

# A tale of two exponentiations in $\mathcal{N} = 8$ supergravity at subleading level

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## Abstract

High-energy massless gravitational scattering in  $\mathcal{N} = 8$  supergravity was recently analyzed at leading level in the deflection angle, uncovering an interesting connection between exponentiation of infrared divergences in momentum space and the eikonal exponentiation in impact parameter space. Here we extend that analysis to the first non trivial sub-leading level in the deflection angle which, for massless external particles, implies going to two loops, i.e. to third post-Minkowskian (3PM) order. As in the case of the leading eikonal, we see that the factorisation of the momentum space amplitude into the exponential of the one-loop result times a finite remainder hides some basic simplicity of the impact parameter formulation. For the conservative part of the process, the explicit outcome is infrared (IR) finite, shows no logarithmic enhancement, and agrees with an old claim in pure Einstein gravity, while the dissipative part is IR divergent and should be regularized, as usual, by including soft gravitational bremsstrahlung. Finally, using recent three-loop results, we test the expectation that eikonal formulation accounts for the exponentiation of the lower-loop results in the momentum space amplitude. This passes a number of highly non-trivial tests, but appears to fail for the dissipative part of the process at all loop orders and sufficiently subleading order in  $\epsilon$ , hinting at some lack of commutativity of the relevant infrared limits for each exponentiation.

# 1 Introduction

The subject of gravitational collisions and radiation has been receiving increased attention in recent years particularly thanks to the amazing experimental breakthroughs in gravitational-wave (GW) detection [1–3]. From a theoretical standpoint one can tackle this problem both at the classical General Relativity (CGR) level, through numerical [4, 5] and analytical [6–9] methods, and at the quantum level using flat spacetime<sup>1</sup> calculations of scattering amplitudes. In this latter approach the non-trivial classical spacetime geometry emerges from the resummation of an infinite number of loop diagrams. While the classical approach goes back to the seventies [11–13], the quantum approach began in the late eighties with the above mentioned work by 't Hooft [10] and independent parallel work by two other groups [14–16] dealing with the transplanckian energy collisions of strings in a generic number  $D$  of macroscopic spacetime dimensions. That approach was further developed in a number of papers [17–25] and extended to the scattering of strings off a stack of D-branes [26, 27]. Many features of CGR, such as deflection angles, time delays and tidal excitations, were neatly recovered and new effects related to the finite string size were uncovered [28, 29](see [30] for a recent review). In even more recent studies the method was extended to the calculation of the gravitational bremsstrahlung [31–36] produced in these “gedanken collisions”. Other groups have used gauge theory and amplitude methods to examine similar issues [37–43].

Although a priori the problem of transplanckian-energy collisions of light particles or strings appears to be unrelated to the one of two coalescing black holes, it has been stressed by Damour [44] that understanding such idealized processes can bring valuable information about the parameters that enter the Effective-One-Body (EOB) potential [6–9] needed for the computation of the waveforms produced in actual black-hole mergers<sup>2</sup>.

Irrespective of their potential usefulness in GW research the problem of high-energy gravitational scattering and radiation also presents considerable theoretical interest. Indeed the original motivations for such a study were quite disconnected from GW physics but rather related to the problem of constructing a unitary gravitational S-matrix and thus an explicit solution to the information puzzle in quantum black-hole physics. So far that program has been only partly successful. It was possible to show how, in the region of large impact parameters (small deflection angles), the violation of tree-level unitarity is cured by loop corrections even in the presence of string-size effects; at the opposite end only a few interesting insights (see e.g. [48]) have been achieved in the regime of small impact parameter (where gravitational collapse is expected to occur) and the precise way unitarity is preserved (if it is) is still somewhat mysterious [49].

The idea of this work is to start investigating such questions in the context of a more manageable theory,  $\mathcal{N} = 8$  supergravity, which, despite being different from CGR, should

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<sup>1</sup>An exception is 't Hooft's 1987 calculation [10] which is carried out assuming a non-trivial background metric.

<sup>2</sup>Recently, impressive amplitude calculations have also been carried out for the collision of massive (and typically non-relativistic) particles up the two-loop (3PM) order [45, 46] and their outcome was incorporated into the EOB potential [47].

share with it the most important large-distance (infrared) features. Hopefully, in this highly supersymmetric context, one will be able to enter even the gravitational collapse regime: after all the famous microscopic understanding of black-hole entropy in string theory [50] does make crucial use of supersymmetry!

In a recent paper [51] we have shown that the exponentiation in impact-parameter space of the leading high-energy ( $s \rightarrow \infty$ ) terms into a leading eikonal phase has non trivial implications for the correction terms (the so-called remainders) to another exponentiation, this time in momentum space, of infrared divergences. And indeed the two- and three-loop remainders of [52] are found to be fully consistent with those implications. Here we extend that analysis to the first subleading correction in the high-energy expansion of the eikonal phase (equivalently a small-deflection-angle expansion). More precisely, we focus on the scattering of transplanckian-energy *massless* particles and check the validity of an extension of the leading eikonal to include additional subleading contributions which can be determined from the already known higher-loop amplitudes in  $\mathcal{N} = 8$  supergravity. For external massless states the even-order loops provide new classical contributions to the eikonal phase. Because of unitarity, they must exponentiate and therefore have to be added to the leading eikonal phase obtained from tree diagrams. By contrast, the odd-order loops provide only quantum contributions and do not need to exponentiate; they must nonetheless be included in the analysis because they mix at higher orders with the classical contributions to reproduce the full scattering amplitude.

Such a procedure allows for a non-trivial consistency check by using again the three-loop results of [52] where we do not expect any new classical contribution. Therefore, all the scattering data up to the first subleading level in the high energy expansion should be reproduced from the eikonal expansion. We find that the check works for all terms except for a mismatch in the non logarithmically enhanced imaginary part of the amplitude at order  $\mathcal{O}(\epsilon^0)$ . More checks of  $b$ -space exponentiation can be performed at all loops for the two leading- $\epsilon$  terms. Although new mismatches are found to occur we notice that they can all be absorbed in a relatively simple, but IR singular, redefinition of the three-loop remainder. Possible origins of these mismatches are discussed.

On the way we will also compute the first classical correction to the eikonal phase (deflection angle) which, for massless-particle collision only occurs [16] at the two-loop (or  $3PM$ ) order and compare it successfully with the one obtained long ago in pure Einstein gravity [17]. The presence of a non-trivial classical correction to the massless  $3PM$  eikonal in  $\mathcal{N} = 8$  supergravity represents a new result. This property is likely to persist also when masses for external particles are introduced even in a supersymmetry preserving way, as done in [53]. In the latter work, it was shown that the  $2PM$  eikonal vanishes in a maximally supersymmetric setup also in the massive case. Moreover, it is possible to perform a probe analysis if one of the masses is much bigger than any other scale in the problem, for instance by using D6-branes as done in [26]: the result for the deflection angle  $\Theta_6$  in Eq. (4.5) of [26] is consistent with the assumption that all classical corrections to the leading eikonal vanish in the probe limit for  $\mathcal{N} = 8$  supergravity. In view of this result, one might have conjectured that the leading eikonal phase (deflection angle) is exact for this theory even when both particles are dynamical; the presence of a non-trivial correction at

two-loop order shows that this is not the case.

We also compute the non-conservative part of the subleading eikonal (the leading being exactly conservative) which should be relevant for understanding the accompanying gravitational radiation directly at the quantum level. Actually, in the soft-graviton limit this should match a calculation already carried out in [36] for  $\mathcal{N} = 8$  supergravity.

The outline of the paper is as follows. In Sect. 2 we discuss the two types of exponentiations and the distinction between classical and quantum contributions at arbitrary loop order. In Sect. 3 we summarize, for completeness, the check presented in [51] that the scattering data up to three loops are consistent with the leading eikonal exponentiation. In Sect. 4 we extend the procedure to subleading terms at high energy and then concentrate on the first subleading correction. Here we find interesting results on the classical corrections at two-loop order and compare them with those obtained in pure Einstein gravity. In Sect. 5 we compare the two exponentiations at different orders in  $\epsilon$  and in the loop expansion. In particular, in Sect. 5.1 we present successful checks for the first two terms in the  $\epsilon$  expansion (for which one can neglect the remainder functions), while in Sect. 5.2 we consider the third and fourth terms in the  $\epsilon$  expansion, which are sensitive to the two and three-loop remainders, and show that a simple (but IR singular) modification of the three-loop remainder cures all the mismatches. Unlike in the previous sections, in Sect. 6 we perform the calculation of the subleading eikonal phase directly in four dimensions. We find agreement for the real phase at order  $\mathcal{O}(\epsilon^0)$  (calculated from arbitrary  $D = 4 - 2\epsilon$ ) and discuss the origin of a disagreement on the imaginary part. In Sect. 7 we summarize our results and discuss some possible interpretation of the mismatches we found between the two exponentiations. In Appendix A we give some useful formulas for the Fourier transforms used in the text and, in Appendix B, we write down for convenience the scattering data at two and three loops extracted from Ref. [52].

## 2 Two different kinds of exponentiation

Amplitudes in  $\mathcal{N} = 8$  supergravity in four spacetime dimensions continue to be at the centre of intense investigation as they provide the ideal laboratory to test ideas and techniques that then can be used also in other, more physical, theories. Over the last few years the UV properties of the  $\mathcal{N} = 8$  four-point amplitudes have been studied to high-loop order, see for instance [54] and references therein. In this paper we will focus on a complementary aspect of the same scattering process: the high energy, small angle (Regge) regime. In terms of the Mandelstam variables<sup>3</sup>

$$s = -(k_1 + k_2)^2, \quad t = -(k_1 + k_4)^2, \quad u = -(k_1 + k_3)^2 \quad ; \quad s + t + u = 0, \quad (2.1)$$

we work in the  $s$ -channel physical region ( $s > 0, t, u < 0$ ) and focus on the near-forward regime  $|t| \ll s$ , hence we also have  $|u| \gg |t|$ . In  $\mathcal{N} = 8$  supergravity the amplitude  $A^{(\ell)}$

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<sup>3</sup>Eq. (2.1) assumes that all external particles are incoming and the mostly plus metric, but the remaining equations of this paper do not require explicitly this convention.

for four-particle scattering at  $\ell$  loops is proportional to the tree-level result. By following the conventions of [52, 55] we write the full amplitude as a formal series

$$A(k_i, \dots) = \sum_{\ell=0}^{\infty} A^{(\ell)}(k_i, \dots) = A^{(0)}(k_i, \dots) \left( 1 + \sum_{\ell=1}^{\infty} \alpha_G^\ell \mathcal{A}^{(\ell)}(t, s) \right), \quad (2.2)$$

where the dots stand for the dependence on the polarizations and flavours of the external states,  $A^{(0)}$  is the tree-level amplitude,  $A^{(\ell)}$  is the  $\ell$ -loop amplitude,  $\mathcal{A}^{(\ell)}$  is its “stripped” counterpart, and

$$\alpha_G \equiv \frac{G}{\pi \hbar} (4\pi \hbar^2)^\epsilon B(\epsilon); \quad B(\epsilon) \equiv \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}, \quad (2.3)$$

where  $G$  is Newton’s constant in  $D = 4 - 2\epsilon$  dimensions.<sup>4</sup> A simplification in  $\mathcal{N} = 8$  supergravity is that the loop expansion can be encoded in a set of “scalar” terms (i.e. the last factor in (2.2)) that depend on  $s$  and  $t$ , but not on the other quantum numbers of the external particles.

We are interested in studying this dynamical factor and in understanding whether there is an infinite subset of contributions that can be expressed in a simple exponential form. A standard approach to find an exponentiation is to use the infrared divergences as guidance: the IR terms in the  $\ell$ -loop amplitude are entirely obtained from the exponentiation in momentum space of the one-loop amplitude. Then it is natural to rewrite (2.2) in the form

$$A(k_i, \dots) = A^{(0)}(k_i, \dots) \exp\left(\alpha_G \mathcal{A}^{(1)}(t, s, \epsilon)\right) \exp\left(\sum_{\ell=2}^{\infty} \alpha_G^\ell F^{(\ell)}(t, s, \epsilon)\right), \quad (2.4)$$

where we explicitly displayed the dependence on the dimensional regularisation<sup>5</sup> parameter  $\epsilon$  of the stripped one-loop amplitude  $\mathcal{A}^{(1)}$  and the remainder function  $F^{(\ell)}$  whose study has been initiated in [56, 57]. As anticipated, this formulation collects all infrared divergent contributions in the exponential of  $\mathcal{A}^{(1)}$ , while all  $F^{(\ell)}$  are expected to be free from infrared divergences, i.e. they are expected to be finite as  $\epsilon \rightarrow 0$ .

A different approach is to look at the forward high-energy kinematics (i.e. the Regge limit  $|t| \ll s$ ). The leading contribution to the  $\ell$ -loop amplitude  $A^{(\ell)}$  scales as  $s^{\ell+2}$  with sub-leading contributions having, modulo logarithms, lower powers of  $s$  and higher powers of  $t$ . As mentioned in the introduction, at sufficiently large  $s$  such a perturbative behavior violates partial wave unitarity ( $Im a_J \geq |a_J|^2$ , where  $J$  is the angular momentum and  $a_J$  is the  $J^{\text{th}}$  partial wave amplitude [58, 59]). Indeed, the behaviour  $A^{(\ell)} \sim s^{\ell+2}$  translates into  $a_J^{(\ell)} \sim s^{\ell+1}$  which cannot satisfy the above inequality at arbitrarily large  $s$ . It turns

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<sup>4</sup> Since the physical dimensions of  $\alpha_G$  depend upon  $\epsilon$ , specifically  $[\alpha_G] \sim [\text{energy}]^{-2+2\epsilon}$ ,  $\mathcal{A}^{(\ell)}$  will have to exhibit the appropriate  $\epsilon$ -dependent dimensions as well.

<sup>5</sup> Because of infrared divergences we have performed all the calculations using dimensional regularization. This procedure has been shown in Ref. [51] to be essential to reproduce the high energy behavior of the scattering amplitude.

out [10, 14–16] that unitarity is explicitly recovered at sufficiently large  $J$  by means of another kind of exponentiation, this time in impact parameter ( $b \sim 2J/\sqrt{s}$ ) –rather than in transverse-momentum– space as in Eq. (2.4).

Let us start to see how this works in the case of the so-called leading eikonal approximation. It is convenient to extract from the tree-level amplitude the leading-energy behaviour

$$A^{(0)}(k_i, \dots) = A_L^{(0)} \hat{A}^{(0)}(k_i, \dots), \quad \text{with} \quad A_L^{(0)} = \frac{8\pi\hbar G s^2}{-t}. \quad (2.5)$$

By construction, in the case of an elastic scattering,  $\hat{A}^{(0)}$  starts with 1 plus terms that are subleading in the  $|t| \ll s$  limit. The leading behaviour  $A_L^{(0)}$  in (2.5) is the only information we need about the tree-level amplitude.

