional regularization.

In the case of the axial quark densities and currents, the totally antisymmetric tensor fields

$$P^{rs}_{\mu\nu\rho\sigma}(x) = \overline{\psi}_r(x)\gamma_{[\mu}\gamma_\nu\gamma_\rho\gamma_{\sigma]}\psi_s(x), \tag{2.8}$$

$$A^{rs}_{\mu\nu\rho}(x) = \overline{\psi}_r(x)\gamma_{[\mu}\gamma_\nu\gamma_{\rho]}\psi_s(x), \tag{2.9}$$

may similarly be taken as a possible representation of these fields in arbitrary dimensions (following common practice, an antisymmetrization over the indices enclosed in square brackets is implied). It may be worth mentioning in passing that the fields (2.8),(2.9) satisfy an exact PCAC relation, in any dimension, involving an evanescent further field, which renormalizes the other fields in the equation and gives rise to the axial anomaly at $D = 4$.

Correlation functions of tensor fields and the fundamental fields can be worked out in perturbation theory in the standard manner. Covariance under the full Lorentz group, including parity, is exactly preserved in these calculations and the required counterterms are Lorentz-covariant polynomials in the external momenta, Kronecker deltas and products of Dirac matrices (if some external Dirac indices are uncontracted). In particular, tensor fields renormalize among themselves.

3. Algebraic properties of the topological charge density

Totally antisymmetric tensors like $(FF)_{\mu\nu\rho\sigma}$ are naturally associated with differential forms. In this particular case, the form is, in any dimension, proportional to the second Chern character and thus has some special algebraic properties.

3.1 BRS variation [8,9]

In presence of the background gauge potential B_μ, the BRS variation of the quantum field q_μ, the ghost fields c, $\bar{c}$ and the quark fields $\psi, \overline{\psi}$ is given by

$$\delta_{\rm BRS} q_\mu = (D_\mu + g_0 \mathrm{Ad}\, q_\mu)\, c, \tag{3.1}$$

$$\delta_{\rm BRS} c = -g_0 c^2, \tag{3.2}$$

$$\delta_{\rm BRS} \bar{c} = \lambda_0 D_\mu q_\mu, \tag{3.3}$$

$$\delta_{\rm BRS} \psi = -g_0 c \psi, \qquad \delta_{\rm BRS} \overline{\psi} = -g_0 \overline{\psi} c. \tag{3.4}$$

Since the background field is not transformed, eq. (3.1) implies

$$\delta_{\rm BRS} A_\mu = g_0 \, (\partial_\mu + \mathrm{Ad}\, A_\mu)\, c, \tag{3.5}$$

which shows that the BRS transformation of the gauge potential A_μ is an infinitesimal gauge transformation.

The BRS variation is an antiderivative with respect to the grading defined by the fermion (ghost plus quark) number, i.e.

$$\delta_{\rm BRS}(fg) = \delta_{\rm BRS} f g + (-1)^n f \delta_{\rm BRS} g \tag{3.6}$$

if f has fermion number n. When acting on differential forms, the rank of the form is often included in the grading and the exterior differential d then anticommutes with $\delta_{\rm BRS}$. In the present context, where the BRS variation eventually acts on tensor fields, this convention however tends to be confusing and is not applied.

3.2 Background gauge variation

Background gauge transformations are generated by classical fields $\omega(x)$ with values in the Lie algebra of $\mathrm{SU}(N)$. The associated gauge variation includes the background field

$$\delta_\omega B_\mu = D_\mu \omega \tag{3.7}$$

and acts on the quantum fields according to

$$\delta_\omega q_\mu = [q_\mu, \omega], \tag{3.8}$$

$$\delta_\omega c = [c, \omega], \qquad \delta_\omega \bar{c} = [\bar{c}, \omega], \tag{3.9}$$

$$\delta_\omega \psi = -\omega\psi, \qquad \delta_\omega \overline{\psi} = \overline{\psi}\omega. \tag{3.10}$$

Gauge and BRS transformations both preserve the total action (2.5) and their commutator $[\delta_{\rm BRS}, \delta_\omega]$ vanishes.

3.3 Descent equations

The Stora–Zumino chain of equations [15–17], which descends from the second Chern character, make the special algebraic properties of the latter explicit (for an introduction to the subject see ref. [19], for example). It is now helpful to introduce the differential forms

$$A = A_\mu {\rm d}x_\mu, \qquad F = \tfrac{1}{2} F_{\mu\nu} {\rm d}x_\mu {\rm d}x_\nu = {\rm d}A + A^2, \tag{3.11}$$

and similarly the forms B and q. Starting from the 4-form

$${\rm tr}\{F^2\} = -\frac{1}{4!} (FF)_{\mu\nu\rho\sigma} {\rm d}x_\mu {\rm d}x_\nu {\rm d}x_\rho {\rm d}x_\sigma, \tag{3.12}$$

a sequence $\phi_3, \phi_2, \phi_1, \phi_0$ of differential forms of decreasing rank may then be constructed satisfying the descent equations

$${\rm tr}\{F^2\} = {\rm d}\phi_3, \tag{3.13}$$

$$\delta_{\rm BRS} \phi_k = {\rm d}\phi_{k-1}, \qquad k = 3, 2, 1, \tag{3.14}$$

$$\delta_{\rm BRS} \phi_0 = 0. \tag{3.15}$$

The particular solution of these equations chosen here,

$$\phi_3 = {\rm tr}\bigl\{A {\rm d}A + \tfrac{2}{3} A^3\bigr\} - g_0 {\rm d}({\rm tr}\{qB\}), \tag{3.16}$$

$$\phi_2 = g_0 \,{\rm tr}\{c {\rm d}A\} - g_0 \delta_{\rm BRS}({\rm tr}\{qB\}) + g_0 {\rm d}({\rm tr}\{cB\}), \tag{3.17}$$

$$\phi_1 = g_0^2 \,{\rm tr}\{c^2 A\} + g_0 \delta_{\rm BRS}({\rm tr}\{cB\}), \tag{3.18}$$

$$\phi_0 = g_0^3 \,\tfrac{1}{3} {\rm tr}\{c^3\}, \tag{3.19}$$

includes several terms proportional to the background field B, which serve to ensure a simple transformation behaviour,

$$\delta_{\omega}\phi_3 = \mathrm{tr}\{\mathrm{d}\omega\,\mathrm{d}B\}, \tag{3.20}$$

$$\delta_{\omega}\phi_k = 0, \quad k = 0, 1, 2, \tag{3.21}$$

under background gauge transformations.

