A Compactification of Moduli Spaces of Monopoles

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1. Introduction

In 1987, A. Floer [8,9] suggested that Yang-Mills-Higgs theory should be studied on asymptotically Euclidean 3-manifolds M and it has been proved a rich subject in recent years. The focus of interest is the geometry of the moduli space of monopoles on M and its applications. In Donaldson [3], the moduli space of SU_2 monopoles on the Euclidean space $\mathbf{R}^3$ is identified as a certain space of rational functions on the complex plane. In Ernst [6], the moduli space $\mathcal{M}^1$ of charge 1 monopoles on M is constructed. By investigating the end of $\mathcal{M}^1$, M is proved diffeomorphic to $\mathbf{R}^3$ if $\|Ric^-\|_{\frac{3}{2}} < \frac{1}{\mu^2}$, where μ is the Sobolev constant of the imbedding of $L^6(M)$ in $L^2_1(M)$ and Ric^- is the negative part of the Ricci curvature. In Wang [28,29], the end $\mathcal{M}^k_{m,\infty}$ of the moduli space $\mathcal{M}^k_m$ of monopoles of charge k and mass m on M is constructed and the structure of $\mathcal{M}^k_{m,\infty}$ is clarified (see also AH [1]). $\mathcal{M}^k_m$ is generally a smooth, orientable manifold with a perturbation.

To apply the theory of monopoles to study the topology of M, a main problem is the non-compactness of moduli space $\mathcal{M}^k_m$. Although it is clearly clarified, $\mathcal{M}^k_m$ itself seems not to have an immediate, natural compactification. Let $\mathcal{C}^k_m$ be the space of Yang-Mills-Higgs configurations of charge $k \in \mathbf{N}$ and mass $m \in \mathbf{R}^+$ on M, $\mathcal{G}^* = Map_*(M, SU_2)$ be the base-preserving gauge group and $\mathcal{B}_{k,m} = \mathcal{C}^k_m/\mathcal{G}^*$. Denote by $Map_k(S^2, S^2)$ the space of maps from S^2 to S^2 of degree k. In this paper, we prove the following:

Theorem: When $H_1(M, \mathbf{Z}) = 0$, the configuration space $\mathcal{B}_{k,m}$ is homotopic to $Map_k(S^2, S^2)$. Moreover, there is an SO_3-equivalence homotopy equivalence $\hat{e} : \mathcal{B}_{k,m} \to Map_k(S^2, S^2)$, such that, $\hat{e}(\mathcal{M}^k_m)$ has a natural compactification

$$\overline{\hat{e}(\mathcal{M}_m^k)} = \hat{e}(\mathcal{M}_m^k) \cup \hat{e}(\mathcal{M}_m^{k-1}) \times S^2 \cdots \cup \hat{e}(\mathcal{M}_m^0) \times (S^2)^k, \qquad (1.1)$$

where SO_3 acts on $Map_k(S^2, S^2)$ by rotating the target S^2.

When $H_1(M, \mathbf{Z}) \neq 0$, $\hat{e}$ is an SO_3-equivalent fibration. The reader may compare (1.1) to the Uhlenbeck compactification of the moduli space of instantons (see cf. DK [5] or FM [11]). The idea of proof consists of two parts which are as follows. (i) As in Taubes [22], for small $\epsilon > 0$, we define the set of ϵ-centers for a monopole $[C] \in \mathcal{M}_m^k$. Then there is the following weak convergence theorem: Let $\{[C_i]\}_{i=1}^\infty \subset \mathcal{M}_m^k$ be a sequence of monopoles on M such that the ϵ-centers of $\{[C_i]\}$ form a bounded set on M for some $\epsilon > 0$. Then $\{[C_i]\}$ has a subsequence $\{[C_{i_\nu}]\}$ which is convergent on M. As a consequence, a sequence $\{[C_i]\}_{i=1}^\infty \subset \mathcal{M}_m^k$ which approaches to the infinity of $\mathcal{M}_m^k$ has always the following properties: there is a subsequence $\{[C_{i_\nu}]\}$, a partition $k = \sum_{j=1}^{l} k_j$ and sequences of centers $\{x_\nu^j\}(j = 1, \cdots, l)$, such that, $\{[C_{i_\nu}]\}$ converges to a k_j-monopole on any given ball $B_r(x_\nu^j)$. (ii) The map $\hat{e}$ is compatible with the non-compactness of $\mathcal{M}_m^k$ as follows. If the sequence of centers $\{x_\nu^j\} \to \infty$ and $\hat{x}_\nu^j = x_\nu^j/|x_\nu^j| \to \hat{x}^j \in S^2$, then $\{\hat{e}([C_{i_\nu}])\}$ correspondnces to a standard blow-up phenomenon in $Map_k(S^2, S^2)$: for example,

$$\lim_{\delta \to 0^+} \lim_{\nu \to \infty} \int_{B_\delta(\hat{x}^j)} \det D\hat{e}([C_{i_\nu}]) = 2k_j\pi, \qquad (1.2)$$

where $D\hat{e}([C_{i_\nu}])$ is the Jacobian of $\hat{e}([C_{i_\nu}])$. Note that $\hat{e}$ is a refined map of that of Taubes [20]. The blow-up is not due to a conformal invariance.

The paper is arranged as follows. In Section 2, we give a brief review of the geometry of $\mathcal{B}_{k,m}$ and $\mathcal{M}_m^k$ and the analytical behaviors of monopoles on M. In section 3, we study the geometry of $\mathcal{B}_{k,m}$ carefully and prove that, there is a fibration $\tilde{e} : \mathcal{B}_{k,m} \to Map_k(S^2, S^2)$ which is a homotopy equivalence if $H_1(M, \mathbf{Z}) = 0$. In Section 4 we give a new proof of the weak convergence theorem (which is also proved in [28]) and a classification of the structure of the end $\mathcal{M}_{m,\infty}^k$ of $\mathcal{M}_m^k$. In section 5, we refine the map $\tilde{e}$ to define $\hat{e}$ and investigate the blow-up phenomenon. In Section 6, we combine the weak convergence theorem and the blow-up phenomenon to define the natural compactification $\overline{\hat{e}(\mathcal{M}_m^k)}$. It is also proved that, when k is in the stable range $k \geq 2$, $\overline{\hat{e}(\mathcal{M}_m^k)}$ define at least a $\mathbf{Z}_2$-fundamental class of dimension $(4k-1)$ if $H_1(M, \mathbf{Q}) = 0$. We do not not discuss the orientablity of

$\overline{\hat{e}(\mathcal{M}_m^k)}$ here. As an application, we will study the extension of the cohomology groups of the space $Map_k(S^2, S^2)$ over the compactification $\overline{\hat{e}(\mathcal{M}_m^k)}$ to define Donaldson-type invariants for M.

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2. Preliminary Results

Let M be an asymptotically Euclidean 3-manifold. In this section, we give a brief review of Yang-Mills-Higgs theory on M and collect some basic results on the geometry of the configuration space $\mathcal{B}_m^k$ and the moduli space $\mathcal{M}_m^k$ and the analytical behaviors of monopoles on M. Without lost of generality, we assume that M is a metric connected sum of a closed, oriented Riemannian 3-manifold X and $\mathbf{R}^3$ over the annulus

$$A_s = \{x \in \mathbf{R}^3 : s \leq |x| \leq 2s\} \tag{2.1}$$

and the metric outside the ball B_{2s} is standard.

2.1 The configuration space $\mathcal{B}_{\mathbf{m}}^{\mathbf{k}}$. We begin by reviewing some basic results on the space of Yang-Mills-Higgs configurations. For further details, the reader may refer to [13,20,8]. By definition, an SU_2 Yang-Mills-Higgs configuration $C = (A, \Phi)$ on M is a couple

$$(A, \Phi) \in (\Omega^1 \oplus \Omega^0)(M, su_2)$$

of a connection A and a Higgs field Φ which has finite energy

$$\underline{a}(C) = ||F_A||_2^2 + ||d_A\Phi||_2^2. \tag{2.2}$$

Here F_A is the curvature and $d_A\Phi$ is the covariant derivative.

A Yang-Mills-Higgs configuration $C = (A, \Phi)$ on M has a mass m and a charge k as follows. Since $d_A\Phi \in L^2$, there is a unique constant $m \geq 0$ such that $m - |\Phi| \in L^6(M)$. m is called the mass of C. The charge k is defined as

$$k = \frac{1}{4\pi m}\int_M Tr(F_A \wedge \nabla_A \Phi) \tag{2.3}$$

when $m \in \mathbf{R}^+$. A theorem of Groisser [13] is that k is an integer.