In order to rewrite the leading energy results in impact parameter space, we first introduce an auxiliary  $(D-2)$ -dimensional momentum  $q$  such that  $q^2 = |t|$ . Then we take the Fourier transform where  $b$  is the conjugate variable to  $q$  and define the leading eikonal phase by [14–16]:

$$2i\delta_0(s, b) = \int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} e^{ibq/\hbar} \frac{iA_L^{(0)}}{2s} = -\frac{iGs}{\epsilon\hbar} \Gamma(1-\epsilon) (\pi b^2)^\epsilon, \quad (2.6)$$

where we used Eq. (A.1). At one loop, we have

$$A^{(1)} = A^{(0)} \alpha_G \mathcal{A}^{(1)} \longrightarrow A_L^{(0)} \alpha_G \left( \frac{-i\pi s}{\epsilon(q^2)^\epsilon} \right) \equiv A_L^{(1)}, \quad (2.7)$$

where in the step indicated by the arrow we focused on the leading term of (4.2) in the Regge (high energy) limit. By going to impact parameter space one gets:

$$\int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} e^{ibq/\hbar} \frac{iA_L^{(1)}}{2s} = \int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} e^{ibq/\hbar} \frac{iA_L^{(0)}}{2s} \alpha_G \frac{-i\pi s}{\epsilon(q^2)^\epsilon} = -\frac{1}{2} (2\delta_0)^2. \quad (2.8)$$

Thus we see that the sum of leading energy contributions of the tree and one-loop amplitudes starts to exponentiate in impact parameter space

$$\int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} e^{ibq/\hbar} \left( \frac{iA_L^{(0)}}{2s} + \frac{iA_L^{(1)}}{2s} + \dots \right) = 2i\delta_0 - \frac{1}{2} (2\delta_0)^2 + \dots = e^{2i\delta_0(s,b)} - 1. \quad (2.9)$$

Such an exponentiation works at all orders and resums all the terms of order  $(Gs)^\ell$ . As a result we have recovered (elastic) unitarity since we managed to lump all the divergent contributions at high energy into a large phase:

$$\frac{iA_L}{2s} = \int d^{D-2}b e^{-ibq/\hbar} (e^{2i\delta_0(s,b)} - 1). \quad (2.10)$$

Note that this leading eikonal resummation should hold at any  $D$  and is thus conceptually unrelated to the exponentiation of infrared divergences. In Sect. 3 we will recall how such

an exponentiation agrees with explicit amplitude calculations up to three loops. In view of extending such an analysis to the first subleading term in Sect. 4 we anticipate here some general considerations about exponentiation in impact-parameter space.

For this purpose it is convenient to associate with the centre of mass energy  $\sqrt{s}$  a length scale:

$$R \equiv (G\sqrt{s})^{\frac{1}{1-2\epsilon}}, \quad \text{i.e.} \quad G\sqrt{s} \equiv R^{D-3}, \quad (2.11)$$

in analogy with the Schwarzschild radius of CGR<sup>6</sup>. In the spirit of [60] we can now express the scaling of different terms at a given loop order in terms of the CGR quantities  $b$  and  $R$  and of Planck's constant. The Fourier transform of the leading energy contribution to the  $\ell$ -loop amplitude scales as:

$$\int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} e^{ibq/\hbar} \frac{iA_L^{(\ell)}}{2s} \sim \left[ \left(\frac{R}{b}\right)^{-2\epsilon} \frac{R\sqrt{s}}{\hbar} \right]^{\ell+1} \quad (2.12)$$

i.e. as the  $(\ell + 1)$ th power of the leading eikonal phase  $\delta_0$  in (2.6):

$$\delta_0 \sim \frac{R\sqrt{s}}{\hbar} \left(\frac{R}{b}\right)^{-2\epsilon} \sim \frac{b\sqrt{s}}{\hbar} \left(\frac{R}{b}\right)^{1-2\epsilon}. \quad (2.13)$$

This confirms that the leading eikonal resums arbitrarily high powers of  $\hbar^{-1}$  into an  $\mathcal{O}(\hbar^{-1})$  phase provided we consider, in order to make contact with CGR,  $R$  and  $b$  as classical quantities. Of particular relevance is the derivative of the eikonal phase with respect to  $b$  since it provides, via a saddle point estimate of the inverse Fourier transform, the classical deflection angle to leading order in  $R/b$ :  $\theta_s \sim \left(\frac{R}{b}\right)^{1-2\epsilon}$ . Such a classical interpretation would fail if the resummation of all the leading powers of  $\hbar^{-1}$  were not to exponentiate. The last term in (2.13) is particularly suggestive since the quantity  $b\sqrt{s}$  can be identified, at the leading eikonal level, with the total angular momentum of the process, assumed to be much larger than  $\hbar$ .

Let us now consider also the subleading energy contributions. The amplitude consists of a sum of terms having powers of  $s$  all the way up to the leading power  $\ell + 1$ . Each one of these terms behaves in impact parameter space as follows (again neglecting possible logarithmic enhancements):

$$\begin{aligned} \int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} e^{ibq/\hbar} \frac{iA^{(\ell)}}{2s} &\sim \sum_{m=0} \hbar^{2m-\ell-1} G^{\ell+1} s^{\ell+1-m} b^{2\epsilon(\ell+1)-2m} \\ &= \sum_{m=0} \left(\frac{R}{b}\right)^{2m-2\epsilon(\ell+1)} \left(\frac{R\sqrt{s}}{\hbar}\right)^{\ell+1-2m}. \end{aligned} \quad (2.14)$$

In the massless case under consideration, and in  $D = 4$ , the amplitude  $A^{(\ell)}$  cannot depend

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<sup>6</sup> The actual Schwarzschild radius  $R_S$  of a black hole of mass  $\sqrt{s}$  differs from  $R$  by a well-known  $\epsilon$ -dependent factor. Note that  $R$  has the dimension of a length for any  $\epsilon$ .

on fractional powers of  $s$ . In particular, it does not contain terms proportional to<sup>7</sup>  $\sqrt{s}$  and so the expansion above is in terms of even powers  $1/b^{2m}$ , while in the massive case all powers of  $1/b$  can (and do) appear. In both the massive and the massless cases, terms proportional to  $1/\hbar$  must be themselves exponentiated through higher-loop contributions and contribute to a classical correction to the eikonal  $\delta$ , while contributions with higher powers of  $1/\hbar$  must be accounted for by the exponentiation of terms appearing at lower-loop order.

In particular, if  $\ell$  is even, the term with  $m = \frac{\ell}{2}$  is a new classical contribution to the eikonal, while the terms with  $m < \frac{\ell}{2}$  reconstruct the exponentiation of terms appearing at a lower-loop order. All other terms with non-negative powers of  $\hbar$  are quantum terms and do not need to exponentiate. If instead  $\ell$  is odd, all terms with  $m \leq \frac{\ell-1}{2}$  contribute to the exponentiation of terms appearing at lower loops, while the terms with  $m \geq \frac{\ell+1}{2}$  are quantum and do not necessarily exponentiate.

In conclusion, terms with  $m < \frac{\ell}{2}$  do not contain new information as far as the classical scattering is concerned and a first ingredient relevant for the classical eikonal (and thus to the deflection angle) appears in the massless case at each even-loop order  $A^{(2\ell)}$  for  $m = \frac{\ell}{2}$ . The odd-loop orders  $A^{(2\ell+1)}$  do not contribute directly to the classical phase or angle. However they still take part in the exponentiation and so are important to extract the correct classical eikonal phase.

On the basis of these considerations we propose the following extension of the leading eikonal to include also subleading contributions:

$$\frac{iA(k_i, \dots)}{2s} \simeq \hat{A}^{(0)}(k_i, \dots) \int d^{D-2}b e^{-ibq/\hbar} \left[ \left(1 + 2i\Delta(s, b)\right) e^{2i\delta(s, b)} - 1 \right], \quad (2.15)$$

where all the terms appearing in  $e^{2i\delta(s, b)}$  are proportional<sup>8</sup> to  $\hbar^{-1}$  while those appearing in the prefactor  $\Delta$  contain the contributions with non-negative powers of  $\hbar$ . The use of  $\simeq$  here and below indicates that the identity (2.15) is restricted to the non-analytic terms as  $q \rightarrow 0$  that capture long-range effects in impact parameter space. Checking the validity of (2.15) will be one of the main themes of the following sections.

### 3 Check of (and constraints from) the leading-eikonal

As argued in the previous section, it is natural to assume that the leading high energy contribution at any loop order is simply captured by taking the Fourier transform of the leading eikonal back to momentum space, see (2.10). In ref. [51], we showed that this equation reproduces the leading terms at two- and three-loop order by using the full

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<sup>7</sup>We take this as an empirical fact whose deeper reason should rest on the fact that each power of  $G$  must be accompanied by an (energy)<sup>2</sup> factor. In the absence of masses, a non-integer power of  $s$  would have to be accompanied by a non-integer power of  $t$  and/or  $u$ , producing a multiple discontinuity excluded by Steinmann-relation-type arguments.

<sup>8</sup>This resembles very much a WKB approximation in which the  $\mathcal{O}(\hbar^{-1})$  exponent contains a classical action satisfying the Hamilton-Jacobi equation. For a review of the relationship between the WKB and eikonal approximations, see e.g. ref. [61].



results for these amplitudes obtained in refs. [52, 55]. This should hold at any order in  $\epsilon$  and not just for the contribution that survives in  $D = 4$ , and we provided evidence of this by checking (2.10) at various orders in the  $\epsilon$  expansion.

Let us now recall how the two exponentiations (2.4) and (2.10) are related. We focus on elastic processes where  $\hat{A}^{(0)}$  is just the identity operator ensuring that the in and the out states have the same polarization and flavour; then, by starting from (2.10), we have

$$\frac{iA_L}{2s} = \int d^{D-2}b e^{-ibq/\hbar} \left( \sum_{\ell=1}^{\infty} \frac{1}{\ell!} (2i\delta_0(s, b))^\ell \right). \quad (3.1)$$

The Fourier transform can be performed term by term thanks to (A.9) and by taking the tree-level result as an overall factor, we obtain

$$\begin{aligned} \frac{iA_L}{2s} &= \frac{iA_L^{(0)}}{2s} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left[ -\frac{iGs}{\epsilon\hbar} \Gamma(1-\epsilon) \left( \frac{4\pi\hbar^2}{q^2} \right)^\epsilon \right]^\ell \frac{\Gamma(\ell\epsilon+1)\Gamma(1-\epsilon)}{\Gamma(1-(\ell+1)\epsilon)} \\ &= \frac{iA_L^{(0)}}{2s} \sum_{\ell=0}^{\infty} \frac{\alpha_G^\ell}{\ell!} \left( \frac{-i\pi s}{\epsilon(q^2)^\epsilon} \right)^\ell G^{(\ell)}(\epsilon), \end{aligned} \quad (3.2)$$

where

$$\begin{aligned} G^{(\ell)}(\epsilon) &= \frac{\Gamma^\ell(1-2\epsilon)\Gamma(1+\ell\epsilon)}{\Gamma^{\ell-1}(1-\epsilon)\Gamma^\ell(1+\epsilon)\Gamma(1-(\ell+1)\epsilon)} \\ &= 1 - \frac{1}{3}\ell(2\ell^2 + 3\ell - 5)\zeta_3\epsilon^3 + \mathcal{O}(\epsilon^4). \end{aligned} \quad (3.3)$$

We can now compare this result with the exponentiation (2.4) and in particular we focus on the two- and three-loop amplitudes that were studied in detail in [52, 55]

$$\frac{1}{2}(\mathcal{A}_L^{(1)})^2 + F_L^{(2)} = \frac{1}{2!} \left( \frac{-i\pi s}{\epsilon(q^2)^\epsilon} \right)^2 G^{(2)}, \quad (3.4)$$

$$\frac{1}{3!}(\mathcal{A}_L^{(1)})^3 + F_L^{(3)} + \mathcal{A}_L^{(1)}F_L^{(2)} = \frac{1}{3!} \left( \frac{-i\pi s}{\epsilon(q^2)^\epsilon} \right)^3 G^{(3)}, \quad (3.5)$$

where on the left-hand side we have the high energy expansion of (2.4) at two and three loops while on the right-hand side we have the corresponding order as it appears in (2.10). Solving for  $F_L^{(2)}$  using  $\mathcal{A}_L^{(1)} = \frac{-i\pi s}{\epsilon(q^2)^\epsilon}$  from Eq. (2.7), we have

$$F_L^{(2)} = \lim_{s \rightarrow \infty} F^{(2)} = \frac{1}{2} \left( \frac{-i\pi s}{\epsilon(q^2)^\epsilon} \right)^2 [G^{(2)}(\epsilon) - 1]. \quad (3.6)$$

Using Eq. (3.3) we obtain

$$F_L^{(2)} = 3\pi^2 s^2 \epsilon \zeta_3 + \mathcal{O}(\epsilon^2, s) \quad (3.7)$$

in agreement with the first line of Eq. (6.5) of ref. [52]. Notice that, in the high energy expansion, the contribution in (3.7) is leading, i.e. at the same level as  $(\mathcal{A}_L^{(1)})^2$ , showing

explicitly that the formulation of (2.4) does not collect all the leading energy terms in the exponential factor.

At the next order in perturbation theory (three loops) the remainder function contains a leading energy contribution also in the IR finite term. From Eq. (3.5) we have

$$F_L^{(3)} = \lim_{s \rightarrow \infty} F^{(3)} = \frac{1}{3!} \left( \frac{-i\pi s}{\epsilon(q^2)^\epsilon} \right)^3 [(G^{(3)} - 1) - 3(G^{(2)} - 1)] . \quad (3.8)$$

Again using (3.3) one obtains

$$F_L^{(3)} = -\frac{2i}{3} \pi^3 s^3 \zeta_3 + \mathcal{O}(\epsilon, s^2) , \quad (3.9)$$

which agrees with the second line<sup>9</sup> of Eq. (6.5) of ref. [52].

## 4 Exponentiation at the first subleading eikonal

In this section we focus on the first subleading-energy correction to the eikonal exponentiation.

As a first step, we need a better approximation to  $\mathcal{A}^{(1)}$ , including the subleading  $\mathcal{O}(t/s)$  corrections. It is possible to perform the massless one-loop box integral for general values of  $\epsilon$  and of the kinematic variables, and then perform the Regge limit of the exact expression up to the desired order in  $t/s$ . A convenient starting point for such an expansion is [62], [63]:

$$\begin{aligned} \epsilon^2 \mathcal{A}^{(1)} = & \quad (-s)^{-\epsilon} \left[ u F\left(\epsilon, 1 + \frac{s}{t}\right) + t F\left(\epsilon, 1 + \frac{s}{u}\right) \right] \\ & + (-t)^{-\epsilon} \left[ u F\left(\epsilon, 1 + \frac{t}{s}\right) + s F\left(\epsilon, 1 + \frac{t}{u}\right) \right] \\ & + (-u)^{-\epsilon} \left[ t F\left(\epsilon, 1 + \frac{u}{s}\right) + s F\left(\epsilon, 1 + \frac{u}{t}\right) \right] , \end{aligned} \quad (4.1)$$

where  $F(\epsilon, z) \equiv {}_2F_1(1, -\epsilon; 1 - \epsilon; z)$ . By following this approach and keeping track carefully of the phases due to the branch cuts of the amplitudes, one finds a closed and rather simple expression for  $\mathcal{A}^{(1)}$ :

$$\begin{aligned} \mathcal{A}^{(1)} &= -\frac{i\pi s}{\epsilon(q^2)^\epsilon} + \mathcal{A}_{SL}^{(1)} + \dots , \\ \mathcal{A}_{SL}^{(1)} &= \frac{q^2(1+2\epsilon)}{\epsilon(q^2)^\epsilon} \left( \log \frac{q^2}{s} + H(\epsilon) \right) - \frac{2q^2(2\epsilon+1)}{\epsilon^2(\epsilon+1)s^\epsilon} \cos^2 \frac{\pi\epsilon}{2} \\ &+ i \frac{\pi q^2}{\epsilon} \left[ \frac{1+\epsilon}{(q^2)^\epsilon} - \frac{1+2\epsilon}{s^\epsilon(1+\epsilon)} \frac{\sin \pi\epsilon}{\pi\epsilon} \right] , \end{aligned} \quad (4.2)$$

---

<sup>9</sup>In [52]  $s$  is taken to be negative. In order to match the different conventions one can use the following dictionary for the Mandelstam variables  $s_{HM} = u$ ,  $u_{HM} = s$ ,  $t_{HM} = t$ , where the subscript HM indicates the variables used in [52].

where  $\mathcal{A}_{SL}^{(1)}$  is the subleading level contribution in the eikonal limit of the stripped amplitude  $\mathcal{A}^{(1)}$ , the dots stand for terms of order  $q^4 s^{-1}$ , and we have defined

$$H(\epsilon) \equiv \psi(-\epsilon) - \psi(1) - 1 + \pi \cot \pi \epsilon, \quad (4.3)$$

where  $\psi(z) = \frac{d \ln(\Gamma(z))}{dz}$  is the Digamma function (for our purposes it is useful to recall that it satisfies  $\psi(1) = -\gamma_E$ , where  $\gamma_E$  is the Euler–Mascheroni constant). Notice that the quantity defined in (4.3) diverges as  $\frac{2}{\epsilon}$  for small  $\epsilon$ . This expression is valid for general values of  $\epsilon$  up to the subleading level in the Regge limit and we checked that in this regime it reproduces the data of [52] where the one-loop result is written explicitly up to  $\mathcal{O}(\epsilon^4)$ .

