Editors' Suggestion

Instantons and Hilbert functions

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We study superpotentials from worldsheet instantons in heterotic Calabi-Yau compactifications for vector bundles constructed from line bundle sums, monads, and extensions. Within a certain class of manifolds and for certain second homology classes, we derive simple necessary conditions for a nonvanishing instanton superpotential. These show that nonvanishing instanton superpotentials are rare and require a specific pattern for the bundle construction. For the class of monad and extension bundles with this pattern, we derive a sufficient criterion for nonvanishing instanton superpotentials based on an affine Hilbert function. This criterion shows that a nonzero instanton superpotential is common within this class. The criterion can be checked using commutative algebra methods only and depends on the topological data defining the Calabi-Yau X and the vector bundle V.

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I. INTRODUCTION

Nonperturbative superpotentials generated from instanton effects play an important role in string theory [1–11] and they form a crucial ingredient for a stability analysis of string vacua and for practically all scenarios of moduli stabilization. It is well known that contributions to the instanton superpotential are proportional to $\exp(-\mathrm{Vol}(C))$, where C is the (calibrated) cycle wrapped by the string or the brane. However, more detailed calculations including the prefactor of this exponential are often difficult to carry out and explicit results are few and far between. In particular, it is not easy to determine whether the instanton superpotential is zero or nonzero.

In this paper, we are concerned with superpotentials from string worldsheet instantons in heterotic compactifications on Calabi-Yau three-folds X with vector bundles $V \to X$. For such compactifications, the instanton superpotential can receive a contribution $W_{\mathcal{C}} \sim \exp(-\mathrm{Vol}(\mathcal{C}))$ from each second homology class $\mathcal{C}$, where all isolated, genus zero holomorphic curves C_i, $i = 1, \ldots, n_{\mathcal{C}}$, in the class $\mathcal{C}$ contribute to the prefactor in $W_{\mathcal{C}}$.

Beasley and Witten [12] have studied linear and half-linear sigma models and have shown that the contributions from the curves C_i sum up to zero, and, hence, that $W_{\mathcal{C}}$ vanishes, under fairly general assumptions (see also Refs. [13–16]). On the other hand, a number of papers [17–20] have produced examples with a nonvanishing $W_{\mathcal{C}}$, thus apparently evading the vanishing theorems of Ref. [12]. There are two obvious resolutions: Either there is a problem with the geometric methods used to calculate the instanton contributions or the examples considered violate one of the assumptions underlying the vanishing theorems of Beasley and Witten. The results of Ref. [20] point to the latter being the correct explanation.

More specifically, one of the assumptions underlying the vanishing theorems is compactness of the instanton moduli space. Unfortunately, this assumption is not easily checked in general. A nice straightforward method, due to Bertolini and Plesser [21], is only available if a Gauged Linear Sigma Model (GLSM) formulation of the model can be found. This limits the models for which the assumptions can be checked with this method and requires, among other things, that the bundle V is given as a monad bundle. However, in Ref. [20] the authors have identified a number of models for which the geometric calculation can be carried out and a GLSM formulation can be found. In all of those cases, the result of the geometric calculation turn

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out to be consistent with the vanishing theorems, as formulated by Bertolini and Plesser.

In the present paper, we would like to invert the logic and assume, based on the evidence in Ref. [20], that the standard geometric methods to calculate instanton superpotential are indeed correct and consistent with the vanishing theorems. By applying these methods we would like to address two main questions.

(i) Can we find simple conditions for the vanishing/nonvanishing of the instanton superpotential based solely on the geometric data (X, V)? These can be thought of as geometric analogues of the Bertolini-Plesser GLSM conditions, but applicable to a wider class of models for which no GLSM description is known.

(ii) How "common" is it for the instanton superpotential to be vanishing or nonvanishing?

As we will see, the first question can be partially answered in terms of simple cohomology conditions and a certain affine Hilbert function which we introduce. Analyzing these, we find that a nonvanishing instanton superpotential only arises within a specific subclass of bundles V, but that it is common to be nonvanishing within this subclass. The plan of the paper is as follows. In the next section, we review the standard geometric method to calculate string instanton superpotentials. As we will see, this method requires explicit knowledge of the isolated, genus zero curves C_i, which can be difficult to determine explicitly. In Sec. III, we introduce a class of (complete intersection) Calabi-Yau manifolds where these curves can be found, at least for certain homology classes $\mathcal{C}$. Basic features of common vector bundle constructions, including line bundle sums, monad bundles, and extension bundles, relevant for our discussion of instantons, are summarized in Sec. IV. The requisite mathematical background on coordinate rings and Hilbert functions is reviewed in Sec. V. In Sec. VI, we formulate the Hilbert function criterion for nonvanishing instanton superpotentials and apply it to a number of examples. We conclude in Sec. VII.

II. GEOMETRIC CALCULATION OF INSTANTON SUPERPOTENTIALS

In this section, we first review a method for calculating instanton superpotentials based on techniques from algebraic geometry (see, for example, Refs. [8,10,11] for more details).

We are working in the context of $E_8 \times E_8$ heterotic string compactifications on Calabi-Yau three-folds to four-dimensional theories with $\mathcal{N} = 1$ supersymmetry. Our main object of interest is the superpotential of the four-dimensional theory generated by string instanton effects.

The basic data which defines the compactification consist of a Calabi-Yau three-fold X and a holomorphic, poly-stable vector bundle $V \to X$ with $c_1(V) = 0$, and a structure group which can be embedded into E_8. In general, there is also another bundle whose structure group embeds into the second E_8 factor and/or five branes wrapping holomorphic curves in X. Details of these further ingredients are not really relevant for our discussion but we would like to ensure that there exist choices of a second bundle or five-branes such that the compactification is anomaly-free and respects supersymmetry. This is guaranteed if we demand that the curve dual to $c_2(TX) - c_2(V)$ is an element of the Mori cone of X for a poly-stable V. In this case, an anomaly-free, supersymmetric completion can, for example, be achieved by wrapping five-branes on a holomorphic curve with class $c_2(TX) - c_2(V)$.

The instanton superpotential W in the resulting four-dimensional theory can be written as a sum $W = \sum_{\mathcal{C}} W_{\mathcal{C}}$ over contributions $W_{\mathcal{C}}$ which are associated with classes $\mathcal{C} \in H_2(X, \mathbb{Z})$ in the second homology of X. We will usually focus on one of these homology classes $\mathcal{C}$ and will attempt to compute $W_{\mathcal{C}}$. The superpotential term $W_{\mathcal{C}}$ receives contributions from the isolated, genus zero holomorphic curves with class $\mathcal{C}$. We denote these curves by C_i, where $i = 1, \ldots, n_{\mathcal{C}}$ and $n_{\mathcal{C}}$ is the genus zero Gromov-Witten invariant. Schematically, the superpotential term $W_{\mathcal{C}}$ can be written as

$$W_{\mathcal{C}} = \left[\sum_{i=1}^{n_{\mathcal{C}}} \mathrm{Pfaff}_{C_i}\right] \exp\left(-\int_{\mathcal{C}} (J + iB)\right) \tag{2.1}$$

where J is a Kähler form on X, B is the NS two-form, and Pfaff_{C_i} is the Pfaffian. Its precise form in terms of differential operators on the curve C_i can be found, for example, in Ref. [6]. The instanton superpotential associated with the class $\mathcal{C}$ is, of course, proportional to the exponent $\exp(-\mathrm{Vol}(\mathcal{C}))$. The (one-loop) prefactor in Eq. (2.1) corresponds to the various contributing isolated, genus zero curves C_i with class $\mathcal{C}$ which are wrapped by instantonic strings.

From a theoretical perspective as well as in the context of physical applications, such as for example in applications to moduli stabilization, it is crucial to know whether the prefactor $\sum_i \mathrm{Pfaff}_{C_i}$ in Eq. (2.1) is zero or nonzero. This is the main question we will address in the present paper.

How can the Pfaffians Pfaff_{C_i} be computed in practice? The key statement [10] underlying the algebraic computation is formulated in terms of the bundle

$$V_i \coloneqq V|_{C_i} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \tag{2.2}$$

and asserts the following equivalence:

$$H^0(V_i) \neq 0 \quad \Leftrightarrow \quad \mathrm{Pfaff}_{C_i} = 0. \tag{2.3}$$

Broadly speaking, the idea is to work out the cohomology on the left-hand side, rather than computing the Pfaffian directly. More specifically, we note that the value of this cohomology does depend on the choice of moduli, that is, on the complex structure moduli of X and on the bundle moduli of V. Here we will generally assume that the complex structure moduli of X have been fixed to suitably generic values and focus on the dependence on the bundle moduli of V, which we denote as $b = (b_\alpha)$. Of course it is possible that the cohomology in (2.3) is nonzero for all values of b. In this case, the Pfaffian, as a function of b, vanishes identically.

