

Tree-level splitting amplitudes for a gluon into four collinear partons

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ABSTRACT: We compute in conventional dimensional regularisation the tree-level splitting amplitudes for a gluon parent which splits into four collinear partons. This is part of the universal infrared behaviour of the QCD scattering amplitudes at next-to-next-to-next-to-leading order (N^3 LO) in the strong coupling constant. Combined with our earlier results for a quark parent, this completes the set of tree-level splitting amplitudes required at this order. We also study iterated collinear limits where a subset of the four collinear partons become themselves collinear.

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1 Introduction

With the discovery of the Higgs boson by the Large Hadron Collider (LHC) at CERN [1, 2], the Standard Model (SM) of particle physics is a complete theory without any free parameters. Current and future collider experiments will be able to test the SM with an increasing level of precision. The requested precision poses a severe challenge to theory and calls for the development of improved techniques for theoretical predictions accurate at the percent level. One of the main approaches to theoretical collider phenomenology is perturbation theory, in which observables are expanded in the small coupling constants of the theory. Here we focus on QCD observables, and consequently we will be concerned with the computation of higher orders in the strong coupling constant α_s of QCD.

Often, leading order predictions in QCD are not reliable. For example it is known that in the case of Higgs production the next-to-leading order corrections almost double the value of the cross section [3, 4]. The next-to-next-to-leading order (NNLO) corrections (in the limit where the top-quark is infinitely heavy) further increase the cross section, and it is only after the inclusion of the next-to-next-to-next-to-leading order

(N³LO) corrections that a reliable estimate of the cross section with a residual uncertainty of only a few percent is obtained [5–7]. This example illustrates that reliable QCD predictions at the percent level can most likely only be achieved after the inclusion of N³LO corrections. So far, however, only very few hadron collider processes are known at this order in perturbation theory. Inclusive N³LO cross sections are known for Higgs in gluon fusion [5–7], bottom-quark fusion [8, 9] and vector boson fusion in the DIS approach [10, 11], as well as double Higgs production [12, 13] and Drell-Yan production (via the intermediate of an off-shell photon) [14]. At the differential level, only the Higgs rapidity and transverse momentum distributions in vector boson fusion in the DIS approximation [10, 15, 16] are known.

One of the reasons progress towards more results at N³LO is difficult lies in the fact that an observable at N^kLO receives contributions from processes with up to k additional partons in the final state. Each such contribution is individually infrared divergent, with divergences arising in particular from the integration over regions of phase space where the emitted partons are either soft or collinear. Several techniques have been developed at NNLO to compute the relevant phase space integrals with generic acceptance cuts [17–50]. With some abuse of language, we refer to these techniques in the following collectively as *subtraction methods*. Developing subtracting methods beyond NNLO will be an important step towards obtaining more predictions with percent accuracy.¹

An important ingredient in the development of subtraction methods is the universal behaviour of the QCD scattering amplitudes in the infrared limits, embodied in universal soft and collinear currents. At NNLO these currents have been computed more than a decade ago, and they include splitting amplitudes for three partons at tree-level and two partons at one-loop, as well as soft currents for the emission of two soft partons at tree-level and one soft parton at one-loop [52–60]. Over the last few years, results have also become available for the universal currents at N³LO. The two-loop currents for the emission of two collinear or one soft parton were obtained in refs. [61–66], and the one-loop current for the emission of three collinear partons is given in refs. [67, 68].² The tree-level currents include the soft current for the emission of three soft partons [70] and the splitting amplitudes for four collinear partons. The latter have been obtained at the amplitude-level in four dimensions in refs. [55, 71, 72]. Recently, we have published the splitting amplitudes for the squared matrix element in dimensional regularisation, in the case where the parent parton is a quark [73]. The main purpose of this paper is to complete the set of tree-level splitting amplitudes at

¹Some of these methods have already been successfully applied at N³LO for processes with a simple final state structure [10, 11, 15, 51].

²One-loop currents with three collinear partons are also known for mixed QCD+QED cases [69].

N³LO by providing analytic results in dimensional regularisation for a gluon splitting into four partons.

This paper is organised as follows: In Sec. 2, we review the collinear limit of tree-level amplitudes and give a precise definition of the quantities that we want to compute. In Sec. 3, we present the main result of our paper, namely the computation of the tree-level splitting amplitudes for a gluon parent to split into four collinear partons. The explicit results are too long to be recorded in this paper and are made available in computer-readable form [74]. In Sec. 4 we study the collinear limit of the splitting amplitudes themselves, and we define new universal objects which appear in these iterated limits. We include several appendices with technical material omitted throughout the main text.

2 Multiple collinear limits

We examine the behaviour of tree-level QCD amplitudes in the limit where a given number of massless partons become collinear. Namely, we consider the scattering of n massless particles with momenta p_i and with flavour, spin and colour quantum numbers f_i , s_i and c_i , respectively, and we analyse the behaviour of the amplitude as m partons of momenta p_1, \dots, p_m become simultaneously collinear to some light-like direction \tilde{P} . In this limit, the leading behaviour is described by the amplitude for the production of a massless particle of momentum \tilde{P} from a scattering of the particles that do not take part in the collinear limit, multiplied by a universal factor, termed the *splitting amplitude*, which depends only on the m partons in the collinear set.

In order to parametrise the approach to the collinear limit, we introduce a light-cone decomposition for all the momenta in the m -parton collinear set,

$$p_i^\mu = x_i \tilde{P}^\mu + k_{\perp i}^\mu - \frac{k_{\perp i}^2}{2x_i} \frac{n^\mu}{\tilde{P} \cdot n}, \quad i = 1, \dots, m, \quad (2.1)$$

where the light-like momentum \tilde{P} specifies the collinear direction, $\tilde{P} \cdot k_{\perp i} = 0$, x_i are the longitudinal momentum fractions with respect to the parent momentum $P^\mu = \sum_{i=1}^m p_i^\mu$ and n^μ is an auxiliary light-like vector such that $n \cdot k_{\perp i} = 0$ and $n \cdot p_i \neq 0 \neq n \cdot \tilde{P}$, and which specifies how the collinear direction is approached. The collinear limit is then defined as the limit in which the transverse momenta $k_{\perp i}$ approach zero at the same rate. This definition of the collinear limit is frame-independent, and it only depends on the collinear direction \tilde{P} and the transverse momenta $k_{\perp i}$. In particular it is independent of the choice of the auxiliary vector n .

The variables that appear in eq. (2.1) are unconstrained apart from on-shellness and transversality, $n \cdot k_{\perp i} = \tilde{P} \cdot k_{\perp i} = 0$, and so the sums of the momentum fractions x_i and

the transverse momenta $k_{\perp i}$ are unconstrained. However, the collinear limit is invariant under longitudinal boosts in the direction of the parent momentum $P = \sum_{i=1}^m p_i$. We trade x_i and $k_{\perp i}$ for new quantities z_i and $\tilde{k}_{\perp i}$ that are boost-invariant in the direction of the parent momentum. In refs. [54, 73], it was shown that a convenient set of such variables is given by

$$z_i = \frac{x_i}{\sum_{j=1}^m x_j} = \frac{p_i \cdot n}{P \cdot n}, \quad \tilde{k}_{\perp i}^\mu = k_{\perp i}^\mu - z_i \sum_{j=1}^m k_{\perp j}^\mu, \quad i = 1, \dots, m. \quad (2.2)$$

It is easy to see that these new variables satisfy the constraints,

$$\sum_{i=1}^m z_i = 1 \quad \text{and} \quad \sum_{i=1}^m \tilde{k}_{\perp i}^\mu = 0. \quad (2.3)$$

From now on, we only work with these variables, and in order to avoid cluttering notation, we shall drop the tilde on the transverse momenta.

In the limit where a subset of massless particles is collinear, a scattering amplitude factorises as [75–77]

$$\begin{aligned} \mathcal{C}_{1\dots m} \mathcal{M}_{f_1\dots f_n}^{c_1\dots c_n; s_1\dots s_n}(p_1, \dots, p_n) \\ = \mathbf{Sp}_{ff_1\dots f_m}^{c, c_1\dots c_m; s, s_1\dots s_m} \mathcal{M}_{ff_{m+1}\dots f_n}^{c, c_{m+1}\dots c_n; s, s_{m+1}\dots s_n}(\tilde{P}, p_{m+1}, \dots, p_n), \end{aligned} \quad (2.4)$$

where $\mathcal{C}_{1\dots m}$ indicates that the equality holds up to terms that are power-suppressed in the collinear limit, while f , s and c respectively denote the flavour, spin and colour indices of the parent particle. The quantity \mathbf{Sp} appearing on the right-hand side is the splitting amplitude, which depends only on the kinematics and the quantum numbers in the collinear set.

For an amplitude whose collinear massless particles occur all in the final state, the factorisation in eq. (2.4) is valid to all orders in perturbation theory.³ Accordingly, also the squared matrix element factorises,

$$|\mathcal{M}_{f_1\dots f_n}(p_1, \dots, p_n)|^2 \equiv \sum_{\substack{(s_1, \dots, s_n) \\ (c_1, \dots, c_n)}} |\mathcal{M}_{f_1\dots f_n}^{c_1\dots c_n; s_1\dots s_n}(p_1, \dots, p_n)|^2, \quad (2.5)$$

where in the short-hand notation of the left-hand side the sum over all spin and colour indices of the matrix element is understood. The factorisation of the squared matrix element can be written as

$$\mathcal{C}_{1\dots m} |\mathcal{M}_{f_1\dots f_n}(p_1, \dots, p_n)|^2 = \left(\frac{2\mu^{2\epsilon} g_s^2}{s_{1\dots m}} \right)^{m-1} \hat{P}_{f_1\dots f_m}^{ss'} \mathcal{T}_{ff_{m+1}\dots f_n}^{ss'}(\tilde{P}, p_{m+1}, \dots, p_n), \quad (2.6)$$

³When the subset of collinear particles contains also initial-state particles, the factorisation in eq. (2.4) is valid in general only for tree amplitudes [78].

where g_s is the strong coupling constant and μ is the scale introduced by dimensional regularisation, and a sum over repeated indices, in this case the spin indices s and s' , is implicit, and we introduced the Mandelstam invariant,

$$s_{1\dots m} \equiv (p_1 + \dots + p_m)^2. \quad (2.7)$$

$\mathcal{T}_{ff_{m+1}\dots f_n}^{ss'}$ denotes the helicity tensor obtained by not summing over the spin indices of the parent parton,

$$\mathcal{T}_{ff_{m+1}\dots f_n}^{ss'} \equiv \sum_{\substack{(s_{m+1}, \dots, s_n) \\ (c, c_{m+1}, \dots, c_n)}} \mathcal{M}_{ff_{m+1}\dots f_n}^{c, c_{m+1} \dots c_n; s, s_{m+1} \dots s_n} \left[\mathcal{M}_{ff_{m+1}\dots f_n}^{c, c_{m+1} \dots c_n; s', s_{m+1} \dots s_n} \right]^*, \quad (2.8)$$

where for brevity we have suppressed the momenta on which the amplitude depends. The tensorial structure of the factorisation in eq. (2.6) is necessary to correctly capture all spin correlations. Due to colour conservation in the hard amplitude there are no non-trivial colour correlations, and we therefore sum over the colour c of the parent parton in eq. (2.8). The quantity $\hat{P}_{f_1 \dots f_m}^{ss'}$ in eq. (2.6) is the (polarised) splitting amplitude for the squared matrix element, which is related to \mathbf{Sp} by

$$\left(\frac{2\mu^{2\epsilon} g_s^2}{s_{1\dots m}} \right)^{m-1} \hat{P}_{f_1 \dots f_m}^{ss'} = \frac{1}{\mathcal{C}_f} \sum_{\substack{(s_1, \dots, s_m) \\ (c, c_1, \dots, c_m)}} \mathbf{Sp}_{ff_1 \dots f_m}^{c, c_1 \dots c_m; s, s_1 \dots s_m} \left[\mathbf{Sp}_{ff_1 \dots f_m}^{c, c_1 \dots c_m; s', s_1 \dots s_m} \right]^*, \quad (2.9)$$

where \mathcal{C}_f is the number of colour degrees of freedom of the parent parton with flavour f , i.e., $\mathcal{C}_g = N_c^2 - 1$ for a gluon and $\mathcal{C}_q = N_c$ for a quark. In eqs. (2.6), (2.9) and henceforth, the dependence of the splitting amplitude on the transverse momenta $\tilde{k}_{\perp i}$ and momentum fractions z_i of the particles in the collinear set is understood. Further, in QCD the flavour of the parent is uniquely determined by the flavours of the particles in the collinear set, thus we suppress the dependence of the splitting amplitude on the left-hand-side of eq. (2.9) on the flavour of the parent parton.