Let $\mathcal{C}_m^k$ denote the space of $L_{1,loc}^2$ Yang-Mills-Higgs configurations of charge k and mass m on M. Endow $\mathcal{C}_m^k$ with the topology of $L_{1,loc}^2$ and that renders continuous the map

$$\mathcal{C}_m^k \mapsto L^2((\Omega^2 \oplus \Omega^1)(M, su_2))$$

defined by F_A and $d_A\Phi$. As Proposition B1.2 of [20], $\mathcal{C}_m^k$ is a paracompact, Hausdorff Fréchet manifold. Let $\mathcal{G} = L_{2,loc}^2(M, SU_2)$ be the gauge group. Then $\mathcal{G}$ acts on $\mathcal{C}_m^k$ continuously as

$$g(C) = (gAg^{-1} + gdg^{-1}, g\Phi g^{-1}). \tag{2.4}$$

As Proposition 1 of [8], $C = (A, \Phi)$ is irreducible if the charge number k is non-zero in the sense that the stablizer of C in $\mathcal{G}$ is trivially ± 1. Notice that an orientation reserving of M correspondences to a sign change of k. We will restrict ourselves to the case $k \in \mathbf{N}$ and $m \in \mathbf{R}^+$. A fundamental theorem of Floer [8] is that the quotient space $\mathcal{B}_m^k = \mathcal{C}_m^k/\mathcal{G}$ is naturally a Hilbert manifold.

A basic regularization process in Yang-Mills-Higgs theory is as follows. Notice that, for $C = (A, \Phi) \in \mathcal{C}_m^k$, it may not be true that $\lim_{x\to\infty} |\Phi| = m$. The idea of the regularization process is to perturb Φ in a certain function space, such that, for the perturbation $\Psi = \Phi + \eta$, there is $\lim_{x\to\infty} |\Psi| = m$. To do this, let H_C' be the Hilbert space of the completion of compactly supported elements in $\Omega^0(M, su_2)$ in the norm given by

$$\|\eta\|_C^2 = \|\nabla_A \eta\|_2^2 + \|[\Phi, \eta]\|_2^2. \tag{2.5}$$

Consider the functional

$$I_C(\eta) = \|d_A(\Phi + \eta)\|_2^2 + \|[\Phi, \eta]\|_2^2 \tag{2.6}$$

on H_C'. As Proposition 4.8 of [13], I_C is a strictly convex, weakly lower semi-continuous and coersive functional on H_C'. Thus, for any $C = (A, \Phi) \in \mathcal{C}_m^k$, there exists a unique $\eta_C \in H_C'$ which minimizes I_C on H_C'. Let $\Psi = \Phi + \eta_C$. Then (A, Ψ) is called the regularization of $C = (A, \Phi)$. As in [13], (A, Ψ) has charge k and mass m. Notice that (A, Ψ) satisfies the second order, elliptic, partial differential equation

$$d_A^* d_A \Psi - [\Phi, [\Phi, \Psi]] = 0. \tag{2.7}$$

A standard a priori estimate implies that $\nabla_A \nabla_A \Psi \in L^2$. As Lemma 3 of [13], Ψ is continuous on M and $|\Psi| \to m$ uniformly at the infinity of M. As Lemma B4.2 of [20], there is the following:

Lemma 2.1: There exists a continuous, $\mathcal{G}$-equivalent function $R : \mathcal{C}^k_m \to [2s, \infty)$ such that where $|x| \geq R(C)$, then $|\Psi|(x) \geq \frac{m}{2}$.

Pick a continuous function R on $\mathcal{C}^k_m$. Notice that, for any $C = (A, \Phi) \in \mathcal{C}^k_m$, $\hat{\Psi}(x) = \Psi(x)/|\Psi(x)|$ is well-defined where $|x| \geq R(C)$. As in [13], the charge number k of C coincides with the degree of the map

$$\hat{\Psi} : S^2_r \subset M \to S^2 \subset su_2 \tag{2.8}$$

where $r \geq R(C)$.

Consider the configuration space $\mathcal{B}^k_m$. Let $\mathcal{G}^*$ be the group be base-preserving gauge transformations on M. Then $\mathcal{B}_{k,m} = \mathcal{C}^k_m/\mathcal{G}^*$ is a principal SO_3-bundle over $\mathcal{B}^k_m$. By Floer's theorem, $\mathcal{B}_{k,m}$ is also a Hilbert manifold. Notice that $\mathcal{G}$ acts on $\mathcal{C}^k_m$ infinitesimally as

$$\xi \mapsto -d_C\xi = -(d_A\xi, [\Phi, \xi]). \tag{2.9}$$

The vector $v_C = d_C\Psi$ correspondences the infinitesimal action of the gauge group

$$\{g = exp(t\Psi) : t \in \mathbf{R}\} \simeq U(1). \tag{2.10}$$

As Theorem 4.1 of [28], there is a natural circle bundle $\tilde{\mathcal{B}}^k_m$ over $\mathcal{B}^k_m$. The space of configurations will be connected to the loop space $\Omega^2_k S^2$ in Section 3.

2.2 The moduli space $\mathcal{M}^{\mathbf{k}}_{\mathbf{m}}$. Sitting inside $\mathcal{B}^k_m$ is the moduli space $\mathcal{M}^k_m$ of monopoles of charge k and mass m on M. By definitiion, a monopole $C = (A, \Phi)$ on M is a minimum of the energy functional (2.2) in $\mathcal{C}^k_m$. Note that

$$\underline{a}(C) = 8k\pi m + \|d_A\Phi - *F_A\|^2_2, \tag{2.11}$$

$C = (A, \Phi) \in \mathcal{C}^k_m$ is a monopoles iff it satisfies the following Bogomolny equation:

$$d_A\Phi = *F_A \tag{2.12}$$

on M. The moduli space $\mathcal{M}^k_m$ is by definition the set of solutions to (2.12) in $\mathcal{C}^k_m$ modulo the gauge group $\mathcal{G}$.

Remark 2.2: Since a monopole $C = (A, \Phi)$ satisfies the equation $d_A^* d_A \Phi = 0$ on M, it is automatically regularized.

Note that a monopole $C = (A, \Phi)$ on M can be considered as a static instanton $C = A + \Phi d\theta$ on the product 4-manifold $Y = M \times S^1$. As in [2] and [28], there is a Baire set of perturbations in the space $\mathcal{T}$ of static Riemannian metrics on Y such that $\mathcal{M}_m^k$ is a smooth manifold by the transversity argument of [10]. By Theorem 4.1 of Wang [28], there is a natural circle bundle $\tilde{\mathcal{M}}_m^k$ over $\mathcal{M}_m^k$ which is quaternionic (i.e. there is an action of quaternions $\mathbf{H}$ on the tangent bundle $T\tilde{\mathcal{M}}_m^k$.) Thus $\tilde{\mathcal{M}}_m^k$ is orientable, so is $\mathcal{M}_m^k$. Denote by $\mathcal{M}_{k,m}$ the space of solutions to (2.12) modulo the gauge group $\mathcal{G}^*$. Then $\mathcal{M}_{k,m}$ is a principal SO_3-bundle over $\mathcal{M}_m^k$. Thus $\mathcal{M}_{k,m}$ is smooth with a perturbation and is orientable.

Remark 2.3: As Theorem 3 of [28], when $H_1(X, \mathbf{Z}) = 0$, the dimension of $\mathcal{M}_m^k$ is $(4k-1)$. In this case, $\mathcal{M}_m^k$ is in fact smooth for a generic perturbation in the space $\mathcal{T}$. This fact is proved in [28] when $m \geq m_0$ for some $m_0 > 0$. In the case m is small, $\mathcal{M}_m^k$ is automatically smooth, see [29].

2.3 Analytical behaviors of monopoles. We list here some basic a priori estimates on the monopole solutions on M. Let $C = (A, \Phi)$ be a monopole on M of charge k and mass m. As well-known, C is gauge equivalent to a smooth solution and $|\Phi| < m$ on M by the maximum principle. As Theorem 11.1 of IV.11 of JT [15], the field strength F_A can be estimated as follows:

Proposition 2.4: $\|F_A\|_{C^0(M)} \leq K$ for some constant $K = K(k, m)$.