A more interesting situation arises when the cohomology in (2.3) vanishes for generic values of b but has a "jumping locus" in bundle moduli space where it acquires a nonzero value. As we will see, such a jumping locus is described by an equation of the form $f_i(b) = 0$, where f_i is a holomorphic function. Since this function f_i and the Pfaffian Pfaff_{C_i} have an identical zero locus they must be proportional. Hence, we can write

$$W_{\mathcal{C}} = \left[\sum_{i=1}^{n_{\mathcal{C}}} \lambda_i f_i(b)\right] \exp\left(-\int_{\mathcal{C}} (J + iB)\right), \quad (2.4)$$

where $\lambda_i \in \mathbb{C}$ are constants.

Unfortunately, we do not currently know how to compute the constants λ_i in Eq. (2.4), at least not with algebraic methods. In fact, these constants are tied up with a rather subtle interpretation [6] of the NS two-form field B. Unfortunately, our ignorance in this respect somewhat obstructs our ability to answer the question about the vanishing of $W_{\mathcal{C}}$. Luckily, not all is lost if the f_i are indeed nontrivial functions of the moduli b, as is frequently the case. Then we have

$$(f_i)_{i=1,\ldots,n_{\mathcal{C}}} \text{linearly independent functions} \quad \Rightarrow \quad W_{\mathcal{C}} \neq 0. \quad (2.5)$$

This is the basic criterion which will underlie much of our discussion. It allows for a definite conclusion if the functions f_i are linearly independent—in this case $W_{\mathcal{C}}$ is a nonzero function. If the f_i are linearly dependent the answer depends on the unknown constants λ_i. If their values are such that they realize the linear dependence relation $\sum_i \lambda_i f_i = 0$ then $W_{\mathcal{C}}$ vanishes, otherwise $W_{\mathcal{C}}$ is still nonzero.

Any computation along the above lines requires, in a first instance, explicit knowledge of the isolated, genus-zero curves[1] C_i in a given class $\mathcal{C}$. Finding these curves can be quite nontrivial, so any concrete progress depends on a setting where these curves can be found. We will now review how this can be done for a certain class of Calabi-Yau manifolds.

[1]We focus on the dominant instanton contributions, which arise from curves with single wrapping.

III. THE CALABI-YAU MANIFOLDS

A. General setup

We consider an ambient space of the form $\mathcal{A} = \mathbb{P}^1 \times \mathcal{B}$, where $\mathcal{B} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$, with homogeneous coordinates $x = (x_0, x_1)$ for the $\mathbb{P}^1$ factor and $y = (y_{\alpha,0}, \ldots, y_{\alpha,n_\alpha})_{\alpha=1,\ldots,m}$ for the other factors. In this ambient space, we define complete intersection Calabi-Yau manifolds (CICYs) X which are specified by a configuration matrix

$$\text{type I}: X \in \left[\begin{array}{c|ccccc} \mathbb{P}^1 & 1 & 1 & 0 & \cdots & 0 \\ \mathcal{B} & q_1 & q_2 & q_3 & \cdots & q_K \end{array}\right],$$
$$\text{type II}: X \in \left[\begin{array}{c|cccc} \mathbb{P}^1 & 2 & 0 & \cdots & 0 \\ \mathcal{B} & q_1 & q_2 & \cdots & q_K \end{array}\right]. \quad (3.1)$$

Every column of the configuration matrix indicates the multidegree of a homogeneous polynomial $P_a = P_a(x, y)$ and the CICY manifold X is the common zero locus of these polynomials. The Calabi-Yau condition, $c_1(TX) = 0$, is equivalent to the degrees in each row of the configuration matrix summing up to the dimension of the projective space plus one. For $\mathbb{P}^1$ this leaves only two possible patterns for the degree and this is how the above two types arise.

The point about these CICY manifolds, as shown in Ref. [18], is that the isolated, genus zero curves in the class $\mathcal{C}$ which corresponds to the first $\mathbb{P}^1$ factor can be determined rather straightforwardly. We briefly review how this works, starting with type I. In this case, the defining polynomials can be written as

$$\begin{aligned} P_1(x, y) &= x_0 Q_1(y) + x_1 Q_2(y), \\ P_2(x, y) &= x_0 Q_3(y) + x_1 Q_4(y), \\ P_a(x, y) &= Q_{a+2}(y) \quad \text{for } a > 2, \end{aligned} \quad (3.2)$$

where Q_1 and Q_2 have multidegree q_1, Q_3 and Q_4 have multidegree q_2, and Q_{a+2} for $a > 2$ has multidegree q_a. For type II, the analogous decompositions are

$$\begin{aligned} P_1(x, y) &= x_0^2 Q_1(y) + x_0 x_1 Q_2(y) + x_1^2 Q_3(y), \\ P_a(x, y) &= Q_{a+2}(y) \quad \text{for } a > 2 \end{aligned} \quad (3.3)$$

where Q_1, Q_2, and Q_3 have multidegree q_1 and Q_{a+2} for $a > 2$ has multidegree q_a. For either type, the defining equations $P_1(x, y) = \cdots = P_K(x, y) = 0$ of the CICY manifold are solved for all $x \in \mathbb{P}^1$ if

$$Q_1(y) = Q_2(y) = \cdots = Q_{K+2}(y) = 0. \quad (3.4)$$

These last equations define a zero-dimensional complete intersection in the space $\mathcal{B}$ which corresponds to a finite number of points Y_i. This finite point set can also be represented by the configuration matrices

$$\text{type I: } \{Y_i\} \in \left[\mathcal{B} \;\middle|\; q_1 \;\; q_1 \;\; q_2 \;\; q_2 \;\; q_3 \;\; \cdots \;\; q_K\right],$$
$$\text{type II: } \{Y_i\} \in \left[\mathcal{B} \;\middle|\; q_1 \;\; q_1 \;\; q_1 \;\; q_2 \;\; \cdots \;\; q_K\right]. \quad (3.5)$$

In this way, we have identified a number of isolated, genus-zero curves $\mathbb{P}^1 \times Y_i \subset X$, where $i = 1, \ldots, n_{\mathcal{C}}$ in the class $\mathcal{C}$ associated with the first $\mathbb{P}^1$ factor. By computing the Gromov-Witten invariant for this class [18], it can be shown that this is indeed the complete set of such curves.

For the calculation of instanton superpotentials along the lines described in Sec. II, we need to find the isolated, genus-zero curves explicitly. The above setup presents us with a straightforward way to do this by solving the Eqs. (3.4) for the loci Y_i of these curves in the "transverse" space $\mathcal{B}$. Note that, while this is conceptually simple, it can still be very difficult to carry out in practice. Finding the exact solutions to Eqs. (3.4) is impossible for anything but the simplest cases and even numerical solutions can be difficult to come by. The alternative algebraic approach we will be formulating is circumventing this problem—it requires no explicit knowledge of the points Y_i.

Finally, we introduce an algebraic descriptions of the above setup. The point set $\{Y_i\}$ is a zero-dimensional algebraic variety but there are two, subtly different ways to think about this. For one, we can think of $\{Y_i\}$ as a projective subvariety of $\mathcal{B}$ and associate with it the projective ideal

$$I = \langle Q_1, \ldots, Q_{K+2} \rangle. \quad (3.6)$$

Alternatively, we can also think about the point set $\{Y_i\}$ as an affine variety. To this end, we focus on the patch U_0 of $\mathcal{B}$ where all $y_{\alpha,0} \neq 0$ and we assume that the defining polynomials Q_a are sufficiently generic such that all points Y_i are contained in U_0. Then, we can think of the point set $\{Y_i\}$ as an affine subvariety of U_0 and associate with it an ideal J which is obtained from I by adding the "localizing" generators $y_{\alpha,0} - 1$. Hence, J is explicitly given by

$$J = \langle Q_1, \ldots, Q_{K+2}, y_{1,0} - 1, \ldots, y_{m,0} - 1 \rangle. \quad (3.7)$$

Associated with the ideals I and J are projective and affine coordinate rings, respectively, and we have the following maps between those rings:

$$\mathbb{C}[y] \overset{r}{\to} S \overset{\ell}{\to} A \quad \text{with} \quad S \coloneqq \frac{\mathbb{C}[y]}{I}, \qquad A \coloneqq \frac{\mathbb{C}[y]}{J}. \quad (3.8)$$

Here, r maps a polynomial in $\mathbb{C}[y]$ to its associated class in S and ℓ is a localization map, effectively carried out by setting all $y_{\alpha,0} = 1$. Note that the affine ring A is, in fact, finite dimensional with dimension equal to $n_{\mathcal{C}}$, the number of points Y_i.

As we will see, these algebraic descriptions of the curve loci $\{Y_i\}$ in terms of coordinate rings are key to our subsequent discussion of instantons. In particular, the rings S and A do not explicitly depend on the points Y_i but merely on the polynomials Q_a. This feature means that out algebraic approach will not rely on the explicit knowledge of these points.