Splitting amplitudes for the squared matrix element have been computed at tree level for the emission of up to three collinear partons in refs. [53, 54], and for the emission of four collinear partons out of a parent quark in ref. [73]. The goal of this paper is to compute the tree-level splitting amplitudes for the squared matrix element for the emission of up to four partons out of a parent gluon, thus completing the set of splitting amplitudes for the emission of up to four collinear partons in QCD.⁴

⁴As in ref. [73], we refer to both \mathbf{Sp} and \hat{P} simply as splitting amplitudes.

Second, it is the metric tensor induced by the D -dimensional Minkowski metric on that space. Finally, it is the sum of all physical polarisations of a gluon with momentum p ,

$$\sum_s \varepsilon_s^\mu(p, n) \varepsilon_s^\nu(p, n) = d^{\mu\nu}(p, n), \quad (3.4)$$

where $\varepsilon_s^\mu(p, n)$ is the polarisation vector for a gluon with momentum p and transverse polarisation $s = 1, \dots, (D-2)$. In principle, we may choose a different reference vector for each gluon, as long as it is not orthogonal to the momentum. In our case, it is convenient to choose all gauge reference vectors to coincide with the reference vector n appearing in the definition of the collinear limit in eq. (2.1). We can then write the collinear factorisation for a parent gluon in terms of Lorentz indices rather than helicities [54],

$$\mathcal{C}_{1\dots m} |\mathcal{M}_{f_1\dots f_n}(p_1, \dots, p_n)|^2 = \left(\frac{2\mu^{2\epsilon} g_s^2}{s_{1\dots m}} \right)^{m-1} \hat{P}_{f_1\dots f_m}^{\mu\nu} \mathcal{T}_{ff_{m+1}\dots f_n, \mu\nu}(\tilde{P}, p_{m+1}, \dots, p_n), \quad (3.5)$$

where quantities with open Lorentz indices are obtained by amputating the polarisation vectors and inserting the polarisation sum in eq. (3.4). Note that in passing from eq. (2.6) to eq. (3.5) we have implicitly used gauge invariance to eliminate the gauge dependent terms in eq. (3.4). Indeed, since physical polarisation states are transverse, only the transverse part of a Lorentz tensor carries physical information. This is because the non-transverse part vanishes upon contraction with a physical polarisation vector. As a consequence, the helicity tensor $\mathcal{T}_{ff_{m+1}\dots f_n}^{\mu\nu}$ in eq. (3.5) can be chosen to satisfy the transversality condition,

$$\tilde{P}_\mu \mathcal{T}_{ff_{m+1}\dots f_n}^{\mu\nu} = \tilde{P}_\nu \mathcal{T}_{ff_{m+1}\dots f_n}^{\mu\nu} = 0. \quad (3.6)$$

With this choice, the complete tensor structure of the splitting amplitude contains terms involving the transverse momenta of the collinear partons [54],

$$\hat{P}_{f_1\dots f_m}^{\mu\nu} = g^{\mu\nu} A_{f_1\dots f_m}^{(g)} + \sum_{i,j=1}^m \frac{\tilde{k}_{\perp i}^\mu \tilde{k}_{\perp j}^\nu}{s_{1\dots m}} B_{ij, f_1\dots f_m}^{(g)}. \quad (3.7)$$

We stress, however, that the splitting amplitude defined in this way does not vanish upon contraction with \tilde{P} .

Since we work in axial gauge, we do not consider the subset of Feynman diagrams where collinear partons are separated by a hard propagator. The sum of all relevant

diagrams can be cast in the form,

$$\mathcal{C}_{1\dots m} |\mathcal{M}_{f_1\dots f_n}(p_1, \dots, p_n)|^2 = \mathcal{C}_{1\dots m} \left[\left(\frac{2\mu^{2\epsilon} g_s^2}{s_{1\dots m}} \right)^{m-1} \left[\mathcal{M}_{ff_{m+1}\dots f_n}^{(n)s} \right]^* V_{f_1\dots f_m}^{(n)ss'}(p_1, \dots, p_m) \mathcal{M}_{ff_{m+1}\dots f_n}^{(n)s'} \right], \quad (3.8)$$

where a sum over the spin indices s, s' of the intermediate state is understood, and we suppress all colour and spin indices of the external partons. Here $\mathcal{M}_{ff_{m+1}\dots f_n}^{(n)s} \equiv \mathcal{M}_{ff_{m+1}\dots f_n}^{(n)s}(P, p_{m+1}, \dots, p_n)$ denotes the sum of all Feynman diagrams with an off-shell leg with momentum P , flavour f and spin s . Note that this subset of Feynman diagrams is by itself not gauge invariant, and the superscript (n) indicates the dependence on the reference vector. The squared off-shell current $V_{f_1\dots f_m}^{(n)ss'}$ may be written as the interference of two colour-dressed off-shell currents,

$$\left(\frac{2\mu^{2\epsilon} g_s^2}{s_{1\dots m}} \right)^{m-1} V_{f_1\dots f_m}^{(n)ss'}(p_1, \dots, p_m) = \frac{1}{\mathcal{C}_f} \sum_{\substack{(s_1, \dots, s_m) \\ (c, c_1, \dots, c_m)}} \left[J_{f_1\dots f_m}^{c, c_1\dots c_m; s' s_1\dots s_m} \right]^* J_{f_1\dots f_m}^{c, c_1\dots c_m; s s_1\dots s_m}, \quad (3.9)$$

where \mathcal{C}_f is defined after eq. (2.9). Note that also $V_{f_1\dots f_m}^{(n)ss'}$ depends on the gauge vector n before the collinear limit is taken. Since the collinear limit is gauge invariant, this dependence disappears in the limit, and the squared off-shell current reduces to the splitting amplitude,

$$\mathcal{C}_{1\dots m} V_{f_1\dots f_m}^{(n)ss'}(p_1, \dots, p_m) = \hat{P}_{f_1\dots f_m}^{ss'}. \quad (3.10)$$

We have computed all gluon-initiated splitting amplitudes up to $m = 4$, and we reproduce all known results for the cases $m = 2$ and 3. The results for $m = 4$ are new and are presented for the first time in this paper. There are four different gluon-initiated splitting amplitudes,

$$g \rightarrow \bar{q}' q' \bar{q} q, \quad g \rightarrow \bar{q} q \bar{q} q, \quad g \rightarrow \bar{q} g g q, \quad g \rightarrow g g g g. \quad (3.11)$$

In the remainder of this section we discuss in more detail the computation of these splitting amplitudes. The explicit results are available in computer-readable form [74].

Let us start by discussing the simplest splitting process, the collinear decay $g \rightarrow \bar{q}' q' \bar{q} q$ with different quark flavours. There are five diagrams that contribute to the off-shell current $J_{\bar{q}' q' \bar{q} q}^\mu$ in eq. (3.9). The diagrams are shown in fig. 1. Going through the steps outlined above, we find that the result for the splitting amplitude $g \rightarrow \bar{q}' q' \bar{q} q$ can be decomposed into an ‘abelian’ and a ‘non-abelian’ part,

$$\hat{P}_{\bar{q}'_1 q'_2 \bar{q}_3 q_4}^{\mu\nu} = \frac{1}{4} C_F \hat{P}_{\bar{q}'_1 q'_2 \bar{q}_3 q_4}^{\mu\nu(\text{ab})} + \frac{1}{4} C_A \hat{P}_{\bar{q}'_1 q'_2 \bar{q}_3 q_4}^{\mu\nu(\text{nab})}, \quad (3.12)$$

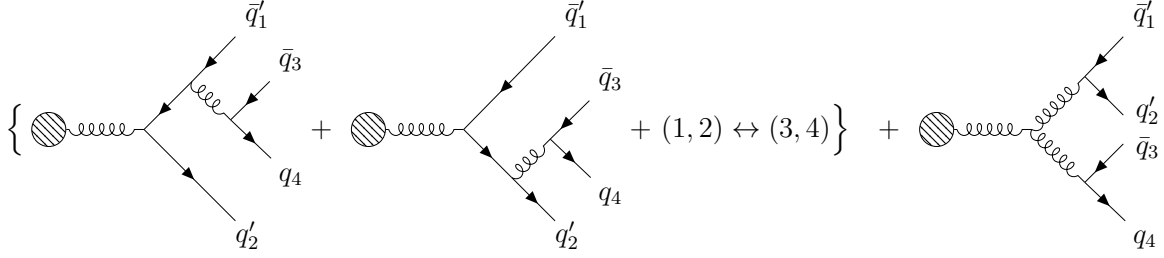


Figure 1. The diagrams contributing to the off-shell current $g \rightarrow \bar{q}'_1 q'_2 \bar{q}_3 q_4$. In the case of identical quarks we also need to include diagrams where the anti-quarks q_1 and q_3 are exchanged.

where the indices carried by the parton label refer to the indices of the momenta and the momentum fractions of the partons, and C_F and C_A denote the quadratic Casimirs of the fundamental and adjoint representations of $SU(N)$,

$$C_F = \frac{N^2 - 1}{2N}, \quad C_A = N. \quad (3.13)$$

The splitting process $g \rightarrow \bar{q}q\bar{q}q$, in which the final state quarks have the same flavour, $q' = q$, includes diagrams where the anti-quarks 1 and 3 are exchanged (or equivalently, where the quarks 2 and 4 are exchanged). This naturally leads to the following representation of the splitting amplitude,

$$\hat{P}_{\bar{q}_1 q_2 \bar{q}_3 q_4}^{\mu\nu} = \left[\hat{P}_{\bar{q}'_1 q'_2 \bar{q}_3 q_4}^{\mu\nu} + (1 \leftrightarrow 3) \right] + \hat{P}_{\bar{q}_1 q_2 \bar{q}_3 q_4}^{\mu\nu}(\text{id}). \quad (3.14)$$

The term in square brackets contains the splitting amplitude in eq. (3.12) in the case of different flavours and the exchange contributions obtained by permuting the external quarks. The last term in eq. (3.14) is new and captures interference contributions from identical quarks. It is again convenient to display the result in terms of colour factors,

$$\hat{P}_{\bar{q}_1 q_2 \bar{q}_3 q_4}^{\mu\nu}(\text{id}) = \frac{1}{2} C_F (C_A - 2C_F) \hat{P}_{\bar{q}_1 q_2 \bar{q}_3 q_4}^{\mu\nu}(\text{id})_1 + \frac{1}{2} C_A (C_A - 2C_F) \hat{P}_{\bar{q}_1 q_2 \bar{q}_3 q_4}^{\mu\nu}(\text{id})_2. \quad (3.15)$$

Since $C_A - 2C_F = \frac{1}{N}$, the interference contributions are colour suppressed.