Let us now discuss how different quantities scale at subleading level following the general discussion of Sect. 2. The first term of Eq. (4.2) is the leading term at high energy discussed in the previous section. The extra  $q^2/s$  factor in  $\mathcal{A}_{SL}^{(1)}$  cancels the Coulomb pole in  $A_L^{(0)}$  and, as a result, we find, after Fourier transforming:

$$\left(\frac{iA^{(1)}}{2s}\right)_{SL} \Rightarrow G^2 s b^{-2+4\epsilon} \sim \left(\frac{R}{b}\right)^{2(1-2\epsilon)}. \quad (4.4)$$

Note that, in agreement with our general discussion in Sect. 2, we obtain a contribution which, unlike the one of (2.13), does not contain an  $\hbar^{-1}$  factor. For the purpose of this paper it is enough to carry out the general discussion up to and including the three-loop order. We have already mentioned the tree and one-loop order. In the latter case (4.4) represents the first term in the expansion of  $\Delta$  that appears in (2.15).

At two loops, we have the following hierarchy of contributions

$$\begin{aligned} \left(\frac{iA^{(2)}}{2s}\right) &\Rightarrow (\delta_0)^3 \sim \left(\frac{b\sqrt{s}}{\hbar} \left(\frac{R}{b}\right)^{1-2\epsilon}\right)^3; \\ (\delta_0 \Delta_1) &\sim \delta_2 \sim \frac{b\sqrt{s}}{\hbar} \left(\frac{R}{b}\right)^{3-6\epsilon} \end{aligned} \quad (4.5)$$

and similarly at three loops:

$$\begin{aligned} \left(\frac{iA^{(3)}}{2s}\right) &\Rightarrow (\delta_0)^4 \sim \left(\frac{b\sqrt{s}}{\hbar} \left(\frac{R}{b}\right)^{1-2\epsilon}\right)^4; \\ (\delta_0)^2 \Delta_1 &\sim (\delta_0 \delta_2) \sim \left(\frac{b\sqrt{s}}{\hbar}\right)^2 \left(\frac{R}{b}\right)^{4(1-2\epsilon)}; \\ \Delta_3 &\sim \left(\frac{R}{b}\right)^{4(1-2\epsilon)}, \end{aligned} \quad (4.6)$$

where  $\Delta_3$  is the next term in the expansion of  $\Delta$ .

Note that at two loops we expect (besides exponentiation of  $\delta_0$ ) a new classical contribution to the eikonal phase  $\delta_2$ , while at three loops (as it was already the case for one

loop) no new classical contribution is expected. On the other hand, at three loops the  $\mathcal{O}(\hbar^{-4})$  and  $\mathcal{O}(\hbar^{-2})$  contributions should properly reconstruct the relevant terms in (2.15).

As already mentioned, for a scattering involving massless particles, the next-to-leading correction to  $\mathcal{A}^{(1)}$  is two powers of centre of mass energy down with respect to the leading contribution. Thus we do not have corrections that scale as  $(R/b)^{1-4\epsilon} R\sqrt{s}/\hbar$  and would provide a classical contribution  $\delta_1$  entering in the full eikonal (this is known to be present for the scattering of massive particles, see e.g. [38, 45, 64]). Instead from the subleading part of  $\mathcal{A}_{SL}^{(1)}$  we obtain the first contribution to  $\Delta$

$$2i\Delta_1 = \int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} e^{ibq/\hbar} \frac{iA_L^{(0)}}{2s} \alpha_G \mathcal{A}_{SL}^{(1)}. \quad (4.7)$$

By using (4.2) we obtain both the real and the imaginary parts of  $\Delta_1$ . Using the formulas for the Fourier transforms in Appendix A we get:

$$Re(2\Delta_1) = \frac{4G^2 s}{\pi b^2} (\pi b^2)^{2\epsilon} (1 + 2\epsilon) \Gamma^2(1 - \epsilon) \left[ -\log\left(\frac{sb^2}{4\hbar^2}\right) + H(\epsilon) + \psi(1 - 2\epsilon) + \psi(\epsilon) \right], \quad (4.8a)$$

$$Im(2\Delta_1) = \frac{4G^2 s}{b^2} (\pi b^2)^{2\epsilon} (1 + \epsilon) \Gamma^2(1 - \epsilon). \quad (4.8b)$$

Note that, while  $Im(2\Delta_1)$  is infrared-finite,  $Re(2\Delta_1)$  is not since  $H(\epsilon) \sim 2\epsilon^{-1}$ . This may look surprising at first. In fact, from (2.15),  $Re(2\Delta_1)$  appears to multiply the  $S$ -matrix by a phase while  $Im(2\Delta_1)$  changes its modulus. However, if we look at things in terms of the  $T$ -matrix ( $T \equiv -i(S - 1) = (A^{(0)} + A^{(1)} + \dots)$ ),  $Im(2\Delta_1)$  comes from (the Fourier transform of) a correction to the phase of  $A^{(0)}$ , while  $Re(2\Delta_1)$  comes from a negative and infrared singular correction to its modulus.

More quantitatively, using the small- $\epsilon$  limit of  $\mathcal{A}_{SL}^{(1)}$  from (4.2), the (singular part of) one-loop suppression of the elastic cross section reads:

$$\sigma_{\text{el}}^{(1)} \sim \sigma_{\text{el}}^{(0)} \left( 1 + 2 \frac{Gq^2}{\pi\epsilon\hbar} (\log(s/q^2) + 1) \right); \quad \sigma_{\text{el}}^{(0)} \sim |A_0|^2, \quad (4.9)$$

and is exactly compensated by the cross section for single-soft-graviton emission. Indeed, the latter is given in terms of  $\sigma_{\text{el}}^{(0)}$  by

$$\frac{1}{\sigma_{\text{el}}^{(0)}} \frac{d\sigma_{\text{inelastic}}}{d\omega} \rightarrow \frac{4G}{\pi\omega\hbar} (s \log s + t \log(-t) + u \log(-u)) \omega^{-2\epsilon}, \quad (4.10)$$

which is nothing but the well known (see e.g. [36])  $D = 4$  expression corrected (up to non-singular terms for  $\epsilon \rightarrow 0$ ) in order to account for  $D = (4 - 2\epsilon)$ -dimensional phase space. Taking the small  $q^2/s$  limit of (4.10) and integrating it over  $\omega$  leads to the (positive) infrared singular inelastic contribution

$$\sigma_{\text{inelastic}} = \int_0 \omega d\omega \frac{d\sigma}{d\omega} \sim -\frac{1}{2\epsilon} \frac{4Gq^2}{\pi\hbar} (\log(s/q^2) + 1) \sigma_{\text{el}}^{(0)}, \quad (4.11)$$

which exactly cancels the singularity in (4.9).

After this digression, we now use the results (4.8a),(4.8b) in (2.15), expand the right-hand side up to order  $G^3$ , and compare it with the eikonal expansion of the two-loop amplitude  $A^{(2)}$  up to subleading level in the eikonal limit. As we discussed in the previous section, the highest power of  $s$  is entirely reproduced by the exponentiation of  $\delta_0$ , so we focus on the next subleading term, which is of order  $G^3 s^2 t$  and yields the first correction  $\delta_2$  to the leading eikonal  $\delta_0$ , so the full classical eikonal  $\delta$  is

$$\delta = \sum_{n=0}^{\infty} \delta_{2n} , \quad \text{where} \quad \delta_{2n} \sim \frac{R\sqrt{s}}{\hbar} \left( \frac{R}{b} \right)^{2n(1-2\epsilon)-2\epsilon} . \quad (4.12)$$

By using this in the perturbative expansion of (2.15) we can derive  $\delta_2$

$$\frac{A_L^{(0)}}{2s} \alpha_G^2 \text{Re} \mathcal{A}_{SL}^{(2)} = \int d^{D-2}b e^{-ibq/\hbar} [-\text{Im}(2\Delta_1)(2\delta_0) + \text{Re}(2\delta_2)] , \quad (4.13)$$

$$\frac{A_L^{(0)}}{2s} \alpha_G^2 \text{Im} \mathcal{A}_{SL}^{(2)} = \int d^{D-2}b e^{-ibq/\hbar} [\text{Re}(2\Delta_1)(2\delta_0) + \text{Im}(2\delta_2)] . \quad (4.14)$$

Since we do not have an expression for the two-loop amplitude that is exact in  $\epsilon$  at this order, we are not able to determine the all- $\epsilon$  form of  $\delta_2$ . By using the results expanded around  $\epsilon = 0$  of [52, 55] we checked that (4.13) is consistent with the following expression for  $\text{Re}(2\delta_2)$

$$\text{Re}(2\delta_2) = \frac{4G^3 s^2}{\hbar b^2} (\pi b^2)^{3\epsilon} \Gamma^3(1-\epsilon) (1 + 6\zeta_3 \epsilon^2 + \dots) . \quad (4.15)$$

In the language of (2.4), this result contains both the contribution from the exponentiation of IR divergences and that from the remainder function  $F_2$ . The first contribution can be calculated exactly in  $\epsilon$  by using (4.2) and one obtains

$$\begin{aligned} \text{Re}(2\delta_2)_{\text{expon}} &= \frac{4G^3 s^2 (\pi b^2)^{3\epsilon}}{\epsilon \hbar b^2} \left[ B^2(\epsilon) \left( \frac{2(1+\epsilon)\Gamma(1-3\epsilon)}{\Gamma(1+2\epsilon)} \right. \right. \\ &\quad \left. \left. - \frac{(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(2+\epsilon)} \frac{\sin \pi\epsilon}{\pi\epsilon} \left( \frac{sb^2}{4\hbar^2} \right)^{-\epsilon} \right) - (1+\epsilon)\Gamma^3(1-\epsilon) \right] . \end{aligned} \quad (4.16)$$

By comparing the  $\epsilon$  expansion of this result with (4.15), which does not contain any log  $s$  terms, it is natural to guess that the contribution of the remainder function should combine with the part proportional to  $B^2(\epsilon)$  in (4.16), slightly modifying the normalisation of the first term and cancelling the contribution of the next term proportional to  $s^{-\epsilon}$ . So we can guess a closed form for the last two factors<sup>10</sup> in (4.15)

$$\begin{aligned} \Gamma^3(1-\epsilon) (1 + 6\zeta_3 \epsilon^2 + \dots) &= \frac{1}{\epsilon} \left( B^2(\epsilon) \frac{(1+2\epsilon)\Gamma(1-3\epsilon)}{\Gamma(1+2\epsilon)} - (1+\epsilon)\Gamma^3(1-\epsilon) \right) \\ &= \frac{\Gamma^3(1-\epsilon)}{\epsilon} \left( \frac{1+2\epsilon}{G^{(2)}(\epsilon)} - (1+\epsilon) \right) , \end{aligned} \quad (4.17)$$

---

<sup>10</sup>The discussion in section 5 uses only on the  $\mathcal{O}(\epsilon)$  part of  $\delta_2$  and so it does not rely on this guess nor on (4.20).

where  $G^{(2)}(\epsilon)$  is defined in (3.3) and the  $(1 + \epsilon)$  term in the second line comes from subtracting the  $-2\delta_0 2Im\Delta_1$  piece in (4.13), which again can be derived exactly in  $\epsilon$ . The contribution from the remainder function is then qualitatively similar, but quantitatively different from the one coming from exponentiation of the one-loop result and can be derived by comparing (4.16) and (4.15) after including the guess (4.17)

$$\begin{aligned} Re(2\delta_2)_{\text{remainder}} &= \frac{4G^3 s^2 (\pi b^2)^{3\epsilon}}{\epsilon \hbar b^2} \left[ B^2(\epsilon) \left( -\frac{\Gamma(1-3\epsilon)}{\Gamma(1+2\epsilon)} \right. \right. \\ &\quad \left. \left. + \frac{(1+2\epsilon)\Gamma(1-2\epsilon) \sin \pi\epsilon}{\Gamma(2+\epsilon) \pi\epsilon} \left( \frac{sb^2}{4\hbar^2} \right)^{-\epsilon} \right) \right]. \end{aligned} \quad (4.18)$$

The need for such a complicated remainder can be understood to follow from a very physical requirement. Since the derivative of  $Re \delta_2$  w.r.t.  $b$  gives a correction to the physical deflection angle, we can reasonably require that it should have a finite classical limit. However, for dimensional reasons, any dependence on  $b^2 s \sim J^2$  needs to be interpreted as a dependence from  $\frac{J^2}{\hbar^2}$  which would lead to a divergent deflection angle in the classical ( $\hbar \rightarrow 0$ ) limit for generic values of  $D$ . As a consequence, the remainder's contribution must have the correct  $b^2 s$ -dependent piece as given in (4.18). This, however, is not enough since that piece is infrared divergent while the remainder, by its definition, is not. The additional term  $-\Gamma(1-3\epsilon)/\Gamma(1+2\epsilon)$  fixes (although not in a unique way) this last problem.

We thus learn that the separation of the full amplitude into an exponential piece and a remainder is hiding a simple physical property. The remainder has to be a complicated function of  $b^2 s$  so that the full amplitude does not depend on it! Or, turning things around, we can say that a simple physical requirement determines a very non trivial structure for the remainder (in analogy with the consequences of exponentiation discussed in [51]).

Turning now to  $Im(\delta_2)$  we find, using (4.14) and again the results of [52, 55]:

$$\begin{aligned} Im(2\delta_2) &= -\frac{4G^3 s^2}{\pi \hbar b^2} (\pi b^2)^{3\epsilon} \frac{(1-2\epsilon)\Gamma^3(1-\epsilon)}{\epsilon} \left[ \left( 1 - 12\epsilon^3 \zeta_3 + \dots \right) \log \left( e^{2\gamma_E} \frac{s b^2}{4\hbar^2} \right) \right. \\ &\quad \left. + \left( 1 - 3\zeta_2\epsilon + (-23\zeta_3 - 32\zeta_2)\epsilon^2 + (-167\zeta_4 - 160\zeta_3 - 64\zeta_2)\epsilon^3 + \dots \right) \right]. \end{aligned} \quad (4.19)$$

A possible guess for the factor in the first line of (4.19) that uses the same function  $G^{(2)}$  encountered before is as follows

$$\left( 1 - 12\epsilon^3 \zeta_3 + \dots \right) = \left( 3 - \frac{2\Gamma(1-3\epsilon)\Gamma(1-\epsilon)\Gamma^2(\epsilon+1)}{\Gamma^2(1-2\epsilon)\Gamma(2\epsilon+1)} \right) = 3 - \frac{2}{G^{(2)}(\epsilon)}, \quad (4.20)$$

while we do not currently have a guess for the factor in the second line of (4.19).