4. Definition and symmetries of the bare vertex functional

The renormalization of the theory with insertions of composite fields will be studied by adding sources for all relevant fields and by discussing the possible structure of the divergent parts of the associated vertex functional. Since the quark fields give rise to only minor complications in this analysis, the pure gauge theory will now first be considered, the modifications required in full QCD being discussed in subsect. 5.6.

4.1 Source terms for the basic fields

Following standard practice, the basic source terms included in the QCD functional integral are

$$(J, q) + (\bar{\eta}, c) + (\bar{c}, \eta) + (K, \delta_{\mathrm{BRS}} q) - (L, \delta_{\mathrm{BRS}} c), \tag{4.1}$$

where $J_{\mu}, \bar{\eta}, \eta, K_{\mu}$ and L are classical source fields with values in the Lie algebra of SU(N). Scalar products of such coloured fields like

$$(J, q) = \int \mathrm{d}^D x \, J^a_{\mu}(x) q^a_{\mu}(x) \tag{4.2}$$

are defined in the obvious way and it is understood that $\bar{\eta}, \eta$ and K_{μ} are fermion fields that anticommute with the ghost fields.

The source terms (4.1) are such that the application of the BRS variation to the sum of terms is equivalent to a change of the source fields. This property is shared by the further source terms introduced below and eventually ensures that the BRS symmetry turns into a symmetry of the vertex functional.

Table 1. Properties of the source fields

Field	Dimension	Ghost no	Field	Dimension	Ghost no
J_μ^a	3	0	E_μ	2	−1
$\bar{\eta}^a$	3	−1	F_μ	1	−2
η^a	3	1	H_0	1	−3
K_μ^a	2	−1	$(H_1)_\mu$	1	−2
L^a	2	−2	$(H_2)_{\mu\nu}$	1	−1
			$(H_3)_{\mu\nu\rho}$	1	0

4.2 Sources for the descendants of the charge density

Appropriate source fields for the totally antisymmetric coefficients $(\phi_k)_{\mu_1\ldots\mu_k}$ of the differential forms (3.16)–(3.19) are classical tensor fields $(H_k)_{\mu_1\ldots\mu_k}$ of the same type. The corresponding source terms,

$$(H_k,\phi_k)=\int \mathrm{d}^D x\,(H_k)_{\mu_1\ldots\mu_k}(x)(\phi_k)_{\mu_1\ldots\mu_k}(x), \tag{4.3}$$

transform under the BRS variation according to†

$$\delta_{\mathrm{BRS}}(H_0,\phi_0)=0, \tag{4.4}$$

$$\delta_{\mathrm{BRS}}(H_k,\phi_k)=(-1)^{k-3}(\mathrm{d}^*H_k,\phi_{k-1}),\quad k=1,2,3, \tag{4.5}$$

where

$$(\mathrm{d}^*H_k)_{\mu_1\ldots\mu_{k-1}}(x)=-\partial_\mu(H_k)_{\mu\mu_1\ldots\mu_{k-1}}(x). \tag{4.6}$$

As will become clear in sect. 5, source terms for two further fields,

$$s_\mu=\mathrm{tr}\{cq_\mu\}\ \text{ and }\ \delta_{\mathrm{BRS}}s_\mu, \tag{4.7}$$

must be included together with the terms (4.3) to be able to renormalize the correlation functions with insertions of $(\phi_1)_\mu$ and $(\phi_2)_{\mu\nu}$.

The fields $s,\delta_{\mathrm{BRS}}s$ and ϕ_k are such that the short-distance singularities generated by their insertion in correlation functions of the basic fields $q,\ldots,\delta_{\mathrm{BRS}}c$ are integrable

† Algebraic consistency requires that source fields with odd fermion number anticommute with the BRS variation of the quantum fields.

at $D = 4$. Additional poles in $1/\epsilon$ are therefore excluded when off-shell correlation functions are considered and there is no difference in these cases between off- and on-shell renormalization.

4.3 Definition of the vertex functional

The complete list of source terms included in the functional integral is thus

$$
\begin{aligned}
&(J, q) + (\bar{\eta}, c) + (\bar{c}, \eta) + (K, \delta_{\mathrm{BRS}} q) - (L, \delta_{\mathrm{BRS}} c) \\
&\qquad + (E, s) - (F, \delta_{\mathrm{BRS}} s) + \sum_{k=0}^{3} (H_k, \phi_k),
\end{aligned}
\tag{4.8}
$$

some relevant properties of the source fields being listed in table 1.

From the partition function $Z[B, J, \ldots, H_3]$ of the theory in presence of the background gauge field and the source terms, the generating functional for the connected correlation functions of the fields $q, \ldots, \phi_3$,

$$
W[B, J, \ldots, H_3] = \ln(Z[B, J, \ldots, H_3]),
\tag{4.9}
$$

is obtained as usual. The Legendre transform

$$
\Gamma[B, Q, \ldots, H_3] = W[B, J, \ldots, H_3] - (J, Q) - (\bar{\eta}, C) - (\overline{C}, \eta),
\tag{4.10}
$$

$$
Q^a_\mu(x) = \frac{\delta W}{\delta J^a_\mu(x)},
\tag{4.11}
$$

$$
C^a(x) = \frac{\delta W}{\delta \bar{\eta}^a(x)}, \qquad \overline{C}^a(x) = -\frac{\delta W}{\delta \eta^a(x)},
\tag{4.12}
$$

in the source fields for q, c and $\bar{c}$ then leads to the vertex functional $\Gamma[B, \ldots, H_3]$ of the theory, all other source fields $E, \ldots, H_3$ and the background field being spectators in this transformation.