B. A few simple examples

It is useful to introduce a few simple examples which can be used to illustrate our method as we go along. We emphasise that the following examples are specifically chosen for their simplicity, particularly a small number, $n_{\mathcal{C}}$, of curves, so that an explicit "on paper" treatment is possible. Our method will of course not be restricted to such simple cases and some more complicated examples will be described later.

1. Example 1: A type I example with two projective factors

Consider the CICY manifold X (number 7867 in the standard list [22,23]) with configuration matrix

$$X \in \left[\begin{array}{c|cccc} \mathbb{P}^1 & 0 & 0 & 1 & 1 \\ \mathbb{P}^6 & 3 & 2 & 1 & 1 \end{array}\right]^{2,68}_{-132} \begin{array}{l} x_0, x_1 \\ y_0, \ldots, y_6 \end{array} \quad (3.9)$$

where the Hodge numbers $h^{1,1}(X)$, $h^{2,1}(X)$ are attached as a superscript and the Euler number as a subscript. The single-wrapping Gromov-Witten invariant associated with the class of the $\mathbb{P}^1$ factor is $n_{\mathcal{C}} = 6$ and the configuration matrix specifying the six loci Y_i of these curves in the transverse space $\mathbb{P}^6$ is

$$\{Y_1, \ldots, Y_6\} \in \left[\mathbb{P}^6 \;\middle|\; 3 \;\; 2 \;\; 1 \;\; 1 \;\; 1 \;\; 1\right] \cong \left[\mathbb{P}^2 \;\middle|\; 3 \;\; 2\right]. \quad (3.10)$$

The last equivalence follows by repeated application of the equivalence $[\mathbb{P}^n|1] \cong \mathbb{P}^{n-1}$. In order to find the points Y_i explicitly, we make a particularly simple choice for the polynomials Q_a, namely

$$Q_1 = y_1^3 - y_0^3, \qquad Q_2 = y_2^2 - y_0^2, \qquad Q_a = y_a \quad \text{for } a = 3, \ldots, 6. \quad (3.11)$$

Then, the six points are given by

$$\{Y_i\} = \{[1 : \alpha^q : (-1)^s : 0 : \cdots : 0] \in \mathbb{P}^6 | q = 0, 1, 2, s = 0, 1\}, \quad (3.12)$$

where $\alpha = \exp(2\pi i/3)$. For the projective and affine coordinate ring of these points we have

$$S = \frac{\mathbb{C}[y_0, \ldots, y_6]}{\langle y_1^3 - y_0^3, y_2^2 - y_0^2, y_3, y_4, y_5, y_6 \rangle} \cong \frac{\mathbb{C}[y_0, y_1, y_2]}{\langle y_1^3 - y_0^3, y_2^2 - y_0^2 \rangle}, \tag{3.13}$$

$$A = \frac{\mathbb{C}[y_0, \ldots, y_6]}{\langle y_1^3 - y_0^3, y_2^2 - y_0^2, y_3, y_4, y_5, y_6, y_0 - 1 \rangle} \cong \frac{\mathbb{C}[y_1, y_2]}{\langle y_1^3 - 1, y_2^2 - 1 \rangle}$$
$$= \mathrm{Span}([1], [y_1], [y_1^2], [y_2], [y_1 y_2], [y_1^2 y_2]). \tag{3.14}$$

In the last expression the square brackets indicate the class in A and we see explicitly that A is six-dimensional. The existence of a basis of A with monomial representatives is a general feature of such affine coordinate rings for zero-dimensional varieties, as we discuss in Sec. V.

2. Example 2: A type II example with two projective factors

The CICY manifold X (with number 7888 in the standard list [22,23]) is defined by the configuration matrix

$$X \in \left[\begin{array}{c|cc} \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^4 & 4 & 1 \end{array}\right]^{2,86}_{-168} \quad \begin{array}{l} x_0, x_1 \\ y_0, \ldots, y_4 \end{array}. \tag{3.15}$$

The single-wrapping Gromov-Witten invariant for the class associated with the $\mathbb{P}^1$ factor is $n_{\mathcal{C}} = 4$ and the loci Y_i of the four curves in $\mathbb{P}^4$ are described by the configuration matrix

$$\{Y_1, Y_2, Y_3, Y_4\} \in \left[\begin{array}{c|cccc} \mathbb{P}^4 & 4 & 1 & 1 & 1 \end{array}\right] \cong \left[\begin{array}{c|c} \mathbb{P}^1 & 4 \end{array}\right]. \tag{3.16}$$

For a simple choice of defining polynomials we can explicitly compute the four points,

$$Q_1 = y_1^4 - y_0^4, \qquad Q_a = y_a \quad \text{for } a = 2, 3, 4$$
$$\Rightarrow \quad Y_q = [1 : i^{q-1} : 0 : 0 : 0 : 0], \quad q = 0, 1, 2, 3. \tag{3.17}$$

The projective and affine coordinate rings of these four points are given by

$$S = \frac{\mathbb{C}[y_0, \ldots, y_4]}{\langle y_1^4 - y_0^4, y_2, y_3, y_4 \rangle} \cong \frac{\mathbb{C}[y_0, y_1]}{\langle y_1^4 - y_0^4 \rangle}, \tag{3.18}$$

$$A = \frac{\mathbb{C}[y_0, \ldots, y_4]}{\langle y_1^4 - y_0^4, y_2, y_3, y_4, y_0 - 1 \rangle} \cong \frac{\mathbb{C}[y_1]}{\langle y_1^4 - 1 \rangle}$$
$$= \mathrm{Span}([1], [y_1], [y_1^2], [y_1^3]). \tag{3.19}$$

3. Example 3: A type I example with three projective factors

For a more complicated type I example with three projective factors we consider the CICY X (number 7804 in the standard list [22,23]) with configuration matrix

$$X \in \left[\begin{array}{c|ccc} \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^2 & 1 & 1 & 1 \\ \mathbb{P}^3 & 3 & 1 & 0 \end{array}\right]^{3,57}_{-108} \quad \begin{array}{l} x_0, x_1 \\ \tilde{y}_0, \tilde{y}_1, \tilde{y}_2 \\ y_0, y_1, y_2, y_3 \end{array}. \tag{3.20}$$

The single-wrapping Gromov-Witten invariant in the $\mathbb{P}^1$ direction is $n_{\mathcal{C}} = 3$ and the loci Y_i of these three curves in $\mathbb{P}^2 \times \mathbb{P}^3$ are described by the configuration matrix

$$\{Y_1, Y_2, Y_3\} \in \left[\begin{array}{c|ccccc} \mathbb{P}^2 & 1 & 1 & 1 & 1 & 1 \\ \mathbb{P}^3 & 3 & 1 & 1 & 0 & 0 \end{array}\right]$$
$$\cong \left[\begin{array}{c|ccc} \mathbb{P}^3 & 3 & 1 & 1 \end{array}\right] \cong \left[\begin{array}{c|c} \mathbb{P}^1 & 3 \end{array}\right]. \tag{3.21}$$

With simple defining equations

$$Q_1 = \tilde{y}_0 y_1^3 - \tilde{y}_0 y_0^3, \qquad Q_2 = \tilde{y}_0 y_2$$
$$Q_3 = \tilde{y}_0 y_3, \qquad Q_4 = \tilde{y}_1, \qquad Q_5 = \tilde{y}_2, \tag{3.22}$$

the three points are explicitly given by

$$Y_q = ([1 : 0 : 0], [1 : \alpha^{q-1} : 0 : 0]), \quad q = 0, 1, 2, \tag{3.23}$$

where $\alpha = \exp(2\pi i/3)$. For the projective and affine coordinate rings of these points we have

$$S = \frac{\mathbb{C}[\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, y_0, y_1, y_2, y_3]}{\langle \tilde{y}_0 y_1^3 - \tilde{y}_0 y_0^3, \tilde{y}_0 y_2, \tilde{y}_0 y_3, \tilde{y}_1, \tilde{y}_2 \rangle} \cong \frac{\mathbb{C}[y_0, y_1]}{\langle y_1^3 - y_0^3 \rangle}, \tag{3.24}$$

$$A = \frac{\mathbb{C}[\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, y_0, y_1, y_2, y_3]}{\langle \tilde{y}_0 y_1^3 - \tilde{y}_0 y_0^3, \tilde{y}_0 y_2, \tilde{y}_0 y_3, \tilde{y}_1, \tilde{y}_2, \tilde{y}_0 - 1, y_0 - 1 \rangle}$$
$$\cong \frac{\mathbb{C}[y_1]}{\langle y_1^3 - 1 \rangle} = \mathrm{Span}([1], [y_1], [y_1^2]). \tag{3.25}$$