Again this should match the result from the exponentiation and remainder contribu-

tions. A long but straightforward calculation gives

$$\begin{aligned}
Im(2\delta_2)_{\text{expon}} &= \frac{8G^3 s^2}{\pi \hbar b^2} (\pi b^2)^{3\epsilon} \frac{(1+2\epsilon)\Gamma^3(1-\epsilon)}{\epsilon^2} \left\{ \frac{Y}{2} + \frac{\Gamma(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)} \right. \\
&\quad \times \left. \left[ \left( \frac{sb^2}{4\hbar^2} \right)^{-\epsilon} \frac{\cos^2 \frac{\pi\epsilon}{2}}{\Gamma(\epsilon+2)} - \frac{\Gamma(1-3\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1+2\epsilon)} X \right] \right\}, \quad (4.21)
\end{aligned}$$

where we have defined:

$$X = \epsilon \left( \pi \cot \pi\epsilon - \log \left( \frac{sb^2}{4\hbar^2} \right) - 1 + \gamma_E + \psi(-\epsilon) + \psi(2\epsilon) + \psi(1-3\epsilon) \right) \quad (4.22)$$

$$Y = \epsilon \left( \pi \cot \pi\epsilon - \log \left( \frac{sb^2}{4\hbar^2} \right) - 1 + \gamma_E + \psi(-\epsilon) + \psi(\epsilon) + \psi(1-2\epsilon) \right). \quad (4.23)$$

On the other hand the leading term in the  $\epsilon$ -expansion of the remainder can be extracted from the imaginary part of (B.3) and gives:

$$Im(2\delta_2)_{\text{remainder}} = -\frac{4s^2 G^3}{\pi \hbar b^2} \left[ \log^2 \left( \frac{sb^2}{4\hbar^2} e^{2\gamma_E} \right) - 2 \log \left( \frac{sb^2}{4\hbar^2} e^{2\gamma_E} \right) - 2 \left( 1 + \frac{\pi^2}{3} \right) \right] + \mathcal{O}(\epsilon). \quad (4.24)$$

One can check that such a remainder gives agreement with (4.19) at  $\epsilon = 0$ . On the other hand, also this time the remainder's contribution has to be highly non trivial in order to reconcile (4.21) with (4.19) at finite  $\epsilon$ . In particular, the power of the  $b^2$ 's dependence of (4.21) has to be cancelled by the remainder leaving just a single (and singular<sup>11</sup>)  $\log(sb^2/4\hbar^2)$  like those appearing in  $X$  and  $Y$  of (4.22) and (4.23). Once more this shows that the separation of the full amplitude into an exponential of the one-loop result and a remainder hides some simple feature of the impact-parameter result.

Let us now discuss some physical consequences of the above results. Notice first that the term of order  $\epsilon^0$  in (4.15)

$$\lim_{\epsilon \rightarrow 0} Re(2\delta_2) = \frac{4G^3 s^2}{\hbar b^2} \quad (4.25)$$

is identical to Eq. (5.26) of [17] where this quantity has been computed for pure gravity.<sup>12</sup> Since we have obtained it for  $\mathcal{N} = 8$  supergravity, this appears to indicate that classical quantities, such as  $Re\delta_2$ , are related only to large-distance physics and are therefore independent of the ultraviolet behavior of the microscopic theory and thus universal.<sup>13</sup> We

<sup>11</sup> Note that, for  $Im(\delta_2)$ , there is nothing wrong with IR divergences since they are related to bremsstrahlung processes.

<sup>12</sup> Before comparing this result to those obtained by other methods, one should be careful about the relation between  $b$  and the actual total angular momentum  $J$  in the process. The calculation of the deflection angle to this subleading level is sensitive to this precise relation.

<sup>13</sup> This universality has been known for sometime [55] for the leading eikonal. We have been informed by Parra-Martinez that it has also been checked at this subleading level for  $4 \leq \mathcal{N} \leq 8$ . A first hint for such universality goes back to [20].

checked (4.15) up to order  $\epsilon^2$  by verifying that (4.13) reproduces the results of the two-loop amplitude in dimensional regularisation [52, 55].

Turning to  $Im\delta_2$ , a few interesting properties of Eqs. (4.15) and (4.19) should be stressed:

- $Im(\delta_2)$  and  $Re(\delta_2)$  both scale like  $G^3 s^2 \hbar^{-1} (b^2)^{-1+3\epsilon}$ .
- Unlike  $Re(\delta_2)$ , which is regular for  $\epsilon \rightarrow 0$ ,  $Im(\delta_2)$  is singular.
- Nonetheless,  $Im(\delta_2)$  does not have  $\mathcal{O}(\epsilon^{-2})$  singularities and its  $\mathcal{O}(\epsilon^{-1})$  term multiplies just the combination  $\log\left(\frac{s b^2}{4\hbar^2} e^{2\gamma_E}\right) + 1$ .
- At  $\mathcal{O}(\epsilon^0)$ ,  $Im(\delta_2)$  develops a term proportional to  $\log\left(\frac{s b^2}{4\hbar^2}\right) \log(\pi b^2)$ .

This is also in line with the findings of Ref. [17] (see Eq. (5.26) there) for the pure gravity case. While  $Re(\delta_2)$  is directly related to a physical observable, the deflection angle,  $Im(\delta_2)$  is related to gravitational bremsstrahlung with its well-known infrared divergences.

It is also amusing to compare  $Im(\delta_2)$  with the Fourier transform of the imaginary part of the full two-loop amplitude. The latter can be found either by adding to (4.19) the known contribution of  $2\delta_0 Re(2\Delta_1)$  according to Eq. (4.14), or by simply starting from the expression given in Appendix B (Eq. (B.1)). The result, up to terms that vanish for  $\epsilon \rightarrow 0$ , can be expressed in a particularly simple form:

$$\int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} e^{ibq/\hbar} \frac{Im A_{SL}^{(2)}}{2s} = -\frac{4G^3 s^2}{\pi\hbar b^2} \left[ \frac{1}{\epsilon^2} (\pi b^2 \exp(\gamma_E + 2/3))^{3\epsilon} - 4 \log\left(\frac{s b^2}{4\hbar^2}\right) + C \right], \quad (4.26)$$

where  $C = -8(\gamma_E + 3/4) - \frac{11}{12}\pi^2 = -19.6649\dots$ . Note that, unlike  $Im(\delta_2)$ , this quantity does have an  $\mathcal{O}(\epsilon^{-2})$  singularity. However this, as well as an  $\mathcal{O}(\epsilon^{-1})$  singularity, only concerns terms involving  $\log(b^2)$  and  $\log^2(b^2)$  and *not*  $\log(sb^2)$ . The latter only occurs at  $\mathcal{O}(\epsilon^0)$ . The presence of a double pole in the amplitude itself arises from the known exponentiation of IR singularities in gravity [65–68]. Denoting the  $\mathcal{O}(\epsilon^m)$  part of the  $\ell$ -loop amplitude by  $\mathcal{A}^{(\ell,m)}$ , one has

$$\mathcal{A}^{(2,-2)} = \frac{1}{2} [\mathcal{A}^{(1,-1)}]^2. \quad (4.27)$$

From Eq. (4.2), one finds

$$\mathcal{A}^{(1,-1)} = -i\pi s + q^2 \left[ \log\left(\frac{s}{q^2}\right) + 1 \right], \quad (4.28)$$

and thus

$$Im \mathcal{A}^{(2,-2)} = \pi q^2 s \log(q^2) + \dots, \quad (4.29)$$

in agreement with Eq. (B.1), where the ellipsis denotes terms analytic in  $q^2$ . Note that this is not inconsistent with the lack of a double  $\epsilon$  pole in  $Im(\delta_2)$ : the latter is in the logarithm of the amplitude, and thus does not contain that part of the two-loop amplitude which results from the exponentiation of lower-order results.



## 5 Comparing the two exponentiations

In ref. [51] (and reviewed in sec. 3) we told the tale of how the exponentiations in impact-parameter space and in momentum space are related for the leading high-energy terms of the amplitude. These exponentiations differ in significant respects: in impact parameter space, the exponentiation starts at tree level with the eikonal phase, and the eikonal phase is IR-divergent. In momentum space, the tree-level amplitude is IR-finite, and the exponentiation starts with the IR-divergent one-loop amplitude. Nevertheless, the first type of exponentiation implies the second, up to an IR-finite correction factor (given by the expression  $G^{(\ell)}(\epsilon)$  in Eq. (3.2)) which determines the leading-order contribution to the remainder function.

In this section, we relate a similar connection between impact-parameter space and momentum space amplitudes at the first subleading level. That is, we show that the proposed extension (2.15) of the eikonal amplitude

$$\frac{iA(k_i, \dots)}{2s} \simeq \hat{A}^{(0)}(k_i, \dots) \int d^{D-2}b e^{-ibq/\hbar} \left[ \left(1 + 2i\Delta(s, b)\right) e^{2i\delta(s, b)} - 1 \right] \quad (5.1)$$

agrees with the expected exponentiation in momentum space at first subleading level in  $q^2/s$ , to at least the first two orders in the Laurent expansion in  $\epsilon$ .

The leading and first subleading contributions are given by

$$\frac{iA_L}{2s} = \hat{A}^{(0)}(k_i, \dots) \int d^{D-2}b e^{-ibq/\hbar} (e^{2i\delta_0} - 1) , \quad (5.2)$$

$$\frac{iA_{SL}}{2s} \simeq \hat{A}^{(0)}(k_i, \dots) \int d^{D-2}b e^{-ibq/\hbar} \left( 2i\Delta_1 \sum_{\ell=1}^{\infty} \frac{(2i\delta_0)^{\ell-1}}{(\ell-1)!} + 2i\delta_2 \sum_{\ell=2}^{\infty} \frac{(2i\delta_0)^{\ell-2}}{(\ell-2)!} \right) . \quad (5.3)$$

We have already considered the leading contribution (5.2) in sec. 3. To compute the subleading contribution (5.3), we use

$$2i\delta_0 = -\frac{iGs}{\epsilon\hbar} \Gamma(1-\epsilon) (\pi b^2)^\epsilon \quad (5.4)$$

together with the expressions for  $\Delta_1$  and  $\delta_2$  obtained in sec. 4

$$\begin{aligned} 2i\Delta_1 &= \frac{4iG^2 s \Gamma^2(1-\epsilon)}{(\pi b^2)^{1-2\epsilon}} \left( (1+2\epsilon) \left[ -\log\left(\frac{sb^2}{4\hbar^2}\right) + H(\epsilon) + \psi(1-2\epsilon) + \psi(\epsilon) \right] + i\pi(1+\epsilon) \right) , \\ 2i\delta_2 &= \frac{4G^3 s^2 \Gamma^3(1-\epsilon)}{\epsilon\hbar(\pi b^2)^{1-3\epsilon}} \left( D_1(\epsilon) \log\left(e^{2\gamma_E} \frac{sb^2}{4\hbar^2}\right) + D_2(\epsilon) \right) , \end{aligned} \quad (5.5)$$

where

$$\begin{aligned} D_1(\epsilon) &= (1-2\epsilon) \left( 3 - \frac{2}{G^{(2)}(\epsilon)} \right) \\ &= 1 - 2\epsilon - 12\zeta_3\epsilon^3 + \mathcal{O}(\epsilon^4) , \\ D_2(\epsilon) &= (1-2\epsilon)L(\epsilon) + i\pi \left( \frac{1+2\epsilon}{G^{(2)}(\epsilon)} - 1 - \epsilon \right) \\ &= (1-2\epsilon)L(\epsilon) + i\pi\epsilon (1 + 6\zeta_3\epsilon^2) + \mathcal{O}(\epsilon^4) \end{aligned} \quad (5.6)$$

and

$$L(\epsilon) = 1 - 3\zeta_2\epsilon + (-23\zeta_3 - 32\zeta_2)\epsilon^2 + (-167\zeta_4 - 160\zeta_3 - 64\zeta_2)\epsilon^3 + \mathcal{O}(\epsilon^4), \quad (5.7)$$

where the terms with  $G^{(2)}(\epsilon)$  are possible guesses to any order in  $\epsilon$  of quantities that are known only up to order  $\epsilon^3$ . Using

$$\frac{iA^{(0)}}{2s} = \frac{iA_L^{(0)}}{2s} \hat{A}^{(0)}(k_i, \dots) = \frac{4\pi i G \hbar s}{q^2} \hat{A}^{(0)}(k_i, \dots) \quad (5.8)$$

together with eqs. (A.9) and (A.10), the computation of (5.3) is straightforward. The  $\ell$ -loop subleading contribution is

$$\begin{aligned} \frac{iA_{SL}^{(\ell)}}{2s} &\simeq \frac{iA^{(0)}}{2s} \frac{\alpha_G^\ell}{\ell!} \left[ \frac{-i\pi s}{\epsilon(q^2)^\epsilon} \right]^\ell \frac{iq^2}{\pi s} G^{(\ell)}(\epsilon) \\ &\times \left\{ (1+2\epsilon) \left[ -\log\left(\frac{s}{q^2}\right) + H(\epsilon) + \psi(1-2\epsilon) + \psi(\epsilon) - \psi(1-(\ell+1)\epsilon) - \psi(\ell\epsilon) \right] \right. \\ &\left. + i\pi(1+\epsilon) + (\ell-1)D_1(\epsilon) \left[ \log\left(e^{2\gamma_E} \frac{s}{q^2}\right) + \psi(1-(\ell+1)\epsilon) + \psi(\ell\epsilon) \right] + (\ell-1)D_2(\epsilon) \right\}. \end{aligned} \quad (5.9)$$

The divergent terms in this expression should match those arising from the IR exponentiation in (2.4). We start by considering the first two terms in the  $\epsilon$  expansion where one can neglect the remainder functions appearing in (2.4). Then in a separate subsection we consider the third and the fourth terms in the  $\epsilon$  expansion: the third order depends on the finite part of  $F^{(2)}$ , while the fourth one receives contributions also from the  $\mathcal{O}(\epsilon)$  term in  $F^{(2)}$  and the finite part of  $F^{(3)}$ .

## 5.1 The first two leading orders in $\epsilon$ at $\ell$ -loop order

As mentioned previously, the eikonal expression (5.1) is only meant to capture the non-analytic contributions to the momentum space amplitude as  $q^2 \rightarrow 0$ . Additional polynomial terms in  $q^2$  will Fourier transform to give  $\delta^{(d-2)}(b)$  function terms (or derivatives thereof) in impact parameter space. To identify all non-analytic terms in (5.9), we must expand  $(q^2)^{-\ell\epsilon} = \exp[-\ell\epsilon \log(q^2)]$  in  $\epsilon$ . In addition we use  $G^{(\ell)}(\epsilon) = 1 + \mathcal{O}(\epsilon^3)$  and Laurent expand the functions

$$\begin{aligned} H(\epsilon) + \psi(1-2\epsilon) + \psi(\epsilon) - \psi(1-(\ell+1)\epsilon) - \psi(\ell\epsilon) &= \left(\frac{\ell+1}{\ell}\right) \frac{1}{\epsilon} - 1 + \mathcal{O}(\epsilon), \\ \psi(1-(\ell+1)\epsilon) + \psi(\ell\epsilon) &= -\frac{1}{\ell\epsilon} - 2\gamma_E + \mathcal{O}(\epsilon). \end{aligned} \quad (5.10)$$

Dropping all the terms in (5.9) that have no  $\log^n(q^2)$ -dependence, we obtain

$$\begin{aligned} \frac{iA_{SL}^{(\ell)}}{2s} &\simeq \frac{iA^{(0)}}{2s} \frac{\alpha_G^\ell}{\ell!} \left( \frac{-i\pi s}{\epsilon} \right)^\ell \frac{iq^2}{\pi s} \left\{ -\ell \log(q^2) \right. \\ &+ \epsilon \left[ \frac{1}{2}\ell(\ell-1)[D_1(0)+1] \log^2(q^2) + \ell[(1-\ell)D_1(0)+1] \log(s) \log(q^2) \right. \\ &\left. \left. + \ell[(1-\ell)D_2(0)-1] \log(q^2) - i\pi\ell \log(q^2) \right] + \mathcal{O}(\epsilon^2) \right\}. \end{aligned} \quad (5.11)$$

By noting that (5.6) implies  $D_1(0) = D_2(0) = 1$  (where the imaginary part of  $D_2(\epsilon)$  only begins at  $\mathcal{O}(\epsilon)$ ), we obtain all the nonanalytic subleading terms through  $\mathcal{O}(1/\epsilon^{\ell-1})$ :

$$\begin{aligned} \frac{iA_{SL}^{(\ell)}}{2s} &\simeq \frac{iA^{(0)}}{2s} \frac{\alpha_G^\ell}{\ell!} \left( \frac{-i\pi s}{\epsilon} \right)^\ell \frac{iq^2}{\pi s} \left\{ -\ell \log(q^2) \right. \\ &+ \epsilon \left[ \ell(\ell-1) \log^2(q^2) - \ell(\ell-2) \log(s) \log(q^2) \right. \\ &\left. \left. - \ell^2 \log(q^2) - i\pi\ell \log(q^2) \right] \right\} + \mathcal{O}(1/\epsilon^{\ell-2}). \end{aligned} \quad (5.12)$$