4.4 BRS symmetry

A little algebra now shows that the BRS symmetry implies [10]

$$
\begin{aligned}
\int \mathrm{d}^D x \, \bigg\{ &\frac{\delta \Gamma}{\delta Q^a_\mu} \frac{\delta \Gamma}{\delta K^a_\mu} - \frac{\delta \Gamma}{\delta C^a} \frac{\delta \Gamma}{\delta L^a} + \lambda_0 (D_\mu Q_\mu)^a \frac{\delta \Gamma}{\delta \overline{C}^a} - E_\mu \frac{\delta \Gamma}{\delta F_\mu} \\
&- (\mathrm{d}^* H_1) \frac{\delta \Gamma}{\delta H_0} + (\mathrm{d}^* H_2)_\mu \frac{\delta \Gamma}{\delta (H_1)_\mu} - (\mathrm{d}^* H_3)_{\mu\nu} \frac{\delta \Gamma}{\delta (H_2)_{\mu\nu}} \bigg\} = 0.
\end{aligned}
\tag{4.13}
$$

The first three terms in this equation are the usual ones deriving from the BRS variation of the basic fields, while all further terms reflect the transformation behaviour of the added fields $s, \ldots, \phi_3$.

4.5 Background gauge transformations

If the coloured source fields are transformed according to

$$\delta_\omega Q_\mu = [Q_\mu, \omega], \tag{4.14}$$

$$\delta_\omega C = [C, \omega], \qquad \delta_\omega \overline{C} = [\overline{C}, \omega], \tag{4.15}$$

$$\delta_\omega K_\mu = [K_\mu, \omega], \tag{4.16}$$

$$\delta_\omega L = [L, \omega], \tag{4.17}$$

the vertex functional is invariant under background gauge transformations up to an inhomogeneous term,

$$\delta_\omega \Gamma[B, \ldots, H_3] = \int \mathrm{d}^D x\, (H_3)_{\mu\nu\rho} \mathrm{tr}\{\partial_\mu \omega \partial_\nu B_\rho\}, \tag{4.18}$$

that derives from the non-invariance of ϕ_3 [cf. eq. (3.20)]. In particular, the vertex functional is gauge invariant beyond the tree level of perturbation theory.

4.6 Shift symmetry

In the background gauge, the QCD action (2.1) is a function of the gauge potential A_μ, while the gauge-fixing and the ghost action depend on both the background and the quantum field. Under an infinitesimal shift

$$\delta_{\mathrm{s}} B_\mu(x) = g_0 \upsilon_\mu(x), \qquad \delta_{\mathrm{s}} q_\mu(x) = -\upsilon_\mu(x), \tag{4.19}$$

of these fields by an arbitrary classical field υ_μ, the total action transforms like

$$\delta_{\mathrm{s}} S_{\mathrm{tot}} = \delta_{\mathrm{BRS}} \left\{ \int \mathrm{d}^D x\, \upsilon_\mu^a [(D_\mu + g_0 \mathrm{Ad}\, q_\mu)\bar{c}]^a \right\}. \tag{4.20}$$

An identity used later, which derives from this property, is

$$\frac{\delta}{\delta B_\mu^a} W[B, 0, \ldots, 0, \mathrm{d}^* H_4] = 0, \tag{4.21}$$

where $(H_4)_{\mu\nu\rho\sigma}$ is any totally antisymmetric tensor source field of rank 4.

5. Renormalization

In the following, the focus will be on the vertex functions with either no or a single insertion of the fields $s, \dots, \phi_3$. The corresponding parts of the vertex functional are denoted by $\Gamma^{(0)}$ and $\Gamma^{(1)}$. Clearly,

$$\Gamma^{(0)}[B, \dots, L] = \Gamma[B, \dots, L, 0, \dots, 0], \tag{5.1}$$

while $\Gamma^{(1)}[B, \dots, H_3]$ coincides with the part of the vertex functional that depends linearly on the source fields $E, \dots, H_3$. The discussion in this section largely follows the one in ref. [18], where the renormalizability of $\Gamma^{(0)}[B, \dots, L]$ was proved. Here the goal is to extend this result to $\Gamma^{(1)}[B, \dots, H_3]$.

5.1 Renormalized vertex functional

The vertex functional is tentatively renormalized by scaling the source fields,

$$\begin{aligned}\Gamma_R[B, Q, \overline{C}, C, K, L, E, \dots, H_3] \\ &= \Gamma[B, Z_3^{1/2}Q, \widetilde{Z}_3^{1/2}\overline{C}, \widetilde{Z}_3^{1/2}C, \widetilde{Z}_3^{1/2}K, Z_3^{1/2}L, \\ &\qquad Z_E(E + X_E \mathrm{d}^*H_2), Z_F(F - X_E H_1), Z_{H_0}H_0, \dots, Z_{H_3}H_3],\end{aligned} \tag{5.2}$$

and by expressing the bare coupling and gauge parameter through the renormalized coupling g and gauge parameter λ according to

$$g_0 = \mu^\epsilon Z_1 Z_3^{-3/2} g, \qquad \mu\text{: normalization mass}, \tag{5.3}$$

$$\lambda_0 = Z_3^{-1}\lambda. \tag{5.4}$$

In these equations, Z_1, Z_3 and $\widetilde{Z}_3$ are the renormalization constants already required for the renormalization of the theory without insertions of the fields $s, \dots, \phi_3$. Some of the other renormalization constants are not independent and satisfy

$$Z_F = Z_E(Z_3\widetilde{Z}_3)^{1/2}, \tag{5.5}$$

$$Z_{H_k} = Z_H(Z_3\widetilde{Z}_3)^{(3-k)/2}, \quad k = 0, \dots, 3. \tag{5.6}$$

The additive renormalizations proportional to X_E are included in eq. (5.2), because the field ϕ_1 mixes with $\delta_{\mathrm{BRS}}s$ and ϕ_2 with $\mathrm{d}s$. A non-zero mixing actually already occurs at one-loop order of perturbation theory. In the following, minimal subtraction (i.e. the MS scheme) is assumed for all renormalization constants.

