Lattice Calculation of D- and B-meson Semileptonic Decays,

using the Clover Action at $\beta = 6.0$ on APE

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Abstract

We present the results of a high statistics lattice calculation of hadronic form factors relevant for $D-$ and $B-$meson semi-leptonic decays into light pseudoscalar and vector mesons. The results have been obtained by averaging over 170 gauge field configurations, generated in the quenched approximation, at $\beta = 6.0$, on a $18^3 \times 64$ lattice, using the $O(a)$-improved SW-Clover action. From the study of the matrix element $< K^-|J_\mu|D^0 >$, we obtain $f_+(0) = 0.78 \pm 0.08$ and from the matrix element $< \bar{K}^{*0}|J_\mu|D^+ >$ we obtain $V(0) = 1.08 \pm 0.22, A_1(0) = 0.67 \pm 0.11$ and $A_2(0) = 0.49 \pm 0.34$. We also obtain the ratios $V(0)/A_1(0) = 1.6 \pm 0.3$ and $A_2(0)/A_1(0) = 0.7 \pm 0.4$. Our predictions for the different form factors are in good agreement with the experimental data, although, in the case of $A_2(0)$, the errors are still too large to draw any firm conclusion.

With the help of the Heavy Quark Effective Theory (HQET) we have also extrapolated the lattice results to B-meson decays. The form factors follow a behaviour compatible with the HQET predictions. Our results are in agreement with a previous lattice calculation, performed at $\beta = 6.4$, using the standard Wilson action.

1 Introduction

There is increasing evidence that quantitative calculations of weak decay amplitudes can be obtained by lattice QCD simulations. Over the last few years, semi-leptonic decays of heavy mesons have been studied on the lattice [1]–[7]. Among these, $D-$meson decays provide a good test of the lattice method, since the relevant CKM matrix element is well constrained by unitarity in the Standard Model, $V_{cs} \simeq 0.975$. The main advantage of the lattice technique is that it is based on first principles only and it does not contain free parameters besides the quark masses and the value of the lattice spacing, both of which are fixed by hadron spectroscopy. Moreover, statistical and systematic errors in lattice simulations can be systematically reduced with increasing computer resources.

In this work, we present a high-statistics study of pseudoscalar-pseudoscalar and pseudoscalar-vector semi-leptonic form factors, performed on the 6.4 Gflops APE machine, by using the $O(a)$-improved Clover Action [8, 9], at $\beta = 6.0$, corresponding to an inverse lattice spacing $a^{-1} \sim 2$ GeV. Our prediction for the form factors, which govern the $D \to K$ and $D \to K^*$ amplitudes are given in the abstract and in table 3. The central values are in remarkable agreement with the experimental results. However, in spite of the large statistics, our predictions still suffer, in some cases, from large statistical errors. A possible explanation for the errors' size is the use of a "thinning" procedure, that will be discussed in detail in sec. 3. This procedure was adopted because our computer memory is not sufficient to store the necessary quark propagators.

Following the suggestion of ref. [5], we have also tried to extrapolate the form factors to $B \to \pi, \rho$ decays, using the scaling laws predicted by the Heavy Quark Effective Theory (HQET) [10]. The final results have large uncertainties, because the statistical errors amplify in the extrapolation. It is reassuring that the present results are in good agreement with those obtained, with the same method, in ref. [5], by using the standard fermion Wilson action, at a smaller value of the lattice spacing, corresponding to $\beta = 6.4$. This gives us confidence on the feasibility of the extrapolation. It should be noticed however, that the form factors are obtained near $q^2 = q^2_{max}$, where q^2 is the momentum of the lepton pair. In order to predict the form factors at all q^2, on current lattices and in the range of the heavy quark masses explored so far, we can only assume pole dominance (or any other simple q^2-dependence). This strongly biases the final results, e.g. the determination of the form factors at $q^2 = 0$. In this respect, the situation is not very different from the quark model approach. In order to improve the accuracy of the predictions, it is necessary to be able to work with heavier quark masses and to increase the range of q^2. This can only be achieved by going to larger lattices. The approach followed here indicates that $B-$meson semi-leptonic decays will be a fruitful area of lattice investigations in the near future.