As in [15], there is also $\|F_A\|_{C^n} \leq K$ for some constant $K = K(k, m, n)$ by the standard boot-strapping argument. To establish the decay property of $C = (A, \Phi)$, we introduce the concept of centers of monopoles on M as in [22] as follows. The concept of centers of monopoles will be used to prove the weak convergence theorem and to analyze the structure of the end of $\mathcal{M}_m^k$ in Section 4. Let δ' be the injectivity of M and $\delta = \min(\delta', 1)$. For any $\epsilon > 0$, define

$$\hat{U}_{C,\epsilon} = \{x \in M : \|F(C)\|^2_{L^2(B_\delta(x))} \geq \epsilon m\},$$
$$U_{C,\epsilon} = \{x \in M : d(x, \hat{U}_{C,\epsilon}) < \delta\}. \tag{2.13}$$

Apparently $U_{C,\epsilon}$ is a bounded set and is not empty when $\epsilon > 0$ is small. As

Lemma C2.9 of [21], $U_{C,\epsilon}$ has the following properties:

Lemma 2.5: (i) The number of components of $U_{C,\epsilon}$ is less than $\frac{4k\pi}{\epsilon}$; (ii) The diameter of each of the components is less than $8k\pi/\delta$.

Let $\{U^j_{C,\epsilon}\}_{j=1}^N$ be the components of $U_{C,\epsilon}$ and $x_j \in U^j_{C,\epsilon}$ $(1 \le j \le N)$. We call $\{x_j\}_{j=1}^N$ a set of ϵ-centers of C. As Lemma C.2.1 and C.3.1 of [22], the centers has the following important property: F_A decays quadratically and $m - |\Phi|$ decays in the first order around $\{x_j\}$.

Theorem 2.6 (Taubes [22]): Let $C = (A, \Phi)$ be a monopole of charge k and mass m on M and $\{x_j : 1 \le j \le N\}$ be a set of ϵ-centers of C. Then there is a constant $K = K(k, m, \epsilon)$, such that,

$$\begin{cases} |F_A|(x) \le K \sum_{j=1}^N \frac{1}{|x-x_j|^2}, \\ m - |\Phi|(x) \le K \sum_{j=1}^N \frac{1}{|x-x_j|}. \end{cases} \tag{2.14}$$

3. Homotopy and Configurations

In this section, we study the geometry and topology of the space of Yang-Mills-Higgs configurations. We show that there are natural smooth maps $\tilde{e} : \mathcal{B}_{k,m} \to Map_k(S^2, S^2)$ and $I : Map_k(S^2, S^2) \to \mathcal{B}_{k,m}$ such that $\tilde{e} \circ I = identity$. Thus $\tilde{e}$ is a fibration and I is an imbedding. We moreover investigate the structure of the fiber of $\tilde{e}$ and prove that $\tilde{e}$ is a homotopy equivalence if $H_1(M, \mathbf{Z}) = 0$. Let $\Omega^2_k S^2$ denote the space of base-preserving maps from S^2 to S^2 of degree k. Notice that both $\tilde{e}$ and I are SO_3-equivalent, $\tilde{e}$ descends on $\tilde{\mathcal{B}}^k_m$ and $\tilde{e} \circ I = identity$ on $\Omega^2_k S^2$. Let $\tilde{\Omega}^2_k S^2 = \Omega^2_k S^2 / S^1$ with S^1 acts on $\Omega^2_k S^2$ by rotating the target S^2. Then $\tilde{e}$ and I descends as $\tilde{e} : \mathcal{B}^k_m \to \tilde{\Omega}^2_k S^2$ and $I : \tilde{\Omega}^2_k S^2 \to \mathcal{B}^k_m$.

Note that $\tilde{e}$ is essentially defined by the regularized Higgs field Ψ and Ψ is continuous on M. We will not distinguish the space of continuous maps from that of smooth ones, since they are the same in the homotopy theory. The construction in this section can be compared to that of Taubes [20]. The map $\tilde{e}$ will be revised in Section 6 to be compatible to the structure of the end $\mathcal{M}^k_{m,\infty}$ of the moduli space $\mathcal{M}^k_m$.

3.1 The geometry of $\mathcal{B}^{\mathbf{k}}_{\mathbf{m}}$. Consider the configuration space $\mathcal{B}_{k,m} = \mathcal{C}^k_m/\mathcal{G}^*$. Notice that $\mathcal{G}^*$ has two normal subgroups $\mathcal{G}^*_0$ and $\mathcal{G}^*_\infty$ as follows:

$$\mathcal{G}^*_0 = \{g \in \mathcal{G}^* : g|_{M^c_{2s}} = 1\},$$

$$\mathcal{G}^*_\infty = \{g \in \mathcal{G}^* : g|_{M_s} = 1\}. \tag{3.1}$$

Let $\mathcal{G}^*_1 = \mathcal{G}^*_0 \cap \mathcal{G}^*_\infty$.

Lemma 3.1: $\mathcal{G}^*$ is a fiber product of $\mathcal{G}^*_0$ and $\mathcal{G}^*_\infty$ over $\mathcal{G}^*_1$: $\mathcal{G}^* = \mathcal{G}^*_0 \times_{\mathcal{G}^*_1} \mathcal{G}^*_\infty$.

Proof: Notice that $A_s = \cup_{s \le r \le 2s} S^2_r$ and the space $Map(S^2, SU_2)$ is path connected. Thus an element $g \in \mathcal{G}^*$ can be always decomposed as $g = g_0 \cdot g_\infty$ for some $g_0 \in \mathcal{G}^*_0$ and $g_\infty \in \mathcal{G}^*_\infty$. Clearly the decomposition is unique up to elements in $\mathcal{G}^*_1$.

To define $\tilde{e}$, for any $[C] \in \mathcal{B}_{k,m}$, we choose a representative $C = (A, \Phi)$ which is in the polar gauge as follows:

$$A_r = \langle A, dr \rangle = 0 \tag{3.2}$$

on the complement M^c_s.

Lemma 3.2: For any $C = (A, \Phi) \in \mathcal{C}^k_m$, there is a unique gauge $g \in \mathcal{G}^*_\infty$ such that $g(C)$ is in the polar gauge.

Proof: Notice that $g(C)$ is in the polar gauge iff $\frac{\partial g}{\partial r} = gA$ on M^c_s. By the existence and uniqueness of the ordinary differential equation

$$\begin{cases} \frac{\partial g}{\partial r} = gA \\ g(s) = 1, \end{cases} \tag{3.3}$$

Lemma 3.2 is proved.

The map $\tilde{e} : \mathcal{B}_{k,m} \to Map_k(S^2, S^2)$ is defined as follows. For any $C = (A, \Phi) \in \mathcal{C}^k_m$, pick $g \in \mathcal{G}^*_\infty$ such that (A, Ψ) is in the polar gauge. Define

$$\tilde{e}(C)(\hat{x}) = \hat{\Psi}(R\hat{x}) \tag{3.4}$$

for $\hat{x} \in S^2$. Where R is a fixed function in Lemma 2.1. Note that $\tilde{e}$ descends onto $\mathcal{B}_{k,m}$ and defines a smooth map. Pick a cut off function on M such that $\beta = 0$ on M^c_{2s}. The map I can be defined as follows:

$$I(e)(x) = (1-\beta)(-[e(\hat{x}), de(\hat{x})], me(\hat{x})) \tag{3.5}$$

as (B1.1) of [20].

Lemma 3.3: *$\tilde{e} \circ I =$ identity.*

Proof: This is because $I(e) = (A(e), \Phi(e))$ defined explicitly as (3.5) has the following two properties: (i) $A_r = 0$, so $I(e)$ is in the polar gauge; (ii) $I(e)$ satisfies the equation $d^*_A d_A \Phi = 0$, it is also regularized.

3.2 The homotopy type of $\mathbf{B}^{\mathbf{k}}_{\mathbf{m}}$. Let us now investigate the structure of the fiber $\tilde{e}^{-1}(e)$ of the map $\tilde{e}$. It will be prove that $\tilde{e}^{-1}(e)$ is contractible if $H_1(X, \mathbf{Z}) = 0$. Thus $\tilde{e}$ is a homotopy equivalence when X is a homology 3-sphere. A theorem of the obstruction theory is needed as follows (see cf. [19]).

Theorem 3.4: Let Y, Z be finite CW-complexes such that Y has trivial homology and Z is simply connected. Then the space $Map_*(Y, Z)$ of base-preserving maps from Y to Z is contractible.