Equations (5.2)–(5.6) are such that the renormalization preserves the form of the BRS identity (4.13), viz.

$$\int \mathrm{d}^D x \left\{ \frac{\delta\Gamma_R}{\delta Q^a_\mu}\frac{\delta\Gamma_R}{\delta K^a_\mu} - \frac{\delta\Gamma_R}{\delta C^a}\frac{\delta\Gamma_R}{\delta L^a} + \lambda(D_\mu Q_\mu)^a \frac{\delta\Gamma_R}{\delta \overline{C}^a} - E_\mu \frac{\delta\Gamma_R}{\delta F_\mu} \right.$$
$$\left. - (\mathrm{d}^* H_1)\frac{\delta\Gamma_R}{\delta H_0} + (\mathrm{d}^* H_2)_\mu \frac{\delta\Gamma_R}{\delta(H_1)_\mu} - (\mathrm{d}^* H_3)_{\mu\nu}\frac{\delta\Gamma_R}{\delta(H_2)_{\mu\nu}} \right\} = 0. \qquad (5.7)$$

Moreover, the shift-symmetry identity (4.21) continues to hold when $W[B,\ldots,H_3]$ is replaced by the generating functional $W_R[B,\ldots,H_3]$ of the renormalized connected correlation functions. In the case of the background gauge symmetry, the transformation law for the renormalized vertex functional,

$$\delta_\omega \Gamma_R[B,\ldots,H_3] = Z_H \int \mathrm{d}^D x\, (H_3)_{\mu\nu\rho} \mathrm{tr}\{\partial_\mu\omega\partial_\nu B_\rho\}, \qquad (5.8)$$

however involves the renormalization constant Z_H, which already shows that Z_H cannot diverge in the limit $D \to 4$ if $\Gamma^{(1)}_R[B,\ldots,H_3]$ is finite.

5.2 Loop expansion of the renormalized vertex functional

The renormalized vertex functional may be expanded in a series

$$\Gamma_R = \sum_{l=0}^{\infty} \Gamma_{R,l} \qquad (5.9)$$

of terms of increasing loop order l, the lowest-order term being

$$\Gamma_{R,0} = -\hat{S}_{\mathrm{tot}}$$
$$+ (K, \hat{\delta}_{\mathrm{BRS}} Q) - (L, \hat{\delta}_{\mathrm{BRS}} C) + (E, \hat{s}) - (F, \hat{\delta}_{\mathrm{BRS}} \hat{s}) + \sum_{k=0}^{3} (H_k, \hat{\phi}_k). \qquad (5.10)$$

All hatted fields in this formula and the hatted total action are obtained from the corresponding expressions in the quantum fields by substituting $q_\mu \to Q_\mu$, $c \to C$, $\bar{c} \to \overline{C}$, $g_0 \to \mu^\epsilon g$ and $\lambda_0 \to \lambda$. The BRS variation

$$\hat{\delta}_{\mathrm{BRS}} Q_\mu = (D_\mu + \mu^\epsilon g \mathrm{Ad}\, Q_\mu) C, \qquad (5.11)$$

$$\hat{\delta}_{\text{BRS}} C = -\mu^{\epsilon} g C^2, \tag{5.12}$$

acting on the source fields must be distinguished from the one acting on the quantum fields, but has identical algebraic properties.

5.3 Proof of finiteness: first steps

The proof of finiteness of $\Gamma^{(1)}_R[B, \ldots, H_3]$ and thus of Z_H proceeds by induction over the loop order l. At a given order n, the induction hypothesis is that the divergences of $\Gamma^{(1)}_{R,l}$ can be canceled at all orders $l < n$ by setting $Z_H = 1$ and by adjusting the l-loop coefficients of Z_E and X_E. The task is then to show that the same is the case at loop order n.

First this requires the structure of the divergent part $\Delta\Gamma^{(1)}_{R,n}$ of $\Gamma^{(1)}_{R,n}$ to be determined for vanishing n-loop terms $Z_{E,n}, X_{E,n}, Z_{H,n}$ of Z_E, X_E and Z_H, their contribution to the vertex functional at this order,

$$\begin{aligned} &Z_{E,n}\{(E, \hat{s}) - (F, \hat{\delta}_{\text{BRS}} \hat{s})\} \\ &\qquad + X_{E,n}\{(\mathrm{d}^* H_2, \hat{s}) + (H_1, \hat{\delta}_{\text{BRS}} \hat{s})\} + Z_{H,n} \sum_{k=0}^{3} (H_k, \hat{\phi}_k), \end{aligned} \tag{5.13}$$

being taken into account in subsect. 5.5.

General principles imply that

$$\Delta\Gamma^{(1)}_{R,n} = \int \mathrm{d}^D x \, p(x), \tag{5.14}$$

where $p(x)$ is a local polynomial in the source fields $B, Q, \ldots, H_3$ and their derivatives, which must have dimension 4, ghost number 0 and be linear in $E, \ldots, H_3$. Partial integration moreover allows any terms with derivatives of these latter fields to be traded for terms in which they appear without derivatives. The field $p(x)$ then inherits the invariance of $\Delta\Gamma^{(1)}_{R,n}$ under Lorentz and background gauge transformations (since $Z_{H,n}$ is, at this point, set to zero).