2 Computation Details

Semi-leptonic decays ($D \to K, K^*$; $B \to D, D^*$; $B \to \pi, \rho$) are described in terms of six independent dimensionless form factors, four of which are important for decay rates into light leptons (see e.g. [2]). For each form factor, the relevant information can be expressed
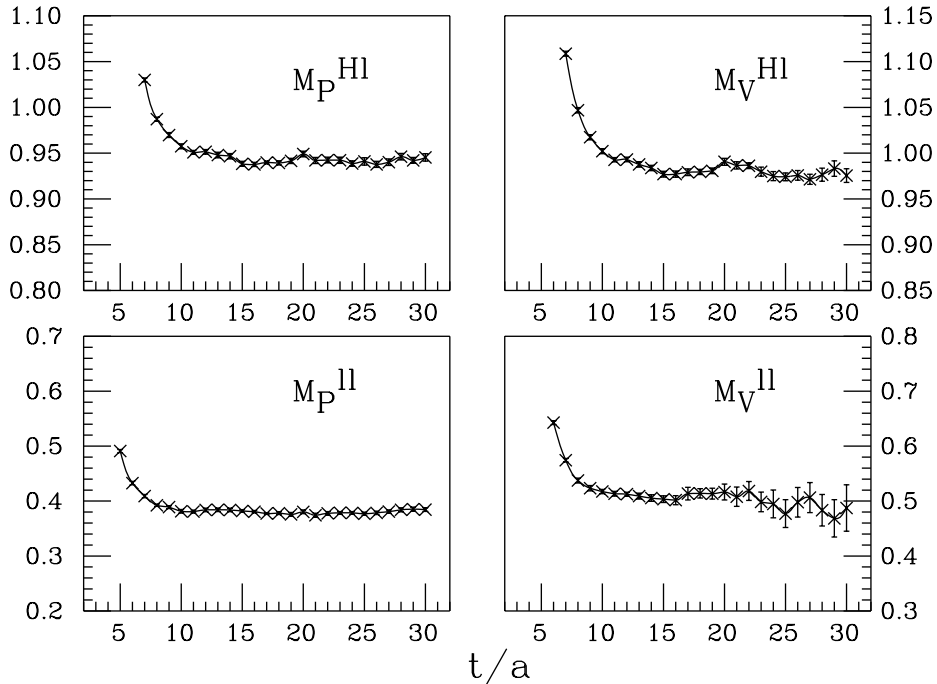
Figure 1: *The effective mass, defined as $M(t)a = \ln(C(t)/C(t+a))$ as a function of t/a. $M_{P,V}^{Hl}$ and $M_{P,V}^{ll}$ refer to the heavy-light or the light-light pseudoscalar and vector mesons respectively. $C(t)$ is the generic, zero momentum two-point correlation function.*

in terms of its value at $q^2 = 0$ and its q^2 dependence. In this section several details of our calculation are given.

We have obtained our results, by averaging over 170 gauge field configurations, generated at $\beta = 6.0$, on a volume $18^3 \times 64$. On this set of configurations, we have computed the quark propagators by using the $O(a)$-improved SW-Clover action [8, 9]. We have considered 3 different values of the Wilson hopping parameter for the light quarks, $K_l = 0.1425$, 0.1432 and 0.1440, and 4 different values for the heavy quarks, $K_H = 0.1150$, 0.1200, 0.1250 and 0.1330. The values of K, corresponding to the zero quark mass and to the strange quark mass (obtained by fixing the pion and the kaon pseudoscalar masses) are $K_{cr} = 0.14545(1)$ and $K_s = 0.1435(2)$ respectively. The value of the charm quark mass, obtained from the $D-$meson mass, corresponds to $K_{ch} = 0.1219(17)$. The inverse lattice spacing, obtained by using the mass of the $\rho-$meson to set the scale, is $a^{-1} = (1.95 \pm 0.08)$ GeV. All two-point functions have been fitted in the interval $t/a = 12 - 28$ and $15 - 28$ for light-light and heavy-light mesons respectively. In fig. 1, we give examples of the effective mass, as a function of time, in the two cases. The statistical errors have been estimated with the jacknife method, by decimating 10 configurations from the total set of 170. Preliminary results obtained with a smaller set of gauge field configurations have been reported in ref. [11].

To extract the form factors, we have computed two- and three-point correlation func-

tions and followed the procedure explained in ref. [5]. The matrix elements have been computed using the pseudoscalar density as source for D and K mesons and the local vector current for the K^* meson[1]. For the weak current we have used the lattice improved local vector and axial vector currents, according to the definition of refs. [12, 13]. The corresponding renormalization constants, Z_V and Z_A, have been fixed to the values $Z_V = 0.824$ and $Z_A = 1.06$, as determined non-perturbatively, using light quark correlation functions [14, 15]. This choice will be justified below.

All matrix elements have been computed by inserting the D-meson source at a time distance $(t_D - t_{K,K^*})/a = 28$. The position of the light meson is fixed at the origin and we have varied the time position of the weak current in the time interval $t_J/a = 10 - 14$.

To study the q^2-dependence of the form factors, we have computed the matrix elements in two different kinematical configurations: in the first case we have taken the $D-$meson at rest, i.e. $\vec{p}_D = \vec{0}$, and $\vec{p}_{K,K^*} \equiv 2\pi/La \cdot (0,0,0), (1,0,0), (1,1,0), (1,1,1)$ and $(2,0,0)$; in the second case we have chosen $\vec{p}_D \equiv 2\pi/La \cdot (1,0,0)$ and $\vec{p}_{K,K^*} \equiv 2\pi/La \cdot (0,0,0)$, $(1,0,0), (-1,0,0), (0,1,0)$. We have therefore nine independent momenta. We have also computed other correlation functions, which are equivalent under the cubic symmetry. All such equivalent cases have been averaged together.

In order to extract the current matrix elements from the three-point functions, we have used two different procedures, denoted by "analytic" and "ratio" methods, discussed in detail in ref. [5]. Within the statistical errors, the above methods are expected to agree, up to $O(a)$ effects. In the present calculation, we find that the two methods yield slightly different results, the differences varying between 2% and 8%. We have taken into account these differences in the evaluation of the final error (see below).