Proof: We show that $Map_*(Y, Z)$ is contractible by proving that any base-preserving map $\tilde{f} : S^n \to Map_*(Y, Z)$ is homotopically trivial as follows. It is equivalent to show that any base-preserving map $f : S^n \times Y \to Z$ is homotopically trivial. Let $f : (S^n, s_0) \times (Y, y_0) \to (Z, z_0)$ be a such map and $\{e^j_i\}$ be a cell-decomposition of Y, where j is the dimension of the cells. To show f is homotopically trivial, construct a homotopy

$$F : S^k \times Y \times I \to Z \tag{3.6}$$

such that

$$\begin{cases} F(s_0, y_0, t) = z_0 \\ F(s, y, 0) = f(s, y) \\ F(s, y, 1) = z_0 \end{cases} \tag{3.7}$$

by induction as follows. Note that $\{e^j_i \times I\}$ is a cell decomposition of $Y \times I$ and F is defined on the lowest skeleton $\{e^0_i \times I\}$. To extend the homotopy F to higher skeleton, the obstructions lie in the cohomology groups

$H^{q+1}(S^n \times Y \times I, \pi_q(Z)), q \in \mathbf{N}$. Since Z is simply connected and Y has trivial homology, the cohomology groups are ordinary singular ones and are all zero. Thus there is no obstruction to the existence of the homotopy F; the map f is homotopically trivial. Theorem 3.4 is proved.

As an application, the gauge group $\mathcal{G}^* = Map_*(M, SU_2)$ is contractible if $H_1(M, \mathbf{Z}) = 0$. Thus $\mathcal{C}_m^k \sim \mathcal{B}_{k,m}$ if X is a homology 3-sphere. Consider the map $\tilde{e} : \mathcal{B}_{k,m} \to Map_k(S^2, S^2)$. For fixed $e \in Map_k(S^2, S^2)$, we will prove that, the fiber $\tilde{e}^{-1}(e)$ consists of equivalence classes of configurations whose regularized Higgs fileds is parallel to e at the infinity of M. Together with the obstruction theory, the so-called gauge fixing techniques will be used to prove that it is contractible.

Similar to (3.4) of Floer [8], for any $e \in Map_k(S^2, S^2)$, define

$$\mathcal{U}_e = \{C = (A, \Phi) \in \mathcal{C}_m^k : [\Psi(x), e(\hat{x})] = 0 \text{ for large } |x|\},$$

$$\mathcal{G}_e^* = \{g \in \mathcal{G}^* : g(x) = exp(f(x)e(\hat{x})) \text{ for large } |x|\}, \tag{3.8}$$

where f is an $L^2_{2,loc}$-function on M. Notice that $\mathcal{G}_e^*$ acts on $\mathcal{U}_e$ freely and continuously.

Lemma 3.5: Restricted on $\tilde{e}^{-1}(e)$, the principal $\mathcal{G}^*$-bundle $\mathcal{C}_m^k \to \mathcal{B}_{k,m}$ reduces to the principal $\mathcal{G}_e^*$-bundle $\mathcal{U}_e \to \tilde{e}^{-1}(e)$.

Proof: Similar to that of Lemma B7.1 of [20] as follows. Notice that $\mathcal{G}^*$ acts on $C = (A, \Phi) \in \mathcal{C}_m^k$ by conjugating the Higgs field Φ. This action reduces to a rotation of the vector Ψ on a large sphere $S_r^2 \subset \mathbf{R}^3$. Thus for any fixed $e \in Map_k(S^2, S^2)$, there is $g \in \mathcal{G}_\infty^*$, such that, $g(\Psi)(x)$ is parallel to $e(\hat{x})$ for large $|x|$. Hence the principal $\mathcal{G}^*$-bundle $\mathcal{C}_m^k \to \mathcal{B}_{k,m}$ can be reduced to $\mathcal{U}_e \to \tilde{e}^{-1}(e)$, which is thus a principal $\mathcal{G}_e^*$-bundle. Lemma 3.5 is proved.

The configuration space $\mathcal{U}_e$ and gauge group $\mathcal{G}_e^*$ have the following interesting geometry: For any $C = (A, \Phi) \in \mathcal{C}_m^k$, let H_C be the Hilbert space of the completion of the compactly supported elements in $(\Omega^1 \oplus \Omega^0)(M, su_2)$ in the norm defined by

$$||\zeta||_C^2 = ||\nabla_A \zeta||_2^2 + ||[\Phi, \zeta]||_2^2. \tag{3.9}$$

Let $C, C' \in \mathcal{U}_e$ and $(a, \phi) = C' - C$. As Lemma B4.1 of [20], (a, ϕ) is almost in the Hilbert space H_C as follows. First $\phi \in H_C'$. Decompose $a = a^L + a^T$

with $a^L = (a \cdot e)e$ for large $|x|$ and denote $\alpha_{C'} = a \cdot e$. As Lemma B4.1 and B4.4, a^T and a^L have the following estimates respectively.

Lemma 3.6: $\nabla_A a^T \in L^2, [\Phi, a^T] \in L^2$ and $d\alpha_{C'} \in L^2$.

Remark that if $(a, \phi) \in H_C$, then $\mathcal{U}_e$ is contractible, since H_C is so. Notice that the gauge group $\mathcal{G}^*_{e,\infty} = \mathcal{G}^*_\infty \cap \mathcal{G}^*_e$ is contractible since elements in $\mathcal{G}^*_{e,\infty}$ can be considered as gauges in $\mathbf{R}^3$. We proceed to prove that $\mathcal{U}_e$ is contractible as follows.It will be shown that, with new gauges in $\mathcal{G}^*_{e,\infty}$, $\nabla\alpha_{C'} \in L^2$. Indeed, $\mathcal{G}^*_{e,\infty}$ acts to preserve the polarization $a = a^L + a^T$ and it acts on a^L as follows. Let $g(x) = exp(f(x)e(\hat{x}))$ for large $|x|$, then

$$\alpha_{g(C')} = \alpha_{C'} - df. \tag{3.10}$$

Notice that, for any $r > 0$, $H^1(B^c_r(0), \mathbf{R}) = 0$. As Lemma B4.5 of [20] or Lemma 4.2 of [8], the following Lemma 3.7 follows from the Hodge theorem.

Lemma 3.7: There exists a continuous map

$$g : \mathcal{U}_e \to \mathcal{G}^*_{e,\infty} \tag{3.11}$$

which induces a continuous map

$$\mathcal{U}_e \to H_C : C' \mapsto g(C')C' - C. \tag{3.12}$$

Thus $\mathcal{U}_e$ is contractible. By Lemma 3.5, $\tilde{e}^{-1}(e)$ is the classifying space for the gauge group $\mathcal{G}^*_e$, $\tilde{e}^{-1}(e) = B\mathcal{G}^*_e$.

Lemma 3.8: $\mathcal{G}^*_e$ is contractible if $H_1(X, \mathbf{Z}) = 0$.

Proof: Similar to that of Theorem 3.4 as follows. Assume that $H_1(X, \mathbf{Z}) = 0$. Let $\tilde{f} : S^n \to \mathcal{G}^*_e$ be a base-preserving map. $\tilde{f}$ is then homotopically trivial as follows. Consider $\tilde{f}$ as a base-preserving map $f : S^n \times M \to SU_2$. By the obstruction theory, $\tilde{f}$ is homotopic to a base-preserving map $\tilde{g} : S^n \to \mathcal{G}^*_{e,\infty}$, since there is no obstruction to deform $f_{|S^n \times M_s}$ into the identity. Since $\mathcal{G}^*_{e,\infty}$ is contractible, Lemma 3.8 is proved.

Corollary 3.9: $\tilde{e}$ is a homotopy equivalence when $H_1(M, \mathbf{Z}) = 0$.

4. A Weak Convergence Theorem

In this section, we prove a weak convergence theorem about sequences of monopoles on M and give a classification of the structure of the end $\mathcal{M}^k_{m,\infty}$ of the moduli space $\mathcal{M}^k_m$. As in [28], $\mathcal{M}^k_m$ is a smooth, orientable manifold with a perturbation. As (3.6) of [1], it has a complete Riemannian metric which is defined by

$$\|\zeta\|^2_C = \|\nabla_A\zeta\|^2_2 + \|[\Phi,\zeta]\|^2_2. \tag{4.1}$$

It is shown that, a sequence $\{[C_i]\} \subset \mathcal{M}^k_m$ has always a converging subsequence $\{C_{i_\nu}\}$ if the centers of $\{[C_i]\}$ form a bounded set on M. As a corollary, $\mathcal{M}^k_{m,\infty}$ has k-regions (open sets) which consists of monopoles which have l $(1 \le l \le k)$ charge of energy concentrated around the infinity of M. As a more detailed picture, a monopole $[C] \in \mathcal{M}^k_{m,\infty}$ is approximately a "gluing" of k_i-monopoles on M with $\sum k_i = k$ whose centers are far away from each other.

The weaker convergence theorem is essentially proved in Wang [28]. The reader may compare also AH [1]. Here we give a proof which is technically simpler. The weak convergence theorem will be a key in Section 6 to prove that $\mathcal{M}^k_m$ indeed has a natural compactification.