All these properties already strongly constrain the form of $p(x)$. Recalling table 1, inspection shows that the field cannot depend on the fields $\overline{C}$, K or L. Moreover, the terms in $p(x)$ depending on the fields E and H_0 must be proportional to $E_\mu(x)\hat{s}_\mu(x)$ and $H_0(x)\hat{\phi}_0(x)$.

5.4 Consequences of the BRS symmetry

Further constraints on $\Delta\Gamma^{(1)}_{R,n}$ derive from the BRS identity (5.7), which holds at all loop orders and all orders in the source fields $E,\ldots,H_3$. At loop order n, and for the terms linear in these fields, the identity together with the induction hypothesis and the leading-order form (5.10) of the vertex functional implies

$$\int \mathrm{d}^D x \left\{ \hat{\delta}_{\mathrm{BRS}} Q^a_\mu \frac{\delta\Delta\Gamma^{(1)}_{R,n}}{\delta Q^a_\mu} + \hat{\delta}_{\mathrm{BRS}} C^a \frac{\delta\Delta\Gamma^{(1)}_{R,n}}{\delta C^a} - E_\mu \frac{\delta\Delta\Gamma^{(1)}_{R,n}}{\delta F_\mu} - (\mathrm{d}^* H_1) \frac{\delta\Delta\Gamma^{(1)}_{R,n}}{\delta H_0} \right.$$
$$\left. + (\mathrm{d}^* H_2)_\mu \frac{\delta\Delta\Gamma^{(1)}_{R,n}}{\delta (H_1)_\mu} - (\mathrm{d}^* H_3)_{\mu\nu} \frac{\delta\Delta\Gamma^{(1)}_{R,n}}{\delta (H_2)_{\mu\nu}} \right\} = 0. \tag{5.15}$$

If only the first two terms were present, the left-hand side of this equation would coincide with $\hat{\delta}_{\mathrm{BRS}}\Delta\Gamma^{(1)}_{R,n}$. The equation thus relates the BRS variation of the terms proportional to E, H_1, H_2 and H_3 to the terms proportional to F, H_0, H_1 and H_2. As a consequence,

$$\Delta\Gamma^{(1)}_{R,n} = z_E \big\{ (E,\hat{s}) - (F, \hat{\delta}_{\mathrm{BRS}}\hat{s}) \big\} + \sum_{k=0}^{3} (H_k, \hat{f}_k), \tag{5.16}$$

where z_E is a (divergent) constant and $\hat{f}_k$, $k = 0,\ldots,3$, some gauge-invariant forms of rank k, with dimension 3 and ghost number $3-k$, satisfying

$$\hat{\delta}_{\mathrm{BRS}} \hat{f}_k = \mathrm{d}\hat{f}_{k-1} \tag{5.17}$$

for all $k = 1,2,3$.

The discussion in appendix A of the descent equations for quantum fields carries over literally to the case of the descent equations (5.17) and shows that these equations have only two linearly independent solutions with the required properties. As a result,

$$\Delta\Gamma^{(1)}_{R,n} = (z_E E + x_E \mathrm{d}^* H_2, \hat{s}) - (z_E F - x_E H_1, \hat{\delta}_{\mathrm{BRS}}\hat{s})$$
$$+ z_H \sum_{k=0}^{3} (H_k, \hat{\phi}_k) - z_H (H_3, \hat{\phi}_3)_{g=0}, \tag{5.18}$$

where x_E and z_H are further (divergent) coefficients.

5.5 Proof of finiteness: final steps

Now when the counterterms (5.13) are included in the vertex functional, all terms on the right of eq. (5.18) except for the last one can be canceled by adjusting the n-loop coefficients of Z_E, X_E and Z_H. Since the uncanceled term only depends on B and H_3, it is a spectator in the Legendre transform that leads from the renormalized vertex functional to the generating functional $W_R[B,\ldots,H_3]$ of the renormalized correlation functions. The latter is therefore finite too at n-loop order apart from this additive divergent term and terms of higher than linear order in the source fields $E,\ldots,H_3$.

Such a divergent term is however excluded by the shift-symmetry relation (4.21) (with $W \to W_R$) and its coefficient z_H must hence be equal to zero. The terms in eq. (5.18) proportional to z_H are thus absent and all divergences at n-loop order can be canceled by setting $Z_H = 1$ and adjusting Z_E and X_E, as was to be shown.

5.6 Inclusion of the quark fields

In presence of the quark fields, ϕ_3 requires an additive renormalization proportional to the flavour-singlet axial current, which is here represented by the tensor field

$$A^s_{\mu\nu\rho}(x) = \sum_{r=1}^{N_{\rm f}} \overline{\psi}_r(x)\gamma_{[\mu}\gamma_\nu\gamma_{\rho]}\psi_r(x). \tag{5.19}$$

After adding source terms for the quark and antiquark fields, their BRS variation and the axial current (5.19), the finiteness of $\Gamma_R^{(1)}$ can then again be proved following the steps taken in the case of the pure gauge theory.

Since A^s is invariant under both the BRS and the background gauge symmetry, there is now a third solution, $\hat{f}_k = \delta_{k3}\hat{A}^s$, of the descent equations (5.17) with all the required properties. The mixing of ϕ_3 with A^s derives from the existence of this additional solution, but a multiplicative renormalization of ϕ_3 remains excluded.

There is, on the other hand, no field that could mix with the axial current. The results obtained in appendix B in fact show that no BRS and gauge invariant 3-form of dimension 3 can be built from the gauge and ghost fields alone. The Lorentz and flavour symmetry then imply that the current must renormalize multiplicatively.

6. Flavour-singlet axial Ward identity

Since the axial anomaly does not require multiplicative renormalization, the relation between the bare and the renormalized fields that appear in the flavour-singlet axial-current conservation equation is slightly simplified. The structure of the equation in the renormalized theory in four dimensions is then easily determined, but there is little new here and the section is included mainly for completeness. All statements made in the following refer to standard QCD with vanishing background field.