In order to obtain the form factors at $q^2 = 0$, for quark masses corresponding to the physical D (B) and K (π) or K^* (ρ) mesons, we have extrapolated the form factors, both in masses and momenta. The following procedure has been used:

1) At fixed heavy quark mass and meson momenta, $\vec{p}_{K,K^*}$, the generic form factor F ($F = f_+$, A_1, A_2 and V) has been extrapolated linearly in the light quark mass, to values corresponding to the strange ($D \to K, K^*$) or massless ($D \to \pi, \rho$) quarks.

2) F has been extrapolated in the mass of the heavy meson, to the $D-$ and $B-$meson masses, using the dependence expected in the HQET (see eqs.(3) below). In order to evaluate the stability of these results with respect to a different extrapolation, we have also extrapolated F in the mass of the heavy meson M_P according to the "naïve" expression $F = \alpha + \beta/M_P$. The differences between the two methods are discussed later on.

3) In order to obtain the form factors at $q^2 = 0$, we have only used the points with $\vec{p}_D = 0$ and $\vec{p}_{K,K^*} = 2\pi/La \cdot (1,0,0)$, which have been extrapolated by assuming meson pole dominance, $F(q^2) = F(0)/(1 - q^2/M_t^2)$. This reduces the uncertainty of the extrapolation because, in most of cases, the point with $\vec{p}_D = 2\pi/La \cdot (0,0,0)$ and $\vec{p}_{K,K^*} = 2\pi/La \cdot (1,0,0)$ corresponds to the smallest q^2. M_t is the mass of the lightest meson exchanged in the t-channel. Thus, the vector and axial meson

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[1]Throughout this paper, K, K^* and D are conventional names to denote the light pseudoscalar and vector mesons and the heavy meson respectively.

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M_{D^*}	$M_{D^{**}}\,1^{++}$	$M_{D^{**}}\,0^{++}$	$M_{D_s^*}$	$M_{D_s^{**}}\,1^{++}$	$M_{D_s^{**}}\,0^{++}$
1.95 ± 0.01	2.29 ± 0.34	2.03 ± 0.19	2.05 ± 0.01	2.40 ± 0.34	2.13 ± 0.19

M_{B^*}	$M_{B^{**}}\,1^{++}$	$M_{B^{**}}\,0^{++}$	$M_{B_s^*}$	$M_{B_s^{**}}\,1^{++}$	$M_{B_s^{**}}\,0^{++}$
5.32 ± 0.01	5.99 ± 0.62	5.46 ± 0.32	5.46 ± 0.01	6.13 ± 0.62	5.60 ± 0.32

Table 1: *Masses (in GeV) of the vector, axial and scalar excitations for the D and B mesons as determined from our simulation. The experimental pseudoscalar masses M_D and M_B are used as an input. These masses have been used to extrapolate the form factors to $q^2 = 0$.*

masses have been used for the extrapolation of f^+, V and A_1, A_2 respectively. M_t has been computed on the lattice, over the same gauge field configurations, at the same heavy and light quark masses used for the three-point functions. In order to obtain the physical $D-$ and $B-$meson masses, we have fitted the vector-pseudoscalar mass difference $\Delta M = M_{P^*} - M_P$ (and similarly for the axial and scalar cases) as $\Delta M = A_M + B_M/M_P$. The results, extrapolated to the strange and massless light quarks, are reported in table 1. These values have been used to extrapolate the form factors at $q^2 = 0$. In this extrapolation, the precise value of M_t is relatively unimportant. For example we have verified that, by using the vector meson mass in all cases, the results change by only a few per cent.

The q^2-dependence of $1/f^+(q^2)$ and $1/A_1(q^2)$ is compared with the meson dominance predictions in fig. 2. The values of the inverse form factors are given as a function of the dimensionless variable q^2/M_t^2, for the values of the heavy and light hopping parameter, $K_H = 0.1250$ and $K_l = 0.1432$. The lines in the figures represent the pole dominance expectations, with the pole masses computed on the lattice. In order to reduce the error due to the extrapolation in q^2, the values of the form factors at zero momentum transfer have been obtained, by fitting only the point closest to $q^2 = 0$, as explained above. The other points give, however, important information on the q^2-dependence of the form factors. As shown in the figure, the q^2 behaviour of the two form factors is compatible with pole dominance predictions. Similar conclusions can be reached also for the form factors $V(q^2)$ and $A_2(q^2)$. Notice that the axial form factors determined by using QCD sum rules [16, 17] do to follow the pole dominance behaviour.

3   Main Sources of Errors

In this section, we briefly describe the main sources of error which are present in our calculation, besides the quenched approximation.