4.1 A priori estimates in the polar gauge. In Section 2.3, we list a few a priori estimates on monopole solutions on M. To prove the weak convergence theorem and also the blow-up phenomenon, we need a priori estimates of monopoles on M which are in the polar gauge. The a priori estimates may have independent interests elsewhere.

Let $C = (A,\Phi) \in \mathcal{C}^k_m$ be a configuration on M which is in the polar gauge. In the local coordinates, write

$$A = \sum_{i=1}^{3} A_i dx^i, F_A = \sum_{1\le i<j\le 3} F_{ij} dx^i \wedge dx^j$$

on M^c_s. As Lemma 2.1 of [25], computing directly from $\sum_i x^i A_i = 0$, there are the following identities:

$$\sum_i x^i F_{ij} = \frac{\partial}{\partial r}(rA_j), \langle \partial_r, d_A\Phi\rangle = \frac{\partial\Phi}{\partial r}. \tag{4.2}$$

As a corollary, $C = (A, \Phi)$ can be estimated as follows: Pointwisely

$$|\frac{\partial}{\partial r}(rA)| \leq r|F_A|, |\frac{\partial \Phi}{\partial r}| \leq |d_A \Phi|. \tag{4.3}$$

Lemma 4.1: Let $C = (A, \Phi)$ be a monopole of charge k and mass m on M which is in the polar gauge. Then, for any $\hat{x} \in S^2$,

$$\lim_{r \to \infty} \Phi(r\hat{x}) = \Phi_m^k(\hat{x}) \tag{4.4}$$

exists and there is a constant $K = K(k, m)$, such that,

$$|\Phi(x) - \Phi_m^k(\hat{x})| \leq \frac{K}{|x|}. \tag{4.5}$$

Proof: By Theorem 2.6 and (4.3), for $r_2 \geq r_1 \geq 2s$,

$$|\Phi(r_2\hat{x}) - \Phi(r_1\hat{x})| \leq \int_{r_1}^{r_2} |d\Phi(s\hat{x})| ds \leq \int_{r_1}^{r_2} \frac{K(k,m)}{s^2} ds \tag{4.6}$$

which is convergent to 0 as $r_2, r_1 \to \infty$. Thus $\Phi_m^k(\hat{x})$ exists. (4.5) is clear by (4.6).

We need gauges both in $\mathcal{G}_0^*$ and $\mathcal{G}_\infty^*$ to estimate A. As Proposition 9.3 of [23], choose gauge in $\mathcal{G}_0^*$, such that,

$$\|A - \Gamma\|_{C^0(M_s)} \leq K\|F_A\|_{L^2(M_{2s})} \leq K(k, m), \tag{4.7}$$

where Γ is a flat connection on M_{2s}. To establish a decay estimate for A, choose a further gauge in $\mathcal{G}_0^*$, such that, Γ is supported on M_s. Note that, this can be done since $\pi_1(M_s^c) = 0$. By (4.7), $\|A\|_{C^0(\partial M_s)} \leq K(k, m)$. Integrating (4.3), there is

$$|A|(x) \leq K(k, m)(1 + \ln \frac{|x|}{s})/|x| \tag{4.8}$$

with C in the polar gauge.

Remark 4.2: It may be true that $|A(x)| \leq K/|x|$ for some constant $K = K(k, m)$ when C is in the polar gauge. Note that $\Phi_m^k(\hat{x})$ is a smooth map from S^2 to S^2 of degree k. Let $\Phi^k(\hat{x}) = \Phi^k_m(\hat{x})/m$. Then Φ^k is independent of m. Ineed, Φ^k is a universal, SO_3-symmetric map from S^2 to S^2 of degree k. It is intuitively as follows. In the "telescope" S^2_∞ which is at the infinity, the images of $[C] \in \mathcal{M}^k_m$ are all centralized and are the same, disregard of where are the centers of $[C]$, or what is exactly the monopole. We will return to this fact and give it a proof in Section 5.1.

4.2 The weak convergence theorem. With the a priori estimates, the weak convergence theorem can be proved as follows.

Theorem 4.3: Let $\{C_i = (A_i, \Phi_i)\}_{i=1}^\infty$ be a sequence of monopoles of charge k and mass m on M which has the following property: for any $\epsilon > 0$, the ϵ-centers of $\{C_i\}$ form a bounded set on M. Then there is a subsequence i_ν of the following significance: there is a sequence of gauges $\{g_{i_\nu}\}$, such that, $\{g_{i_\nu}(C_{i_\nu})\}_{\nu=1}^\infty$ converges to $C \in \mathcal{C}^k_m$ on M in C^∞-norm.

Proof: We will prove that $\{[C_i]\}$ has a subsequence $\{[C_{i_\nu}]\}$ which is convergent in the C^0-norm. The C^∞-convergence follows then from a standard boot-strapping argument (see cf. [23]). (i) As (4.7), choose gauges in $\mathcal{G}^*_0$ such that

$$\|A_i - \Gamma\|_{C^0(M_s)} \le K\|F(A_i)\|_{L^2(M_{2s})} \le K(k, m), \tag{4.9}$$

where Γ is a flat connection on M_{2s} which is supported on M_s. Thus $\{C_i\} \subset C^0(M_s)$ is bounded. By boot-strapping, $\{(A_i, \Phi_i)\} \subset C^1(M_s)$ is also bounded. Thus $\{C_i\} \subset C^0(M^c_s)$ is bounded and equi-continuous, there is a subsequence $\{C_{i_\nu}\}$ of $\{C_i\}$ which is convergent in $C^0(M_s)$. For convenience, denote by $C_\nu = C_{i_\nu}$.

(ii) Choose gauges in $\mathcal{G}^*_\infty$ such that $\{C_\nu\}$ are in the polar gauge. Notice that the centers of $\{C_\nu\}$ form a bounded set, there is a constant K, which is independent of ν, such that,

$$|A_\nu|(x) \le K(1 + \ln\frac{|x|}{s})/|x|. \tag{4.10}$$

Thus $\{C_\nu\}$ is bounded in $C^0(M^c_s)$. Notice that, by differentiating (4.2), $\{\nabla C_\nu\}$ can be similarly estimated as (4.8). Thus $\{C_\nu\} \subset C^1(M^c_s)$ is also bounded. Hence $\{C_\nu\}$ has a subsequence which converges in $C^0(M^c_s)$. Denote again $\{C_\nu\}$ the subsequence. Combining (i), $\{C_\nu\}$ converges in $C^0(M)$.

(iii) Let $C = (A, \Phi)$ denote the limit. Notice that C is smooth on M by boot-strapping and it satisfies the Bogomolny equation on M. Note that, there is a Soblev constant μ, such that,

$$\|m - |\Phi_\nu|\|_{L^6} \leq \mu \|\nabla_{A_\nu} \Phi_\nu\|_{L^2} \tag{4.11}$$

which is uniformly bounded, C has mass m. Notice that, the centers of $\{C_\nu\}$ is a bounded subset, there is a constant K, which is independent of ν , such that,

$$m - |\Phi_\nu| \leq \frac{K}{|x|}, |F_{A_\nu}| \leq \frac{K}{|x|^2} \tag{4.12}$$

by Theorem 2.6. As in [13], C has charge k. Theorem 4.5 is proved.

4.3 The structure of $\mathcal{M}^{\mathbf{k}}_{\mathbf{m},\infty}$. Let $\{[C_i]\} \subset \mathcal{M}^k_m$ be a sequence of monopoles on M. By theorem 4.3, the sequence may approaches to the infinity of $\mathcal{M}^k_m$ only when there is a sequence of centers $\{x^i\}$ of $\{C_i\}$ drifting away to the infinity of M. Let $\{x^i_j\}^{l_i}_{j=1}$ be a set of centers of C_i. Consider the limit of this sequence of sets of centers as $i \to \infty$. There is first a classification of the end $\mathcal{M}^k_{m,\infty}$ of $\mathcal{M}^k_m$ as follows.

Proposition 4.4: The end $\mathcal{M}^k_{m,\infty}$ is a union of k regions $\mathcal{M}^k_{m,1}, \cdots, \mathcal{M}^k_{m,k}$ which consists monopoles which have l charges of energy concentrated around the infinity of M.