6.1 Renormalized fields

The renormalized fields participating in the Ward identity are

$$(F\,{}^*\!F)_{\rm R} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\left\{F^a_{[\mu\nu}F^a_{\rho\sigma]} + \tfrac{1}{3}Z_{FA}\partial_{[\mu}A^s_{\nu\rho\sigma]}\right\}, \tag{6.1}$$

$$(A^s_\mu)_{\rm R} = \tfrac{1}{6}\epsilon_{\mu\nu\rho\sigma}Z_A A^s_{\nu\rho\sigma}, \tag{6.2}$$

$$(mP)_{\rm R} = \tfrac{1}{24}\epsilon_{\mu\nu\rho\sigma}Z_m Z_P \sum_{r=1}^{N_{\rm f}} m_{0,r}P^{rr}_{\mu\nu\rho\sigma}. \tag{6.3}$$

In these equations, the renormalization constants Z_m, Z_A and Z_P are for the quark masses and for the flavour-singlet axial current and density, while $Z_{FA} = {\rm O}(g^4)$ is a mixing coefficient. Minimal subtraction is assumed, as before, and the contraction with the Levi–Civita symbol is performed only after passing to $D = 4$ dimensions.

The anomalous dimensions of $(A^s_\mu)_{\rm R}$ and $(mP)_{\rm R}$ are

$$\gamma_A = (-\epsilon g + \beta)\frac{\partial \ln Z_A}{\partial g}, \tag{6.4}$$

$$\gamma_{mP} = (-\epsilon g + \beta)\frac{\partial \ln(Z_m Z_P)}{\partial g} = \gamma_m + \gamma_P, \tag{6.5}$$

where $\beta = -b_0 g^3 - b_1 g^5 + \ldots$ denotes the β-function at $\epsilon = 0$. Since $(F\,{}^*\!F)_{\rm R}$ mixes with $\partial_\mu(A^s_\mu)_{\rm R}$, the associated anomalous dimension is a 2×2 matrix,

$$\gamma_F = \begin{pmatrix} 0 & \gamma_{FA} \\ 0 & \gamma_A \end{pmatrix}, \qquad \gamma_{FA} = (-\epsilon g + \beta)\frac{\partial Z_{FA}}{\partial g}Z_A^{-1}, \tag{6.6}$$

acting on these fields.

6.2 Renormalization-group-invariant (RGI) fields

RGI fields are related to the renormalized ones through finite renormalization factors chosen such that the anomalous dimensions vanish. In the case of a multiplet $(\mathcal{O}_k)_{\rm R}$, $k = 1, \ldots, n$, of fields with anomalous-dimension matrix γ, the RGI fields are given by

$$(\mathcal{O}_k)_{\rm RGI} = \sum_{l=1}^{n} \mathcal{R}_{kl}(\mathcal{O}_l)_{\rm R}, \tag{6.7}$$

where $\mathcal{R}$ is an $n \times n$ matrix satisfying

$$\beta \frac{\partial \mathcal{R}}{\partial g} + \mathcal{R}\gamma = 0 \tag{6.8}$$

plus some conventional boundary condition at $g = 0$. Apart from having vanishing anomalous dimension, RGI fields are independent of the renormalization scheme and any relations among them are therefore universally valid.

In the case of the fields considered here, the boundary condition $\lim_{g\to 0} \mathcal{R} = 1$ can be imposed and the RGI fields are then given by

$$(F{}^*F)_{\rm RGI} = (F{}^*F)_{\rm R} + \mathcal{X}_{FA}\partial_\mu(A^s_\mu)_{\rm R}, \tag{6.9}$$

$$(A^s_\mu)_{\rm RGI} = \mathcal{X}_A(A^s_\mu)_{\rm R}, \tag{6.10}$$

$$(mP)_{\rm RGI} = \mathcal{X}_{mP}(mP)_{\rm R}, \tag{6.11}$$

where

$$\mathcal{X}_O = \exp\biggl\{-\int_0^g {\rm d}h\,\frac{\gamma_O(h)}{\beta(h)}\biggr\}, \qquad O = A, mP, \tag{6.12}$$

$$\mathcal{X}_{FA} = -\mathcal{X}_A \int_0^g {\rm d}h\,\frac{\gamma_{FA}(h)}{\beta(h)\mathcal{X}_A(h)} \tag{6.13}$$

(the integrals are all absolutely convergent, since the anomalous dimensions γ_A, γ_{mP} and γ_{FA} are of order g^4). The factor $\mathcal{X}_{mP}$ can, incidentally, also be determined by matching the normalizations of the axial and scalar quark densities as in ref. [2], for example.

6.3 Ward identity

In terms of the RGI fields, and for any product $\mathcal{O}$ of fields at non-zero distances from x, the flavour-singlet Ward identity assumes the form

$$\left\langle\{\partial_\mu(A^s_\mu)_{\rm RGI}(x)+k_1(mP)_{\rm RGI}(x)+k_2(F\,{}^*\!F)_{\rm RGI}(x)\}\mathcal{O}\right\rangle=0. \tag{6.14}$$

Since the renormalization group excludes a dependence of the coefficients k_1 and k_2 on the gauge coupling, their values

$$k_1=-2, \qquad k_2=\frac{N_{\rm f}}{16\pi^2}, \tag{6.15}$$

coincide with the ones obtained at 1-loop order of perturbation theory.