Next, let us discuss the splitting amplitude $g \rightarrow \bar{q}qgg$. The Feynman diagrams contributing to the off-shell current $J_{\bar{q}qgg}^\mu$ are shown in fig. 2. As usual, we can decompose the splitting amplitude into contributions from different colour factors as follows,

$$\hat{P}_{\bar{q}_1 g_2 g_3 q_4}^{\mu\nu} = \frac{1}{2} C_F^2 \hat{P}_{\bar{q}_1 g_2 g_3 q_4}^{\mu\nu}(\text{ab}) + \frac{1}{2} C_A^2 \hat{P}_{\bar{q}_1 g_2 g_3 q_4}^{\mu\nu}(\text{nab})_1 + \frac{1}{2} C_A C_F \hat{P}_{\bar{q}_1 g_2 g_3 q_4}^{\mu\nu}(\text{nab})_2. \quad (3.16)$$

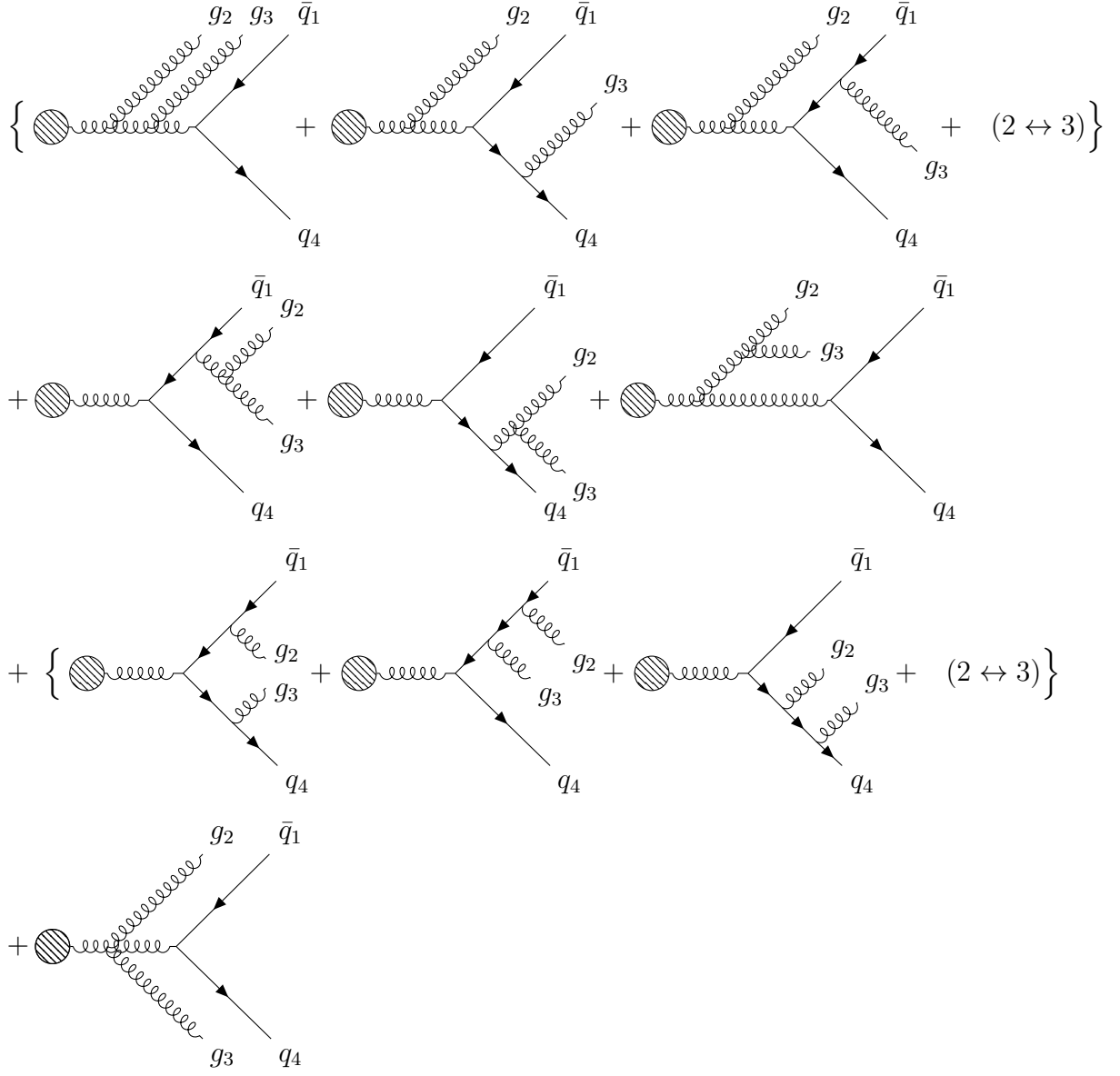


Figure 2. The diagrams contributing to the collinear decay $g \rightarrow \bar{q}_1 g_2 g_3 q_4$.

It is possible to express each colour coefficient in eq. (3.16) in a reduced form by exploiting the symmetry under the exchange of the two external partons,

$$\hat{P}_{\bar{q}_1 g_2 g_3 q_4}^{\mu\nu(X)} = \left(\hat{P}_{\bar{q}_1 g_2 g_3 q_4}^{\mu\nu(X) \text{ symm.}} + (1 \leftrightarrow 4) \right) + (2 \leftrightarrow 3), \quad (3.17)$$

where $(X) \in \{(ab), (nab)_1, (nab)_2\}$.

Finally, let us discuss the pure gluon splitting process $g \rightarrow gggg$, which poses a challenge due to the large degree of Bose symmetry under the exchange of the external gluons. The diagrams contributing to the decay are shown in fig. 3. An important step in the computation of the splitting amplitude was to take into account symmetries between different permutations of the four external gluons in order to minimise the number of terms. We can write $\hat{P}_{gggg}^{\mu\nu}$ in a symmetrised form as

$$\hat{P}_{g_1 g_2 g_3 g_4}^{\mu\nu} = \hat{P}_{g_1 g_2 g_3 g_4}^{\mu\nu \text{ symm.}} + (11 \text{ permutations of } g_1 g_2 g_3 g_4). \quad (3.18)$$

The above permutations do not include orderings of the external gluons which leave the first diagram in fig. 3 invariant.

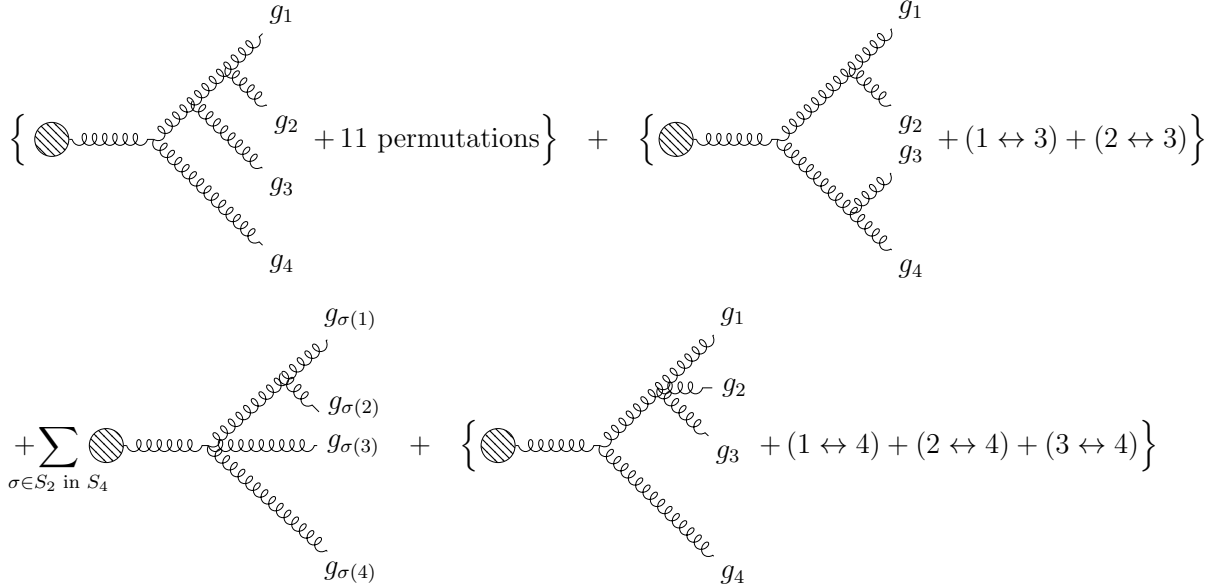


Figure 3. The diagrams contributing to the splitting process $g \rightarrow g_1 g_2 g_3 g_4$. In the third diagram we sum over the 6 possible pairings of the partons in the three-gluon vertex.

4 Nested collinear limits

In this section we analyse the collinear limit of the splitting amplitudes themselves, i.e., we study their behaviour in the limit where a subset of collinear partons is more collinear than the others. To be concrete, let us consider a collection of m partons with

flavour indices $\{f_1, \dots, f_{m'}, \dots, f_m\}$ and momenta $\{p_1, \dots, p_{m'}, \dots, p_m\}$, with $m' < m$. We always think of these partons as being part of an on-shell n -point amplitude $\mathcal{M}_{f_1 \dots f_n}$ involving $(n-m)$ additional coloured partons. Our goal is to study the behaviour of the amplitude in the limit where $\{p_1, \dots, p_{m'}\}$ become collinear to some lightlike direction \tilde{P}' , and $\{\tilde{P}', p_{m'+1}, \dots, p_m\}$ are collinear to another lightlike direction \tilde{P} . Depending on the order in which the different collinear limits are taken, there are two different scenarios of how such a kinematic configuration can be reached, referred to as *iterated* and *strongly-ordered* collinear limits in ref. [73]. The (splitting) amplitudes factorise in the same way in each of the limits, and the factorisation involves the same universal quantities in both cases [73]. We therefore focus here only on the strongly-ordered limit from now on.

We start by giving a precise definition of the strongly-ordered collinear limit. We perform separate light-cone decompositions in each of the m - and m' -parton sets. For the m -parton set, we will use the notations and conventions of eq. (2.1). For the m' -parton subset we write

$$p_i^\mu = y_i \tilde{P}'^\mu + \kappa_{\perp i}^\mu - \frac{\kappa_{\perp i}^2}{2y_i} \frac{n'^\mu}{n' \cdot \tilde{P}'}, \quad i = 1, \dots, m', \quad (4.1)$$

with $n'^2 = \tilde{P}'^2 = \tilde{P}' \cdot \kappa_{\perp i} = n' \cdot \kappa_{\perp i} = 0$. The momenta \tilde{P} and \tilde{P}' indicate the directions to which the partons in each set become collinear. We stress that at this point the lightcone directions \tilde{P}' and n' in eq. (4.1) are not related to the quantities \tilde{P} and n in eq. (2.1). However, without loss of generality, we may choose $n' = n$, and we work in the axial gauge where the reference vectors of all external and internal gluons are n . For more details about the parametrisation of the strongly-ordered collinear limit, we refer to ref. [73]. With this setup, the strongly-ordered collinear limit is defined in analogy with the ordinary collinear limit in Sec. 2: the vectors $k_{\perp i}^\mu$ and $\kappa_{\perp i}^\mu$ parametrise the transverse distance to the planes spanned by (\tilde{P}, n) and (\tilde{P}', n) , respectively. The strongly-ordered collinear limit where the m' -parton subset is more collinear than the m -parton set is defined as the limit where both $k_{\perp i}^\mu$ and $\kappa_{\perp i}^\mu$ approach zero, but the $\kappa_{\perp i}^\mu$ tend to zero faster than the $k_{\perp i}^\mu$. We can implement the operation of taking this limit by a uniform rescaling of the transverse momenta in each collinear set by a different parameter,

$$k_{\perp i} \rightarrow \lambda k_{\perp i}, \quad \kappa_{\perp i} \rightarrow \lambda' \kappa_{\perp i}, \quad (4.2)$$

and keeping the dominant singular terms of order $1/\lambda'^{2(m'-1)}\lambda^{2(m-m')}$ in the limit $\lambda, \lambda' \rightarrow 0$ with $\lambda \gg \lambda'$.