A monopole $[C] \in \mathcal{M}^k_{m,l}$ is thus approximately a "gluing" of a charge $(k-l)$ monopole which has energy concentrated around M_{2s} and a charge l monopole in $\mathbf{R}^3$ whose energy is concentrated far away from the origin. Hence $\mathcal{M}^k_m$ has the following "stratified" structure: $\mathcal{M}^k_m = \cup^k_{l=0} \mathcal{M}^k_{m,l}$. By investigating further the decomposition of the sequence of sets of centers $\{x^i_j\}^{l_i}_{j=1}$ as $i \to \infty$, we have the following:

Theorem 4.5: Let $\{[C_i]\} \subset \mathcal{M}^k_m$ be a sequence of monopoles on M which approaches to the infinity of $\mathcal{M}^k_m$. Then there is a subsequence i_ν, a partition $k = \sum^l_{j=1} k_j$ and sequences of centers $x^j_\nu (j = 1, \cdots, l)$, such that, $\{[C_{i_\nu}]\}$ converges to a k_j-monopole on any given ball $B_r(x^j_\nu) \subset M$.

5. A Blow-up Phenomenon

In this section, we revise $\tilde{e}$ to define another SO_3-equivalent map

$$\hat{e} : B_{k,m} \to Map_k(S^2, S^2) \tag{5.1}$$

which is a homotopy equivalence if $H_1(M, \mathbf{Z}) = 0$ and is compatible with the geometry of $\mathcal{M}^k_{m,\infty}$. Indeed, the non-compactness of $\mathcal{M}^k_m$ correpondences to a standard blow-up phenomenon in the space $Map_k(S^2, S^2)$. We will show that $\hat{e}(\mathcal{M}^k_m)$ has a natural compactification in Section 6.

5.1 A blow-up phenomenon. Let us first examine some simplest examples of the blow-up phenomenon to motivate the definition of $\hat{e}$ which is slightly technical. Recall that $\tilde{e}$ is defined as (3.4), where R is a fixed function on $\mathcal{C}^k_m$ in Lemma 2.1. Fix

$$R(C) = \inf_{\tilde{R}}\{\tilde{R}(C) : |\Psi|(x) \geq \frac{m}{2}, |x| \geq \tilde{R}(C)\}. \tag{5.2}$$

Note that R descends on $\mathcal{B}^k_m$ as a continuous function.

Consider the simplest case $k = 1$. Let $\{[C_i]\} \subset \mathcal{M}^1_m$ be a sequence of monopoles on M approaches to the infinity of $\mathcal{M}^k_m$. By Theorem 4.5, for large i, $[C_i]$ is approximately a 1-monopole in $\mathbf{R}^3$ with center far away from the origin O. Note that, in the polar gauge, the standard 1-monopole $C = (A, \Phi)$ in $\mathbf{R}^3$ with zero at O can be written as

$$\begin{cases} A = (\frac{1}{r} - \frac{1}{\sinh r})\hat{x} \times \sigma \cdot dx \\ \Phi = (\frac{1}{\tanh r} - \frac{1}{r})\hat{x} \end{cases} \tag{5.3}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the standard basis of su_2. Let x_i be the center of C_i. Then C_i is approximately $(A(m(x - x_i)), m\Phi(m(x - x_i)))$ in the polar gauge.

Consider the map $\tilde{e}$. Note that $\{C_i\}$ are regularized. With $\{C_i\}$ in the polar gauge, $\tilde{e}([C_i])$ can be also defined as follows. Note that there is $\lambda_i \leq \lambda(m)$ such that $|\Phi_i(x)| \geq \frac{m}{2}$ with $|x - x_i| \geq \lambda_i$,

$$\tilde{e}([C_i])(\hat{x}) = \hat{\Phi}_i(r_i\hat{x}) \simeq \frac{r_i\hat{x} - x_i}{|r_i\hat{x} - x_i|} \tag{5.4}$$

on S^2, where $r_i = |x_i| + \lambda_i$.

Proposition 5.1: Let $[C_i] \in \mathcal{M}^1_m, i \in \mathrm{N}$ be a sequence of monopoles on M which approaches to the infinity of $\mathcal{M}^1_m$. Then there is a subsequence

$\{[C_{i_\nu}]\}$ of $\{[C_i]\}$ such that

(i) $\{\hat{x}_{i_\nu}\}$ converges to some $\hat{x}_0 \in S^2$;

(ii) $\tilde{e}([C_{i_\nu}])(\hat{x}) \to \frac{\hat{x}-\hat{x}_0}{|\hat{x}-\hat{x}_0|}$ for $\hat{x} \neq \hat{x}_0$;

(iii) $\lim_{\delta \to 0^+} \lim_{\nu \to \infty} \int_{B_\delta(\hat{x}_0)} det(D\tilde{e})([C_{i_\nu}]) = 2\pi$.

Proof: (i) is clear since S^2 is compact. To prove (ii), notice that

$$\tilde{e}([C_{i_\nu}])(\hat{x}) \simeq \frac{(1+\epsilon_{i_\nu})\hat{x} - \hat{x}_{i_\nu}}{|(1+\epsilon_{i_\nu})\hat{x} - \hat{x}_{i_\nu}|}, \tag{5.5}$$

where $\epsilon_i = \frac{\lambda_i}{|x_i|}$. Since $\epsilon_i \to 0$, (ii) is proved. Notice that (5.5) blows up the small neighborhood $B_\delta(\hat{x}_0)$ into the hemisphere, Lemma 5.1 is proved.

Remark: The blow-up phenomenon is not due to a conformal invariance, such as the case of Yang-Mills fields. In fact, a straightforword computation gives

$$\lim_{\delta \to 0^+} \lim_{\nu \to \infty} \int_{B_\delta(\hat{x}_0)} |\nabla \tilde{e}|^2([C_{i_\nu}]) = \infty. \tag{5.6}$$

When $k \geq 2$, the simplest case is that the sequence $\{[C_i]\} \subset \mathcal{M}_m^k$ have all the energy drifting to the infinity of M in the following pattern: there is a constant $D > 0$, such that, the centers $\{x_i^j\}_{j=1}^{l_i}$ of $[C_i]$ form a set of diameter $d_i \leq D$. In this case, $[C_i]$ is approximately a monopole in $\mathbf{R}^3$ whose centers are $\{x_i^j\}$. As the case $k = 1$, there is a constant $\lambda_i \leq \lambda(k, m)$, such that, $|\Phi|(x) \geq \frac{m}{2}$ when $|x - x_i^j| \geq \lambda_i (1 \leq j \leq l_i)$. Since C_i is regularized,

$$\tilde{e}([C_i])(\hat{x}) = \hat{\Phi}_i(r_i \hat{x}) \tag{5.7}$$

where $r_i = \max_{1 \leq j \leq l_i} |x_i| + \lambda_i$ with C_i in the polar gauge.

Proposition 5.2: Let $[C_i] \in \mathcal{M}_m^k, i \in \mathbf{N}$ be a sequence of monopoles on M whose centers approach to the infinity of M and form a set of diameter $d_i \leq D$. Then there is a subsequence $\{[C_{i_\nu}]\}$ of $\{[C_i]\}$ which has the following properties:

(i) $\{\hat{x}_{i_\nu}\}$ converges to some $\hat{x}_0 \in S^2$;

(ii) For any $\hat{x} \neq \hat{x}_0$, $\tilde{e}([C_{i_\nu}])(\hat{x})$ converges to a universal map $\Phi^k(\frac{\hat{x}-\hat{x}_0}{|\hat{x}-\hat{x}_0|})$;

(iii) $\lim_{\delta \to 0^+} \lim_{\nu \to \infty} \int_{B_\delta(\hat{x}_0)} det(D\tilde{e})([C_{i_\nu}]) = 2k\pi$.

Proof: (i) holds as that of Lemma 5.1. (ii) Consider the corresponding subsequence $C_\nu = C_{i_\nu}$. The convergence of $\tilde{e}([C_i])(\hat{x})$ can be argued as follows. (A) When $[C_\nu]$ is approximately the gluing of k single monopoles in $\mathbf{R}^3$ whose zeros are far apart from each other as that of Definition 7.1 of IV.7 of JT [15], $C_\nu = (A_\nu, \Phi_\nu)$ can be explicitly given. In this case, $\tilde{e}([C_\nu])(\hat{x})$ can be shown explicitly to converge to $\Phi^k(\frac{\hat{x}-\hat{x}_0}{|\hat{x}-\hat{x}_0|})$ at $\hat{x} \neq \hat{x}_0$. The proof is lengthy but not difficult. (B) When $[C_\nu]$ is not approximately the gluing of k separate 1-monopoles, it is encircled by a domain $\mathcal{D}^k_\nu$ in $\mathcal{M}^k_m$ which consists of monopoles which are approximately a gluing of separate single monopoles. This can be envisaged as follows. For example, when $[C_\nu]$ has two infinitesimally closed centers, the break-up of one center from another in all directions form a domain encircling $[C_\nu]$. The situation is similar to that of Theorem 4.5. Since $\tilde{e}(\mathcal{D}^k_\nu)$ converges to a point by (A), $\tilde{e}([C_\nu])(\hat{x})$ converges as the case of (A). Φ^k is a universal map means that it is independent of the sequence $\{[C_i]\}$.