4. Example 4: A type II example with three projective factors

Our final example is a CICY X (number 7881 in the standard list [22,23]) with configuration matrix

$$X \in \left[\begin{array}{c|cc} \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 2 & 0 \\ \mathbb{P}^3 & 3 & 1 \end{array}\right]^{3,75}_{-144} \quad \begin{array}{l} x_0, x_1 \\ y_0, y_1 \\ \tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3 \end{array} \tag{3.26}$$

and a single-wrapping genus zero Gromov-Witten invariant for the class associated with the first $\mathbb{P}^1$ factor of $n_{\mathcal{C}} = 2$. The loci $Y_i \in \mathbb{P}^1 \times \mathbb{P}^3$ of the two curves in the transverse space are described by the configuration matrix

$$\{Y_1, Y_2\} \in \left[\begin{array}{c|cccc} \mathbb{P}^1 & 2 & 0 & 0 & 0 \\ \mathbb{P}^3 & 3 & 1 & 1 & 1 \end{array}\right] \cong \left[\begin{array}{c|c} \mathbb{P}^1 & 2 \end{array}\right]. \tag{3.27}$$

With simple choices for the defining equations, these two points are easily computed:

$$Q_1 = \tilde{y}_0^3 y_1^2 - \tilde{y}_0^3 y_0^2, \qquad Q_2 = \tilde{y}_1, \qquad Q_3 = \tilde{y}_2,$$
$$Q_4 = \tilde{y}_3 \qquad \Rightarrow \qquad Y_\pm = ([1:\pm 1],[1:0:0:0]). \tag{3.28}$$

The projective and affine coordinate rings of these two points are

$$S = \frac{\mathbb{C}[y_0, y_1, \tilde{y}_0, \tilde{y}_1, \tilde{y}_2 \tilde{y}_3]}{\langle \tilde{y}_0^3 y_1^2 - \tilde{y}_0^3 y_0^2, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3 \rangle} \cong \frac{\mathbb{C}[y_0, y_1]}{\langle y_1^2 - y_0^2 \rangle}, \tag{3.29}$$

$$A = \frac{\mathbb{C}[y_0, y_1, \tilde{y}_0, \tilde{y}_1, \tilde{y}_2 \tilde{y}_3]}{\langle \tilde{y}_0^3 y_1^2 - \tilde{y}_0^3 y_0^2, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3, y_0 - 1, \tilde{y}_0 - 1 \rangle} \cong \frac{\mathbb{C}[y_1]}{\langle y_1^2 - 1 \rangle}$$
$$= \mathrm{Span}([1],[y_1]). \tag{3.30}$$

IV. THE BUNDLE

Our next step is the construction of vector bundles $V \to X$ over the CICY manifolds introduced in the previous section. There are, of course, many ways to construct such bundles. Here we focus on three standard methods, namely, line bundle sums, extension bundles, and monad bundles. We consider each of these classes in turn and discuss how they relate to the geometric method for instanton calculations outlined in Sec. II.

A. Line bundle sums

Recall that we are working with CICY manifolds $X \subset \mathcal{A}$ in an ambient space of the form $\mathcal{A} = \mathbb{P}^1 \times \mathcal{B}$, with $\mathcal{B}$ an arbitrary product of projective factors. Line bundles on X are denoted by $\mathcal{O}_X(k, \hat{k})$, where k is the degree in the $\mathbb{P}^1$ direction and $\hat{k}$ the multidegree in the factors of $\mathcal{B}$. As our vector bundle we take a rank $r \leq 8$ line bundle sum

$$V = \bigoplus_{a=1}^{r} \mathcal{O}_X(k_a, \hat{k}_a). \tag{4.1}$$

As usual, we impose $c_1(V) = 0$ so that an embedding into E_8 is possible and this is equivalent to

$$c_1(V) = 0 \quad \Leftrightarrow \quad \sum_{a=1}^{r} k_a = \sum_{a=1}^{r} \hat{k}_a = 0. \tag{4.2}$$

To guarantee bundle supersymmetry we require that there is a locus in the Kahler moduli space where the slopes of all line bundles vanish. Finally, we require that the curve dual to $c_2(TX) - c_2(V)$ is in the Mori cone of X so that there exists a supersymmetric, anomaly-free completion of the model. These conditions impose further constraints on the line bundle integers k_a and $\hat{k}_a$ which can be easily worked out. We refrain from doing so as the details are not relevant for our discussion of instanton effects.

From Eq. (2.3), we need to consider the bundles $V_i = V|_{C_i} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ in order to calculate the Pfaffians. Remembering that the curves C_i are given by a point in $\mathcal{B}$ times the first $\mathbb{P}^1$ factor, these bundles are easily computed by restricting the line bundles to the degrees in the $\mathbb{P}^1$ direction,

$$V_i = V|_{C_i} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) = \bigoplus_{a=1}^{r} \mathcal{O}_{\mathbb{P}^1}(k_a - 1). \tag{4.3}$$

Recall that the cohomology dimensions for line bundles on $\mathbb{P}^1$ are governed by the formulas

$$h^0(\mathcal{O}_{\mathbb{P}^1}(k)) = \begin{cases} k+1 & \text{for} \quad k \geq 0 \\ 0 & \text{for} \quad k < 0 \end{cases},$$
$$h^1(\mathcal{O}_{\mathbb{P}^1}(k)) = \begin{cases} 0 & \text{for} \quad k \geq 0 \\ -k-1 & \text{for} \quad k < 0 \end{cases}. \tag{4.4}$$

This implies immediately that

$$h^0(V_i) = \sum_{\{a | k_a \geq 0\}} k_a. \tag{4.5}$$

Combining this result with Eqs. (2.3) and (4.2) leads to a very simple criterion for the vanishing of the instanton superpotential,

$$\text{at least one } k_a \neq 0 \quad \Leftrightarrow \quad W_{\mathcal{C}} = 0. \tag{4.6}$$

In other words, the only cases which lead to nonvanishing instanton superpotentials are the ones where all line bundles restrict trivially to the curves C_i.

In conclusion, for line bundle sums we have a rather simple and satisfactory criterion for the vanishing of the instanton superpotential $W_{\mathcal{C}}$. However, note that line bundle sums typically do have moduli and represent special "split loci" in a moduli space of bundles which generically have a non-Abelian structure group. The vanishing of $W_{\mathcal{C}}$ for a line bundle sum does not necessarily imply that $W_{\mathcal{C}}$ remains zero once we move away from the line bundle locus in moduli space. To address this problem we need to consider other bundle constructions which allow for non-Abelian structure groups.

B. Monad and extension bundles

Extensions and monads are two standard methods to construct bundles with a non-Abelian structure group. We would now like to consider these two classes and summarize how they relate to instanton superpotential calculations.

The monad and extension bundles will be built from two line bundle sums

$$A = \bigoplus_{\alpha=1}^{r_A} \mathcal{O}_X(a_\alpha, \hat{a}_\alpha), \qquad B = \bigoplus_{\beta=1}^{r_B} \mathcal{O}_X(b_\beta, \hat{b}_\beta), \tag{4.7}$$

where we recall that the first entries a_α, b_β denote the degree in the $\mathbb{P}^1$ direction and $\hat{a}_\alpha$, $\hat{b}_\beta$ are the multidegrees in the transverse space $\mathcal{B}$. It is also useful to introduce the restrictions of these line bundle sums to the curves C_i, tensored with $\mathcal{O}_{\mathbb{P}^1}(-1)$, since these bundles determine the properties of the instantons,

$$A_i \coloneqq A|_{C_i} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) = \bigoplus_\alpha \mathcal{O}_{\mathbb{P}^1}(a_\alpha - 1),$$
$$B_i \coloneqq B|_{C_i} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) = \bigoplus_\beta \mathcal{O}_{\mathbb{P}^1}(b_\beta - 1). \tag{4.8}$$

In terms of the above line bundle sums, monad and extension bundles $V \to X$ are defined by short exact sequences and their properties are summarized in the following table.