The leading behaviour of an amplitude in the strongly-ordered collinear limit is described by a factorisation formula very similar to eq. (2.6) [73],

$$\begin{aligned} \mathcal{C}_{(1\dots m')\dots m} \mathcal{C}_{1\dots m'} |\mathcal{M}_{f_1\dots f_n}(p_1, \dots, p_n)|^2 &= \left(\frac{2g_s^2 \mu^{2\epsilon}}{s_{1\dots m'}} \right)^{m'-1} \left(\frac{2g_s^2 \mu^{2\epsilon}}{s_{[1\dots m']\dots m}} \right)^{m-m'} \\ &\times \hat{P}_{f_1\dots f_{m'}}^{hh'} \hat{H}_{f_{(1\dots m')}f_{m'+1}\dots f_m}^{hh';ss'} \mathcal{T}_{ff_{m+1}\dots f_n}^{ss'}(\tilde{P}, p_{m'+1}, \dots, p_n), \end{aligned} \quad (4.3)$$

where

$$s_{[1\dots m']\dots m} = (\tilde{P}' + p_{m'+1} + \dots + p_m)^2. \quad (4.4)$$

The functions $\hat{P}_{f_1\dots f_{m'}}^{hh'}$ and $\mathcal{T}_{ff_{m+1}\dots f_n}^{ss'}$ are the splitting amplitude and the helicity tensor introduced in Sec. 2. The *splitting tensor* $\hat{H}_{f_{(1\dots m')}f_{m'+1}\dots f_m}^{hh';ss'}$ is new. It is obtained by squaring the amplitude-level splitting amplitude without summing over the helicities of one of the partons in the collinear set (cf. eq. (2.9)),

$$\begin{aligned} &\left(\frac{2g_s^2 \mu^{2\epsilon}}{s_{[1\dots m']\dots m}} \right)^{m-m'} \hat{H}_{f_{(1\dots m')}f_{m'+1}\dots f_m}^{hh';ss'} \\ &= \frac{1}{\mathcal{C}_f} \sum_{\substack{(s_{m'+1}, \dots, s_m) \\ (c_{m'+1}, \dots, c_m)}} \mathbf{Sp}_{f_{(1\dots m')}f_{m'+1}\dots f_m}^{c, c_{m'+1}\dots c_m; s, h, s_{m'+1}\dots s_m} \left[\mathbf{Sp}_{f_{(1\dots m')}f_{m'+1}\dots f_m}^{c, c_{m'+1}\dots c_m; s', h', s_{m'+1}\dots s_m} \right]^*, \end{aligned} \quad (4.5)$$

where \mathcal{C}_f is defined after eq. (2.9). Just like in Sec. 2 we suppress the dependence of all splitting amplitudes and tensors on their arguments. The factorisation of the squared amplitude in the strongly-ordered limit can be cast in the form of a factorisation of the splitting amplitude itself,

$$\mathcal{C}_{1\dots m'} \hat{P}_{f_1\dots f_m}^{ss'} = \left(\frac{s_{[1\dots m']\dots m}}{s_{1\dots m'}} \right)^{m'-1} \hat{P}_{f_1\dots f_{m'}}^{hh'} \hat{H}_{f_{(1\dots m')}f_{m'+1}\dots f_m}^{hh';ss'}. \quad (4.6)$$

By comparing eqs. (2.6) and (4.3), it is easy to see that upon summing over the helicities (h, h') the splitting tensor reduces to an ordinary splitting amplitude,

$$\delta^{hh'} \hat{H}_{f_{(1\dots m')}f_{m'+1}\dots f_m}^{hh';ss'} = \hat{P}_{f_{(1\dots m')}f_{m'+1}\dots f_m}^{ss'}. \quad (4.7)$$

In the following we refer to the partons with spin indices (s, s') and (h, h') as the *parent* and *sub-parent*, respectively. Depending on the flavour of the parent and the sub-parent, the structure of the splitting tensor can be further simplified. The case where the parent is a quark was considered in ref. [73]. Here we only consider the case where the parent is a gluon. Using a similar argument as for the splitting amplitude in

Sect. 3, we can trade helicity indices (s, s') for Lorentz indices (μ, ν) and write eq. (4.3) in the equivalent form,

$$\begin{aligned} \mathcal{C}_{(1\dots m')\dots m} \mathcal{C}_{1\dots m'} |\mathcal{M}_{f_1\dots f_n}(p_1, \dots, p_n)|^2 &= \left(\frac{2g_s^2 \mu^{2\epsilon}}{s_{1\dots m'}} \right)^{m'-1} \left(\frac{2g_s^2 \mu^{2\epsilon}}{s_{[1\dots m']\dots m}} \right)^{m-m'} \\ &\times \hat{P}_{f_1\dots f_{m'}}^{hh'} \hat{H}_{f_{(1\dots m')}f_{m'+1}\dots f_m}^{\alpha\beta;\mu\nu} \mathcal{T}_{ff_{m+1}\dots f_n, \mu\nu}(\tilde{P}, p_{m'+1}, \dots, p_n), \end{aligned} \quad (4.8)$$

If the sub-parent is a quark, helicity must be conserved, and the splitting tensor is diagonal in the spin indices (h, h') of the sub-parent. Equation (4.7) then implies

$$\mathcal{C}_{1\dots m'} \hat{P}_{f_1\dots f_m}^{\mu\nu} = \left(\frac{s_{[1\dots m']\dots m}}{s_{1\dots m'}} \right)^{m'-1} \langle \hat{P}_{f_1\dots f_{m'}} \rangle \hat{P}_{qf_{m'+1}\dots f_m}^{\mu\nu}, \quad (4.9)$$

where $\langle \hat{P}_{f_1\dots f_{m'}} \rangle$ denotes the unpolarised splitting amplitude,

$$\langle \hat{P}_{f_1\dots f_m} \rangle \equiv \frac{1}{N_{\text{pol}}} \delta_{hh'} \hat{P}_{f_1\dots f_m}^{hh'}, \quad (4.10)$$

and N_{pol} denotes the number of physical polarisation states for the parent parton. We work in conventional dimensional regularisation (CDR), where the quarks and gluons have 2 and $(D-2)$ polarisation states, respectively.

If also the sub-parent is a gluon, we can use a similar argument to that of Sect. 3 to trade in the helicity indices (h, h') for Lorentz indices (α, β) by amputating external polarisation vectors and contracting with polarisation tensors. Since only transverse polarisations are physical, only the transverse part of a Lorentz tensor carries physical information. Thus, we can write eq. (4.8) in the equivalent form,

$$\begin{aligned} \mathcal{C}_{(1\dots m')\dots m} \mathcal{C}_{1\dots m'} |\mathcal{M}_{f_1\dots f_n}(p_1, \dots, p_n)|^2 &= \left(\frac{2g_s^2 \mu^{2\epsilon}}{s_{1\dots m'}} \right)^{m'-1} \left(\frac{2g_s^2 \mu^{2\epsilon}}{s_{[1\dots m']\dots m}} \right)^{m-m'} \\ &\times \hat{P}_{f_1\dots f_{m'}, \alpha\beta}^{\alpha\beta} \hat{H}_{f_{(1\dots m')}f_{m'+1}\dots f_m}^{\alpha\beta;\mu\nu} \mathcal{T}_{ff_{m+1}\dots f_n, \mu\nu}(\tilde{P}, p_{m'+1}, \dots, p_n), \end{aligned} \quad (4.11)$$

and the relation in eq. (4.7) becomes

$$\hat{H}_{f_{(1\dots m')}f_{m'+1}\dots f_m}^{\alpha\beta;\mu\nu} d_{\alpha\beta}(\tilde{P}', n) = \hat{P}_{f_{(1\dots m')}f_{m'+1}\dots f_m}^{\mu\nu} + \text{gauge terms}. \quad (4.12)$$

In App. B we show that the most general tensor structure of the splitting tensor is

$$\begin{aligned}
\hat{H}_{gf_{m'+1}\dots f_m}^{\alpha\beta;\mu\nu} &= d^{\alpha\beta}(\tilde{P}, n) g^{\mu\nu} A_{gf_{m'+1}\dots f_m}^{(g)} + \sum_{i,j=m'+1}^m \frac{k_{\perp i}^\mu k_{\perp j}^\nu}{S_{[1\dots m']\dots m}} d^{\alpha\beta}(\tilde{P}, n) B_{ij,gf_{m'+1}\dots f_m}^{(g)} \\
&+ \sum_{i,j=m'+1}^m \frac{k_{\perp i}^\alpha k_{\perp j}^\beta}{S_{[1\dots m']\dots m}} g^{\mu\nu} C_{ij,gf_{m'+1}\dots f_m}^{(g)} + \sum_{i,j,k,l=m'+1}^m \frac{k_{\perp i}^\mu k_{\perp j}^\nu k_{\perp k}^\alpha k_{\perp l}^\beta}{S_{[1\dots m']\dots m}^2} D_{ijkl,gf_{m'+1}\dots f_m}^{(g)} \\
&+ \sum_{i,j=m'+1}^m \frac{g^{\alpha\mu} k_{\perp i}^\beta k_{\perp j}^\nu + g^{\beta\nu} k_{\perp i}^\alpha k_{\perp j}^\mu + g^{\alpha\nu} k_{\perp i}^\beta k_{\perp j}^\mu + g^{\beta\mu} k_{\perp i}^\alpha k_{\perp j}^\nu}{S_{[1\dots m']\dots m}} E_{ij,gf_{m'+1}\dots f_m}^{(g)} \\
&+ (g^{\alpha\nu} g^{\beta\mu} + g^{\alpha\mu} g^{\beta\nu}) F_{gf_{m'+1}\dots f_m}^{(g)}. \tag{4.13}
\end{aligned}$$

At this point we have to make some comments about eq. (4.13). First, let us discuss the symmetry properties of the splitting tensor $\hat{H}_{gf_{m'+1}\dots f_m}^{\alpha\beta;\mu\nu}$. From its definition in eq. (4.5) it follows that the splitting tensor must be symmetric under the exchange $(\mu, \alpha) \leftrightarrow (\nu, \beta)$. In eq. (4.3) it is contracted with $\hat{P}^{\alpha\beta}$ and $\mathcal{T}^{\mu\nu}$, which are symmetric tensors at tree level. Hence, only the part of $\hat{H}_{gf_{m'+1}\dots f_m}^{\alpha\beta;\mu\nu}$ that is individually symmetric under $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$ enters the factorisation in eq. (4.3), and we only present here the part of the splitting tensor with this enlarged symmetry. Second, we see that eq. (4.13) involves a mixture of metric tensors $g^{\rho\sigma}$ and polarisation tensors $d^{\rho\sigma}(\tilde{P}, n)$. The tensor structure given in app. B involves only polarisation tensors, which would make the splitting tensor explicitly transverse. However, similar to the case of the splitting amplitude discussed in Sec. 3 (cf. eq. (3.7)), some of the gauge-dependent terms drop out in the contraction in eq. (4.11). In particular, we can perform the replacements,

$$d^{\rho\sigma}(\tilde{P}, n) \leftrightarrow -g^{\rho\sigma}, \quad (\rho, \sigma) \in \{(\mu, \nu), (\alpha, \mu), (\beta, \nu), (\alpha, \nu), (\beta, \mu)\}. \tag{4.14}$$

For $(\rho, \sigma) = (\mu, \nu)$, the equivalence between the polarisation tensor and the metric tensor follows from eq. (3.6), while the other cases follow from relations like

$$\begin{aligned}
\hat{P}_{f_1\dots f_{m'}, \alpha\beta} d^{\alpha\mu}(\tilde{P}, n) d^{\beta\nu}(\tilde{P}, n) \mathcal{T}_{ff_{m+1}\dots f_n, \mu\nu} &= \hat{P}_{f_1\dots f_{m'}, \alpha\beta} g^{\alpha\mu} g^{\beta\nu} \mathcal{T}_{ff_{m+1}\dots f_n, \mu\nu}, \\
\hat{P}_{f_1\dots f_{m'}, \alpha\beta} d^{\alpha\mu}(\tilde{P}, n) k_{\perp i}^\beta k_{\perp j}^\nu \mathcal{T}_{ff_{m+1}\dots f_n, \mu\nu} &= -\hat{P}_{f_1\dots f_{m'}, \alpha\beta} g^{\alpha\mu} k_{\perp i}^\beta k_{\perp j}^\nu \mathcal{T}_{ff_{m+1}\dots f_n, \mu\nu}.
\end{aligned} \tag{4.15}$$

We have checked that our results for the quadruple splitting amplitudes have the correct behaviour in all strongly-ordered collinear limits, i.e., they satisfy eq. (4.6) for $m' = 2$ and 3. The strongly-ordered limit of the quadruple splitting amplitudes involves the splitting tensors with two or three collinear particles in the final state. The relevant

splitting tensors for two collinear partons can be found in ref. [31],

$$\begin{aligned}\hat{H}_{\bar{q}q}^{hh';\mu\nu} &= \frac{1}{2}\delta^{hh'}\hat{P}_{\bar{q}q}^{\mu\nu}, \\ \hat{H}_{gg}^{\alpha\beta;\mu\nu} &= 2C_A \left[\frac{1-z}{2z} (g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu}) + \frac{z}{1-z}g^{\mu\nu}\frac{k_{\perp}^{\alpha}k_{\perp}^{\beta}}{k_{\perp}^2} - z(1-z)\frac{k_{\perp}^{\mu}k_{\perp}^{\nu}}{k_{\perp}^2}d^{\alpha\beta}(\tilde{P}, n) \right],\end{aligned}\tag{4.16}$$

where we set $k_{\perp} = k_{1\perp} = -k_{2\perp}$. Note that, compared to ref. [31], we express the splitting tensor $\hat{H}_{gg}^{\alpha\beta;\mu\nu}$ in a form that is individually symmetric in (μ, ν) and (α, β) . In addition, the two-parton splitting tensor $\hat{H}_{gg}^{\alpha\beta;\mu\nu}$ is special in that certain tensor structures do not appear. The coefficients $A_{gg}^{(g)}$ and $D_{3333,gg}^{(g)}$ are subleading in the collinear limit, while $E_{33,gg}^{(g)}$ vanishes in the explicitly symmetric form (4.13). The relevant splitting tensors for three collinear partons are new and are given in ref. [73] and in App. C.