If the Ward identity is written in terms of the minimally subtracted field $(F\,{}^*\!F)_{\rm R}$ instead of $(F\,{}^*\!F)_{\rm RGI}$, as in refs. [2–5], the equation becomes

$$\left\langle\{\partial_\mu(A^s_\mu)_{\rm R'}(x)+k_1(mP)_{\rm RGI}(x)+k_2(F\,{}^*\!F)_{\rm R}(x)\}\mathcal{O}\right\rangle=0, \tag{6.16}$$

$$(A^s_\mu)_{\rm R'}=(\mathcal{X}_A+k_2\mathcal{X}_{FA})(A^s_\mu)_{\rm R}. \tag{6.17}$$

In this renormalization scheme, the anomalous dimensions satisfy

$$\gamma'_A=-k_2\gamma'_{FA}, \tag{6.18}$$

as already noted in ref. [5], and the finite renormalization factor $\mathcal{X}_A+k_2\mathcal{X}_{FA}$ can be computed by requiring (6.16) to hold in the massless theory [2].

7. Concluding remarks

The fact that the topological charge density does not require multiplicative renormalization derives from its algebraic properties, namely that it coincides with the exterior differential of a gauge-variant local 3-form, the Chern–Simons form. Eventually the normalization of the density is fixed by the inhomogeneous gauge transformation behaviour of the latter.

A straightforward argumentation along this line is however not possible in perturbation theory in view of the required gauge fixing. Use had instead to be made of the background gauge and the BRS symmetry, whose application to the Chern–Simons form generates a chain of forms with increasing ghost number. All these forms must be included in the renormalization process and a multiplicative renormalization of the Chern–Simons form (and thus of the charge density) is then seen to be excluded by the symmetries of the QCD vertex functional. The other forms however require multiplicative renormalization and some additive renormalization as well.

Specific renormalization properties like the one discussed in this paper can depend on the chosen regularization of the theory. Simple expressions for the topological charge density in lattice QCD, for example, need not be exactly representable through a discrete version of the Chern–Simons form and may consequently require multiplicative renormalization. After renormalization and removal of the regularization, the RGI form (6.14) of the flavour-singlet chiral Ward identity however holds in these cases too.

Appendix A. Notation

A.1 Gauge group

The Lie algebra of the gauge group $\mathrm{SU}(N)$ may be identified with the space of all complex antihermitian $N\times N$ matrices with vanishing trace. If T^a, $a=1,\ldots,N^2-1$, is a basis of such matrices satisfying

$$\mathrm{tr}\{T^aT^b\} = -\tfrac{1}{2}\delta^{ab}, \tag{A.1}$$

the general element X of the Lie algebra is given by $X = X^aT^a$ with real components $X^a = -2\,\mathrm{tr}\{XT^a\}$ (repeated indices are automatically summed over).

While the quark fields are assumed to be in the fundamental representation of the gauge group, the gauge and ghost fields take values in its Lie algebra. The adjoint action of the latter on itself is defined by

$$\mathrm{Ad}\,X\cdot Y = [X,Y] = f^{abc}X^aY^bT^c, \tag{A.2}$$

where f^{abc} are the $\mathrm{SU}(N)$ structure constants in the chosen basis of group generators.

A.2 Dimensional regularization

The theory is defined in the standard manner in $D = 4-2\epsilon$ Euclidean dimensions. Lorentz indices run from 0 to 3 in $D=4$ dimensions and formally to $D-1$ in arbitrary dimensions, i.e. the trace of the Kronecker delta $\delta_{\mu\nu}$ is equal to D.

The Dirac matrices γ_μ in D dimensions are formal objects satisfying

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \tag{A.3}$$

By taking products and linear combinations, the Dirac matrices generate an infinite dimensional linear space. The trace $\mathrm{tr}\{\cdot\}$ is a mapping from this space to the space of polynomials in Kronecker deltas, which is implicitly defined by its linearity and cyclicity, the normalization convention

$$\mathrm{tr}\{1\} = 4, \tag{A.4}$$

the Dirac algebra (A.3) and the rule that products of odd numbers of Dirac matrices have vanishing trace.

In $D = 4$ dimensions, the Dirac matrices are assumed to be Hermitian and the fifth Dirac matrix is taken to be

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3, \tag{A.5}$$

but no attempt is made to assign a meaning to γ_5 in arbitrary dimensions. The same applies to the Levi–Civita symbol $\epsilon_{\mu\nu\rho\sigma}$, which is normalized such that $\epsilon_{0123} = 1$.

A.3 Differential forms

Differential forms $f(x)$ of rank n are homogeneous polynomials

$$f(x) = f(x)_{\mu_1\ldots\mu_n} \mathrm{d}x_{\mu_1} \ldots \mathrm{d}x_{\mu_n} \tag{A.6}$$

in the Grassmann algebra generated by the anticommuting symbols $\mathrm{d}x_\mu$. The coefficients $f(x)_{\mu_1\ldots\mu_n}$ may be real, complex or take values in the Lie algebra of $\mathrm{SU}(N)$, for example.

The exterior differential d acts on such forms according to

$$\mathrm{d}f(x) = \mathrm{d}x_\mu \partial_\mu f(x). \tag{A.7}$$

Clearly, $\mathrm{d}^2 = 0$ and

$$\mathrm{d}(f(x)g(x)) = \mathrm{d}f(x)g(x) + (-1)^n f(x)\mathrm{d}g(x) \tag{A.8}$$

if $f(x)$ has rank n.

Appendix B. Solution of the descent equations

The goal in this appendix is to find the general solution of the descent equations

$$\delta_{\text{BRS}} f_k = \mathrm{d} f_{k-1}, \quad k = 1, 2, 3, \tag{B.1}$$

in the space of local gauge-invariant forms f_k of rank $k = 0, \ldots, 3$, with ghost number $3 - k$ and dimension 3, which can be composed from the fields B^a_μ, q^a_μ, c^a and their derivatives.