	Monads	Extensions
Sequence	$0 \to V \to A \overset{F}{\to} B \to 0$	$0 \to A \to V \to B \to 0$
Map	$F \in H^0(B \otimes A^*)$	$\delta \in \mathrm{Ext}^1(B,A) \cong H^1(A \otimes B^*)$
$\mathrm{rk}(V)$	$r_A - r_B$	$r_A + r_B$
$c_1(V)$	$c_1(A) - c_1(B)$	$c_1(A) + c_1(B)$
$H^0(V_i)$	$\mathrm{Ker}(H^0(A_i) \overset{\delta_i}{\to} H^0(B_i))$	$\mathrm{Ker}(H^0(B_i) \overset{\delta_i}{\to} H^1(A_i))$

For either construction, we should impose that $\mathrm{r}(V) \leq 8$ and $c_1(V) = 0$ which leads to certain constraints on the line bundle integers. Further constraints arise from bundle superymmetry and the anomaly conditions but there is no need to discuss these in detail. It is worth noting that the cohomology dimensions which appear in the last row can be easily computed from Eq. (4.4) and are given by

$$h^0(A_i) = \sum_{\{\alpha | a_\alpha \geq 0\}} a_\alpha, \qquad h^0(B_i) = \sum_{\{\beta | b_\beta \geq 0\}} b_\beta,$$
$$h^1(A_i) = -\sum_{\{\alpha | a_\alpha \leq 0\}} a_\alpha. \tag{4.9}$$

In analogy with Eq. (4.8), we also introduce the restriction

$$V_i \coloneqq V|_{C_i} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \tag{4.10}$$

of V to the curve C_i. Since the index $\chi(V_i) = c_1(V) = 0$ vanishes from the index theorem we conclude that

$$h^0(V_i) = h^1(V_i). \tag{4.11}$$

Next, consider the long exact sequence in cohomology associated with the monad sequence restricted to C_i,

$$\begin{array}{l} 0 \to H^0(V_i) \to H^0(A_i) \overset{\delta_i}{\to} H^0(B_i) \\ \quad \to H^1(V_i) \to H^1(A_i) \to H^1(B_i) \to 0 \end{array}. \tag{4.12}$$

Combining this sequence with the equality (4.11) shows that whenever $h^0(A_i) \neq h^0(B_i)$ we must have $h^0(V_i) \neq 0$. The analogous long exact sequence for extensions

$$\begin{array}{l} 0 \to H^0(A_i) \to H^0(V_i) \to H^0(B_i) \\ \quad \overset{\delta_i}{\to} H^1(A_i) \to H^1(V_i) \to H^1(B_i) \to 0 \end{array}, \tag{4.13}$$

together with Eq. (4.11) leads to a similar conclusion. For $h^0(B_i) \neq h^1(A_i)$ we must necessarily have $h^0(V_i) \neq 0$. Combining these observations with the criterion (2.3) then proves the simple vanishing statement

$$\left\{ \begin{array}{ll} h^0(A_i) \neq h^0(B_i) & \text{for monads} \\ h^0(B_i) \neq h^1(A_i) & \text{for extensions} \end{array} \right\} \Rightarrow W_C = 0. \tag{4.14}$$

In other words, all cases with a nonzero instanton superpotential must necessarily satisfy

$$h^0(A_i) = h^0(B_i) \text{ for monads,}$$
$$h^0(B_i) = h^1(A_i) \text{ for extensions,} \tag{4.15}$$

and, from now on, we will assume these relations are satisfied. Then, we can think of the maps δ_i as square matrices and introduce the determinants

$$f_i = \det(\delta_i). \tag{4.16}$$

Clearly, $H^0(V_i) \neq 0$ if and only if $f_i = 0$ and, hence, the f_i are the functions of the same name which we have introduced in Sec. II and which enter the criterion (2.5). The maps δ_i can be computed by restricting the monad map F or the extension map δ to the cycle C_i, and then working out the induced map on cohomology. In cases where the monad and extension maps descend from ambient space polynomials, on which we focus here, this always leads to functions f_i which can be expressed as

$$f_i = f|_{C_i}, \qquad f \in \mathbb{C}[y]_k, \tag{4.17}$$

that is, as a restriction to C_i of polynomials f with a certain multidegree k in the directions of the transverse space $\mathcal{B}$. Different choices of f with this multidegree reflect different points in the bundle moduli space—we can think of the coefficients of a general $f \in \mathbb{C}[y]_k$ as (some of the) bundle moduli b. Note that this considerably simplifies the structure of the discussion. All we need to know is the multidegree k in order to determine the crucial maps f_i. It can be computed from the line bundle integers $\hat{a}_\alpha$ and $\hat{b}_\beta$ but the precise relation depends on the case. Our subsequent discussion is largely independent of these details and merely starts with Eq. (4.17). Some examples of the relation between k and the line bundle integers are provided in Sec. VI B.

V. COORDINATE RINGS AND HILBERT FUNCTIONS

In this section we review some basic mathematical facts about zero-dimensional varieties and their coordinate rings and Hilbert functions. A useful mathematical reference for some of this material is [24].

We briefly recall the algebraic setup which we have already introduced in Sec. III A. For a product $\mathcal{B} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ of projective spaces with homogeneous coordinates $y = (y_{\alpha,0}, \ldots, y_{\alpha,n_\alpha})_{\alpha=1,\ldots,m}$ we have the associated multigraded coordinate ring $\mathbb{C}[y]$, with multidegrees denoted by $k = (k_1, \ldots, k_m)$. It is also useful to introduce the standard open patch U_0 of $\mathcal{B}$ where all $y_{\alpha,0} \neq 0$. Assume we have a zero-dimensional variety consisting of a finite number of points $\{Y_1, \ldots, Y_n\} \subset \mathcal{B}$. In the context of instanton calculations, these points are of course the loci of the isolated, genus-zero curves in the transverse space $\mathcal{B}$. We can think of this point set as a projective subvariety of $\mathcal{B}$ which is then described by a projective ideal $I \subset \mathbb{C}[y]$. Alternatively, if all points Y_i are contained in U_0 we can think of it as an affine variety whose associated ideal $J = I + \langle y_{1,0} - 1, \ldots, y_{m,0} - 1\rangle$ is obtained from I by adding the localizing polynomials $y_{\alpha,0} - 1$.

The map $r\colon \mathbb{C}[y] \to S$ introduced in Eq. (3.8) is defined by $r(f) = [f]$, that is, it takes the class of a polynomial within $S = \mathbb{C}[y]/I$. The map $\ell\colon S \to A$ is the localization map which, in practice, amounts to setting all $y_{\alpha,0} = 1$. The affine coordinate ring $A = \mathbb{C}[y]/J$ is finite dimensional and its dimension $\dim(A) = n$ equals the number of points it describes. It is also known [24] that is has a basis with monomial representatives.

A. Hilbert functions

The rings $\mathbb{C}[y]$ and S are multigraded and they have standard Hilbert functions. For the ring S, the Hilbert function h_S and the Hilbert series H_S are defined by

$$h_S(k) = \dim(S_k), \qquad H_S(t) = \sum_k h_S(k) t^k. \qquad (5.1)$$

In other words, the Hilbert function gives the dimension of each multidegree part S_k of S while the Hilbert series is simply the generating series for the Hilbert function (where $t^k = t_1^{k_1} \cdots t_m^{k_m}$). For sufficiently large degrees k, the Hilbert function is described by a polynomial—the so-called Hilbert polynomial—whose degree equals the dimension of the associated variety. Since we are concerned with a zero-dimensional variety, the Hilbert function becomes a constant for large k which is, in fact, equal to the number n of points,

$$h_S(k) \to n \quad \text{for } k \gg 1. \qquad (5.2)$$

There are standard methods to compute the Hilbert function h_S, in particular by using syzygies in cases where the variety is a complete intersection. Since h_S is not the main object of interest for our discussion we refrain from providing further details (see, for example, Ref. [25]).

The affine ring A is not graded but filtered, with the filtration induced by the subalgebras $A_{\leq k}$ of elements with multidegree less or equal than k. The affine Hilbert function and series for A are somewhat less common and are defined by

$$h_A(k) \coloneqq \dim(A_{\leq k}), \qquad H_A(t) = \sum_k h_A(k) t^k. \qquad (5.3)$$

From Eq. (3.8), we have $A_{\leq k} = \ell(S_k)$, which implies that

$$h_S(k) \geq h_A(k) \qquad (5.4)$$

for all k. Unfortunately, equality does not always hold since the map $\ell|_{S_k}$ is not necessarily injective. Since A has a finite basis with monomial representatives, it is clear that h_A has the same asymptotic behavior as h_S, namely

$$h_A(k) \to n \quad \text{for } k \gg 1. \qquad (5.5)$$

How can the affine Hilbert function h_A be computed? The following provides a basic algorithm.

(1) Compute a Groebner basis $G = (g_i)$ of J.
(2) Compute a monomial basis $B = (b_i)$ of (class representatives of) A by collecting all monomials not contained in $\langle \mathrm{LT}(g_i)\rangle$, where $\mathrm{LT}(g_i)$ denotes the leading term of g_i as induced by the ordering chosen in the Groebner basis computation.
(3) Select a monomial basis (m_i) of $\mathbb{C}[y]_k$ and compute its remainders m_i^G relative to the Groebner basis G. These remainders are linear combinations of the basis B.
(4) Find the dimension of the space spanned by the remainders m_i^G. This dimension equals $h_A(k)$.

B. Examples

Let us illustrate Hilbert functions and their computation by continuing with the example from Sec. III B.