5 Conclusions

In this paper, we have computed the quadruple-collinear splitting amplitudes for a gluon parent in CDR. Combined with our previous results for a quark parent [73], this completes the set of tree-level splitting amplitudes describing all collinear singularities for the emission of up to four collinear partons at N³LO. Our results are available in computer-readable form online [74]. We have also considered the strongly-ordered limit when a subset of the four collinear partons become collinear to each other, and we have derived the corresponding factorisation formulæ. Our results satisfy the expected factorisations in all strongly-ordered limits, which provides a strong check on the correctness of our computations.

Our results are an important building block towards understanding the complete infrared structure of massless QCD amplitudes at N³LO, which is a cornerstone to construct a subtraction method at this order. Indeed, firstly, the purely virtual infrared singularities of massless amplitudes with up to three loops are completely known [80–85]. Secondly, when our results are combined with the results for the one-loop emission of up to three collinear particles [52–58, 60, 67, 68]⁶ and the two-loop splitting amplitudes for two collinear partons [61–63], they provide a complete description of all collinear singularities up to N³LO. Finally, soft emissions are known for the tree-level emission of up to three soft partons [54, 70, 86, 87], and at one and two loops for the emission of a single soft gluon [56, 58, 59, 64–66]. The soft current describing the emission of a pair of two soft partons at one-loop, however, is still missing. For the future,

⁶But for the one-loop collinear splitting amplitude $q \rightarrow ggq$, which at present is unknown.

it would be interesting to compute this current and to complete the description of all infrared singularities of massless QCD amplitudes at N³LO.

Acknowledgements

CD and MM acknowledge the hospitality of the ETH Zurich, and MM also acknowledges the hospitality of the TH Department of CERN, at various stages of this work. This work was supported in part by the ERC starting grant 637019 ‘‘MathAm’’ (CD), the FRIA grant of the Fonds National de la Recherche Scientifique (FNRS), Belgium (MM), the European Research Council (ERC) under grant agreement No 694712 (PertQCD) and the Swiss National Science Foundation (SNF) under contract agreement No 179016 (VD, RH, AL).

A The iterated collinear limit

The strongly-ordered amplitude [73],

$$\hat{P}_{f_1\dots f_m}^{\text{s.o.};ss'} = \hat{P}_{f_1\dots f_{m'}}^{hh'} \hat{H}_{f_{(1\dots m')}f_{m'+1}\dots f_m}^{hh';ss'} \quad (\text{A.1})$$

depends on the quantum numbers and light-cone kinematics of both the m -parton collinear set and its m' -parton subset. It is obtained by summing over the helicities (h, h') of the parent parton of the collinear subset. In case the sub-parent with helicities (h, h') is a quark, following the factorisation in eq. (4.9), the strongly-ordered splitting amplitude has the same tensor structure as an ordinary splitting amplitude. In case both the parent with helicities (s, s') and the sub-parent are gluons, the strongly-ordered amplitude has a similar tensor structure as that of a gluon splitting amplitude,

$$\hat{P}_{f_1\dots f_m}^{\text{s.o.};\mu\nu} = g^{\mu\nu} A_{f_1\dots f_m}^{\text{s.o.}} + \sum_{i,j=1}^{m'} \frac{\kappa_{\perp i}^\mu \kappa_{\perp j}^\nu}{s_{1\dots m'}} B_{ij,f_1\dots f_m}^{\text{s.o.}} + \sum_{k,l=m'+1}^m \frac{k_{\perp k}^\mu k_{\perp l}^\nu}{s_{m'+1\dots m}} C_{kl,f_1\dots f_m}^{\text{s.o.}} \quad (\text{A.2})$$

It depends on the transverse momenta $\kappa_{\perp i}^\mu$ of the collinear subset, $i = 1, \dots, m'$ and those of the other $(m - m')$ -collinear partons unaffected by the strongly-ordered limit, $k_{\perp j}^\mu$ with $j = m' + 1, \dots, m$.

The strongly-ordered splitting amplitude can be obtained by performing the m' -parton iterated limit on the m -parton splitting amplitude,

$$\hat{P}_{f_1\dots f_m}^{\text{s.o.};\mu\nu} = \left(\frac{s_{1\dots m'}}{s_{[1\dots m']\dots m}} \right)^{m'-1} \mathcal{C}_{1\dots m'} \hat{P}_{f_1\dots f_m}^{\mu\nu} \quad (\text{A.3})$$

The steps involved are essentially the same as for a quark parent, which was considered in ref. [73]. However, as a first step, we need to apply the relation

$$k_{\perp i}^{\mu} = \zeta_i K^{\mu} + \kappa_{\perp i}^{\mu} + \frac{K \cdot \kappa_{\perp i}}{\alpha} \frac{n^{\mu}}{n \cdot \tilde{P}}, \quad i = 1, \dots, m', \quad (\text{A.4})$$

where $K^{\mu} = \sum_{j=1}^{m'} k_{\perp j}^{\mu} = -\sum_{j=m'+1}^m k_{\perp j}^{\mu}$ and $\alpha = \sum_{j=1}^{m'} z_j$. Then, we perform the steps outlined in App. C of ref. [73].

Comparing the strongly-ordered limit obtained from eq. (A.3) with that obtained from eq. (A.1), works as a strong check on the splitting amplitudes. This was done to check all possible two- and three-parton sub-limits.

B Tensor structure of gluon parent and sub-parent splitting tensors

In this appendix, we detail the general tensor structure of the gluon-initiated splitting tensor $\hat{H}_{gf_{m'+1} \dots f_m}^{\alpha\beta;\mu\nu}$ with a gluon sub-parent, defined in Sec. 4. This tensor appears as the collinear limit of the helicity tensor $\mathcal{T}^{\alpha\beta}$, defined in eq. (2.8). Therefore, we begin by considering an n -parton tree-level QCD process with its associated helicity tensor $\mathcal{T}_{f_1 \dots f_n}^{\alpha\beta}(p_1, \dots, p_n)$, where parton 1 carries the Lorentz indices (α, β) . Using the same kind of argument as for eq. (3.8), our gauge choice allows us to consider only diagrams that have a propagator carrying the momentum $P = \sum_{i=1}^m p_i$ of the parent gluon. This directly implies that the sub-diagrams splitting the parent parton into m partons will factorise (cf. eq. (3.8)), the only difference being that the spin indices of parton 1 are not summed over,

$$\begin{aligned} \mathcal{E}_{1 \dots m} \mathcal{T}_{f_1 \dots f_n}^{\alpha\beta}(p_1, \dots, p_n) = & \quad (\text{B.1}) \\ \mathcal{E}_{1 \dots m} \left[\left(\frac{2\mu^{2\epsilon} g_s^2}{s_{1 \dots m}} \right)^{m-1} \left[\mathcal{M}_{gf_{m+1} \dots f_n}^{(n)\mu} \right]^* H_{f_1 \dots f_m}^{(n)\alpha\beta;\mu\nu}(p_1, \dots, p_m) \mathcal{M}_{gf_{m+1} \dots f_n}^{(n)\nu} \right]. \end{aligned}$$

The squared off-shell current $H_{f_1 \dots f_m}^{(n)\alpha\beta;\mu\nu}$ is the interference of the colour-dressed off-shell currents splitting the parent parton into m collinear partons (cf. eq. (3.9)), with the Lorentz indices of parton 1 and the parent parton open,

$$\left(\frac{2\mu^{2\epsilon} g_s^2}{s_{1 \dots m}} \right)^{m-1} H_{f_1 \dots f_m}^{(n)\alpha\beta;\mu\nu} = \frac{1}{\mathcal{C}_f} \sum_{\substack{(s_2, \dots, s_m) \\ (c, c_1, \dots, c_m)}} \left[\mathcal{J}_{f_1 \dots f_m}^{c, c_1 \dots c_m; \nu\beta s_2 \dots s_m} \right]^* \mathcal{J}_{f_1 \dots f_m}^{c, c_1 \dots c_m; \mu\alpha s_2 \dots s_m}, \quad (\text{B.2})$$

where we have suppressed the dependence on the momenta. As usual the superscript (n) indicates the dependence on the reference vector. Given the definition in eq. (B.2),

$H_{f_1 \dots f_m}^{(n) \alpha \beta; \mu \nu}$ is symmetric under the simultaneous exchange of the Lorentz indices of parton 1 and the parent parton,

$$H_{f_1 \dots f_m}^{(n) \alpha \beta; \mu \nu} = H_{f_1 \dots f_m}^{(n) \beta \alpha; \nu \mu} . \quad (\text{B.3})$$

This means that the most general tensor structure for $H_{f_1 \dots f_m}^{(n) \alpha \beta; \mu \nu}$ is given by

$$\begin{aligned} H_{gf_1 \dots f_m}^{(n) \alpha \beta; \mu \nu} &= g^{\alpha \beta} g^{\mu \nu} \overline{A}_{gf_1 \dots f_m}^{(g)} + \sum_{i,j=1}^m \frac{p_i^\mu p_j^\nu}{s_{1 \dots m}} g^{\alpha \beta} \overline{B}_{ij, gf_1 \dots f_m}^{(g)} + \sum_{i,j=1}^m \frac{p_i^\alpha p_j^\beta}{s_{1 \dots m}} g^{\mu \nu} \overline{C}_{ij, gf_1 \dots f_m}^{(g)} \\ &+ \sum_{i,j,k,l=1}^m \frac{p_i^\mu p_j^\nu p_k^\alpha p_l^\beta}{s_{1 \dots m}^2} \overline{D}_{ijkl, gf_1 \dots f_m}^{(g)} + \sum_{i,j=1}^m \frac{g^{\alpha \mu} p_i^\beta p_j^\nu + g^{\beta \nu} p_i^\alpha p_j^\mu}{s_{1 \dots m}} \overline{E}_{1ij, gf_1 \dots f_m}^{(g)} \\ &+ \sum_{i,j=1}^m \frac{g^{\alpha \nu} p_i^\beta p_j^\mu + g^{\beta \mu} p_i^\alpha p_j^\nu}{s_{1 \dots m}} \overline{E}_{2ij, gf_{m'+1} \dots f_m}^{(g)} + g^{\alpha \mu} g^{\beta \nu} \overline{F}_{1gf_1 \dots f_m}^{(g)} \\ &+ g^{\alpha \nu} g^{\beta \mu} \overline{F}_{2gf_1 \dots f_m}^{(g)} + \text{gauge terms} , \end{aligned} \quad (\text{B.4})$$

where ‘gauge terms’ on the right-hand-side denote all terms proportional to the axial gauge vector n . By the same arguments as for the tensor structure of the gluon-initiated splitting amplitude, the coefficients $\overline{A}_{gf_1 \dots f_m}^{(g)}$ through $\overline{F}_{2gf_1 \dots f_m}^{(g)}$ in eq. (B.4) are dimensionless and of order zero in the collinear limit. In addition, we have the following symmetries, which follow directly from eq. (B.3),

$$\begin{aligned} \overline{B}_{ij, gf_1 \dots f_m}^{(g)} &= \overline{B}_{ji, gf_1 \dots f_m}^{(g)} , \\ \overline{C}_{ij, gf_1 \dots f_m}^{(g)} &= \overline{C}_{ji, gf_1 \dots f_m}^{(g)} , \\ \overline{D}_{ijkl, gf_1 \dots f_m}^{(g)} &= \overline{D}_{jilk, gf_1 \dots f_m}^{(g)} . \end{aligned} \quad (\text{B.5})$$

Since only the transverse part of a Lorentz tensor holds physical information (cf. Secs. 3 and 4), we multiply $H_{gf_1 \dots f_m}^{(n) \alpha \beta; \mu \nu}$ by spin-polarisation tensors,

$$d_{\mu\tilde{\mu}}(P, n) d_{\nu\tilde{\nu}}(P, n) d_{\alpha\tilde{\alpha}}(p_1, n) d_{\beta\tilde{\beta}}(p_1, n) H_{f_1 \dots f_m}^{(n) \tilde{\alpha} \tilde{\beta}; \tilde{\mu} \tilde{\nu}} . \quad (\text{B.6})$$

Next, we take the m -parton collinear limit of $H_{f_1 \dots f_m}^{(n) \alpha \beta; \mu \nu}$ to obtain the splitting tensor. Equation (B.1) can then be written as

$$\mathcal{C}_{1 \dots m} \mathcal{T}_{f_1 \dots f_n}^{\alpha \beta}(p_1, \dots, p_n) = \left(\frac{2\mu^{2\epsilon} g_s^2}{s_{1 \dots m}} \right)^{m-1} \hat{H}_{f_1 \dots f_m}^{\alpha \beta; \mu \nu} \mathcal{T}_{gf_{m+1} \dots f_n, \mu \nu}(\tilde{P}, p_{m+1}, \dots, p_n) . \quad (\text{B.7})$$

By computing explicitly the contractions of the tensors on the right-hand-side of eq. (B.4) with the spin-polarisation tensors, we obtain

$$\begin{aligned}
d_{\mu\alpha}(p_i, n) g^{\alpha\beta} d_{\nu\beta}(p_i, n) &= -d_{\mu\nu}(\tilde{P}, n) + \dots, \\
d^\mu{}_\nu(p_j, n) p_i^\nu &= -k_{\perp i}^\mu + \frac{z_i}{z_j} k_{\perp j}^\mu + \dots, \\
d^{\mu\nu}(P, n) P_\nu &= 0 + \dots, \\
d^{\mu\nu}(P, n) n_\nu &= 0, \\
d^\mu{}_\nu(p_i, n) \tilde{P}^\nu &= \frac{1}{z_i} k_{\perp i}^\mu + \dots, \\
d^\mu{}_\nu(\tilde{P}, n) p_i^\nu &= -k_{\perp i}^\mu + \dots,
\end{aligned} \tag{B.8}$$

where in the ellipses we have suppressed sub-leading terms in the m -parton collinear limit. Thus, the transverse part of the splitting tensor in eq. (B.7) contains only combinations of spin-polarisation tensors and transverse momenta.