(iii) When $[C_\nu]$ is the gluing of k separate single monopoles, (iii) can still explicitly proved. To prove the general case, notice that Φ^k is a degree k map on S^2 by Remark 4.2. By (ii), on the complement $B^c_\delta(\hat{x}_0)$,

$$\lim_{\delta\to 0^+} \lim_{\nu\to\infty} \int_{B^c_\delta(\hat{x}_0)} det(D\tilde{e})([C_\nu]) = 2k\pi. \tag{5.8}$$

(We need in fact only to check the case $k = 1$, which is proved in Proposition 5.1.) Notice that $\tilde{e}([C_\nu])$ has degree k, (iii) is proved.

5.2 The ê-map. When the sequence $\{[C_i]\} \subset \mathcal{M}^k_m$ approaches to the inifinity of $\mathcal{M}^k_m$ with centers $\{x^j_i\}_{j=1}^{j=l_i}$ approaches in different directions and at different rates to the infinity of M, the map $\tilde{e}$ no longer guarantees a canonical blow-up as those in Proposition 5.1 and 5.2. We revise therefore $\tilde{e}$ to define a map $\hat{e} : \mathcal{B}_{k,m} \to Map_k(S^2, S^2)$ which is compatible with the geometry of $\mathcal{M}^k_{m,\infty}$. The definition of $\hat{e}$ is slightly technical due to the complexity of the geometry of $\mathcal{M}^k_{m,\infty}$, compare [23] and [17].

Let $C = (A, \Phi) \in \mathcal{C}^k_m$ be a configuration on M. Recall that the regularized (A, Ψ) has the following property: Ψ is continuous on M and $|\Psi|(x) \to m$ uniformly as $x \to \infty$. Let

$$\Omega_C = \{x \in M : |\Psi|(x) \leq \frac{m}{2}\}. \tag{5.9}$$

Then $\Omega_C \subset M$ is a compact domain. Denote by $\{\Omega_C^j\}_{j=1}^l$ the path components of Ω_C and d_j the diameter of Ω_C^j. For convenience, we pick $x^j \in \mathbf{R}^3$ in the following "canonical" way: If $\Omega_C^j \cap M_s = \phi$, x^j is the center of mass of Ω_C^j:

$$x^j = \int_{\Omega_C^j} x dx^3 / vol(\Omega_C^j). \tag{5.10}$$

If $\Omega_C^j \cap M_s \neq \phi$, x^j is the center of mass of $\Omega_C^j \setminus M_s$.

To deal with the case $x^j \to \infty$ and $\{x^j\}$ fall apart, for each j, encircle x^j with a ball $B_{r_j}(y_j) \subset \mathbf{R}^3$ as follows. When $x^j = O$, take $y_j = O$ and $r_j = d_j$. When $x^j \neq 0$,

$$y_j = (|x^j| - \lambda_j)\hat{x}^j, \lambda_j = \min_{i \neq j}(\frac{1}{4}|x^i - x^j|, \frac{1}{4}|x^j|) \tag{5.11}$$

and $r_j = d_j + \lambda_j$. Note that $\{B_{r_j}(y_j)\}$ do not intersect each other and $B_{2s}(0)$ if $x^j \to \infty$ and $\{x^j\}$ fall apart.

Define the connected sum $S^2_{2s}(0) \# S^2_{r_1}(y_j) \# \cdots \# S^2_{r_l}(y_l)$ by ballooning each $S^2_{r_j}(y_j)$ to $S^2_{2s}(0)$ as follows. If $B_{r_j}(y_j) \cap B_{2s}(0)$ is not empty, $S^2_{2s}(0) \# S^2_{r_j}(y_j)$ is the boundary of $\bar{B}_{2s}(0) \cup \bar{B}_{r_j}(y_j)$. If $B_{2s}(0) \cap B_{r_i}(y_j) = \phi$, then $S^2_{2s}(0) \# S^2_{r_j}(y_j)$ is connected by the tube in $\mathbf{R}^3$ which has small radius $\delta_j = \min(\frac{1}{|x^j|}, \frac{1}{4})$ and is centered along the line $L_j : \{x = t\hat{x}^j : t \in \mathbf{R}\}$. Note that, when $B_{r_i}(y_i) \cap B_{r_j}(y_j) \neq \phi$ for some i and j, we consider $S^2_{2s}(0) \# \cdots \# S^2_{r_l}(y_l)$ as an immersed 2-sphere in $\mathbf{R}^3$. Fix a parametriztion

$$\mu : S^2 \to S^2_{2s}(0) \# \cdots \# S^2_{r_l}(y_l) \tag{5.12}$$

such that, if $x^j \neq O$,

$$\mu(\hat{x}^j) = (|x^j| + d_j)\hat{x}^j. \tag{5.13}$$

The map $\hat{e}$ is then defined as follows. Let $[C] \in \mathcal{M}^k_m$ be a configuration on M. Choose a representative $C = (A, \Phi)$ such that C is in the polar gauge. Define

$$\hat{e}'([C]) = \hat{\Psi} : S^2_{2s}(0) \# \cdots \# S^2_{r_l}(y_l) \to S^2 \tag{5.14}$$

and $\hat{e} = \hat{e}' \circ \mu$. $\hat{e}$ is clearly SO_3-equivalent.

For any $e \in Map_k(S^2, S^2)$, define $I(e) \in \mathcal{B}_{k,m}$ as (3.5). Recall that $I(e)$ is regularized and is in the polar gauge. Notice that $|\Phi| = m$ on M^c_{2s}, $x^j = O$. It implies that $\hat{e} \circ I$ is a diffeomorphism. Thus $\hat{e}$ is a fibration and I is an imbedding.

Proposition 5.3: $\hat{e}$ is a homotopy equivalence if $H_1(M, \mathbf{Z}) = 0$.

Proof: Similar to that of Corollary 3.9.

Consider the restriction of $\hat{e}$ on $\mathcal{M}^k_m$. Let $[C] \in \mathcal{M}^k_m$ be a monopole on M. Consider the set Ω^j_C. By Theorem 2.6, the diameter od Ω^j_C has a bound $D = D(k, m)$. Let $C = (A, \Phi)$ be in the polar gauge. Notice that $\hat{e}'_{|S^2_{r_j}(y_j)} = \hat{\Phi}$. As Proposition 5.2, there is the following standard blow-up phenomenon.

Proposition 5.4: Let $\{[C_i]\} \subset \mathcal{M}^k_m$ be a sequence of monopoles on M which has a sequence of centers $\{x_i\}$ approaches to the infinity of M and carries away l charges of energy. Assume that $\hat{x}_i \to \hat{x}_0 \in S^2$. Then there is the following:

$$\lim_{i \to \infty} \hat{e}'([C_i])(x) \to \Phi^l\left(\frac{\hat{x} - \hat{x}_0}{|\hat{x} - \hat{x}_0|}\right) \tag{5.15}$$

where $x = y_j + r_j \hat{x} \in S^2_{r_j}(y_j)$, $\hat{x} \neq \hat{x}_0$. Moreover,

$$\lim_{\delta \to 0^+} \lim_{i \to \infty} \int_{B_\delta(\hat{x}_0)} det(D\hat{e})([C_i]) = 2l\pi. \tag{5.16}$$

6. The Comapctification of $\hat{\mathbf{e}}(\mathcal{M}^{\mathbf{k}}_{\mathbf{m}})$

In this section, we prove that $\hat{e}(\mathcal{M}^k_m) \subset Map_k(S^2, S^2)$ has a natural compactification. The compactification is based on the weak convergence theorem 4.3 and the fact that the map $\hat{e}$ defines a canonical blow-up. Let $\overline{\hat{e}(\mathcal{M}^k_m)}$ denote the compactification. We prove also that, when $H_1(X, \mathbf{Q}) = 0$, $\overline{\hat{e}(\mathcal{M}^k_m)}$ defines at least a $\mathbf{Z}_2$-fundamental class of dimension $(4k-1)$ when k is in the stable range $k \geq 2$. We do not discuss the orientation of $\overline{\hat{e}(\mathcal{M}^k_m)}$ here.

6.1 The compactification of $\hat{e}(\mathcal{M}^k_m)$. Let us now prove that the space $\hat{e}(\mathcal{M}^k_m)$ has a natural compactification. As in DK [5] and FM [11], we call a pair $(\hat{e}([C]), (\hat{x}^1, \cdots, \hat{x}^l))$ an ideal image if $[C] \in \mathcal{M}^{k-l}_m$ and $(\hat{x}^1, \cdots, \hat{x}^l)$ is an unordered l-tuple of points of S^2.