Now let us check this against the expected exponentiation in momentum space

$$\frac{iA}{2s} = \frac{iA^{(0)}}{2s} \exp\left(\alpha_G \mathcal{A}^{(1)}\right) \exp\left(\sum_{\ell=2}^{\infty} \alpha_G^\ell F^{(\ell)}\right), \quad (5.13)$$

where

$$\mathcal{A}^{(1)} = \frac{1}{\epsilon(q^2)^\epsilon} \left[ -i\pi s + q^2 \left( \log\left(\frac{s}{q^2}\right) + 1 \right) \right] + \frac{q^2}{(q^2)^\epsilon} \left[ -\log^2\left(\frac{s}{q^2}\right) + i\pi \log\left(\frac{s}{q^2}\right) \right] + \mathcal{O}(\epsilon). \quad (5.14)$$

Since the remainder function  $F^{(\ell)}$  is IR-finite and only begins at two-loop order, the first two terms in the Laurent expansion of the  $\ell$ -loop amplitude are completely dictated by the one-loop amplitude

$$\frac{iA^{(\ell)}}{2s} = \frac{iA^{(0)}}{2s} \frac{\alpha_G^\ell}{\ell!} (\mathcal{A}^{(1)})^\ell + \mathcal{O}(1/\epsilon^{\ell-2}). \quad (5.15)$$

Substituting Eq. (5.14) into (5.15), we obtain for the leading level  $\ell$ -loop amplitude

$$\frac{iA_L^{(\ell)}}{2s} = \frac{iA^{(0)}}{2s} \frac{\alpha_G^\ell}{\ell!} \left( \frac{-i\pi s}{\epsilon(q^2)^\epsilon} \right)^\ell + \mathcal{O}(1/\epsilon^{\ell-2}) \quad (5.16)$$

agreeing with the leading level eikonal expression (3.2) to this order in  $\epsilon$ . For the subleading level  $\ell$ -loop amplitude, we get

$$\begin{aligned}
\frac{iA_{SL}^{(\ell)}}{2s} &= \frac{iA^{(0)}}{2s} \frac{\alpha_G^\ell}{\ell!} \left( \frac{-i\pi s}{\epsilon(q^2)^\epsilon} \right)^\ell \frac{iq^2\ell}{s\pi} \left\{ \log\left(\frac{s}{q^2}\right) + 1 \right. \\
&\quad \left. + \epsilon \left[ -\log^2\left(\frac{s}{q^2}\right) + i\pi \log\left(\frac{s}{q^2}\right) \right] \right\} + \mathcal{O}(1/\epsilon^{\ell-2}) \\
&= \frac{iA^{(0)}}{2s} \frac{\alpha_G^\ell}{\ell!} \left( \frac{-i\pi s}{\epsilon} \right)^\ell \frac{iq^2}{s\pi} \left\{ \ell \log(s) + \ell - \ell \log(q^2) \right. \\
&\quad \left. + \epsilon \left[ -\ell \log^2(s) + i\pi\ell \log(s) + \ell(\ell-1) \log^2(q^2) \right. \right. \\
&\quad \left. \left. - \ell(\ell-2) \log(s) \log(q^2) - \ell^2 \log(q^2) - i\pi\ell \log(q^2) \right] \right\} + \mathcal{O}\left(\frac{1}{\epsilon^{\ell-2}}\right).
\end{aligned} \tag{5.17}$$

Comparing the  $\log^n(q^2)$ -dependent terms of this expression with (5.12) we find perfect agreement.

## 5.2 The first four leading orders in $\epsilon$ at $\ell$ -loop order

So far we exploited only the knowledge of the one-loop amplitude in evaluating (2.4), but thanks to the explicit results of [52] we can extend the comparison between the two exponentiations at the subleading level in the eikonal limit to the first four terms in the  $\epsilon$  expansion.

Let us start by analysing in some detail the three-loop case. The leading term of  $A^{(3)}/(2s)$  scales as  $s^4$  and, as discussed in section 3, it is entirely reproduced by the exponentiation of  $\delta_0$ . The subleading contribution  $A_{SL}^{(3)}/(2s)$  scales, after Fourier transform to impact parameter space, as  $(Gs/\hbar)^2(R/b)^2 \log^{n-1}(b^2)$  and, as discussed before, we focus on the long-range contributions, i.e. the terms with  $n \geq 1$ . From the scaling above it is clear that such terms grows too quickly with the energy (and is too singular in the classical limit) to be absorbed in a contribution  $\delta_3$  to the total eikonal or in a contribution  $\Delta_3$  to the prefactor  $\Delta$ . Thus they must be reproduced by the leading and the subleading eikonal data, as dictated by (5.3). Then, by separating the real and the imaginary parts, we have<sup>14</sup>

$$\frac{A_L^{(0)}}{2s} \alpha_G^3 \text{Re} \mathcal{A}_{SL}^{(3)} = \int d^{D-2}b e^{-ibq/\hbar} \left[ -\frac{1}{2} (2\delta_0)^2 \text{Re}(2\Delta_1) - (2\delta_0) \text{Im}(2\delta_2) \right] \tag{5.19}$$

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<sup>14</sup>We can write the three-loop consistency condition dictated by assuming the eikonal exponentiation (2.15) in the momentum space language as follows:

$$\frac{A_L^{(0)}}{2s} \alpha_G^3 \mathcal{A}^{(3)} = \frac{1}{2} \int d^{D-2}b e^{-ibq/\hbar} \left[ (2\delta_0)^2 (2\Delta_1) \right] + i \int \frac{d^{D-2}k}{(2\pi\hbar)^{D-2}} \frac{(4\pi G\hbar s)^2}{(q-k)^2 k^2} \alpha_G^2 \mathcal{A}^{(2)}(s, k^2). \tag{5.18}$$

and similarly for the imaginary part

$$\frac{A_L^{(0)}}{2s} \alpha_G^3 \text{Im} \mathcal{A}_{SL}^{(3)} = \int d^{D-2} b e^{-ibq/\hbar} \left[ -\frac{1}{2} (2\delta_0)^2 \text{Im}(2\Delta_1) + (2\delta_0) \text{Re}(2\delta_2) \right]. \quad (5.20)$$

The left-hand side of these equations can be extracted from the full three-loop  $\mathcal{N} = 8$  4-point amplitude recently derived in [52]. The relevant terms in the Regge regime up to the first subleading level in the Regge limit are summarised in Appendix B. The right-hand side is obtained by using (2.6) for  $\delta_0$ , (4.8) for  $\Delta_1$ , (4.15) for  $\text{Re}(\delta_2)$ , and (4.19) for  $\text{Im}(\delta_2)$ . The relation (5.20) is easier to check since  $\text{Re}(\delta_2)$  is simpler than  $\text{Im}(\delta_2)$ . The left-hand side is given by the five imaginary terms of the subleading (i.e. proportional to  $s^2$ ) contribution in (B.2). We checked that the eikonal exponentiation on the right-hand side of (5.20) reproduces exactly these terms.

We performed a similar check for (5.19). Now the left-hand side involves eighteen terms which are the real contributions to the  $s^2$  part of (B.2). The structure of the answer is more complicated and includes contributions enhanced by a factor of  $\log(s)$ . By comparing this result with the prediction on the right-hand side coming from the eikonal exponentiation we find agreement for all terms but one. In particular all divergent terms as  $\epsilon \rightarrow 0$  and all terms proportional to  $\log^n(q^2)$  with  $n \geq 2$  match. However by going all the way down to the lowest order contribution (i.e of  $\mathcal{O}(G^4 s^3/b^2)$  with no  $\log s$  enhancement) we find a mismatch, which, in momentum space, reads:

$$(\text{lhs} - \text{rhs})_{Eq. (5.19)} = \frac{16 G^4 s^3}{3 \hbar^2} (3\zeta_3 - \pi^2) \log(q^2). \quad (5.21)$$

From (5.21) we see that the mismatch is sensitive to the two-loop contribution proportional to  $\epsilon \log q^2$  and to the three-loop contribution proportional to  $\log q^2$ . Suppose one were to modify these terms in the amplitude

$$\tilde{\mathcal{A}}^{(2)} = \mathcal{A}^{(2)} + i\pi\epsilon c_2 s q^2 \log q^2 + \dots, \quad \tilde{\mathcal{A}}^{(3)} = \mathcal{A}^{(3)} + \pi^2 c_3 s^2 q^2 \log q^2 + \dots, \quad (5.22)$$

where the  $\mathcal{A}$ 's on the right-hand side are those given in (B.1) and (B.2) and the dots stand for further analytic contributions or higher order terms in  $\epsilon$ . This would change the remainder functions from the ones given in (B.3) and (B.4) to

$$\tilde{F}^{(2)} = F^{(2)} + i\pi\epsilon c_2 s q^2 \log q^2, \quad \tilde{F}^{(3)} = F^{(3)} + \pi^2 (c_3 - c_2) s^2 q^2 \log q^2, \quad (5.23)$$

and the eikonal in (5.5) to

$$\tilde{\delta}_2 = \delta_2 - \frac{4iG^3 s^2 \epsilon \Gamma^3(1-\epsilon)}{\hbar(1-2\epsilon)(\pi b^2)^{1-3\epsilon}} + \mathcal{O}(\epsilon^2). \quad (5.24)$$

The tilde'd quantities now satisfy the consistency check (5.19), provided that the parameters appearing in (5.22) satisfy the constraint

$$c_3 = c_2 - \frac{4}{3} (3\zeta_3 - \pi^2). \quad (5.25)$$

This modification, however, turns out to be insufficient to cure a mismatch at higher-loop order, as we shall now argue.

We can follow the logic of (5.1) and use the first four terms in the  $\epsilon$ -expansion of the  $\ell$ -loop result for (5.3) as a check of remainder functions proposed in (5.23). The  $\ell$ -loop eikonal prediction (5.3) for the subleading amplitude still does not agree with the (IR-divergent) prediction of the momentum-space exponentiation (2.4), even when using the modified remainder functions (5.23). Furthermore, this mismatch is independent of the choice for the residual parameter  $c_2$ , which is thus unfixed by these checks. The mismatch first appears at order  $1/\epsilon^{\ell-3}$  (for  $\ell > 3$ ), and has the following pattern:

$$(\text{lhs} - \text{rhs})_{Eq. (5.3)} \sim \frac{i\pi s q^2 \log q^2}{\epsilon^{\ell-3}(\ell-4)!}, \quad (5.26)$$

where the proportionality constant is independent of  $\ell$  and all the quantities are calculated using (5.22)–(5.25). Amazingly, the mismatch (5.26) could be avoided for all  $\ell$  by the following further redefinition of the three-loop remainder function

$$\hat{F}^{(3)} = \tilde{F}^{(3)} + 2\pi^2 s^2 q^2 \frac{\zeta_3}{\epsilon}. \quad (5.27)$$

Such a redefinition, however, is not allowed if all infrared divergences are captured by the exponentiation of the one-loop result as assumed in (2.4).

It is difficult to assess the meaning of the few mismatches we found when weighed against the large number of successful checks. One possibility is that factorization can slightly break in the non-conservative contributions to the amplitude since, by themselves, they do not carry a physical meaning. If so, one should check whether some inconsistency is still present after computing a more physical quantity such as an infrared-finite inclusive cross section. Another, perhaps more interesting possibility, is that the two results have different regimes of validity depending on whether the IR cutoff is the lowest energy scale in the problem or not. We will add further comments on this point in the final section.

## 6 The $D = 4$ eikonal using a momentum cutoff

So far we have regularized infrared divergences by using dimensional regularization and have checked exponentiation in impact parameter space at the leading and first subleading level in  $t/s$  and at different orders in the small- $\epsilon$  expansion. We have then obtained the  $D = 4$  results by taking, at the end, the  $\epsilon \rightarrow 0$  limit.

In this section, we try to make a more direct connection with the approach of [17] by deriving again  $\delta_2$  while staying all the time in  $D = 4$  supplemented with a low-momentum cutoff. We will show that the  $D = 4$  result for the real part of  $\delta_2$  agrees with the one obtained in the previous section while this does not appear to be the case for its imaginary part. We will give an interpretation for these two contrasting results.

We will start again from the exact expression (4.1) and first perform a small- $\epsilon$  expansion for a generic kinematics. A straightforward calculation leads to:

$$\begin{aligned} \mathcal{A}^{(1)} &= \frac{1}{\epsilon} \left[ s \log \frac{-s}{\mu^2} + t \log \frac{-t}{\mu^2} + u \log \frac{-u}{\mu^2} \right] \\ &+ \left[ u \log \frac{-s}{\mu^2} \log \frac{-t}{\mu^2} + t \log \frac{-s}{\mu^2} \log \frac{-u}{\mu^2} + s \log \frac{-u}{\mu^2} \log \frac{-t}{\mu^2} \right], \end{aligned} \quad (6.1)$$

which agrees with the known result (see, e.g. [69]). As in the previous sections we specify the Riemann sheet along the positive real  $s$ -axis by taking  $\log(-s) = \log s - i\pi$ . Using also  $s + t + u = 0$  to eliminate  $u$  we get:

$$\mathcal{A}^{(1)} = -i\pi s \left( \frac{1}{\epsilon} - \log \frac{-t}{\mu^2} \right) - i\pi t \log \frac{s+t}{-t} - s \log \frac{s}{s+t} \left( \frac{1}{\epsilon} - \log \frac{-t}{\mu^2} \right) + t \log \frac{-t}{s+t} \left( \frac{1}{\epsilon} - \log \frac{s}{\mu^2} \right). \quad (6.2)$$

Up to now this expression is exact. We now expand it for  $s \gg |t|$  keeping only terms up to  $\mathcal{O}(t)$  (and neglecting those of  $\mathcal{O}(t^2/s)$ ) to get

$$\mathcal{A}^{(1)} = -i\pi s \left( \frac{1}{\epsilon} - \log \frac{-t}{\mu^2} \right) - i\pi t \log \frac{s}{-t} + t \left( \frac{1}{\epsilon} - \log \frac{-t}{\mu^2} \right) + t \log \frac{-t}{s} \left( \frac{1}{\epsilon} - \log \frac{s}{\mu^2} \right). \quad (6.3)$$

As a double check, we can extract the terms of order  $\frac{1}{\epsilon}$  and of order  $\epsilon^0$  of Eq. (4.2) and show that Eq. (6.3) is exactly reproduced.

We now get rid of  $\epsilon$  by introducing an infrared momentum cutoff  $\lambda$  through the relation:

$$\frac{1}{\epsilon} \equiv \log \frac{\lambda^2}{\mu^2} \Rightarrow \frac{1}{\epsilon} - \log \frac{-t}{\mu^2} = -\log \frac{-t}{\lambda^2} \quad ; \quad \frac{1}{\epsilon} - \log \frac{s}{\mu^2} = -\log \frac{s}{\lambda^2}. \quad (6.4)$$

We then arrive at

$$\mathcal{A}^{(1)} \sim i\pi(s+t) \log \frac{-t}{\lambda^2} - t \log \frac{-t}{\lambda^2} \left( \log \frac{s}{\lambda^2} - 1 \right) - i\pi t \log \frac{s}{\lambda^2} + t \log^2 \frac{s}{\lambda^2} \quad (6.5)$$

and note that all dependence on  $\mu$  has also disappeared as a consequence of UV finiteness.