3.1   The "Thinning" Procedure

"Thinning" means that, when computing the correlation functions, one uses only one point out of N_{th}, in each spatial direction. In our case $N_{th} = 3$. Thus, for example, the
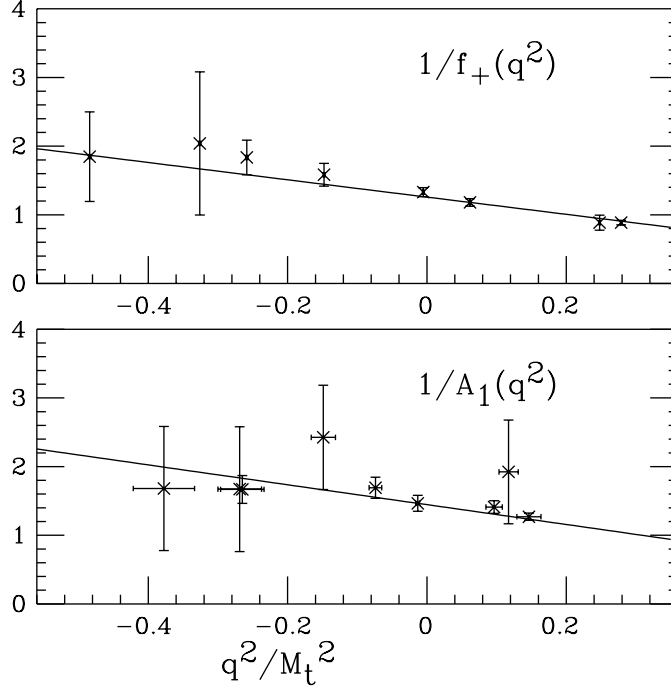
Figure 2: $1/f_+(q^2)$ and $1/A_1(q^2)$ as a function of the dimensionless variable q^2/M_t^2. The heavy and light Wilson parameters correspond to $K_H = 0.1250$ and $K_l = 0.1432$ respectively. The lines represent the pole dominance approximation.

Fourier transform of the two-point correlation function is defined as

$$C(t, \vec{p}) = \mathcal{N}_T \sum_{\vec{x}_T} e^{-i\vec{p}\cdot\vec{x}_T} C(t, \vec{x}_T) \tag{1}$$

where $x_T = 0, 3, 6, ...$ in each direction and $\mathcal{N}_T$ is a suitable normalization factor. This procedure is necessary when, as in our case, the computer memory is not sufficient to store the full quark propagators. There is a systematic error introduced by thinning, because we cannot eliminate high momentum components in the correlation functions. For $N_{th} = 3$, it is possible to show that, for each spatial direction, two higher momentum components (p_1 and p_2), besides the smallest one (p_0), give a contribution to the correlation function. p_1 and p_2 are related to p_0 by the simple relation:

$$p_m = p_0 + \frac{2\pi}{La} \cdot \left(\frac{L}{3}\right) \cdot m \quad , \quad m = 1, 2 \tag{2}$$

The systematic error introduced by thinning is expected to be negligible at large time distances, since the contribution of the unwanted higher energy states is exponentially suppressed in time. On the other hand, the resulting signal may be noisier, because we use a small sample of the lattice points.

In the case of the two-point correlation functions, we have been able to directly compare "thinned" and "non-thinned" correlation functions, computed on the same set of gauge

configurations, and for momenta $\vec{p} = 2\pi/La \cdot (0,0,0)$ and $(1,0,0)$. For the pseudoscalar and vector meson correlations, no observable statistical or systematic effect has been detected. However, the statistical noise introduced by thinning may be larger, when we compute three-point correlation functions, simply because, in the latter case, we have to thin twice. Thus, the thinning procedure could be responsible for the quite large statistical errors which have been found, in spite of the high statistical sample used in this calculation. Another reason could be the small spatial volume of our lattice.

3.2 $O(g^2a)$ and $O(a^2)$ Effects

Using the Clover action, discretization errors are of $O(g^2a)$ and $O(a^2)$. In the case of light quarks, these effects have been shown to be much smaller than in the case of the standard Wilson action [12]. In the charm quark mass region, at $\beta \sim 6.0$, however, discretization errors may still be important also in the Clover case.

An estimate of the lattice artefacts can be obtained by comparing values of lattice renormalization constants, computed from different matrix elements, in a non-perturbative way [12]. In ref. [18], the renormalization constants of the local vector and axial vector currents, $V_\mu^L = \bar{\psi}_1 \gamma_\mu \psi_2$ and $A_\mu^L = \bar{\psi}_1 \gamma_\mu \gamma_5 \psi_2$, have been determined, at $\beta = 6.0$, as a function of the quark masses, m_1 and m_2, using the Ward identity method [12, 14]. A variation of about $10 - 15\%$, on the values of Z_V and Z_A, has been observed, for heavy quark masses between $m_H a = 0.3$ and 0.9, which is the range used in the present study. The renormalization of the quark fields, proposed in ref. [19] in order to reduce $O(m_H a)$ effects, enters, in our case, only at $O(m_H^2 a^2)$, since we are using an improved action and improved operators [8, 9]. Since the effects that we have observed are linear in $m_H a$, the KLM correcting factor [19] is of no help in our case. We interpret the residual mass dependence as an effect of $O(\alpha_s a)$. Similar results have been observed by the UKQCD Collaboration, which works with a Clover action, at $\beta = 6.2$ [6]. For the above reasons, we have used, through this paper, the non-perturbative values of Z_V and Z_A, as determined in refs. [14, 15], for light quark masses. A $10 - 15\%$ of systematic error is then certainly present in our final results, due to residual $O(\alpha_s a)$ effects.