1. Example 1

Recall that this example involves six points in $\mathbb{P}^6$ described by the configuration matrix

$$\{Y_1, \ldots, Y_6\} \in \left[\mathbb{P}^6 \,\middle|\, 3\ \ 2\ \ 1\ \ 1\ \ 1\ \ 1\right] \cong \left[\mathbb{P}^2 \,\middle|\, 3\ \ 2\right], \qquad (5.6)$$

and with coordinate rings

$$S \cong \frac{\mathbb{C}[y_0, y_1, y_2]}{\langle y_1^3 - y_0^3, y_2^2 - y_0^2 \rangle},$$
$$A \cong \frac{\mathbb{C}[y_1, y_2]}{\langle y_1^3 - 1, y_2^2 - 1 \rangle} = \mathrm{Span}([1], [y_1], [y_1^2], [y_2], [y_1 y_2], [y_1^2 y_2]). \quad (5.7)$$

Using standard methods, the Hilbert series, and Hilbert function for S are obtained as

$$H_S(t_1) = \frac{1 + 2t_1 + 2t_1^2 + t_1^3}{1 - t_1} \Rightarrow \quad h_S(k) = \begin{cases} 2k+1 & \text{for} \quad k < 3 \\ 6 & \text{for} \quad k \geq 3 \end{cases}. \quad (5.8)$$

To compute the affine Hilbert function we can follow the above algorithm. First we need to compute a Groebner basis G for the ideal

$$J = \langle y_1^3 - y_0^3, y_2^2 - y_0^2, y_0 - 1 \rangle. \quad (5.9)$$

In lexicographic ordering, the Groebner basis and its leading terms are

$$G = (y_0 - 1, y_1^3 - 1, y_2^2 - 1) \quad \Rightarrow \quad \langle \mathrm{LT}(g_i) \rangle = \langle y_0, y_1^3, y_2^2 \rangle. \quad (5.10)$$

Collecting terms not contained in $\langle \mathrm{LT}(g_i) \rangle$, we find

$$B = (1, y_1, y_1^2, y_2, y_1 y_2, y_1^2 y_2), \quad (5.11)$$

and this is indeed the monomial basis for A given in Eq. (5.7). Next, we compute the monomial basis and its remainders. Let us look at $k = 2$. A monomial basis for $\mathbb{C}[y]_2$ is simply

$$(m_i) = (y_0^2, y_0 y_1, y_0 y_2, y_1^2, y_1 y_2, y_2^2). \quad (5.12)$$

Reducing this modulo (5.10), we find

$$(m_i^G) = (1, y_1, y_2, y_1^2, y_1 y_2, 1). \quad (5.13)$$

Since the space spanned by the remainders is five dimensional we have $h_A(2) = 5$. Continuing along those lines it is straightforward to verify that $h_A = h_S$, so in this case the two Hilbert functions coincide.

2. Example 2

This example involves four points in $\mathbb{P}^4$ with configuration matrix

$$\{Y_1, Y_2, Y_3, Y_4\} \in \left[\mathbb{P}^4 \;\middle|\; 4 \;\; 1 \;\; 1 \;\; 1 \right] \cong \left[\mathbb{P}^1 \;\middle|\; 4 \right] \quad (5.14)$$

and coordinate rings

$$S \cong \frac{\mathbb{C}[y_0, y_1]}{\langle y_1^4 - y_0^4 \rangle},$$
$$A \cong \frac{\mathbb{C}[y_1]}{\langle y_1^4 - 1 \rangle} = \mathrm{Span}([1], [y_1], [y_1^2], [y_1^3]). \quad (5.15)$$

The Hilbert series and function for S are given by

$$H_S(t_1) = \frac{1 + t_1 + t_1^2 + t_1^3}{1 - t_1} \Rightarrow \quad h_S(k) = \begin{cases} k+1 & \text{for} \quad k < 3 \\ 4 & \text{for} \quad k \geq 3 \end{cases}. \quad (5.16)$$

A quick inspection of the monomial basis for A in Eq. (5.15) shows that $h_A = h_S$, so again the Hilbert functions coincide.

3. Example 3

This example involves three points in $\mathcal{B} = \mathbb{P}^2 \times \mathbb{P}^3$ described by a configuration matrix

$$\{Y_1, Y_2, Y_3\} \in \left[\begin{array}{c|ccccc} \mathbb{P}^2 & 1 & 1 & 1 & 1 & 1 \\ \mathbb{P}^3 & 3 & 1 & 1 & 0 & 0 \end{array} \right], \quad (5.17)$$

and with associated coordinate rings

$$S = \frac{\mathbb{C}[\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, y_0, y_1, y_2, y_3]}{\langle \tilde{y}_0 y_1^3 - \tilde{y}_0 y_0^3, \tilde{y}_0 y_2, \tilde{y}_0 y_3, \tilde{y}_1, \tilde{y}_2 \rangle},$$
$$A = \frac{\mathbb{C}[\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, y_0, y_1, y_2, y_3]}{\langle \tilde{y}_0 y_1^3 - \tilde{y}_0 y_0^3, \tilde{y}_0 y_2, \tilde{y}_0 y_3, \tilde{y}_1, \tilde{y}_2, \tilde{y}_0 - 1, y_0 - 1 \rangle}. \quad (5.18)$$

The Hilbert series for S is a bit more complicated

$$H_S(t) = \frac{1 - t_1 t_2^5 + 2t_1 t_2^4 - t_1 t_2^3 + t_1 t_2^2 - 2t_1 t_2}{(1 - t_1)(1 - t_2)^4} = 1 + t_1 + 4t_2 + 2t_1 t_2 + \cdots \quad (5.19)$$

and we have expanded only up to terms of degree $k \leq (1, 1)$. The affine Hilbert function can be computed algorithmically, as discussed, and the result is schematically shown in Fig. 1. We note from Eq. (5.19) that $h_S(0, 1) = 4$ while Fig. 1 indicates that $h_A(0, 1) < 3$, in fact, $h_A(0, 1) = 2$. This is an example where the two Hilbert functions do not coincide—the map $\ell|_{S_{(0,1)}}$ is not injective.

4. Example 4

For this example, we have two points in $\mathcal{B} = \mathbb{P}^1 \times \mathbb{P}^3$ with configuration matrix

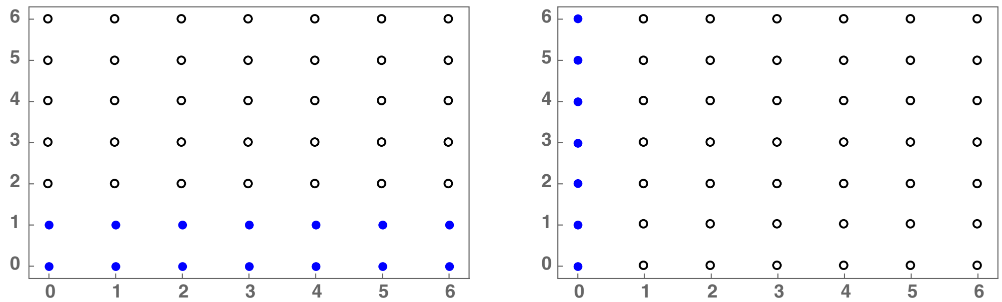


FIG. 1. Results for affine Hilbert function $h_A(k)$ in the $k = (k_1, k_2)$ plane (with k_1 on the horizontal axis and k_2 on the vertical axis) for example 3 with $n_C = 3$ (left) and example 4 with $n_C = 2$ (right). Blue points indicate degrees for which $h_A(k) < n_C$ and empty points satisfy $h_A(k) = n_C$.

$$\{Y_1, Y_2\} \in \left[\begin{array}{c|cccc} \mathbb{P}^1 & 2 & 0 & 0 & 0 \\ \mathbb{P}^3 & 3 & 1 & 1 & 1 \end{array}\right], \tag{5.20}$$

and coordinate rings

$$S = \frac{\mathbb{C}[y_0, y_1, \tilde{y}_0, \tilde{y}_1, \tilde{y}_2\tilde{y}_3]}{\langle \tilde{y}_0^3 y_1^2 - \tilde{y}_0^3 y_0^2, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3 \rangle},$$
$$A = \frac{\mathbb{C}[y_0, y_1, \tilde{y}_0, \tilde{y}_1, \tilde{y}_2\tilde{y}_3]}{\langle \tilde{y}_0^3 y_1^2 - \tilde{y}_0^3 y_0^2, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3, y_0 - 1, \tilde{y}_0 - 1 \rangle}. \tag{5.21}$$

The Hilbert series for S is given by

$$H_S(k) = \frac{1 - t_1^2 t_2^3}{(1 - t_1)^2 (1 - t_2)} = 1 + 2t_1 + t_2 + 2t_1 t_2 + \cdots \tag{5.22}$$

and the results for the affine Hilbert function is schematically shown in Fig. 1. It turns out that in this case $h_A = h_S$ so the two Hilbert functions coincide.

C. Evaluation from coordinate rings

Recall that our goal is to use the criterion (2.5) for the nonvanishing of the instanton superpotential. This requires us to work out the functions f_i which are proportional to the Pfaffians. We have seen in Eq. (4.17) that they can be obtained from $f_i = f(Y_i)$, that is, by evaluating functions $f \in \mathbb{C}[y]_k$ of a certain multidegree k at the loci Y_i of the curves C_i. This is straightforward in principle but might not be easy to carry out in practice since the points Y_i may be difficult to compute. We will now propose an alternative method to calculate f_i which does not rely on the explicit knowledge of the points Y_i but uses the affine coordinate ring A instead. See Ref. [24] for mathematical details underlying this approach.