This gives, for the one-loop amplitude,

$$\frac{iA^{(1)}}{2s} = \frac{iA_L^{(0)}}{2s} \frac{G}{\pi\hbar} \mathcal{A}^{(1)} \sim -\frac{4\pi G^2 s^2}{q^2} \log \frac{q^2}{\lambda^2} - 4\pi G^2 s \log \frac{s}{q^2} + 4iG^2 s \log \frac{q^2}{\lambda^2} \left( \log \frac{s}{\lambda^2} - 1 \right), \quad (6.6)$$

where we have used (2.5) for the tree amplitude  $A_L^{(0)}$ .

Using formulae from Appendix A the ( $D = 4$ ) Fourier transform of the first term is given by

$$\int \frac{d^2q}{(2\pi\hbar)^2} e^{iqb/\hbar} \left( -\frac{4\pi G^2 s^2}{q^2} \log \frac{q^2}{\lambda^2} \right) = -\frac{1}{2} \left( \frac{Gs}{\hbar} \right)^2 \log^2 \left( \frac{b^2 \lambda^2}{\hbar^2} \right) = \frac{1}{2} (2i\delta_0)^2, \quad (6.7)$$

where, in this section,  $2i\delta_0$  is the  $\epsilon \rightarrow 0$  limit of the Fourier transform of the tree amplitude  $iA_L^{(0)}/(2s)$  given in Eq. (2.5) with the above identification  $\epsilon^{-1} = \log(\lambda^2/\mu^2)$ . The Fourier transform of the second term in (6.6) gives

$$\int \frac{d^2q}{(2\pi\hbar)^2} e^{iqb/\hbar} \left( -4\pi G^2 s \log \frac{q^2}{s} \right) = Im(2\Delta_1) = \frac{4G^2 s}{b^2}, \quad (6.8)$$

Finally, the Fourier transform of the third term is equal to

$$\int \frac{d^2q}{(2\pi\hbar)^2} e^{iqb/\hbar} \left( 4iG^2 s \log \frac{q^2}{\lambda^2} \right) \left( \log \frac{s}{\lambda^2} - 1 \right) = 2iRe\Delta_1 = -\frac{4iG^2 s}{\pi b^2} \left( \log \frac{s}{\lambda^2} - 1 \right). \quad (6.9)$$

In conclusion, we have checked (to this order) the exponentiation of the leading eikonal and we have determined the real and imaginary part of  $\Delta_1$  that we rewrite here:

$$Re(2\Delta_1) = -\frac{4G^2 s}{\pi b^2} \left( \log \frac{s}{\lambda^2} - 1 \right) \quad ; \quad Im(2\Delta_1) = \frac{4G^2 s}{b^2}. \quad (6.10)$$

Comparing the above results with the  $\epsilon \rightarrow 0$  limit of those obtained in (4.8a) and (4.8b) we note that there is agreement in the latter case ( $Im(2\Delta_1)$ ) but not in the former ( $Re(2\Delta_1)$ ). The mismatch looks quite substantial since (4.8a) produces, as  $\epsilon \rightarrow 0$ , a  $\log b^2$  term which is clearly absent in (6.10). We will argue that the origin of these two contrasting results is related to the fact that  $Im(2\Delta_1)$  is infrared finite while  $Re(2\Delta_1)$  is infrared divergent.

As a first guess one might argue that we have taken too quickly the  $\epsilon \rightarrow 0$  limit in computing the subleading one-loop amplitude. This however is not the case: a direct expansion of the one-loop amplitude (4.2) shows that no  $\log^2 q^2$  term is generated in the  $\epsilon \rightarrow 0$  limit. As a consequence, a two-dimensional Fourier transform cannot produce a  $\log b^2$  contribution. Therefore the reason for the discrepancy must be found in the order in which one performs the Fourier transform itself. And indeed, if one performs *first* the Fourier transform in  $2 - 2\epsilon$  dimensions and *then* takes the limit, the  $\log b^2$  term does come out as in (4.8a). The relevant maths is perfectly exemplified by the function:

$$\frac{q^{-2\epsilon}}{\epsilon^2} \left( \epsilon \log(q^2/s) + 2 \right) \quad (6.11)$$

whose  $\epsilon \rightarrow 0$  limit has  $\log q^2$  but no  $\log^2 q^2$  terms, and whose Fourier transform at finite  $\epsilon$  develops a  $\log b^2$  contribution in that same limit.

Our conclusion is that for infrared-divergent terms one has to work all the time within a consistent regularization scheme. One such scheme is usually assumed to be dimensional regularization –which we have also adopted– while introducing a straight momentum cutoff is not obviously a consistent scheme (one could try instead to work in a finite box and then take the limit as done in lattice gauge theories). In any case one should compare physical infrared-finite quantities in both schemes. For these reasons in the rest of this section we shall limit ourselves to the calculation of  $Re(\delta_2)$  for which only the knowledge of the infrared-safe  $Im(\Delta_1)$  is needed.



Starting from (2.4) and keeping only terms up to order  $\epsilon^0$  in  $\mathcal{A}^{(1)}$  and the leading terms in  $F^{(\ell)}$  we have at order  $G^3$

$$\frac{iA^{(2)}}{2s} = \frac{iA_L^{(0)}}{2s} \frac{G^2}{\pi^2 \hbar^2} \left( \frac{1}{2} (\mathcal{A}^{(1)})^2 + F^{(2)} \right), \quad (6.12)$$

which should be compared with the corresponding expansion of the eikonal exponentiation

$$(1 + 2i\Delta_1) e^{2i\delta_0 + 2i\delta_2} \sim \frac{1}{3!} (2i\delta_0)^3 - (2\delta_0)(2\Delta_1) + 2i\delta_2 + \dots \quad (6.13)$$

For the reasons explained above we will compare only the imaginary part of these two equations. Starting from the first term in Eq. (6.12), we have

$$\frac{iA_L^{(0)}}{2s} \frac{G^2}{2\pi^2 \hbar^2} (\mathcal{A}^{(1)})^2 \sim -\frac{2\pi i G^3 s^3}{\hbar q^2} \log^2 \frac{q^2}{\lambda^2} + \frac{4G^3 s^2}{\hbar} \left[ i\pi \log^2 \frac{q^2}{\lambda^2} - i\pi \log \frac{q^2}{\lambda^2} \log \frac{s}{\lambda^2} \right], \quad (6.14)$$

where we focused on the leading and the first subleading contributions in the Regge limit.

We can extract the expression for the second term in Eq. (6.12) from [52]. By focusing on the imaginary terms that are the relevant ones at high energy one gets<sup>15</sup>

$$\frac{iA_L^{(0)}}{2s} \frac{G^2}{\pi^2 \hbar^2} F^{(2)} \sim -\frac{2\pi i G^3 s^2}{\hbar} \left( \log \frac{q^2}{\lambda^2} - \log \frac{s}{\lambda^2} \right)^2 - \frac{4\pi i G^3 s^2}{\hbar} \log q^2. \quad (6.15)$$

The leading term in  $s$  comes from the first term in (6.14) whose Fourier transform is

$$\int \frac{d^2 q}{(2\pi \hbar)^2} e^{iqb/\hbar} \left( -\frac{2i\pi G^3 s^3}{\hbar q^2} \log^2 \frac{q^2}{\lambda^2} \right) = \frac{i}{3!} \left( \frac{Gs}{\hbar} \log \left( \frac{b^2 \lambda^2}{\hbar^2} \right) \right)^3 = \frac{1}{3!} (2i\delta_0)^3 \quad (6.16)$$

in agreement with the first term of Eq. (6.13). Note that in the subleading terms the contributions proportional to  $\log^2 q^2 \log s$  cancel. The Fourier transform of the rest gives

$$\int \frac{d^2 q}{(2\pi \hbar)^2} e^{iqb/\hbar} \left( \frac{2\pi i G^3 s^2}{\hbar} \log^2 \frac{q^2}{\lambda^2} - \frac{4\pi i G^3 s^2}{\hbar} \log \frac{q^2}{\lambda^2} \right) = \frac{4i G^3 s^2}{\hbar b^2} \log \left( \frac{b^2 \lambda^2}{\hbar^2} \right) + \frac{4i G^3 s^2}{\hbar b^2}. \quad (6.17)$$

Using now:

$$(2i\delta_0)(-Im 2\Delta_1) = \frac{4i G^3 s^2}{\hbar b^2} \log \left( \frac{b^2 \lambda^2}{\hbar^2} \right), \quad (6.18)$$

as well as the imaginary part of (6.13), we immediately find:

$$Re(2\delta_2) = \frac{4G^3 s^2}{\hbar b^2}. \quad (6.19)$$

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<sup>15</sup>The terms relevant at high energy can be extracted from Eq. (6.1) of Ref. [52], where all the factors of  $\log(x)$  should be replaced by  $\log(x) - i\pi$ . Then one can check that Eq. (6.1) of Ref. [52] agrees with the result of [55]. We would like to thank J. M. Henn for a clarifying discussion on this point.

Happily, the value of  $Re(2\delta_2)$  coincides with the one obtained in (4.25) and with Eq. (5.26) of Ref. [17]. (As expected, a similar agreement does not hold for the term of order  $\epsilon^0$  in  $Im(2\delta_2)$  whose explicit calculation we omit.)

We finally note that, if we expand up to order  $\epsilon^0$  the quantity

$$(2i\delta_0)(-Im(2\Delta_1)) = -\frac{4iG^3s^2(\pi b^2)^{3\epsilon}\Gamma^3(1-\epsilon)(1+\epsilon)}{\epsilon\hbar b^2} \quad (6.20)$$

needed in the calculation of  $Re(2\delta_2)$ , we get

$$-\frac{4iG^3s^2}{\hbar b^2} \left( \frac{1}{\epsilon} + 3\log(\pi b^2) - 3\psi(1) + 1 \right) \quad (6.21)$$

whose term with  $\log b^2$  differs by a factor 3 from the one of (6.18), while, as mentioned, the results for  $Re(2\delta_2)$  agree. This is due to the following reason: the interference terms  $\epsilon \times 1/\epsilon$  that we neglected in calculating (6.18) in  $D = 4$  are identical to the corresponding interference terms neglected in (6.14). This happens because the  $1/\epsilon$  contribution is a constant in both cases and so the Fourier transform acts non-trivially only on the  $\mathcal{O}(\epsilon)$  term mapping exactly the  $\mathcal{O}(\epsilon)$  contribution of  $\mathcal{A}^{(1)}$  into that of  $\Delta_1$ . Once more the same cancellation does not occur for  $Im(2\delta_2)$ .

## 7 Summary and Outlook

Four-point amplitudes in  $\mathcal{N} = 8$  supergravity are known with a great degree of precision. In this work we set up a systematic approach for the analysis of these loop amplitudes in the Regge regime where the momentum transferred is much smaller than the centre of mass energy. A first result is that, even in this highly supersymmetric setup, some of the contributions that grow polynomially with the energy are not accounted for by the exponentiation of the leading eikonal (2.6) alone. Instead they give rise to a new classical contribution ( $2i\delta_2$  in (5.5)) that modifies the leading eikonal at 3PM order, i.e.  $(R/b)^2$  in  $D \rightarrow 4$  and in the Regge regime  $R \ll b$ , where  $b$  is the impact parameter and  $R$  is a scale related to the energy of the process (2.11). Corrections at 2PM order are absent in massless theories, see the comment after (2.14), but it is interesting to notice that in a maximally supersymmetric setup, they are absent also when the external states are massive [53]. Our results show that this cancellation, motivated by supersymmetry, does not survive at higher orders when both particles are dynamical. Further corrections at 5PM order, i.e.  $(R/b)^4$ , are expected and should be extracted from the sub-subleading terms in the four-loop amplitude.

Notice that these power-like contributions are different from the most logarithmically enhanced terms discussed in [70, 71]. In theories with only spin 1 particles, the dominant terms in the Regge limit are proportional to  $(\log^2(t/s))^\ell$  at  $\ell$  loops. By contrast, in gravity theories these terms take the form  $(t \log^2(t/s))^\ell$ , and thus become increasingly power-suppressed in  $t$  as the loop order increases. Nevertheless, an algorithm exists for deriving them at arbitrary order [70, 71], and they should be resummed in order to describe

the scattering process for values of the impact parameter  $b$  that are closer to  $R$  (even before reaching Planckian scales).

The main property of the classical eikonal is that it should exponentiate, see in (2.15): in this way the full amplitude has the expected classical limit  $\hbar \rightarrow 0$ , where the only singular term is a WKB-like exponential, see [60] for a closely related discussion. Contrary to what happens for the leading eikonal and for the 2PM correction when this is present, the 3PM result ( $2\delta_2$ ) contains both a real and an imaginary part. The real part is directly related to physical observables such as the deflection angle and the Shapiro time delay and so one would expect it to be free of IR divergences. This is the case in our result since the infrared divergent term in the real part of the two-loop amplitude is cancelled in the subtraction (4.13) yielding an IR finite result for  $Re(2\delta_2)$ . The imaginary part of the eikonal is IR divergent and it would be very interesting to study a physical observable, such as an inclusive cross section, which is sensitive to  $Im(2\delta_2)$ , so as to check how the cancellation of the IR divergences works at higher order, generalising for instance the discussion after Eq. (4.9) at two loops.

There is another interesting aspect related to IR divergences that we analysed in some detail: the relation between the IR exponentiation in momentum space (2.4) and the eikonal exponentiation in impact parameter space (2.15). At leading level in the Regge regime the two expressions match in a non-trivial way in the common regime of validity for any value of the dimensional regularisation parameter  $\epsilon$  as already discussed [51]. The leading eikonal is universal, i.e. it does not depend on the presence of supersymmetry and is the same for all gravity theories that at large distances reduce to CGR. Then the relation between the two exponentiations provides an easy set of predictions for the terms of the  $\ell$ -loop gravitational amplitudes that scale as  $s^{\ell+1}$  for small  $t$ . In this paper we extended this logic to the subleading terms in the Regge regime. At this order the amplitudes depend on the details of the theory and we focused on the case of  $\mathcal{N} = 8$  supergravity.<sup>16</sup> By using the explicit results of [52] we compared the two exponentiations at all loops for the first four terms in the  $\epsilon \rightarrow 0$  expansion. As discussed in section 5, there is an impressive agreement between the eikonal prediction (2.15) and the explicit results of [52] that satisfy perfectly the IR exponentiation in momentum space (2.4). However there is a mismatch for one term appearing at the lowest power of  $1/\epsilon$  and the lowest power of  $\log(q^2)$  accessible with the current data. At three-loop order the mismatch appears in the IR finite part, see (5.21): then a correction in the  $\mathcal{O}(\epsilon)$  part of the 3PM eikonal or the finite part of the three-loop amplitude can restore the agreement at three loops between the two exponentiations. However the tension resurfaces at four loops and higher in the terms  $\mathcal{O}(\epsilon^{4-\ell})$ , see (5.26). What is most puzzling is that such a mismatch indicates a breakdown of either the eikonal or the IR exponentiation. It may be that one has to restrict the comparison of the two results only to physical/IR finite observables. Understanding this point better is of course of great interest and would probably require to specify better the regime of validity of both formulas. The standard approach to amplitude calculations is to fix the kinematics, including the Mandelstam variables, and take the small  $\epsilon$  expansion

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<sup>16</sup>However it is interesting to notice that  $Re(\delta_2)$  seems to be universal, see the comment after (4.25).

to focus around  $D = 4$ . This implies that the IR regulator is the smallest scale in the problem. In the eikonal approach we kept  $\epsilon$  fixed (even when small) and then considered all values of the exchanged momentum  $|q|$ : actually the most important contributions to the large distance physics ( $b \gg R$ ) relevant to the Regge regime are those that are divergent as  $|q| \rightarrow 0$ . It would be interesting to understand whether the discrepancy mentioned above is related to the different kinematics where the two exponentiations are valid <sup>17</sup>. Clarifying this point may be relevant beyond the  $\mathcal{N} = 8$  case studied in this work, since now, even for the physically interesting case of the massive scattering in CGR, the focus is on 3PM and higher order corrections [45, 46] where such subtleties may play some role.