In addition, we can take into account that in eq. (B.7) the splitting tensor is contracted with the symmetric tensor $\mathcal{T}_{gf_{m+1}\dots f_n}^{\mu\nu}$, which allows us to discard terms in $\hat{H}_{f_1\dots f_m}^{\alpha\beta;\mu\nu}$ that are anti-symmetric under the exchange $\mu \leftrightarrow \nu$. This leads to the following expression for the splitting tensor,

$$\begin{aligned}
\hat{H}_{gf_1\dots f_m}^{\alpha\beta;\mu\nu} &= -d^{\alpha\beta} d^{\mu\nu} A_{gf_1\dots f_m}^{(g)} + \sum_{i,j=1}^m \frac{k_{\perp i}^\mu k_{\perp j}^\nu}{s_{1\dots m}} d^{\alpha\beta} B_{ij,gf_1\dots f_m}^{(g)} \\
&- \sum_{i,j=1}^m \frac{k_{\perp i}^\alpha k_{\perp j}^\beta}{s_{1\dots m}} d^{\mu\nu} C_{ij,gf_1\dots f_m}^{(g)} + \sum_{i,j,k,l=1}^m \frac{k_{\perp i}^\mu k_{\perp j}^\nu k_{\perp k}^\alpha k_{\perp l}^\beta}{s_{1\dots m}^2} D_{ijkl,gf_1\dots f_m}^{(g)} \\
&- \sum_{i,j=1}^m \frac{d^{\alpha\mu} k_{\perp i}^\beta k_{\perp j}^\nu + d^{\beta\nu} k_{\perp i}^\alpha k_{\perp j}^\mu + d^{\alpha\nu} k_{\perp i}^\beta k_{\perp j}^\mu + d^{\beta\mu} k_{\perp i}^\alpha k_{\perp j}^\nu}{s_{1\dots m}} E_{ij,gf_1\dots f_m}^{(g)} \\
&+ (d^{\alpha\mu} d^{\beta\nu} + d^{\alpha\nu} d^{\beta\mu}) F_{gf_1\dots f_m}^{(g)},
\end{aligned} \tag{B.9}$$

where we have suppressed the dependence of the spin-polarisation tensor on the gauge vectors \tilde{P} and n . The scalar coefficients appearing in eq. (B.9) are linear combinations of the coefficients in eq. (B.4). To keep the symmetry of this expression, the coefficients now satisfy

$$\begin{aligned}
B_{ij,gf_1\dots f_m}^{(g)} &= B_{ji,gf_1\dots f_m}^{(g)}, \\
C_{ij,gf_1\dots f_m}^{(g)} &= C_{ji,gf_1\dots f_m}^{(g)}, \\
D_{ijkl,gf_1\dots f_m}^{(g)} &= D_{jikl,gf_1\dots f_m}^{(g)} = D_{ijlk,gf_1\dots f_m}^{(g)} = D_{jilk,gf_1\dots f_m}^{(g)}.
\end{aligned} \tag{B.10}$$

We can now immediately obtain eq. (4.13) from eq. (B.9) by replacing some of the polarisation tensors by metric tensors following the discussion in Sec. 4.

C The three-parton splitting tensors

There are three gluon-initiated splitting tensors with three partons in the collinear set,

$$g \rightarrow \bar{q}qg, \quad g \rightarrow g\bar{q}q, \quad g \rightarrow ggg. \quad (\text{C.1})$$

The first is obtained by not summing over the helicities (h, h') of the quark (or equivalently, by charge-conjugation, the anti-quark) in the collinear set. It is proportional to the three-parton splitting amplitude $\hat{P}_{\bar{q}qg}^{\mu\nu}$ given in ref. [54],

$$\hat{H}_{\bar{q}qg}^{hh';\mu\nu} = \frac{1}{2} \delta^{hh'} \hat{P}_{\bar{q}qg}^{\mu\nu}. \quad (\text{C.2})$$

The other two have the tensor structure of eq. (4.13). The three-gluon splitting tensor $\hat{H}_{ggg}^{\alpha\beta;\mu\nu}$ is too lengthy to be presented on paper, but we make it available in computer-readable form [74]. In the remainder of this section, we provide explicit results for the splitting tensor $\hat{H}_{g\bar{q}q}^{\alpha\beta;\mu\nu}$. We can write it in terms of an ‘abelian’ and ‘non-abelian’ part,

$$\hat{H}_{g\bar{q}q}^{\alpha\beta;\mu\nu} = \frac{1}{2} C_F \hat{H}_{g\bar{q}q}^{\alpha\beta;\mu\nu \text{ (ab)}} + \frac{1}{2} C_A \hat{H}_{g\bar{q}q}^{\alpha\beta;\mu\nu \text{ (nab)}}. \quad (\text{C.3})$$

We add a subscript (12) to denote the on-shell momentum of the gluon sub-parent. Furthermore, we define the shorthand,

$$z_{1\dots j} = z_1 + \dots + z_j, \quad \bar{z}_i = 1 - z_i, \quad k_{\perp 1\dots j} = k_{\perp 1} + \dots + k_{\perp j}. \quad (\text{C.4})$$

In what follows, we have eliminated z_{12} and $k_{\perp 12}$ using the constraints. The sub-energies $s_{[ij]k}$ are defined in eq. (4.4). The coefficients in eq. (4.13) belonging to the ‘abelian’ piece of $\hat{H}_{g(12)\bar{q}3q4}^{\alpha\beta;\mu\nu}$ are given by

$$A_{g(12)\bar{q}3q4}^{(g) \text{ (ab)}} = -\frac{(s_{[12]3} + s_{[12]4})^2}{2s_{[12]3}s_{[12]4}}, \quad (\text{C.5})$$

$$B_{33,g(12)\bar{q}3q4}^{(g) \text{ (ab)}} \equiv B_{34,g(12)\bar{q}3q4}^{(g) \text{ (ab)}} \equiv B_{43,g(12)\bar{q}3q4}^{(g) \text{ (ab)}} \equiv B_{44,g(12)\bar{q}3q4}^{(g) \text{ (ab)}} = -\frac{2s_{[12]34}}{s_{[12]3}s_{[12]4}}, \quad (\text{C.6})$$

$$C_{33,g(12)\bar{q}3q4}^{(g) \text{ (ab)}} = -\frac{2s_{[12]34} (z_4 s_{[12]3} - \bar{z}_4 s_{[12]4})^2}{s_{[12]3}^2 s_{[12]4}^2 (1 - z_{34})^2}, \quad (\text{C.7})$$

$$C_{44,g(12)\bar{q}3q4}^{(g) \text{ (ab)}} = -\frac{2s_{[12]34} (z_3 s_{[12]4} - \bar{z}_3 s_{[12]3})^2}{s_{[12]3}^2 s_{[12]4}^2 (1 - z_{34})^2}, \quad (\text{C.8})$$

$$C_{34,g(12)\bar{q}3q4}^{(g) \text{ (ab)}} \equiv C_{43,g(12)\bar{q}3q4}^{(g) \text{ (ab)}} = \frac{2s_{[12]34} (z_3 s_{[12]4} - \bar{z}_3 s_{[12]3}) (z_4 s_{[12]3} - \bar{z}_4 s_{[12]4})}{s_{[12]3}^2 s_{[12]4}^2 (1 - z_{34})^2}, \quad (\text{C.9})$$

$$D_{3333,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} = -\frac{8z_4^2}{s_{[12]4}^2(1-z_{34})^2}, \quad (\text{C.10})$$

$$D_{4444,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} = -\frac{8z_3^2}{s_{[12]3}^2(1-z_{34})^2}, \quad (\text{C.11})$$

$$D_{3334,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} \equiv D_{3343,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} = -\frac{8z_4\bar{z}_3}{s_{[12]4}^2(1-z_{34})^2}, \quad (\text{C.12})$$

$$D_{3433,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} \equiv D_{4333,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} = -\frac{8z_4\bar{z}_4}{s_{[12]3}s_{[12]4}(1-z_{34})^2}, \quad (\text{C.13})$$

$$D_{4434,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} \equiv D_{4443,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} = -\frac{8z_3\bar{z}_4}{s_{[12]3}^2(1-z_{34})^2}, \quad (\text{C.14})$$

$$D_{3444,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} \equiv D_{4344,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} = -\frac{8z_3\bar{z}_3}{s_{[12]3}s_{[12]4}(1-z_{34})^2}, \quad (\text{C.15})$$

$$\begin{aligned} D_{3434,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} &\equiv D_{4343,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} \equiv D_{4334,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} \equiv D_{3443,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} \\ &= 4 \frac{z_3\bar{z}_4 + z_4\bar{z}_3 - 1}{s_{[12]3}s_{[12]4}(1-z_{34})^2}, \end{aligned} \quad (\text{C.16})$$

$$D_{4433,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} = -\frac{8\bar{z}_4^2}{s_{[12]3}^2(1-z_{34})^2}, \quad (\text{C.17})$$

$$D_{3344,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} = -\frac{8\bar{z}_3^2}{s_{[12]4}^2(1-z_{34})^2}, \quad (\text{C.18})$$

$$E_{33,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} = -\frac{2z_4}{s_{[12]4}(1-z_{34})}, \quad (\text{C.19})$$

$$E_{44,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} = -\frac{2z_3}{s_{[12]3}(1-z_{34})}, \quad (\text{C.20})$$

$$E_{34,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} = -\frac{2\bar{z}_4}{s_{[12]3}(1-z_{34})}, \quad (\text{C.21})$$

$$E_{43,g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} = -\frac{2\bar{z}_3}{s_{[12]4}(1-z_{34})}, \quad (\text{C.22})$$

$$F_{g(12)\bar{q}_3q_4}^{(g) \text{ (ab)}} = -1, \quad (\text{C.23})$$

while the ‘non-abelian’ coefficients read

$$A_{g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = \frac{1}{2}, \quad (\text{C.24})$$

$$B_{33,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = \frac{s_{[12]34}(s_{34} + s_{[12]4})}{s_{34}s_{[12]3}s_{[12]4}}, \quad (\text{C.25})$$

$$B_{34,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} \equiv B_{43,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = \frac{s_{[12]34}(s_{[12]34} + s_{34})}{2s_{34}s_{[12]3}s_{[12]4}}, \quad (\text{C.26})$$

$$B_{44,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = \frac{s_{[12]34} (s_{34} + s_{[12]3})}{s_{34}s_{[12]3}s_{[12]4}}, \quad (\text{C.27})$$

$$C_{33,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = -\frac{s_{[12]34} (\bar{z}_4 (2z_4s_{34} + s_{[12]4}) + z_4s_{[12]3})}{s_{34}s_{[12]3}s_{[12]4}(1 - z_{34})^2}, \quad (\text{C.28})$$