3.3 Extrapolations of the Form Factors

As explained in the previous section, the form factors, computed in a range of heavy quark masses around the charm, must be extrapolated in $1/M_P$ and q^2, in order to obtain predictions for the physical form factors at zero momentum transfer. There is also an extrapolation in the light quark mass, but this is quite smooth and unlikely to be a source of an important uncertainty, within the present statistical accuracy. The extrapolation of the form factors in $1/M_P$ is more delicate. There are arguments, based on HQET, which allow the expansion of the form factors in inverse powers of the heavy meson mass M_P. On the basis of HQET, up to $O(1/M_P^2)$ and up to logarithmic corrections, one expects

the following behaviour for the relevant form factors [10]:

$$\frac{f^+}{\sqrt{M_P}} = \gamma_+ \times \left(1 + \frac{\delta_+}{M_P}\right) \qquad \frac{V}{\sqrt{M_P}} = \gamma_V \times \left(1 + \frac{\delta_V}{M_P}\right)$$
$$A_1\sqrt{M_P} = \gamma_1 \times \left(1 + \frac{\delta_1}{M_P}\right) \qquad \frac{A_2}{\sqrt{M_P}} = \gamma_2 \times \left(1 + \frac{\delta_2}{M_P}\right)$$

$$(3)$$

The expansions given in eqs. (3) are valid, in the limit of large heavy quark mass, at fixed momentum $\vec{p}$ of the light meson (in the frame where the heavy meson is at rest) when $|\vec{p}| \ll M_P$. The above conditions are always satisfied for q^2_{max}, when the initial and final mesons are both at rest. In our simulation, they are also satisfied for the points corresponding to $\vec{p} = 2\pi/La \cdot (1,0,0)$ (these have been used in order to obtain all our final predictions).

In sec. 4, we show that the dependence of the form factors on the heavy meson mass, M_P, is compatible with the HQET predictions. In order to evaluate the stability of the results with respect to a different extrapolation, we have also used a "naïve" scaling law of the form:

$$f^+, V, A_1, A_2 = \alpha \times \left(1 + \frac{\beta}{M_P}\right) \qquad (4)$$

For $D-$mesons, we find that the differences between the different extrapolations, eq.(3) or (4), are quite small ($\leq 2\%$) and completely negligible with respect to the statistical errors. The results for $B-$meson decays will be reported in sec. 4. In this case the differences are larger, of the order of $10 - 20\%$, but still small with respect to the statistical errors, which amplify in the extrapolation. Thus, we are not able to distinguish between the two different behaviours.

We now turn the discussion to the q^2 extrapolation. In the range of heavy quark masses considered in this study, the q^2-dependence of the form factors is compatible with pole dominance predictions. This range corresponds approximately to a heavy meson mass of the order of the $D-$meson mass and we also have points at $q^2 \sim 0$. Therefore, as far as the D decays are concerned, the extrapolation to $q^2 = 0$ does not represent an important source of theoretical uncertainty. The extrapolation in $1/M_P$ to the $B-$meson mass results in momentum transfers close to the maximum one. In order to then predict the form factors at small q^2, we have been forced to assume pole dominance. Since the range of the extrapolation is very large[2], this strongly biases our final results, e.g. the determination of the form factors at $q^2 = 0$. In particular, we have used pole dominance also in those cases, where it is in contradiction with the results of QCD sum rules calculations [16, 17]. In order to improve the situation, it is necessary to be able to work with heavier quark masses and to increase the range of q^2. This can only be achieved by going to larger lattices (at larger values of β).

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[2]While the typical $\sqrt{q^2}$ on our lattice is at most 1.6 GeV, the extrapolation in the heavy quark mass will bring us to $\sqrt{q^2} = 4.5$ GeV ($q^2 = M_B^2 - 2M_B\sqrt{M_\pi^2 + (2\pi/18\,a)^2} + M_\pi^2$), and similarly for $B \to \rho$.

3.4 The "Analytic" and "Ratio" Methods

In order to extract the weak current matrix elements from the three-point functions, we used two different procedures, denoted by "analytic" and "ratio" methods. The two procedures are discussed in detail in ref. [5]. The "ratio" methods means that, at each fixed time distance, we divide the three-point correlation function by the two relevant two-point functions of the D and $K(K^*)$ meson (with corresponding momentum) in order to cancel the exponential time-dependence of the three-point correlation function. The "analytic" method differs from the previous one because, instead of dividing by the two-point correlation functions, computed at different momenta, we divide by the corresponding analytical expressions, with the source matrix elements and the meson energies (computed from the meson masses, see eqs.(6) and (8) below) taken from the fit of the two-point functions, at zero momentum.

When both the initial and final mesons are at rest these two methods are practically equivalent and lead to almost identical results. However, when the meson momenta are different from zero, within the statistical fluctuations, the two methods are expected to agree only up to $O(a)$ effects.

At $\beta = 6.2$ and with a lattice volume $24^3 \times 48$, it was found that discretization errors seem to be reduced, by using, for the two-point correlation functions, the "lattice" dispersion relation of a free boson

$$\bar{C}(t, \vec{p}) = \frac{Z}{2\sinh E} e^{-Et},$$ (5)

where

$$E = \frac{2}{a}\text{arcsinh}\left(\sqrt{\sinh^2(\frac{ma}{2}) + \sum_{i=1,3}\sin^2(\frac{p_i a}{2})}\right)$$ (6)

and $\vec{p}$ is the momentum of the meson [20]. The same is true in our case. We have verified this point, by studying the ratio $R(t) = C(t, \vec{p})/\bar{C}(t, \vec{p})$. $C(t, \vec{p})$ is the two-point correlation function of a meson with momentum $\vec{p}$, as computed in our simulation. $\bar{C}(t, \vec{p})$ is the expression in eq. (5), with Z and m taken from the fit of the zero momentum correlation to $\bar{C}(t, \vec{p} = \vec{0})$. At large time distances, we find that $R(t)$ is close to one (typically 1.05 ± 0.01). This compares favourably to the value 1.10 ± 0.02 in the case where, instead of eq. (5), we use the standard expression

$$\hat{C}(t, \vec{p}) = \frac{Z}{2E} e^{-Et},$$ (7)

with

$$E = \sqrt{m^2 + |\vec{p}|^2}.$$ (8)

For this reason, in all our analysis, we have fitted the two point functions to $\bar{C}(t, \vec{p})$, as defined in eq. (5).