First recall that the affine coordinate ring A, associated with the point set $\{Y_1, \ldots, Y_n\}$, is a finite-dimensional vector space of dimension n. We can define a linear map by

$$\mu : \mathbb{C}[y] \to \mathrm{End}(A), \qquad \mu(f)(a) \coloneqq [f]a, \tag{5.23}$$

where $[f] = \ell \circ r(f)$ is the class of the polynomial f in A. Hence, for every polynomial $f \in \mathbb{C}[y]$ the image $\mu(f)$ is a linear map on A which acts simply by the multiplication in the ring A. Since the ring multiplication is commutative, we have

$$\mu(f)\mu(\tilde{f}) = \mu(\tilde{f})\mu(f) \tag{5.24}$$

for all $f, \tilde{f} \in \mathbb{C}[y]$. In other words, all linear maps on A obtained in this way commute with each other. The main mathematical statement we will be relying on is the following [24]:

$$\{f(Y_1), \ldots, f(Y_n)\} = \{\text{eigenvalues of } \mu(f)\}. \tag{5.25}$$

This means, the crucial quantities $f_i = f(Y_i)$ proportional to the Pfaffians are given by the eigenvalues of the linear map $\mu(f) : A \to A$. Moreover, all maps $\mu(f)$ obtained for f ranging in $\mathbb{C}[y]_k$ commute and, hence, can be simultaneously diagonalized.

This discussion allows us to reformulate our original problem of linear (in)dependence of f_i in terms of the properties of polynomials in the coordinate ring A. These properties can be studied using standard methods of commutative algebra and Hilbert series. As a result the criterion for a nonvanishing superpotential can be stated using the Hilbert function as will be considered in the next section.

VI. A HILBERT FUNCTION CONDITION FOR INSTANTONS

We are now ready to combine our various observations and formulate a condition for a nonzero instanton superpotential W_C, based on the affine Hilbert function. After stating the condition in general, we apply it to a range of examples.

A. The general condition

From our main criterion (2.5), we need to decide whether or not the quantities $f_i = f(Y_i)$, where $i = 1, \ldots, n_{\mathcal{C}}$, viewed as functions of bundle moduli b, are linearly independent. A practical way to reformulate this is to choose a basis $(f_I)_{I=1,\ldots,N}$ of $\mathbb{C}[y]_k$ and consider the $N \times n_{\mathcal{C}}$ matrix $M_{Ii} = f_I(Y_i)$. In terms of this matrix, the criterion (2.5) can be reformulated as

$$\mathrm{rk}(M) = n_{\mathcal{C}} \quad \Rightarrow \quad W_{\mathcal{C}} \neq 0. \tag{6.1}$$

Let us point out that here it is assumed that the polynomial $f \in \mathbb{C}[y]_k$ is generic in the sense that we span the entire space $\mathbb{C}[y]_k$ as we vary its coefficients. In other words, f can be expanded in the basis of $(f_I)_{I=1,\ldots,N}$ with all basis elements present in the expansion. Otherwise, if only $N' < N$ basis elements appear in the expansion of f, we have to restrict $\mathbb{C}[y]_k$ to the subspace spanned by these basis elements. The matrix M must now be constructed using the basis elements $(f_I)_{I=1,\ldots,N'}$ and is of the size $N' \times n_{\mathcal{C}}$. However, the condition (6.1) remains the same.

Now consider the linear maps $\mu(f_I){:}\, A \to A$, as defined in the previous section. All these maps are simultaneously diagonalizable and the eigenvalues of $\mu(f_I)$ are precisely the entries $(M_{I1}, \ldots, M_{In_{\mathcal{C}}})$ of the Ith row of M. Hence, it follows that

$$\begin{aligned} \mathrm{rk}(M) &= \dim(\mu(\mathbb{C}[y]_k)) = \dim(\ell \circ r(\mathbb{C}[y]_k) \\ &= \dim(A_{\leq k}) = h_A(k). \end{aligned} \tag{6.2}$$

This means the criterion (6.1) can be rewritten in terms of the affine Hilbert function and then reads

$$h_A(k) = n_{\mathcal{C}} \quad \Rightarrow \quad W_{\mathcal{C}} \neq 0. \tag{6.3}$$

This is our main result. We can use the affine Hilbert function of the coordinate ring A, which describes the locations of the curves C_i in the transverse space, to decide whether the instanton superpotential $W_{\mathcal{C}}$ is nonzero. To do this, we have to determine the relevant multidegree k for the bundle V in question. For common constructions, such as extension and monad bundles, this degree can usually be read off from the defining data of the bundle. Some explicit examples of this are provided below. The simple conclusion is that, whenever the affine Hilbert function $h_A(k)$ takes its maximal value $n_{\mathcal{C}}$ (equal to the number of curves C_i), the instanton superpotential must be nonzero. For cases with $h_A(k) < n_{\mathcal{C}}$ we cannot draw a definite conclusion and $W_{\mathcal{C}}$ can be zero or nonzero, depending on the undetermined constants λ_i in Eq. (2.4). Note that the criterion (6.3) does not depend on the precise locations of the points Y_i, which might be difficult to compute from the polynomial equations (3.4). It depends only on the Hilbert function of the coordinate ring A, which can be computed using methods of commutative algebra. The above result leads to a general picture for the nonvanishing of the instanton superpotential. First of all, we see from Eqs. (4.14) that "most" patterns which arise in common bundle constructions, such as monads and extensions, lead to a vanishing superpotential. However, there are specific patterns, characterized by the conditions (4.15), where the superpontial can be nonzero. For such cases, the answer depends on a multidegree k which can be extracted from the relevant bundle construction. The superpotential is nonzero if the Hilbert function criterion (6.3) is satisfied. As Eq. (5.5) shows, this criterion will be satisfied for sufficiently large k. This means, within the subclass of bundles characterized by Eq. (4.15), a nonvanishing instanton superpotential is the "typical" situation. We would now like to illustrate this general picture with a number of examples.

B. Examples

To set the scene, we indicate how the crucial multidegree k can be extracted from a given bundle construction. Consider a monad or extension bundle constructed from the line bundle sums

$$\begin{aligned} A &= \begin{pmatrix} \pm c & 0 & \cdots & 0 \\ \hat{a}_1 & \hat{a}_2 & \cdots & \hat{a}_{r_A} \end{pmatrix}, \\ B &= \begin{pmatrix} c & 0 & \cdots & 0 \\ \hat{b}_1 & \hat{b}_2 & \cdots & \hat{b}_{r_B} \end{pmatrix} \end{aligned} \tag{6.4}$$

where each column contains the multidegree of a line bundle, with the first row the degree in the $\mathbb{P}^1$ direction and the other rows the multidegree in the transverse space $\mathcal{B}$. The upper sign in the (1,1) entry of A is for monads, the lower sign for extensions, and c is a positive integer. Note that for either case the condition (4.15) is satisfied, so we have indeed a pattern where the instanton contribution can be nonvanishing. A quick calculation shows that the multidegree k for this pattern is given by

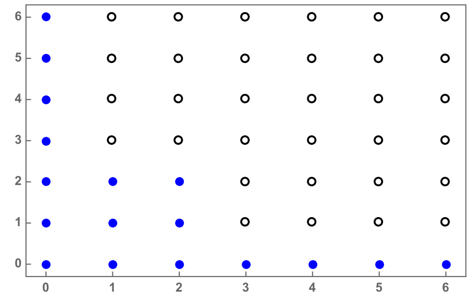
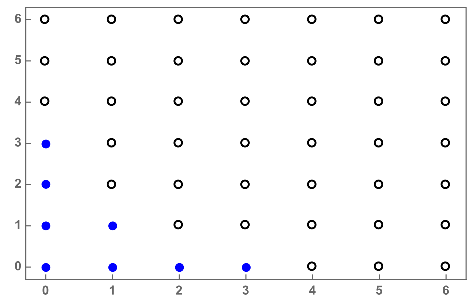
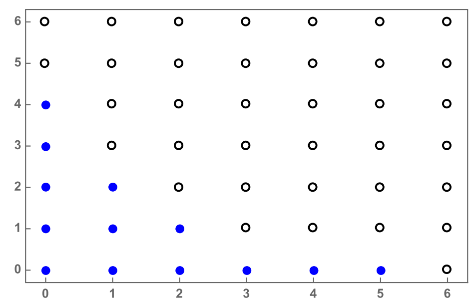
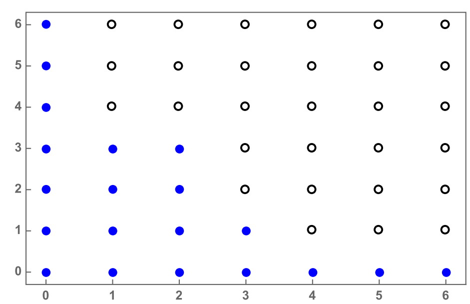
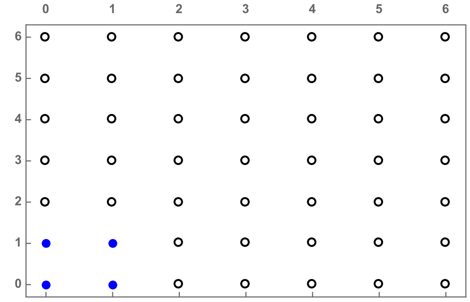
$$k = \pm c(\hat{b}_1 - \hat{a}_1), \tag{6.5}$$

with the upper sign for monads and the lower sign for extensions. Similar relations can be derived for other patterns. In the following, we will not be specific about this relation but rather present our examples in terms of the multidegree k. In this way, the results are applicable to a wide range of bundles, using equations such as (6.5). We begin by revisiting our "running" examples, introduced in Sec. III B.