Let $\{[C_i]\} \subset \mathcal{M}^k_m$ be a sequence of monopoles on M and $(\hat{e}([C]), (\underbrace{\hat{x}^1, \cdots, \hat{x}^1}_{k_1}, \cdots, \underbrace{\hat{x}^t, \cdots, \hat{x}^t}_{k_t})$ with $\sum k_j = l$ be an ideal image. We call $\{\hat{e}([C_i])\}$ weakly converges to $(\hat{e}([C]), (\hat{x}^1, \cdots, \hat{x}^t))$ if $\{[C_i]\}$ has sequences of centers $\{x^j_i\}^t_{j=1}$, such that, $x^j_i \to \infty$ and carries away k_j charges of energy, $\hat{x}^j_i \to \hat{x}^j$, and

(i) $\{\hat{e}([C_i])\}$ converges wealky (i.e. on any compact subset of M) to $\hat{e}([C])$;

(ii) $\hat{e}([C_i])\}$ has the standard blow-up at $\{\hat{x}^j\}$ as follows:

$$\hat{e}'([C_i])(x) \to \Phi^{k_j}(\frac{\hat{x} - \hat{x}^j}{|\hat{x} - \hat{x}^j|}) \tag{6.1}$$

for $y_j + r_j\hat{x} \in S^2_{r_j}(y_j)$ with $\hat{x} \neq \hat{x}^j$ and

$$\lim_{\delta \to 0^+} \lim_{i \to \infty} \int_{B_\delta(\hat{x}^j)} det D\hat{e}([C_i]) = 2k_j\pi. \tag{6.2}$$

We thus define the compactificatin $\overline{\hat{e}(\mathcal{M}^k_m)}$ of $\hat{e}(\mathcal{M}^k_m)$ to be

$$\overline{\hat{e}(\mathcal{M}^k_m)} = \hat{e}(\mathcal{M}^k_m) \cup \hat{e}(\mathcal{M}^{k-1}_m) \times S^2 \cup \cdots \hat{e}(\mathcal{M}^0_m) \times (S^2)^l. \tag{6.3}$$

with the topology given by the weak convergence. It is easy to see that this topology is second-countable, Hausdorff and metrizable. $\hat{e}(\mathcal{M}^k_m)$ is embedded as an open subset of $\overline{\hat{e}(\mathcal{M}^k_m)}$. More generally, the induced topology of the different strata $\hat{e}(\mathcal{M}^{k-l}_m)$ is the usual one.

Remark 6.1: Note that $\hat{e}(\mathcal{M}^k_m)$ is not necessarily dense in the compactification. We may also define $\overline{\hat{e}(\mathcal{M}^k_m)}$ to be the completion of $\hat{e}(\mathcal{M}^k_m)$ in the space of ideal images with the topology given above. It seems that it does not make a difference for our purpose.

Remark 6.2: Note that a 0-monopole $C = (A, \Phi)$ has zero energy, $F_A = 0, d_A\Phi = 0$. Thus $|\Phi| \equiv m$ and Φ defines a splitting

$$M \times su_2 = (M \times \mathbf{R}) \oplus \Phi^{\perp} \tag{6.4}$$

where $M \times \mathbf{R}$ is generated by Φ and $\Phi^{\perp}$ is the complement. Note that A defines a flat connection on $\Phi^{\perp}$,

$$\mathcal{M}_m^0 \simeq Hom(\pi_1 M, U(1))/\pm 1 \tag{6.5}$$

where $\{\pm 1\}$ acts as the flipping on the diagnal of $B \in SU_2$. (6.5) is well-known as the "pillow case". It has dimension $b_1 = \dim H^1(M, \mathbf{R})$. Analytically, C is gauge equivalent to $(\omega, m\tau)$, where ω is a harmonic 1-form supported on M_s. Note that $(\omega, m\tau)$ is in the polar gauge, $\hat{e}(\mathcal{M}_m^0) = \tau$ which is a point.

Theorem 6.3: The space $\overline{\hat{e}(\mathcal{M}_m^k)}$ is compact.

Proof: Let $\{[C_i]\}$ be a sequence in $\overline{\hat{e}(\mathcal{M}_m^k)}$. Passing to a subsequence, $\{[C_i]\}$ is in $\hat{e}(\mathcal{M}_m^{k-l}) \times (S^2)^l$ for some $l \leq k$. Note that S^2 is compact. By Theorem and 4.3 and 4.5, the sequence $\hat{e}([C_i])$ with $[C_i] \subset \mathcal{M}_m^{k-l}$ has the following properties: it has a subsequence, denoted again as $\{\hat{e}([C_i])\}$, such that, (i) $\{[C_i]\}$ convergens weakly on M to a $k-l-l'$ monopole on M, (ii) there is a partition $k - l - l' = \sum_{j=1}^N k_j$ and sequences of centers $\{x_i^j\}$ $(j = 1, \cdots, N)$, such that, $\{x_i^j\} \to \infty$, $\{x_i^j\}$ carries away k_j charges of energy to the infinity of M and $\hat{x}_i^j \to \hat{x}^j \in S^2$. By Proposition 5.4, $\{\hat{e}([C_i])\}$ converges in $\hat{e}(\mathcal{M}_m^{k-l-l'}) \times (S^2)^{l'}$. Theorem 6.3 is proved.

6.2 The fundamental class $\overline{\hat{\mathbf{e}}(\mathcal{M}_{\mathbf{m}}^{\mathbf{k}})}$. To ensure that $\overline{\hat{e}(\mathcal{M}_m^k)}$ represents a fundamental class of dimension $(4k-1)$ for $k \geq 2$, we need to prove two things: (i) The map $\hat{e}$ is an imbedding on an open subset of $\mathcal{M}_m^k$, thus $\hat{e}(\mathcal{M}_m^k)$ has dimension $(4k-1)$; (ii) In the stable range $k \geq 2$, the lower stratas $\mathcal{M}_m^{k-l} \times (S^2)^l$ have codimension greater or equal than 2 in $\overline{\hat{e}(\mathcal{M}_m^k)}$. We do not discuss the orientablity of $\overline{\hat{e}(\mathcal{M}_m^k)}$ here.

Lemma 6.4. Assume that $H_1(X, \mathbf{Q}) = 0$. Then, for any $k \in \mathbf{N}$, there is an open set $\mathcal{U} \subset \mathcal{M}_m^k$ on which $\hat{e}$ is an imbedding.

Proof: Consider the open set $\mathcal{U}$ which consists of monopoles $[C] \in \mathcal{M}_m^k$ whose energy is concentrated around k different points $x_1, \cdots, x_k$ which are far away from each other and the compact set M_{2s}. Then $[C]$ is close to a 0-monopole on M_{2s}. When $H_1(M, \mathbf{Q}) = 0$, as in the case in [28], $\mathcal{U}$ is diffeomorphic to the open set $\mathcal{U}'$ which consists of monopoles in $\mathbf{R}^3$ whose energy is concentrated around the k different points $x_1, \cdots, x_k \in \mathbf{R}^3$ and $[C]$ is approximately the gluing of the k single monopoles which have zeros at x_i respectively. Similar to (5.4), $\hat{e}'$ and its differentiation can be explicitly identified on each sphere $S^2_{r_j}(y_j)$. $\hat{e}'$ is an imbedding on $\mathcal{U}$.

Lemma 6.5: When $H_1(M, \mathbf{Q}) = 0$ and $k \geq 2$, the lower stratas of $\overline{\hat{e}(\mathcal{M}^k_m)}$ have codimension greater or equal than 2 in $\overline{\hat{e}(\mathcal{M}^k_m)}$.

Proof: By Lemma 6.4, when $H_1(M, \mathbf{Q}) = 0$ and $k \geq 2$, the lower strata $\hat{e}(\mathcal{M}^{k-l}_m) \times (S^2)^l$ with $l < k$ has dimension

$$4(k-l) - 1 + 2l \leq (4k-1) - 2 \tag{6.6}$$

and the lowest strata $\hat{e}(\mathcal{M}^0_m) \times (S^2)^k$ has dimension $2k \leq (4k-1) - 2$. Lemma 6.5 is proved.

Corollary 6.6: When $H_1(M, \mathbf{Q}) = 0$ and $k \geq 2$, $\overline{\hat{e}(\mathcal{M}^k_m)}$ defines a $\mathbf{Z}_2$-fundamental class of dimension (4k-1).

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