Using $\bar{C}(t, \vec{p})$, the "ratio" and "analytic" methods yield slightly different results. In order to exhibit the differences, we give in table 2 the corresponding sets of values for the form factors $f_+(0)$, $V(0)$, $A_1(0)$ and $A_2(0)$, as well as for the ratios V/A_1 and A_2/A_1 at zero momentum transfer, for the $D \to K$ and $D \to K^*$ decays. From the entries in the

	$f_+(0)$	$V(0)$	$A_1(0)$
analytic	0.81 ± 0.07	1.07 ± 0.21	0.66 ± 0.11
ratio	0.75 ± 0.06	1.09 ± 0.22	0.67 ± 0.11

	$A_2(0)$	$V(0)/A_1(0)$	$A_2(0)/A_1(0)$
analytic	0.44 ± 0.34	1.63 ± 0.29	0.67 ± 0.43
ratio	0.52 ± 0.29	1.62 ± 0.29	0.79 ± 0.35

Table 2: *Semi-leptonic form factors at zero momentum transfer for $D \rightarrow K$ and $D \rightarrow K^*$ decays. The results have been obtained by using the method called "analytic" or "ratio" to extract the form factors.*

table one can see that the differences are larger for the decay in the pseudoscalar channel and they are always between 2% and $\sim 8\%$[3]. In our final results, the differences between the two methods have been added in quadrature with the statistical errors. Had we used the standard expression $\hat{C}(t, \vec{p})$, we would had found larger differences between the "ratio" and "analytic" methods, which in the worst case have been evaluated to be 11%, see also ref. [5].

4   Physics Results

4.1 $D-$Meson Decays

Our best estimates for the form factors and partial widths are those given in the abstract and in tables 3 and 4. In these tables, we present our results, together with other calculations and experimental determinations of the form factors. Our central values for the four relevant form factors are in good agreement with the experimental data (typically to within less than 10%), although we still suffer, in some cases, from sizeable statistical errors, especially in the case of A_2. Considering that lattice calculations do not contain free parameters, we find this agreement remarkable. We also observe (tables 3 and 4) that predictions from QCD sum rules calculations are in agreement with lattice calculations and experimental determinations, whereas quark models fail to describe the $D \rightarrow K^*$ decay. Besides the results reported in the tables, we have also obtained $f_+^\pi(0)/f_+^K(0) = 1.02 \pm 0.03$.

4.2   Extrapolation to $B-$Meson Decay

At the values of lattice spacing currently used in numerical simulations, we are unable to study directly the b quarks. However, as discussed above, in order to obtain indirect information on $B-$meson semi-leptonic decays, we can follow the strategy suggested in ref. [5]: we study the form factors in the region of the charm quark mass and then extrapolate the results to the bottom mass by using the scaling behaviour predicted by the HQET, eqs. (3). In order to reduce the uncertainty due to the extrapolation, one

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[3]For the form factor A_2, the difference is of the order of 15%. In this case, however, the statistical errors are so large that the possibility of drawing any conclusion is prevented.

	Reference	$\mathbf{f_+(0)}$	$\mathbf{V(0)}$	$\mathbf{A_1(0)}$
EXP	Average [21]	0.77 ± 0.04	1.16 ± 0.16	0.61 ± 0.05
LAT	**This work**	0.78 ± 0.08	1.08 ± 0.22	0.67 ± 0.11
	LMMS [3]	0.63 ± 0.08	0.86 ± 0.10	0.53 ± 0.03
	BKS [4]	$0.90 \pm 0.08 \pm 0.21$	$1.43 \pm 0.45 \pm 0.49$	$0.83 \pm 0.14 \pm 0.28$
	ELC [5]	$0.60 \pm 0.15 \pm 0.07$	0.86 ± 0.24	0.64 ± 0.16
	UKQCD [6]	$0.67^{+0.07}_{-0.08}$	$1.01^{+0.30}_{-0.13}$	$0.70^{+0.07}_{-0.10}$
	LANL [7]	0.73 ± 0.05	1.27 ± 0.08	0.66 ± 0.03
SR	BBD [16]	$0.60^{+0.15}_{-0.10}$	1.10 ± 0.25	0.50 ± 0.15
QM	WSB [22]	0.76	1.23	0.88
	ISGW [23]	0.8	1.1	0.8
	GS [24]	0.69	1.5	0.73
	Reference	$\mathbf{A_2(0)}$	$\mathbf{V(0)/A_1(0)}$	$\mathbf{A_2(0)/A_1(0)}$
EXP	Average [21]	0.45 ± 0.09	1.90 ± 0.25	0.74 ± 0.15
LAT	**This work**	0.49 ± 0.34	1.6 ± 0.3	0.7 ± 0.4
	LMMS [3]	0.19 ± 0.21	1.6 ± 0.2	0.4 ± 0.4
	BKS [4]	$0.59 \pm 0.14 \pm 0.24$	$1.99 \pm 0.22 \pm 0.33$	$0.7 \pm 0.16 \pm 0.17$
	ELC [5]	$0.40 \pm 0.28 \pm 0.04$	1.3 ± 0.2	0.6 ± 0.3
	UKQCD [6]	$0.66^{+0.10}_{-0.15}$	$1.4^{+0.5}_{-0.2}$	0.9 ± 0.2
	LANL [7]	0.44 ± 0.16	1.83 ± 0.09	0.74 ± 0.19
SR	BBD [16]	0.60 ± 0.15	2.2 ± 0.2	1.2 ± 0.2
QM	WSB [22]	1.15	1.4	1.3
	ISGW [23]	0.8	1.4	1.0
	GS [24]	0.55	2.0	0.8