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## A Useful Fourier transforms to impact parameter space

In this Appendix we derive the Fourier transforms into impact parameter space that we have used in this paper. The basic starting formula is <sup>18</sup>:

$$\int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} e^{ibq/\hbar} \left(\frac{q^2}{\hbar^2}\right)^\nu = \frac{2^{2\nu}}{\pi^{1-\epsilon}} \frac{\Gamma(1+\nu-\epsilon)}{\Gamma(-\nu)(b^2)^{\nu+1-\epsilon}} ; \quad D-4 = -2\epsilon. \quad (\text{A.1})$$

It can be rewritten as follows:

$$\int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} e^{ibq/\hbar} \sum_{n=0}^{\infty} \frac{\nu^n}{(n+1)!} \log^{n+1} \left(\frac{q^2}{\hbar^2}\right) = -\frac{f(\nu)(\pi b^2)^\epsilon}{\pi b^2} \sum_{n=0}^{\infty} \frac{(-1)^n \nu^n}{n!} \log^n b^2, \quad (\text{A.2})$$

where

$$f(\nu) = 2^{2\nu} \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} \equiv \sum_{m=0}^{\infty} \frac{f^{(m)}}{m!} \nu^m. \quad (\text{A.3})$$

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<sup>17</sup>See ref. [72–75] for studies relating the exponentiation of infrared singularities to known properties of the Regge limit in a gauge theory context.

<sup>18</sup>For non integer values of  $D-2$  this is defined, as usual, via analytic continuation from all positive integer values of that same quantity.

The first few coefficients of the above sum are:

$$f^{(0)} = 1 \quad ; \quad f^{(1)} = \log 4 + 2\psi(1) \quad ; \quad f^{(2)} = (\log 4 + 2\psi(1))^2 \quad ; \quad \psi(1) = -\gamma_E. \quad (\text{A.4})$$

Inserting the expansion in Eq. (A.3) in Eq. (A.2) we get

$$\int \frac{d^{D-2}q}{(2\pi\hbar)^2} e^{iq/\hbar} \log^{n+1} \left( \frac{q^2}{\hbar^2} \right) = -(n+1) \frac{(\pi b^2)^\epsilon}{\pi b^2} \sum_{m=0}^n \binom{n}{m} f^{(m)} (-1)^{n-m} \log^{n-m} b^2. \quad (\text{A.5})$$

For  $n = 0$  we get

$$\int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} e^{iq/\hbar} \log \left( \frac{q^2}{\hbar^2} \right) = -\frac{(\pi b^2)^\epsilon}{\pi b^2}, \quad f^{(0)} = 1. \quad (\text{A.6})$$

For  $n = 1$  we get

$$\begin{aligned} \int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} e^{iq/\hbar} \log^2 \left( \frac{q^2}{\hbar^2} \right) &= -\frac{2(\pi b^2)^\epsilon}{\pi b^2} (-f^{(0)} \log b^2 + f^{(1)}) = \frac{2(\pi b^2)^\epsilon}{\pi b^2} \log \frac{b^2}{e^{f^{(1)}}} \\ &= \frac{2(\pi b^2)^\epsilon}{\pi b^2} \left( \log \frac{b^2}{4} - 2\psi(1) \right). \end{aligned} \quad (\text{A.7})$$

For  $n = 2$  we get

$$\begin{aligned} \int \frac{d^{D-2}q}{(2\pi\hbar)^2} e^{iq/\hbar} \log^3 \left( \frac{q^2}{\hbar^2} \right) &= -\frac{3(\pi b^2)^\epsilon}{\pi b^2} (f^{(0)} \log^2 b^2 - 2f^{(1)} \log b^2 + f^{(2)}) \\ &= -\frac{3(\pi b^2)^\epsilon}{\pi b^2} \left( \log^2 \frac{b^2}{e^{f^{(1)}}} - (f^{(1)})^2 + f^{(2)} \right) \\ &= -\frac{3(\pi b^2)^\epsilon}{\pi b^2} \log^2 \frac{b^2}{4e^{2\psi(1)}}. \end{aligned} \quad (\text{A.8})$$

In the main text we are also using the inverse Fourier transform (from  $b$  to  $q$ -space) which can be easily derived from the above results using the well known properties of the Fourier transform. As an example the analog of (B.1) reads:

$$\int d^{D-2}b e^{-ibq/\hbar} (b^2)^{-\nu} = \frac{\pi^{\frac{D-2}{2}}}{2^{2\nu+2-D}} \frac{\Gamma\left(\frac{D}{2} - 1 - \nu\right)}{\Gamma(\nu)} \left( \frac{q^2}{\hbar^2} \right)^{1+\nu-\frac{D}{2}} \quad (\text{A.9})$$

from which we can derive another useful relation

$$\begin{aligned} \int d^{D-2}b e^{-ibq/\hbar} (b^2)^{-\nu} \log b^2 &= \frac{\pi^{\frac{D-2}{2}}}{2^{2\nu+2-D}} \frac{\Gamma\left(\frac{D}{2} - 1 - \nu\right)}{\Gamma(\nu)} \left( \frac{q^2}{\hbar^2} \right)^{1+\nu-\frac{D}{2}} \\ &\quad \times \left[ \log \left( \frac{4\hbar^2}{q^2} \right) + \psi\left(\frac{D}{2} - 1 - \nu\right) + \psi(\nu) \right]. \end{aligned} \quad (\text{A.10})$$

## B Results of Henn and Mistlberger

In this appendix we write the eikonal limit of the three-loop  $\mathcal{N} = 8$  4-point amplitude recently derived in [52] up to order  $\epsilon^0$  in dimensional regularization. We write also the two-loop result up to order  $\epsilon^2$  included in the same paper. With respect to [52], we write the result in the  $s$ -forward channel, i.e. with  $s > 0$  and  $t, u < 0$  and, for simplicity, in the equations below we set the dimensional regularization scale to one  $\mu = 1$ . As mentioned in the main text, we focus only on the non-analytic terms as  $|t| = q^2 \rightarrow 0$  as they are the only ones yielding a long-range interaction in the impact parameter space and so are captured by the eikonal exponentiation (2.15). We organise the formulas by writing first the leading eikonal terms (proportional to  $s^\ell$  at  $\ell$  loops) and then the first subleading term. For each of the two contributions we order the various terms according to the power  $n$  of  $\log^n(q^2)$ .

At two-loop order we have

$$\begin{aligned}
\mathcal{A}^{(2)} = & s^2 \left\{ -\frac{\pi^2}{3} \epsilon^2 \log^4(q^2) + \frac{2\pi^2}{3} \epsilon \log^3(q^2) - \pi^2 \log^2(q^2) + \log(q^2) \left( \frac{\pi^2}{\epsilon} - 6\pi^2 \zeta_3 \epsilon^2 \right) \right\} \\
& + s^2 q^2 \left\{ \frac{2i\pi}{15} \epsilon^2 \log^5(q^2) + \log^4(q^2) \left[ \left( \frac{\pi^2}{3} + \frac{4i\pi}{3} \right) \epsilon^2 - \frac{1}{3} i\pi \epsilon \right] \right. \\
& + \log^3(q^2) \left[ \epsilon^2 \left( -\frac{7i\pi^3}{9} - \frac{4\pi^2}{3} - \frac{8i\pi}{3} - \frac{8i\pi}{3} \log s \right) + \left( -\frac{2\pi^2}{3} - \frac{4i\pi}{3} \right) \epsilon + \frac{2i\pi}{3} \right] \\
& + \log^2(q^2) \left[ \epsilon^2 (41i\pi \zeta_3 + 5i\pi^3) - \frac{i\pi}{\epsilon} + \epsilon \left( \frac{7i\pi^3}{6} + 2\pi^2 + 4i\pi + 4i\pi \log s \right) + \pi^2 \right] \\
& + \log(q^2) \left[ +\frac{i\pi}{\epsilon^2} + \frac{-\pi^2 + 2i\pi}{\epsilon} - \frac{7i\pi^3}{6} - 2\pi^2 - 4i\pi - 4i\pi \log s \right. \\
& \left. \left. - \epsilon (35i\pi \zeta_3 + 5i\pi^3) + \epsilon^2 \left( -86i\pi \zeta_3 - 12i\pi \zeta_3 \log s - \frac{31i\pi^5}{15} \right) \right] \right\} + \dots
\end{aligned} \tag{B.1}$$

and at three-loop order we have

$$\begin{aligned}
\mathcal{A}^{(3)} = & s^3 \left\{ -\frac{3i\pi^3}{4} \log^3(q^2) + \frac{3i\pi^3 \log^2(q^2)}{4\epsilon} - \frac{i\pi^3 \log(q^2)}{2\epsilon^2} \right\} \\
& + s^2 q^2 \left\{ -\frac{9\pi^2}{8} \log^4(q^2) + \log^3(q^2) \left[ \frac{5\pi^2}{4\epsilon} + \frac{3i\pi^3}{4} - \frac{3\pi^2}{4} + \frac{3}{4} \pi^2 \log s \right] \right. \\
& + \log^2(q^2) \left[ -\frac{\pi^2}{\epsilon^2} - \frac{3}{4} \frac{i\pi^3 + \pi^2 + \pi^2 \log s}{\epsilon} + \frac{11\pi^4}{8} - \frac{9i\pi^3}{4} + \frac{9\pi^2}{2} + \frac{9\pi^2}{2} \log s \right] \\
& + \log(q^2) \left[ \frac{\pi^2}{2\epsilon^3} + \frac{i\pi^3 + 3\pi^2 + \pi^2 \log s}{2\epsilon^2} \right. \\
& \left. \left. + \frac{\frac{3i\pi^3}{2} - 3\pi^2 - \frac{11\pi^4}{12} - 3\pi^2 \log s}{\epsilon} - \frac{35\pi^4}{6} - \frac{91\pi^2}{2} \zeta_3 \right] \right\} + \dots
\end{aligned} \tag{B.2}$$

Finally, we can derive the IR divergent part of the four-loop amplitude by using the exponentiation (2.4). For this it is sufficient to know the two-loop remainder function up

to order  $\epsilon$  and that of three-loop remainder function at order  $\mathcal{O}(\epsilon^0)$  and both these results are provided in the ancillary files of [52]. Once translated in our  $s$ -channel convention ( $s > 0, t, u < 0$ ), they read

$$\begin{aligned}
F^{(2)} = & \pi^2 s^2 3\epsilon \zeta_3 + \pi s q^2 \left\{ -\frac{5i}{12} \epsilon \log^4(q^2) + \left[ \frac{i}{3} + \epsilon \left( i \log s + \frac{\pi}{2} - i \right) \right] \log^3(q^2) \right. \\
& + \left[ -\frac{\pi}{2} + i - i \log s + \epsilon \left( -\frac{i \log^2 s}{2} + \log s \left( i - \frac{\pi}{2} \right) + \frac{5i\pi^2}{12} + i + \frac{\pi}{2} \right) \right] \log^2(q^2) \\
& + \left[ i \log^2 s + (\pi - 2i) \log s - \frac{2i\pi^2}{3} - \pi - 2i + \epsilon \left( -\frac{i \log^3 s}{3} + \left( i - \frac{\pi}{2} \right) \log^2 s \right. \right. \\
& \left. \left. + \left( \frac{i\pi^2}{2} + 2i + \pi \right) \log s - 33i\zeta(3) + \frac{\pi^3}{6} + 2i - \frac{7i\pi^2}{2} + \pi \right) \right] \log(q^2) \\
& + \frac{1}{6} \log s \left( (-2i \log s - 3\pi + 6i) \log s + 4i\pi^2 + 6\pi + 12i \right) + 4i\zeta(3) + \frac{\pi^3}{4} \\
& + \frac{2i\pi^2}{3} + \pi + 2i + \epsilon \left( \frac{\zeta_3}{2} (50i \log s + 19\pi + 50i) + \frac{1}{12} \log s \left( \log s \left( 3i \log^2 s \right. \right. \right. \\
& \left. \left. + 6(\pi - 2i) \log s - 36i + \pi(-18 - 11i\pi) \right) - 2\pi(18 + \pi(4\pi - 13i)) - 72i \right) \\
& \left. \left. + \frac{151i\pi^4}{180} + \frac{5\pi^3}{6} + \frac{13i\pi^2}{6} - 3\pi - 6i \right) \right\} + \dots
\end{aligned} \tag{B.3}$$

and

$$\begin{aligned}
F^{(3)} = & -\frac{2}{3} i\pi^3 s^3 \zeta(3) + \pi^2 s^2 q^2 \left\{ \frac{1}{12} \log^4(q^2) + \left( -\frac{1}{3} \log s + \frac{i\pi}{6} + \frac{1}{3} \right) \log^3(q^2) \right. \\
& + \left( \frac{\log^2 s}{2} - \left( 1 + \frac{i\pi}{2} \right) \log s - \frac{\pi^2}{3} + \frac{i\pi}{2} - 1 \right) \log^2(q^2) \\
& + \left( -\frac{\log^3 s}{3} + \left( 1 + \frac{i\pi}{2} \right) \log^2 s + \left( 2 + \frac{2\pi^2}{3} - i\pi \right) \log s - 4\zeta(3) \right) \log(q^2) \\
& + \frac{\log^4 s}{12} - \frac{1}{6} i(\pi - 2i) \log^3 s - \frac{1}{6} (6 - 3i\pi + 2\pi^2) \log^2 s - i\pi(\zeta(3) - 1) \\
& \left. + 8\zeta_3 + \frac{5\zeta_5 - 2\zeta_3^2}{\pi^2} + \frac{2\pi^2}{3} - \frac{i\pi^3}{4} - \frac{841\pi^4}{5670} - 2 \right\} + \dots
\end{aligned} \tag{B.4}$$

Then the non-analytic part of  $\mathcal{A}_4$  reads

$$\begin{aligned}
\mathcal{A}^{(4)} = & s^4 \pi^4 \left\{ -\frac{\log q^2}{6\epsilon^3} + \frac{\log^2 q^2}{3\epsilon^2} - \frac{4 \log^3 q^2}{9\epsilon} \right\} \\
& + s^3 q^2 \pi^3 \left\{ \frac{10i}{9\epsilon} \log^4 q^2 + \left[ -\frac{8i}{9\epsilon^2} + \frac{4(\pi - 2i \log s)}{9\epsilon} \right] \log^3 q^2 \right. \\
& + \left[ \frac{i}{2\epsilon^3} - \frac{\pi - 2i - 2i \log s}{3\epsilon^2} - \frac{16i \log s + 5i\pi^2 + 8\pi + 16i}{6\epsilon} \right] \log^2 q^2 \\
& \left. + \left[ \frac{-i}{6\epsilon^4} + \frac{\pi - 4i - 2i \log s}{6\epsilon^3} + \frac{16i \log s + 5i\pi^2 + 8\pi + 16i}{12\epsilon^2} + \frac{173i\zeta(3) + 21i\pi^2}{6\epsilon} \right] \log q^2 \right\}. \tag{B.5}
\end{aligned}$$

## References

- [1] **Virgo, LIGO Scientific** Collaboration, B. P. Abbott *et al.*, “Observation of Gravitational Waves from a Binary Black Hole Merger,” *Phys. Rev. Lett.* **116** (2016) no. 6, 061102, [arXiv:1602.03837 \[gr-qc\]](#).
- [2] **Virgo, LIGO Scientific** Collaboration, B. P. Abbott *et al.*, “GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence,” *Phys. Rev. Lett.* **116** (2016) no. 24, 241103, [arXiv:1606.04855 \[gr-qc\]](#).
- [3] **VIRGO, LIGO Scientific** Collaboration, B. P. Abbott *et al.*, “GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2,” *Phys. Rev. Lett.* **118** (2017) no. 22, 221101, [arXiv:1706.01812 \[gr-qc\]](#).
- [4] J. Lange *et al.*, “Parameter estimation method that directly compares gravitational wave observations to numerical relativity,” *Phys. Rev.* **D96** (2017) no. 10, 104041, [arXiv:1705.09833 \[gr-qc\]](#).
- [5] J. Blackman, S. E. Field, M. A. Scheel, C. R. Galley, C. D. Ott, M. Boyle, L. E. Kidder, H. P. Pfeiffer, and B. Szilágyi, “Numerical relativity waveform surrogate model for generically precessing binary black hole mergers,” *Phys. Rev.* **D96** (2017) no. 2, 024058, [arXiv:1705.07089 \[gr-qc\]](#).
- [6] A. Buonanno and T. Damour, “Effective one-body approach to general relativistic two-body dynamics,” *Phys. Rev.* **D59** (1999) 084006, [arXiv:gr-qc/9811091 \[gr-qc\]](#).
- [7] A. Buonanno and T. Damour, “Transition from inspiral to plunge in binary black hole coalescences,” *Phys. Rev.* **D62** (2000) 064015, [arXiv:gr-qc/0001013](#).
- [8] T. Damour, “Coalescence of two spinning black holes: An effective one-body approach,” *Phys. Rev.* **D64** (2001) 124013, [arXiv:gr-qc/0103018](#).