1. Example 1

Our first example is for CICY manifold 7867 in the ambient space $\mathcal{A} = \mathbb{P}^1 \times \mathbb{P}^6$ defined by the configuration matrix (3.9). Its Picard number is $h^{1,1}(X) = 2$ so k is, in fact, just a single degree in this case. We have $n_{\mathcal{C}} = 6$

TABLE I. Affine Hilbert function $h_A(k)$ in the $k = (k_1, k_2)$ plane (with k_1 on the horizontal axis and k_2 on the vertical axis). Blue points have $h_A(k) < n_{\mathcal{C}}$ whereas empty points indicate that $h_A(k) = n_{\mathcal{C}}$.

Configuration	#	$n_{\mathcal{C}}$	h_A
$\left[\begin{array}{c\|cccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 \\ \mathbb{P}^3 & 2 & 0 & 1 & 1 \\ \mathbb{P}^3 & 0 & 2 & 1 & 1 \end{array}\right]^{3,35}_{-64}$	6771	32	
$\left[\begin{array}{c\|cccccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{P}^4 & 1 & 0 & 2 & 1 & 1 & 0 \\ \mathbb{P}^4 & 0 & 1 & 0 & 1 & 1 & 2 \end{array}\right]^{3,39}_{-72}$	7208	8	
$\left[\begin{array}{c\|cccc} \mathbb{P}^1 & 2 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 1 & 1 & 1 \\ \mathbb{P}^4 & 2 & 1 & 1 & 1 \end{array}\right]^{3,45}_{-84}$	7585	24	
$\left[\begin{array}{c\|ccc} \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^2 & 2 & 0 & 1 \\ \mathbb{P}^3 & 1 & 2 & 1 \end{array}\right]^{3,46}_{-86}$	7610	32	
$\left[\begin{array}{c\|cccccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{P}^4 & 1 & 0 & 1 & 1 & 1 & 1 \\ \mathbb{P}^4 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right]^{3,47}_{-88}$	7636	6	

curves and the Hilbert function $h_A = h_S$, computed in Eq. (5.8), together with the criterion (6.3), shows that

$$k \geq 3 \quad \Rightarrow \quad W_{\mathcal{C}} \neq 0. \tag{6.6}$$

This illustrates our earlier statements that, within patterns of bundle constructions satisfying Eqs. (4.15), nonzero instanton superpotentials are common.

2. Example 2

This is CICY manifold 7888 in the ambient space $\mathcal{A} = \mathbb{P}^1 \times \mathbb{P}^4$ with configuration matrix (3.15). The Picard number is $h^{1,1}(X) = 2$, so again k is a single degree, and there are $n_{\mathcal{C}} = 4$ curves. From the associated affine Hilbert function (5.16) ($h_A = h_S$ in this case) we conclude that

$$k \geq 3 \quad \Rightarrow \quad W_{\mathcal{C}} \neq 0. \tag{6.7}$$

3. Example 3

The CICY manifold 7804 is defined in the ambient space $\mathcal{A} = \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$ with configuration matrix (3.20). There are $n_{\mathcal{C}} = 3$ curves and since $h^{1,1}(X) = 3$ we know that $k = (k_1, k_2)$ is a bidegree. The affine Hilbert function in this case has been plotted in Fig. 1 and it indicates that

$$k_2 \geq 2 \quad \Rightarrow \quad W_{\mathcal{C}} \neq 0. \tag{6.8}$$

Again, we see that the instanton superpotential is nonvanishing for "most" bidegrees $k = (k_1, k_2)$.

4. Example 4

CICY manifold 7881 is defined in the ambient space $\mathcal{A} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$ with configuration matrix (3.26). It has $n_{\mathcal{C}} = 2$ curves and Picard number $h^{1,1}(X) = 3$, so that $k = (k_1, k_2)$ is a bidegree. The associated affine Hilbert function, plotted in Fig. 1, shows that

$$k_1 \geq 1 \quad \Rightarrow \quad W_{\mathcal{C}} \neq 0. \tag{6.9}$$

The above examples have been chosen for their relative simplicity, particularly a small number $n_{\mathcal{C}}$ of curves. We have computed the affine Hilbert function for a number of more complicated examples with Picard number $h^{1,1}(X) = 3$, using the algorithm described in Sec. VA. The results are shown in Table I. For all cases, $k = (k_1, k_2)$ is a bidegree and blue points in the figures correspond to bidegrees with $h_A(k) < n_{\mathcal{C}}$ while empty points indicate $h_A(k) = n_{\mathcal{C}}$. From our main criterion (6.3) all bidegrees k with empty points in those plots lead to a nonvanishing instanton superpotential $W_{\mathcal{C}}$.

VII. CONCLUSION

In this paper, we have studied string instanton superpotentials for heterotic Calabi-Yau compactifications. Our main goal has been to find conditions for the vanishing/nonvanishing of the instanton superpotential $W_{\mathcal{C}}$ associated with a second homology class $\mathcal{C}$ of the Calabi-Yau manifold X. We have considered bundles $V \to X$ constructed from line bundle sums, monads, and extensions.

For line bundle sums we have found a simple criterion, Eq. (4.6), for the vanishing/nonvanishing of the instanton superpotential $W_{\mathcal{C}}$. It shows that nonvanishing instanton superpotentials for line bundle sums requires a special class of line bundles, which become trivial when restricted to the curves C_i, but that within this class, the superpotential is nonvanishing.

For bundles with non-Abelian structure groups, constructed from monads or extensions, the picture is somewhat more complicated. If certain cohomology dimensions of the constituent line bundles are not equal, as in Eq. (4.14), the instanton superpotential $W_{\mathcal{C}}$ vanishes. On the other hand, if these dimensions are equal, as in Eq. (4.15), the superpotential can be vanishing or nonvanishing.

In such cases, a criterion for nonvanishing superpotentials can be formulated in terms of an affine Hilbert function. This Hilbert function, h_A, is associated with the coordinate ring A which describes the loci Y_i of the $n_{\mathcal{C}}$ curves C_i in a transverse space. What we have shown [see Eq. (6.3)] is that whenever $h_A(k) = n_{\mathcal{C}}$, the instanton superpotential is nonzero. Here k is a multidegree which can be read off from the specific bundle construction. The asymptotic behavior $h_A(k) \to n_{\mathcal{C}}$ for large k means that a nonvanishing instanton superpotential is a common feature within this class.

The first observation from these results is that nonvanishing instanton superpotentials are rare, in the sense that they require a specific pattern when constructing the bundle V. However, within the class of bundles following this pattern, the superpotential is either always nonzero (for line bundle sums) or it is frequently nonzero (for monads and extensions). These observations may well provide useful guidance for model building, particularly in view of moduli stabilization.

There are several interesting directions to pursue. The current formulation of our Hilbert function criterion depends on an ambient space of the form $\mathcal{A} = \mathbb{P}^1 \times \mathcal{B}$, so that we can talk about the loci Y_i of the curves C_i in the transverse space $\mathcal{B}$ and introduce their associated coordinate ring A. It would be desirable to generalize this condition so that it can be applied to more general manifolds, possibly by introducing a coordinate ring associated with the union of all curves C_i. It is currently not clear how to formulate this.

Another deficit is that the criterion (6.3) only works in one direction. If $h_A(k) < n_{\mathcal{C}}$ we are not able to draw a definite conclusion. Unfortunately, improving on this requires knowledge of the constants of proportionality λ_i in the instanton superpotential (2.4), which are hard to compute. Moreover, it is interesting that the condition for vanishing/nonvanishing instanton superpotentials depends on the degrees k in the transverse space, while the compactness criterion of Bertolini-Plesser depends on the degree of the line bundles of the curve class under investigation. However, the two degrees are linked by anomaly cancellation and supersymmetry conditions, and we have shown for examples in which a GLSM description exists that the two condition give the same result. It would be interesting to show the equality of the two approaches algebraically.

Finally, our discussion has been limited to certain homology classes, related to $\mathbb{P}^1$ factors in the ambient space, and it would be desirable to remove this limitation.

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