Table 3: *Semi-leptonic form factors for $D \to K$ and $D \to K^*$ decays. The label EXP refers to the experimental results and the labels LAT, QM and SR correspond to lattice, quark model and sum rules calculations respectively.*

	Reference	$\Gamma(D \to K)$	$\Gamma(D \to K^*)$	$\dfrac{\Gamma_L(D \to K^*)}{\Gamma_T(D \to K^*)}$
EXP	Average [21, 25]	9.0 ± 0.5	5.1 ± 0.5	1.15 ± 0.17
LAT	**This work**	9.1 ± 2.0	6.9 ± 1.8	1.2 ± 0.3
	LMMS [3]	5.8 ± 1.5	5.0 ± 0.9	1.51 ± 0.27
	ELC [5]	$5.4 \pm 3.0 \pm 1.4$	6.4 ± 2.8	1.4 ± 0.3
	UKQCD [6]	$7.0 \pm 1.6 \pm 0.4$	$6.0^{+0.8}_{-1.6}$	$1.06 \pm 0.16 \pm 0.02$
SR	BBD [16]	6.4 ± 1.4	3.8 ± 1.5	0.86 ± 0.06
QM	WSB [22]	8.8	9.7	0.92
	ISGW [23]	8.5	9.2	1.09
	GS [24]	7.1	$-$	$-$
	Reference	$\Gamma(D^0 \to \pi^-)$	$\Gamma(D^0 \to \rho^-)$	$\Gamma(D_s \to \phi)$
EXP	Average [26]	$0.9^{+0.3}_{-0.2}$	$-$	4.0 ± 0.6
LAT	**This work**	0.8 ± 0.2	0.6 ± 0.2	6.4 ± 1.1
	LMMS [3]	0.5 ± 0.2	0.40 ± 0.09	4.4 ± 0.6
	ELC [5]	$0.5 \pm 0.3 \pm 0.1$	$0.6 \pm 0.3 \pm 0.1$	$-$
	UKQCD [6]	$0.52 \pm 0.18 \pm 0.04$	0.43 ± 0.11	$-$
SR	Ball [17]	0.39 ± 0.08	0.12 ± 0.03	$-$
QM	WSB [22]	0.72	0.68	7.9
	ISGW [23]	0.38	0.46	$-$

Table 4: *Semi-leptonic partial widths (in units of $10^{10}s^{-1}$) for $D \to K$, K^*, π, ρ and ϕ, using $V_{cs} = 0.975$ and $V_{cd} = 0.222$. The ratio of the longitudinal to transverse polarization partial widths for $D \to K^*$ is also given. The experimental values of $\Gamma(D^0 \to \pi^-)$, and $\Gamma(D_s \to \phi)$ have been computed by taking the corresponding branching ratios and meson life-times from ref. [26].*
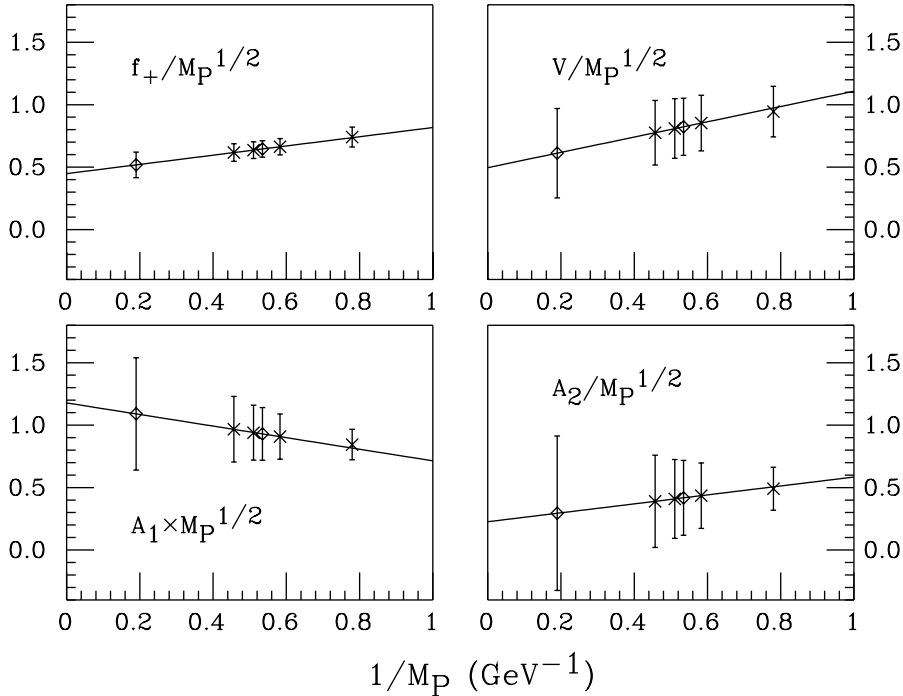
Figure 3: *Extrapolation of the form factors to the D- and B-meson masses (diamonds) using the predictions of the HQET. The points corresponding to the lightest heavy quark mass have not been used in the fits.*

could also compute the form factors in the static limit, i.e. the limit in which the heavy quark mass is infinite. This determination is not available yet.