B.1 Gauge-covariant exterior differential

Let f be any differential form of rank n with values in the space of $N \times N$ matrices of elements of a complex Grassmann algebra. The gauge-covariant exterior differential d^B acts on f according to

$$\mathrm{d}^B f = \mathrm{d} f + B f + (-1)^{n+1} f B. \tag{B.2}$$

If f is a gauge-covariant expression in the basic fields, i.e. if

$$\delta_\omega f = [f, \omega], \tag{B.3}$$

its differential $\mathrm{d}^B f$ is gauge-covariant too. Moreover,

$$\mathrm{d}^B(\mathrm{d}^B f) = [G, f], \qquad G = \mathrm{d}B + B^2, \tag{B.4}$$

$$\mathrm{d}^B G = 0, \tag{B.5}$$

$$\mathrm{d}^B(fg) = \mathrm{d}^B f g + (-1)^n f \mathrm{d}^B g, \tag{B.6}$$

$$\mathrm{d}\,(\mathrm{tr}\{f\}) = \mathrm{tr}\{\mathrm{d}^B f\}. \tag{B.7}$$

Using these rules, the differential of the trace of any gauge-covariant polynomial in the basic fields can be worked out and yields expressions of the same type.

B.2 Gauge-invariant forms

The dimension, ghost number and gauge invariance of the differential forms f_k implies that they are linear combinations of terms of the form $\mathrm{tr}\{\mathcal{O}\}$, where $\mathcal{O}$ is a product of three of the fields B_μ, q_μ, c or of one of these fields and another one with a derivative acting on it. Moreover, none of the Lorentz indices of the fields may be contracted, i.e. $\mathcal{O}$ must be a product of $B, q, c, \mathrm{d}B, \mathrm{d}q$ and $\mathrm{d}c$. The gauge-invariant terms with ghost number $0, \ldots, 3$,

$$h_1 = \mathrm{tr}\{q^3\}, \qquad h_2 = \mathrm{tr}\{q\,\mathrm{d}^B q\}, \qquad h_3 = \mathrm{tr}\{qG\}, \tag{B.8}$$

$$h_4 = \mathrm{tr}\{cq^2\}, \qquad h_5 = \mathrm{tr}\{c\,\mathrm{d}^B q\}, \qquad h_6 = \mathrm{tr}\{\mathrm{d}^B c q\}, \qquad h_7 = \mathrm{tr}\{cG\}, \tag{B.9}$$

$$h_8 = \mathrm{tr}\{c^2 q\}, \qquad h_9 = \mathrm{tr}\{c\,\mathrm{d}^B c\}, \tag{B.10}$$

$$h_{10} = \mathrm{tr}\{c^3\}, \tag{B.11}$$

are then easily found by applying a gauge variation to the general linear combination of all possible terms.

B.3 General solution of the descent equations

Recalling the discussion in subsect. 3.3, the sequence of forms

$$f_k = \phi_k - \delta_{k3}\ \phi_3|_{g_0=0}\,, \quad k = 0, \ldots, 3, \tag{B.12}$$

is easily shown to satisfy the descent equations (B.1) and all other requirements too. An obvious second solution is

$$f_k = \delta_{k1}\delta_{\mathrm{BRS}} s + \delta_{k2}\mathrm{d}s, \qquad s = \mathrm{tr}\{cq\}, \tag{B.13}$$

but there are no further linearly independent solutions.

The proof of this statement begins by noting that the forms f_k must be linear combinations

$$f_3 = \sum_{k=1}^{3} c_k h_k, \quad f_2 = \sum_{k=4}^{7} c_k h_k, \quad \ldots \tag{B.14}$$

of the forms $h_1, \ldots, h_{10}$ with some coefficients $c_1, \ldots, c_{10}$. Since

$$\phi_3 - \ \phi_3|_{g_0=0} = \tfrac{2}{3} g_0^3 h_1 + g_0^2 h_2 + 2 g_0 h_3, \tag{B.15}$$

$$\mathrm{d}s = h_5 + h_6, \tag{B.16}$$

the coefficients c_3 and c_5 can be nullified by subtracting a linear combination of the solutions (B.12),(B.13) and it remains to be shown that the descent equations imply the vanishing of all coefficients $c_1, \ldots, c_{10}$ if $c_3 = c_5 = 0$.

Noting

$$\delta_{\rm BRS} q = {\rm d}^B c + g_0[q,c], \tag{B.17}$$

$$\delta_{\rm BRS} {\rm d}^B q = [G,c] + g_0[{\rm d}^B q, c] - g_0\{{\rm d}^B c, q\}, \tag{B.18}$$

some algebra yields

$$\delta_{\rm BRS} h_1 = 3\,{\rm tr}\{{\rm d}^B c q^2\}, \tag{B.19}$$

$$\delta_{\rm BRS} h_2 = {\rm tr}\{{\rm d}^B c {\rm d}^B q - 2 g_0 {\rm d}^B c q^2 - [q,c]G\}, \tag{B.20}$$

for the forms with ghost number 0, while

$$ {\rm d}h_4 = {\rm tr}\{{\rm d}^B c q^2 + [q,c]{\rm d}^B q\}, \tag{B.21}$$

$$ {\rm d}h_6 = -{\rm tr}\{{\rm d}^B c {\rm d}^B q + [q,c]G\}, \tag{B.22}$$

$$ {\rm d}h_7 = {\rm tr}\{{\rm d}^B c G\}. \tag{B.23}$$

The matching of the independent terms on the two sides of the equation $\delta_{\rm BRS} f_3 = {\rm d}f_2$ then shows that $c_1 = c_2 = c_4 = c_6 = c_7 = 0$ and thus $f_2 = f_3 = 0$. Finally, since the forms

$$ {\rm d}h_8 = {\rm tr}\{c^2 {\rm d}^B q - [q,c]{\rm d}^B c\}, \tag{B.24}$$

$$ {\rm d}h_9 = {\rm tr}\{{\rm d}^B c {\rm d}^B c - 2c^2 G\}, \tag{B.25}$$

$$ {\rm d}h_{10} = 3\,{\rm tr}\{c^2 {\rm d}^B c\}, \tag{B.26}$$

are linearly independent, the remaining coefficients c_8, c_9, c_{10} must vanish too.

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