$$C_{44,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = -\frac{s_{[12]34} (\bar{z}_3 (2z_3s_{34} + s_{[12]3}) + z_3s_{[12]4})}{s_{34}s_{[12]3}s_{[12]4}(1 - z_{34})^2}, \quad (\text{C.29})$$

$$C_{34,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} \equiv C_{43,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = -\frac{s_{[12]34}}{(1 - z_{34})^2} \left(\frac{\bar{z}_4\bar{z}_3 + z_4z_3}{s_{[12]3}s_{[12]4}} + \frac{\bar{z}_3 + z_4}{2s_{34}s_{[12]4}} + \frac{\bar{z}_4 + z_3}{2s_{34}s_{[12]3}} \right),$$

$$D_{3333,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = -\frac{8z_4^2 (s_{34} + s_{[12]4})}{s_{34}^2s_{[12]4}(1 - z_{34})^2}, \quad (\text{C.30})$$

$$D_{4444,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = -\frac{8z_3^2 (s_{34} + s_{[12]3})}{s_{34}^2s_{[12]3}(1 - z_{34})^2}, \quad (\text{C.31})$$

$$D_{3334,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} \equiv D_{3343,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = -\frac{8z_4\bar{z}_3 (s_{34} + s_{[12]4})}{s_{34}^2s_{[12]4}(1 - z_{34})^2}, \quad (\text{C.32})$$

$$D_{3433,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} \equiv D_{4333,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = \frac{4z_4\bar{z}_4}{s_{[12]4}(1 - z_{34})^2} \left(\frac{s_{34} + 2s_{[12]4}}{s_{34}^2} + \frac{s_{34} + s_{[12]4}}{s_{34}s_{[12]3}} \right), \quad (\text{C.33})$$

$$D_{4434,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} \equiv D_{4443,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = -\frac{8z_3\bar{z}_4 (s_{34} + s_{[12]3})}{s_{34}^2s_{[12]3}(1 - z_{34})^2}, \quad (\text{C.34})$$

$$D_{3444,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} \equiv D_{4344,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = \frac{4z_3\bar{z}_3}{s_{[12]4}(1 - z_{34})^2} \left(\frac{s_{34} + 2s_{[12]4}}{s_{34}^2} + \frac{s_{34} + s_{[12]4}}{s_{34}s_{[12]3}} \right), \quad (\text{C.35})$$

$$D_{3434,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} \equiv D_{4343,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} \equiv D_{4334,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} \equiv D_{3443,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} \\ = 2 \frac{\bar{z}_4\bar{z}_3 + z_3z_4}{(1 - z_{34})^2} \left(\frac{2}{s_{34}^2} + \frac{s_{[12]34}}{s_{[12]3}s_{[12]4}s_{34}} \right), \quad (\text{C.36})$$

$$D_{4433,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = -\frac{8\bar{z}_4^2 (s_{34} + s_{[12]3})}{s_{34}^2s_{[12]3}(1 - z_{34})^2}, \quad (\text{C.37})$$

$$D_{3344,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = -\frac{8\bar{z}_3^2 (s_{34} + s_{[12]4})}{s_{34}^2s_{[12]4}(1 - z_{34})^2}, \quad (\text{C.38})$$

$$E_{33,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = \frac{2z_4^2(s_{[12]3} + s_{34}) - 4z_3z_4s_{[12]4}}{(1 - z_{34})z_{34}s_{34}} \left(\frac{1}{s_{[12]4}} + \frac{1}{s_{34}} \right) + \frac{2z_4^2s_{[12]3}}{(1 - z_{34})z_{34}s_{34}^2}, \quad (\text{C.39})$$

$$E_{44,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = \frac{2z_3^2(s_{[12]4} + s_{34}) - 4z_3z_4s_{[12]3}}{(1 - z_{34})z_{34}s_{34}} \left(\frac{1}{s_{[12]3}} + \frac{1}{s_{34}} \right) + \frac{2z_3^2s_{[12]4}}{(1 - z_{34})z_{34}s_{34}^2}, \quad (\text{C.40})$$

$$E_{34,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = \frac{s_{34} + s_{[12]4}}{2s_{34}s_{[12]3}z_{34}} - \frac{(s_{34} + s_{[12]3})}{2s_{34}s_{[12]4}} \left(\frac{4z_4\bar{z}_3}{(1 - z_{34})z_{34}} + \frac{1}{z_{34}} \right)$$

$$+ \frac{4\bar{z}_3}{s_{34}(1-z_{34})z_{34}} \left(z_3 - \frac{1}{2}z_4 - \frac{s_{[12]3}z_4 - s_{[12]4}z_3}{s_{34}} \right), \quad (\text{C.41})$$

$$E_{43,g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = \frac{s_{34} + s_{[12]3}}{2s_{34}s_{[12]4}z_{34}} - \frac{(s_{34} + s_{[12]4})}{2s_{34}s_{[12]3}} \left(\frac{4z_3\bar{z}_4}{(1-z_{34})z_{34}} + \frac{1}{z_{34}} \right) \\ + \frac{4\bar{z}_4}{s_{34}(1-z_{34})z_{34}} \left(z_4 - \frac{1}{2}z_3 + \frac{s_{[12]3}z_4 - s_{[12]4}z_3}{s_{34}} \right), \quad (\text{C.42})$$

$$F_{g(12)\bar{q}_3q_4}^{(g) \text{ (nab)}} = \frac{1}{4} - \frac{(z_3s_{[12]4} - z_4s_{[12]3})^2}{s_{34}^2z_{34}^2} + \frac{(s_{34} + s_{[12]4})(s_{34} - z_3s_{[12]4})}{2s_{34}s_{[12]3}z_{34}} - \frac{s_{34} + s_{[12]3}}{2s_{[12]4}} \\ + \frac{s_{[12]4}(z_{34}(4-3z_4) - 8z_3^2)}{2s_{34}z_{34}^2} + \frac{3(1-z_{34})}{2z_{34}} + \frac{z_3z_4}{z_{34}^2} + (3 \leftrightarrow 4). \quad (\text{C.43})$$

References

- [1] ATLAS collaboration, G. Aad et al., *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*, *Phys. Lett. B* **716** (2012) 1–29, [[1207.7214](#)].
- [2] CMS collaboration, S. Chatrchyan et al., *Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC*, *Phys. Lett. B* **716** (2012) 30–61, [[1207.7235](#)].
- [3] D. Graudenz, M. Spira and P. M. Zerwas, *QCD corrections to Higgs boson production at proton proton colliders*, *Phys. Rev. Lett.* **70** (1993) 1372–1375.
- [4] M. Spira, A. Djouadi, D. Graudenz and P. M. Zerwas, *Higgs boson production at the LHC*, *Nucl. Phys. B* **453** (1995) 17–82, [[hep-ph/9504378](#)].
- [5] C. Anastasiou, C. Duhr, F. Dulat, F. Herzog and B. Mistlberger, *Higgs Boson Gluon-Fusion Production in QCD at Three Loops*, *Phys. Rev. Lett.* **114** (2015) 212001, [[1503.06056](#)].
- [6] C. Anastasiou, C. Duhr, F. Dulat, E. Furlan, T. Gehrmann, F. Herzog et al., *High precision determination of the gluon fusion Higgs boson cross-section at the LHC*, *JHEP* **05** (2016) 058, [[1602.00695](#)].
- [7] B. Mistlberger, *Higgs boson production at hadron colliders at N^3LO in QCD*, *JHEP* **05** (2018) 028, [[1802.00833](#)].
- [8] C. Duhr, F. Dulat and B. Mistlberger, *Higgs production in bottom-quark fusion to third order in the strong coupling*, [1904.09990](#).
- [9] C. Duhr, F. Dulat, V. Hirschi and B. Mistlberger, *Higgs production in bottom quark fusion: Matching the 4- and 5-flavour schemes to third order in the strong coupling*, [2004.04752](#).
- [10] F. A. Dreyer and A. Karlberg, *Vector-Boson Fusion Higgs Production at Three Loops in QCD*, *Phys. Rev. Lett.* **117** (2016) 072001, [[1606.00840](#)].
- [11] F. A. Dreyer and A. Karlberg, *Vector-Boson Fusion Higgs Pair Production at N^3LO* , *Phys. Rev. D* **98** (2018) 114016, [[1811.07906](#)].
- [12] L.-B. Chen, H. T. Li, H.-S. Shao and J. Wang, *Higgs boson pair production via gluon fusion at N^3LO in QCD*, *Phys. Lett. B* **803** (2020) 135292, [[1909.06808](#)].
- [13] L.-B. Chen, H. T. Li, H.-S. Shao and J. Wang, *The gluon-fusion production of Higgs boson pair: N^3LO QCD corrections and top-quark mass effects*, *JHEP* **03** (2020) 072, [[1912.13001](#)].
- [14] C. Duhr, F. Dulat and B. Mistlberger, *The Drell-Yan cross section to third order in the strong coupling constant*, [2001.07717](#).

- [15] L. Cieri, X. Chen, T. Gehrmann, E. W. N. Glover and A. Huss, *Higgs boson production at the LHC using the q_T subtraction formalism at N^3LO QCD*, *JHEP* **02** (2019) 096, [[1807.11501](#)].
- [16] F. Dulat, B. Mistlberger and A. Pelloni, *Precision predictions at N^3LO for the Higgs boson rapidity distribution at the LHC*, *Phys. Rev.* **D99** (2019) 034004, [[1810.09462](#)].
- [17] T. Binoth and G. Heinrich, *An automatized algorithm to compute infrared divergent multiloop integrals*, *Nucl. Phys.* **B585** (2000) 741–759, [[hep-ph/0004013](#)].
- [18] C. Anastasiou and A. Daleo, *Numerical evaluation of loop integrals*, *JHEP* **10** (2006) 031, [[hep-ph/0511176](#)].
- [19] S. Catani and M. Grazzini, *An NNLO subtraction formalism in hadron collisions and its application to Higgs boson production at the LHC*, *Phys. Rev. Lett.* **98** (2007) 222002, [[hep-ph/0703012](#)].
- [20] R. Boughezal, X. Liu and F. Petriello, *N -jettiness soft function at next-to-next-to-leading order*, *Phys. Rev.* **D91** (2015) 094035, [[1504.02540](#)].
- [21] J. Gaunt, M. Stahlhofen, F. J. Tackmann and J. R. Walsh, *N -jettiness Subtractions for NNLO QCD Calculations*, *JHEP* **09** (2015) 058, [[1505.04794](#)].
- [22] A. Gehrmann-De Ridder, T. Gehrmann and E. W. N. Glover, *Antenna subtraction at NNLO*, *JHEP* **09** (2005) 056, [[hep-ph/0505111](#)].
- [23] A. Daleo, T. Gehrmann and D. Maitre, *Antenna subtraction with hadronic initial states*, *JHEP* **04** (2007) 016, [[hep-ph/0612257](#)].
- [24] A. Gehrmann-De Ridder, T. Gehrmann and E. W. N. Glover, *Gluon-gluon antenna functions from Higgs boson decay*, *Phys. Lett.* **B612** (2005) 49–60, [[hep-ph/0502110](#)].
- [25] A. Gehrmann-De Ridder, T. Gehrmann and E. W. N. Glover, *Quark-gluon antenna functions from neutralino decay*, *Phys. Lett.* **B612** (2005) 36–48, [[hep-ph/0501291](#)].
- [26] A. Daleo, A. Gehrmann-De Ridder, T. Gehrmann and G. Luisoni, *Antenna subtraction at NNLO with hadronic initial states: initial-final configurations*, *JHEP* **01** (2010) 118, [[0912.0374](#)].
- [27] T. Gehrmann and P. F. Monni, *Antenna subtraction at NNLO with hadronic initial states: real-virtual initial-initial configurations*, *JHEP* **12** (2011) 049, [[1107.4037](#)].
- [28] R. Boughezal, A. Gehrmann-De Ridder and M. Ritzmann, *Antenna subtraction at NNLO with hadronic initial states: double real radiation for initial-initial configurations with two quark flavours*, *JHEP* **02** (2011) 098, [[1011.6631](#)].
- [29] A. Gehrmann-De Ridder, T. Gehrmann and M. Ritzmann, *Antenna subtraction at NNLO with hadronic initial states: double real initial-initial configurations*, *JHEP* **10** (2012) 047, [[1207.5779](#)].