Our results show that the dependence of the form factors on the heavy meson mass, M_P, is compatible with the HQET predictions, see eq.(3), as first observed in ref. [5]. This dependence is shown in fig. 3, where the four relevant form factors, extrapolated to the chiral limit in the light quark mass, are given, as a function of $1/M_P$ (crosses). The form factors were computed with $\vec{p}_D = 0$ and $\vec{p}_{K,K^*} = 2\pi/La(1,0,0)$. The values interpolated/extrapolated to the D and B meson masses (diamonds) are also given.

From the values of fig. 3, we can compute the form factors at $q^2 = 0$. They have been obtained by assuming the pole meson dominance behaviour, using the meson masses given in table 1. The corresponding results are presented in table 5, labelled "b", together with the results of other theoretical determinations. As discussed earlier, in order to evaluate the stability of these results with respect to a different extrapolation, we also give the values (labelled as "a") obtained with the naïve scaling laws, eq. (4). In the case of the $D-$meson the difference between the HQET scaling laws and naïve scaling is immaterial. For the $B-$meson, the differences are smaller than the statistical errors (typically $10 - 20\%$), so that we are not able to distinguish between the two behaviours. We find reassuring, however, that the results for $B-$decays are in good agreement with the results of ref. [5], where the standard fermion Wilson action and a smaller lattice

	Reference	$f_+(0)$	$V(0)$	$A_1(0)$
LAT	**This work "a"**	0.29 ± 0.06	0.45 ± 0.22	0.29 ± 0.16
	This work "b"	0.35 ± 0.08	0.53 ± 0.31	0.24 ± 0.12
	ELC "a" [5]	$0.26 \pm 0.12 \pm 0.04$	0.34 ± 0.10	0.25 ± 0.06
	ELC "b" [5]	$0.30 \pm 0.14 \pm 0.05$	0.37 ± 0.11	0.22 ± 0.05
SR	Ball [17]	0.26 ± 0.02	0.6 ± 0.2	0.5 ± 0.1
QM	WSB [22]	0.33	0.33	0.28
	ISGW [23]	0.09	0.27	0.05
	Reference	$A_2(0)$	$V(0)/A_1(0)$	$A_2(0)/A_1(0)$
LAT	**This work "a"**	0.24 ± 0.56	2.0 ± 0.9	0.8 ± 1.5
	This work "b"	0.27 ± 0.80	2.6 ± 1.9	1.0 ± 3.1
	ELC "a" [5]	$0.38 \pm 0.18 \pm 0.04$	1.4 ± 0.2	$1.5 \pm 0.5 \pm 0.2$
	ELC "b" [5]	$0.49 \pm 0.21 \pm 0.05$	1.6 ± 0.3	$2.3 \pm 0.7 \pm 0.2$
SR	Ball [17]	0.4 ± 0.2	–	–
QM	WSB [22]	0.28	1.2	1.0
	ISGW [23]	0.02	5.4	0.4

Table 5: *Semi-leptonic form factors for $B \to \pi$ and ρ. The label "a" refers to the naïve extrapolation in $1/M_P$, eq. (3), and label "b" to the extrapolation given in eq. (4). To extrapolate to zero momentum transfer we have used the masses of table 1. We have taken the ISGW form factors, as extrapolated to $q^2 = 0$, in ref. [17].*

spacing, corresponding to $\beta = 6.4$, were used. Finally, we observe that we find values for A_1 and A_2 smaller than those obtained by QCD sum rules, cf. table 5. The reason is probably due to the fact that we have assumed pole dominance, whereas QCD sum rules find that the axial form factors are flat in q^2.

From the values of the form factor $f_+(0)$ given in table 5, and by assuming meson dominance for the q^2-dependence, we can give an estimate of the $B \to \pi$ decay rate. To evaluate the errors, we have allowed the form factor to vary in all possible ways by one σ within the statistical errors and to vary in all possible ways among the values obtained with different extrapolations in $1/M_P$, i.e. fits "a" and "b". We obtain

$$\Gamma(B \to \pi l \nu_l) = |V_{ub}|^2 (8 \pm 4) \times 10^{12} s^{-1}, \qquad (9)$$

that corresponds to the branching ratio

$$B(B \to \pi l \nu_l) = |V_{ub}|^2 (12 \pm 6) \qquad (10)$$

when the value $\tau_B = (1.49 \pm 0.12) \times 10^{-12}$ sec. is used for the B−meson lifetime. On the other hand, with our present accuracy, the errors on the form factors for the $B \to \rho$ decay are still too large to get an estimate of the corresponding branching ratio. A accurate determination of this quantity can only be achieved by working with heavier quark masses and by going to larger lattices.

Acknowledgements

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