- [9] T. Damour, P. Jaranowski, and G. Schafer, “On the determination of the last stable orbit for circular general relativistic binaries at the third postNewtonian approximation,” *Phys. Rev.* **D62** (2000) 084011, [arXiv:gr-qc/0005034](#) [gr-qc].
- [10] G. ’t Hooft, “Graviton Dominance in Ultrahigh-Energy Scattering,” *Phys. Lett.* **B198** (1987) 61–63.
- [11] P. D. D’Eath, “High Speed Black Hole Encounters and Gravitational Radiation,” *Phys. Rev.* **D18** (1978) 990.
- [12] S. J. Kovacs and K. S. Thorne, “The Generation of Gravitational Waves. 3. Derivation of Bremsstrahlung Formulas,” *Astrophys. J.* **217** (1977) 252–280.
- [13] S. J. Kovacs and K. S. Thorne, “The Generation of Gravitational Waves. 4. Bremsstrahlung,” *Astrophys. J.* **224** (1978) 62–85.
- [14] D. Amati, M. Ciafaloni, and G. Veneziano, “Superstring Collisions at Planckian Energies,” *Phys. Lett.* **B197** (1987) 81.
- [15] I. J. Muzinich and M. Soldate, “High-Energy Unitarity of Gravitation and Strings,” *Phys. Rev.* **D37** (1988) 359.
- [16] D. Amati, M. Ciafaloni, and G. Veneziano, “Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions,” *Int. J. Mod. Phys.* **A3** (1988) 1615–1661.
- [17] D. Amati, M. Ciafaloni, and G. Veneziano, “Higher Order Gravitational Deflection and Soft Bremsstrahlung in Planckian Energy Superstring Collisions,” *Nucl. Phys.* **B347** (1990) 550–580.
- [18] H. L. Verlinde and E. P. Verlinde, “Scattering at Planckian energies,” *Nucl. Phys.* **B371** (1992) 246–268, [arXiv:hep-th/9110017](#) [hep-th].
- [19] D. Amati, M. Ciafaloni, and G. Veneziano, “Planckian scattering beyond the semiclassical approximation,” *Phys. Lett.* **B289** (1992) 87–91.
- [20] A. Bellini, M. Ademollo, and M. Ciafaloni, “Superstring one loop and gravitino contributions to Planckian scattering,” *Nucl. Phys.* **B393** (1993) 79–94, [arXiv:hep-th/9207113](#) [hep-th].
- [21] D. Amati, M. Ciafaloni, and G. Veneziano, “Effective action and all order gravitational eikonal at Planckian energies,” *Nucl. Phys.* **B403** (1993) 707–724.
- [22] S. B. Giddings, M. Schmidt-Sommerfeld, and J. R. Andersen, “High energy scattering in gravity and supergravity,” *Phys. Rev.* **D82** (2010) 104022, [arXiv:1005.5408](#) [hep-th].

- [23] D. N. Kabat and M. Ortiz, “Eikonal quantum gravity and Planckian scattering,” *Nucl. Phys.* **B388** (1992) 570–592, [arXiv:hep-th/9203082](#) [hep-th].
- [24] A. K. Collado, P. Di Vecchia, R. Russo, and S. Thomas, “The subleading eikonal in supergravity theories,” *JHEP* **10** (2018) 038, [arXiv:1807.04588](#) [hep-th].
- [25] A. Koemans Collado and S. Thomas, “Eikonal Scattering in Kaluza-Klein Gravity,” *JHEP* **04** (2019) 171, [arXiv:1901.05869](#) [hep-th].
- [26] G. D’Appollonio, P. Di Vecchia, R. Russo, and G. Veneziano, “High-energy string-brane scattering: Leading eikonal and beyond,” *JHEP* **11** (2010) 100, [arXiv:1008.4773](#) [hep-th].
- [27] G. D’Appollonio, P. Vecchia, R. Russo, and G. Veneziano, “Microscopic unitary description of tidal excitations in high-energy string-brane collisions,” *JHEP* **11** (2013) 126, [arXiv:1310.1254](#) [hep-th].
- [28] D. Amati, M. Ciafaloni, and G. Veneziano, “Can Space-Time Be Probed Below the String Size?,” *Phys. Lett.* **B216** (1989) 41.
- [29] G. D’Appollonio, P. Di Vecchia, R. Russo, and G. Veneziano, “Regge behavior saves String Theory from causality violations,” *JHEP* **05** (2015) 144, [arXiv:1502.01254](#) [hep-th].
- [30] G. Veneziano, “High-energy collisions of particles, strings, and branes,” *Les Houches Lect. Notes* **97** (2015) 397–415.
- [31] A. Gruzinov and G. Veneziano, “Gravitational Radiation from Massless Particle Collisions,” *Class. Quant. Grav.* **33** (2016) no. 12, 125012, [arXiv:1409.4555](#) [gr-qc].
- [32] M. Ciafaloni, D. Colferai, F. Coradeschi, and G. Veneziano, “Unified limiting form of graviton radiation at extreme energies,” *Phys. Rev.* **D93** (2016) no. 4, 044052, [arXiv:1512.00281](#) [hep-th].
- [33] A. Laddha and A. Sen, “Observational Signature of the Logarithmic Terms in the Soft Graviton Theorem,” *Phys. Rev.* **D100** (2019) no. 2, 024009, [arXiv:1806.01872](#) [hep-th].
- [34] B. Sahoo and A. Sen, “Classical and Quantum Results on Logarithmic Terms in the Soft Theorem in Four Dimensions,” *JHEP* **02** (2019) 086, [arXiv:1808.03288](#) [hep-th].
- [35] M. Ciafaloni, D. Colferai, and G. Veneziano, “Infrared features of gravitational scattering and radiation in the eikonal approach,” *Phys. Rev.* **D99** (2019) no. 6, 066008, [arXiv:1812.08137](#) [hep-th].

- [36] A. Addazi, M. Bianchi, and G. Veneziano, “Soft gravitational radiation from ultra-relativistic collisions at sub- and sub-sub-leading order,” *JHEP* **05** (2019) 050, [arXiv:1901.10986 \[hep-th\]](#).
- [37] S. Melville, S. G. Naculich, H. J. Schnitzer, and C. D. White, “Wilson line approach to gravity in the high energy limit,” *Phys. Rev.* **D89** (2014) no. 2, 025009, [arXiv:1306.6019 \[hep-th\]](#).
- [38] A. Luna, S. Melville, S. G. Naculich, and C. D. White, “Next-to-soft corrections to high energy scattering in QCD and gravity,” *JHEP* **01** (2017) 052, [arXiv:1611.02172 \[hep-th\]](#).
- [39] R. Akhouri, R. Saotome, and G. Sterman, “High Energy Scattering in Perturbative Quantum Gravity at Next to Leading Power,” [arXiv:1308.5204 \[hep-th\]](#).
- [40] N. E. J. Bjerrum-Bohr, J. F. Donoghue, B. R. Holstein, L. Plante, and P. Vanhove, “Light-like Scattering in Quantum Gravity,” *JHEP* **11** (2016) 117, [arXiv:1609.07477 \[hep-th\]](#).
- [41] N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté, and P. Vanhove, “General Relativity from Scattering Amplitudes,” *Phys. Rev. Lett.* **121** (2018) no. 17, 171601, [arXiv:1806.04920 \[hep-th\]](#).
- [42] N. E. J. Bjerrum-Bohr, A. Cristofoli, and P. H. Damgaard, “Post-Minkowskian Scattering Angle in Einstein Gravity,” [arXiv:1910.09366 \[hep-th\]](#).
- [43] G. Kälin and R. A. Porto, “From Boundary Data to Bound States,” [arXiv:1910.03008 \[hep-th\]](#).
- [44] T. Damour, “High-energy gravitational scattering and the general relativistic two-body problem,” *Phys. Rev.* **D97** (2018) no. 4, 044038, [arXiv:1710.10599 \[gr-qc\]](#).
- [45] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, “Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order,” *Phys. Rev. Lett.* **122** (2019) no. 20, 201603, [arXiv:1901.04424 \[hep-th\]](#).
- [46] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, “Black Hole Binary Dynamics from the Double Copy and Effective Theory,” [arXiv:1908.01493 \[hep-th\]](#).
- [47] A. Antonelli, A. Buonanno, J. Steinhoff, M. van de Meent, and J. Vines, “Energetics of two-body Hamiltonians in post-Minkowskian gravity,” *Phys. Rev.* **D99** (2019) no. 10, 104004, [arXiv:1901.07102 \[gr-qc\]](#).

- [48] D. Amati, M. Ciafaloni, and G. Veneziano, “Towards an S-matrix Description of Gravitational Collapse,” *JHEP* **02** (2008) 049, [arXiv:0712.1209 \[hep-th\]](#).
- [49] M. Ciafaloni, D. Colferai, and G. Falcioni, “Unitarity alternatives in the reduced-action model for gravitational collapse,” *JHEP* **1109** (2011) 044, [arXiv:1106.5628 \[hep-th\]](#).
- [50] A. Strominger and C. Vafa, “Microscopic origin of the Bekenstein-Hawking entropy,” *Phys. Lett.* **B379** (1996) 99–104, [arXiv:hep-th/9601029 \[hep-th\]](#).
- [51] P. Di Vecchia, A. Luna, S. G. Naculich, R. Russo, G. Veneziano, and C. D. White, “A tale of two exponentiations in  $\mathcal{N} = 8$  supergravity,” *Phys. Lett.* **B798** (2019) 134927, [arXiv:1908.05603 \[hep-th\]](#).
- [52] J. M. Henn and B. Mistlberger, “Four-graviton scattering to three loops in  $\mathcal{N} = 8$  supergravity,” *JHEP* **05** (2019) 023, [arXiv:1902.07221 \[hep-th\]](#).
- [53] S. Caron-Huot and Z. Zahraee, “Integrability of Black Hole Orbits in Maximal Supergravity,” *JHEP* **07** (2019) 179, [arXiv:1810.04694 \[hep-th\]](#).
- [54] Z. Bern, J. J. Carrasco, W.-M. Chen, A. Edison, H. Johansson, J. Parra-Martinez, R. Roiban, and M. Zeng, “Ultraviolet Properties of  $\mathcal{N} = 8$  Supergravity at Five Loops,” *Phys. Rev.* **D98** (2018) no. 8, 086021, [arXiv:1804.09311 \[hep-th\]](#).
- [55] C. Boucher-Veronneau and L. J. Dixon, “ $N \geq 4$  Supergravity Amplitudes from Gauge Theory at Two Loops,” *JHEP* **12** (2011) 046, [arXiv:1110.1132 \[hep-th\]](#).
- [56] S. G. Naculich, H. Nastase, and H. J. Schnitzer, “Two-loop graviton scattering relation and IR behavior in  $N=8$  supergravity,” *Nucl. Phys.* **B805** (2008) 40–58, [arXiv:0805.2347 \[hep-th\]](#).
- [57] A. Brandhuber, P. Heslop, A. Nasti, B. Spence, and G. Travaglini, “Four-point Amplitudes in  $N=8$  Supergravity and Wilson Loops,” *Nucl. Phys.* **B807** (2009) 290–314, [arXiv:0805.2763 \[hep-th\]](#).
- [58] M. Soldate, “Partial Wave Unitarity and Closed String Amplitudes,” *Phys. Lett.* **B186** (1987) 321–327.
- [59] S. B. Giddings and R. A. Porto, “The Gravitational S-matrix,” *Phys. Rev.* **D81** (2010) 025002, [arXiv:0908.0004 \[hep-th\]](#).
- [60] D. A. Kosower, B. Maybee, and D. O’Connell, “Amplitudes, Observables, and Classical Scattering,” *JHEP* **02** (2019) 137, [arXiv:1811.10950 \[hep-th\]](#).
- [61] S. Weinberg, *Lectures on Quantum Mechanics*. Cambridge University Press, 2 ed., 2015.

- [62] Z. Bern, L. J. Dixon, and D. A. Kosower, “Dimensionally regulated pentagon integrals,” *Nucl. Phys.* **B412** (1994) 751–816, [arXiv:hep-ph/9306240](#) [[hep-ph](#)].
- [63] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein, and J. S. Rozowsky, “On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences,” *Nucl. Phys.* **B530** (1998) 401–456, [arXiv:hep-th/9802162](#) [[hep-th](#)].
- [64] A. Koemans Collado, P. Di Vecchia, and R. Russo, “Revisiting the 2PM eikonal and the dynamics of binary black holes,” [arXiv:1904.02667](#) [[hep-th](#)].
- [65] S. Weinberg, “Infrared photons and gravitons,” *Phys. Rev.* **140** (1965) B516–B524.
- [66] S. G. Naculich and H. J. Schnitzer, “Eikonal methods applied to gravitational scattering amplitudes,” *JHEP* **05** (2011) 087, [arXiv:1101.1524](#) [[hep-th](#)].
- [67] C. D. White, “Factorization Properties of Soft Graviton Amplitudes,” *JHEP* **05** (2011) 060, [arXiv:1103.2981](#) [[hep-th](#)].
- [68] R. Akhouri, R. Saotome, and G. Sterman, “Collinear and Soft Divergences in Perturbative Quantum Gravity,” *Phys. Rev.* **D84** (2011) 104040, [arXiv:1109.0270](#) [[hep-th](#)].
- [69] D. C. Dunbar and P. S. Norridge, “Calculation of graviton scattering amplitudes using string based methods,” *Nucl. Phys.* **B433** (1995) 181–208, [arXiv:hep-th/9408014](#) [[hep-th](#)].
- [70] J. Bartels, L. N. Lipatov, and A. Sabio Vera, “Double-logarithms in Einstein-Hilbert gravity and supergravity,” *JHEP* **07** (2014) 056, [arXiv:1208.3423](#) [[hep-th](#)].
- [71] A. Sabio Vera, “Double-logarithms in  $\mathcal{N} = 8$  supergravity: impact parameter description & mapping to 1-rooted ribbon graphs,” *JHEP* **07** (2019) 080, [arXiv:1904.13372](#) [[hep-th](#)].
- [72] V. Del Duca, C. Duhr, E. Gardi, L. Magnea, and C. D. White, “An infrared approach to Reggeization,” *Phys. Rev.* **D85** (2012) 071104, [arXiv:1108.5947](#) [[hep-ph](#)].
- [73] V. Del Duca, C. Duhr, E. Gardi, L. Magnea, and C. D. White, “The Infrared structure of gauge theory amplitudes in the high-energy limit,” *JHEP* **12** (2011) 021, [arXiv:1109.3581](#) [[hep-ph](#)].
- [74] S. Caron-Huot, E. Gardi, J. Reichel, and L. Vernazza, “Infrared singularities of QCD scattering amplitudes in the Regge limit to all orders,” *JHEP* **03** (2018) 098, [arXiv:1711.04850](#) [[hep-ph](#)].
- [75] S. Caron-Huot, E. Gardi, and L. Vernazza, “Two-parton scattering in the high-energy limit,” *JHEP* **06** (2017) 016, [arXiv:1701.05241](#) [[hep-ph](#)].