- [30] J. Currie, E. W. N. Glover and S. Wells, *Infrared Structure at NNLO Using Antenna Subtraction*, *JHEP* **04** (2013) 066, [[1301.4693](#)].
- [31] G. Somogyi, Z. Trocsanyi and V. Del Duca, *Matching of singly- and doubly-unresolved limits of tree-level QCD squared matrix elements*, *JHEP* **06** (2005) 024, [[hep-ph/0502226](#)].
- [32] G. Somogyi, Z. Trocsanyi and V. Del Duca, *A Subtraction scheme for computing QCD jet cross sections at NNLO: Regularization of doubly-real emissions*, *JHEP* **01** (2007) 070, [[hep-ph/0609042](#)].
- [33] G. Somogyi and Z. Trocsanyi, *A Subtraction scheme for computing QCD jet cross sections at NNLO: Regularization of real-virtual emission*, *JHEP* **01** (2007) 052, [[hep-ph/0609043](#)].
- [34] G. Somogyi and Z. Trocsanyi, *A Subtraction scheme for computing QCD jet cross sections at NNLO: Integrating the subtraction terms. I.*, *JHEP* **08** (2008) 042, [[0807.0509](#)].
- [35] U. Aglietti, V. Del Duca, C. Duhr, G. Somogyi and Z. Trocsanyi, *Analytic integration of real-virtual counterterms in NNLO jet cross sections. I.*, *JHEP* **09** (2008) 107, [[0807.0514](#)].
- [36] G. Somogyi, *Subtraction with hadronic initial states at NLO: An NNLO-compatible scheme*, *JHEP* **05** (2009) 016, [[0903.1218](#)].
- [37] P. Bolzoni, S.-O. Moch, G. Somogyi and Z. Trocsanyi, *Analytic integration of real-virtual counterterms in NNLO jet cross sections. II.*, *JHEP* **08** (2009) 079, [[0905.4390](#)].
- [38] P. Bolzoni, G. Somogyi and Z. Trocsanyi, *A subtraction scheme for computing QCD jet cross sections at NNLO: integrating the iterated singly-unresolved subtraction terms*, *JHEP* **01** (2011) 059, [[1011.1909](#)].
- [39] V. Del Duca, G. Somogyi and Z. Trocsanyi, *Integration of collinear-type doubly unresolved counterterms in NNLO jet cross sections*, *JHEP* **06** (2013) 079, [[1301.3504](#)].
- [40] G. Somogyi, *A subtraction scheme for computing QCD jet cross sections at NNLO: integrating the doubly unresolved subtraction terms*, *JHEP* **04** (2013) 010, [[1301.3919](#)].
- [41] M. Czakon, *A novel subtraction scheme for double-real radiation at NNLO*, *Phys. Lett. B* **693** (2010) 259–268, [[1005.0274](#)].
- [42] M. Czakon, *Double-real radiation in hadronic top quark pair production as a proof of a certain concept*, *Nucl. Phys. B* **849** (2011) 250–295, [[1101.0642](#)].
- [43] M. Czakon and D. Heymes, *Four-dimensional formulation of the sector-improved residue subtraction scheme*, *Nucl. Phys. B* **890** (2014) 152–227, [[1408.2500](#)].

- [44] M. Czakon, A. van Hameren, A. Mitov and R. Poncelet, *Single-jet inclusive rates with exact color at $\mathcal{O}(\alpha_s^4)$* , *JHEP* **10** (2019) 262, [[1907.12911](#)].
- [45] F. Caola, K. Melnikov and R. Roentsch, *Nested soft-collinear subtractions in NNLO QCD computations*, *Eur. Phys. J.* **C77** (2017) 248, [[1702.01352](#)].
- [46] F. Caola, M. Delto, H. Frellesvig and K. Melnikov, *The double-soft integral for an arbitrary angle between hard radiators*, *Eur. Phys. J.* **C78** (2018) 687, [[1807.05835](#)].
- [47] M. Delto and K. Melnikov, *Integrated triple-collinear counter-terms for the nested soft-collinear subtraction scheme*, *JHEP* **05** (2019) 148, [[1901.05213](#)].
- [48] F. Caola, K. Melnikov and R. Röntschi, *Analytic results for color-singlet production at NNLO QCD with the nested soft-collinear subtraction scheme*, *Eur. Phys. J.* **C79** (2019) 386, [[1902.02081](#)].
- [49] F. Caola, K. Melnikov and R. Röntschi, *Analytic results for decays of color singlets to gg and $q\bar{q}$ final states at NNLO QCD with the nested soft-collinear subtraction scheme*, *Eur. Phys. J. C* **79** (2019) 1013, [[1907.05398](#)].
- [50] M. Cacciari, F. A. Dreyer, A. Karlberg, G. P. Salam and G. Zanderighi, *Fully Differential Vector-Boson-Fusion Higgs Production at Next-to-Next-to-Leading Order*, *Phys. Rev. Lett.* **115** (2015) 082002, [[1506.02660](#)].
- [51] J. Currie, T. Gehrmann, E. W. N. Glover, A. Huss, J. Niehues and A. Vogt, *N^3LO corrections to jet production in deep inelastic scattering using the Projection-to-Born method*, *JHEP* **05** (2018) 209, [[1803.09973](#)].
- [52] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B425** (1994) 217–260, [[hep-ph/9403226](#)].
- [53] J. M. Campbell and E. W. N. Glover, *Double unresolved approximations to multiparton scattering amplitudes*, *Nucl. Phys.* **B527** (1998) 264–288, [[hep-ph/9710255](#)].
- [54] S. Catani and M. Grazzini, *Infrared factorization of tree level QCD amplitudes at the next-to-next-to-leading order and beyond*, *Nucl. Phys.* **B570** (2000) 287–325, [[hep-ph/9908523](#)].
- [55] V. Del Duca, A. Frizzo and F. Maltoni, *Factorization of tree QCD amplitudes in the high-energy limit and in the collinear limit*, *Nucl. Phys.* **B568** (2000) 211–262, [[hep-ph/9909464](#)].
- [56] Z. Bern, V. Del Duca and C. R. Schmidt, *The Infrared behavior of one loop gluon amplitudes at next-to-next-to-leading order*, *Phys. Lett.* **B445** (1998) 168–177, [[hep-ph/9810409](#)].
- [57] D. A. Kosower and P. Uwer, *One loop splitting amplitudes in gauge theory*, *Nucl. Phys.* **B563** (1999) 477–505, [[hep-ph/9903515](#)].

- [58] Z. Bern, V. Del Duca, W. B. Kilgore and C. R. Schmidt, *The infrared behavior of one loop QCD amplitudes at next-to-next-to leading order*, *Phys. Rev.* **D60** (1999) 116001, [[hep-ph/9903516](#)].
- [59] S. Catani and M. Grazzini, *The soft gluon current at one loop order*, *Nucl. Phys.* **B591** (2000) 435–454, [[hep-ph/0007142](#)].
- [60] D. A. Kosower, *Multiple singular emission in gauge theories*, *Phys. Rev.* **D67** (2003) 116003, [[hep-ph/0212097](#)].
- [61] Z. Bern, L. J. Dixon and D. A. Kosower, *Two-loop $g \rightarrow gg$ splitting amplitudes in QCD*, *JHEP* **08** (2004) 012, [[hep-ph/0404293](#)].
- [62] S. D. Badger and E. W. N. Glover, *Two loop splitting functions in QCD*, *JHEP* **07** (2004) 040, [[hep-ph/0405236](#)].
- [63] C. Duhr, T. Gehrmann and M. Jaquier, *Two-loop splitting amplitudes and the single-real contribution to inclusive Higgs production at N^3LO* , *JHEP* **02** (2015) 077, [[1411.3587](#)].
- [64] C. Duhr and T. Gehrmann, *The two-loop soft current in dimensional regularization*, *Phys. Lett.* **B727** (2013) 452–455, [[1309.4393](#)].
- [65] Y. Li and H. X. Zhu, *Single soft gluon emission at two loops*, *JHEP* **11** (2013) 080, [[1309.4391](#)].
- [66] L. J. Dixon, E. Herrmann, K. Yan and H. X. Zhu, *Soft gluon emission at two loops in full color*, *JHEP* **05** (2020) 135, [[1912.09370](#)].
- [67] S. Catani, D. de Florian and G. Rodrigo, *The Triple collinear limit of one loop QCD amplitudes*, *Phys. Lett.* **B586** (2004) 323–331, [[hep-ph/0312067](#)].
- [68] S. Badger, F. Buciuini and T. Peraro, *One-loop triple collinear splitting amplitudes in QCD*, *JHEP* **09** (2015) 188, [[1507.05070](#)].
- [69] G. F. R. Sborlini, D. de Florian and G. Rodrigo, *Triple collinear splitting functions at NLO for scattering processes with photons*, *JHEP* **10** (2014) 161, [[1408.4821](#)].
- [70] S. Catani, D. Colferai and A. Torrini, *Triple (and quadruple) soft-gluon radiation in QCD hard scattering*, *JHEP* **01** (2020) 118, [[1908.01616](#)].
- [71] T. G. Birthwright, E. W. N. Glover, V. V. Khoze and P. Marquard, *Multi-gluon collinear limits from MHV diagrams*, *JHEP* **05** (2005) 013, [[hep-ph/0503063](#)].
- [72] C. Duhr, *Applications of twistor methods in QCD*, Master’s thesis, Université catholique de Louvain, 2006.
- [73] V. Del Duca, C. Duhr, R. Haindl, A. Lazopoulos and M. Michel, *Tree-level splitting amplitudes for a quark into four collinear partons*, *JHEP* **02** (2020) 189, [[1912.06425](#)].

- [74] “Quadruple Collinear Splitting Amplitudes supplementary material.”
https://people.phys.ethz.ch/~pheno/quadruple_collinear.
- [75] D. Amati, R. Petronzio and G. Veneziano, *Relating Hard QCD Processes Through Universality of Mass Singularities*, *Nucl. Phys.* **B140** (1978) 54–72.
- [76] D. Amati, R. Petronzio and G. Veneziano, *Relating Hard QCD Processes Through Universality of Mass Singularities. 2.*, *Nucl. Phys.* **B146** (1978) 29–49.
- [77] R. K. Ellis, H. Georgi, M. Machacek, H. D. Politzer and G. G. Ross, *Factorization and the Parton Model in QCD*, *Phys. Lett.* **78B** (1978) 281–284.
- [78] S. Catani, D. de Florian and G. Rodrigo, *Space-like (versus time-like) collinear limits in QCD: Is factorization violated?*, *JHEP* **07** (2012) 026, [[1112.4405](#)].
- [79] D. de Florian and M. Grazzini, *The Structure of large logarithmic corrections at small transverse momentum in hadronic collisions*, *Nucl. Phys.* **B616** (2001) 247–285, [[hep-ph/0108273](#)].
- [80] S. Catani, *The Singular behavior of QCD amplitudes at two loop order*, *Phys. Lett.* **B427** (1998) 161–171, [[hep-ph/9802439](#)].
- [81] G. F. Sterman and M. E. Tejeda-Yeomans, *Multiloop amplitudes and resummation*, *Phys. Lett.* **B552** (2003) 48–56, [[hep-ph/0210130](#)].
- [82] T. Becher and M. Neubert, *Infrared singularities of scattering amplitudes in perturbative QCD*, *Phys. Rev. Lett.* **102** (2009) 162001, [[0901.0722](#)].
- [83] E. Gardi and L. Magnea, *Factorization constraints for soft anomalous dimensions in QCD scattering amplitudes*, *JHEP* **03** (2009) 079, [[0901.1091](#)].
- [84] O. Almeldid, C. Duhr and E. Gardi, *Three-loop corrections to the soft anomalous dimension in multileg scattering*, *Phys. Rev. Lett.* **117** (2016) 172002, [[1507.00047](#)].
- [85] O. Almeldid, C. Duhr, E. Gardi, A. McLeod and C. D. White, *Bootstrapping the QCD soft anomalous dimension*, *JHEP* **09** (2017) 073, [[1706.10162](#)].
- [86] A. Bassetto, M. Ciafaloni and G. Marchesini, *Jet Structure and Infrared Sensitive Quantities in Perturbative QCD*, *Phys. Rept.* **100** (1983) 201–272.
- [87] S. Catani, *New Techniques for Calculating Higher-Order QCD Corrections*, in *Proceedings, Workshop on New Techniques for Calculating Higher Order QCD Corrections, Zurich, Switzerland, 16-18 Dec 1992*, (Zürich), 1992.