3 NNLO corrections in 4 dimensions

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3.1 Introduction

Currently, four-dimensional techniques applied to higher order calculations are under active investigation [1–8]. The main motivation for this is the need of simplifying perturbative calculations necessary to cope with the precision requirements of the future LHC and FCC experiments.

In this contribution, I review the FDR approach [9] to the computation of NNLO corrections in 4 dimensions. In particular, I describe how fully inclusive NNLO final state quark-pair corrections [10]

$$\sigma^{NNLO} = \sigma_B + \sigma_V + \sigma_R \qquad \text{with} \qquad \begin{cases} \sigma_B = \int d\Phi_n \sum_{\text{spin}} |A_n^{(0)}|^2 \\ \sigma_V = \int d\Phi_n \sum_{\text{spin}} \left\{ A_n^{(2)} (A_n^{(0)})^* + A_n^{(0)} (A_n^{(2)})^* \right\} \\ \sigma_R = \int d\Phi_{n+2} \sum_{\text{spin}} \left\{ A_{n+2}^{(0)} (A_{n+2}^{(0)})^* \right\} \end{cases} \tag{3.73}$$

are computed in FDR by directly enforcing gauge invariance and unitarity in the definition of the regularized UV and IR divergent integrals. The IR divergent parts of the amplitudes are depicted in Fig. C.2 and $d\Phi_m := \delta\left(P - \sum_{i=1}^m p_i\right) \prod_{i=1}^m d^4 p_i \delta_+(p_i^2)$.

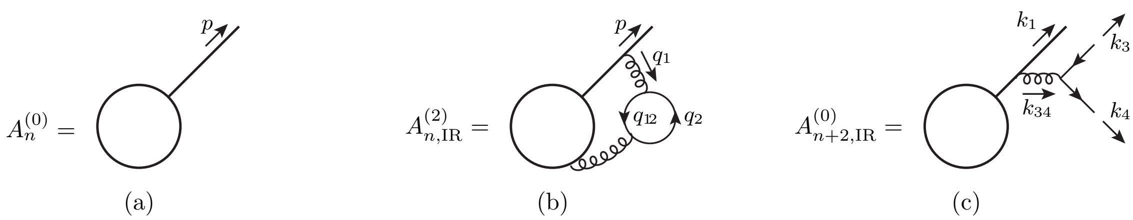

Fig. C.2: The lowest order amplitude (a), the IR divergent final-state virtual quark-pair correction (b) and the IR divergent real component (c). The empty circle stands for the emission of $n-1$ particles. Additional IR finite corrections are created if the gluons with momenta q_1 and k_{34} are emitted by off-shell particles contained in the empty circle.

In Sec. 3.2, I recall the basics of FDR. The next Sections deal with its use in the contest of the calculation of σ^{NNLO} in Eq. (3.73).

3.2 FDR integration and loop integrals

The main idea of FDR can be sketched out with the help of a simple one-dimensional example [11]. More details can be found in the relevant literature [9, 10, 12–16]. Let's assume one needs to define the UV divergent integral

$$I = \lim_{\Lambda \to \infty} \int_0^\Lambda dx \frac{x}{x+M}, \tag{3.74}$$

where M stands for a physical energy scale. FDR identifies the UV divergent pieces in terms of integrands which do not depend on M, the so called FDR vacua, and separates them by rewriting

$$\frac{x}{x+M} = 1 - \frac{M}{x} + \frac{M^2}{x(x+M)}. \tag{3.75}$$

The first term in the r.h.s. of Eq. (3.75) is the vacuum responsible for the linear $\mathcal{O}(\Lambda)$ UV divergence of I and $1/x$ generates its $\ln\Lambda$ behavior. By *definition* of FDR integration, both divergent contributions need to be subtracted from Eq. (3.74). The subtraction of the $\mathcal{O}(\Lambda)$ piece is performed over the full integration domain $[0,\Lambda]$, while the logarithmic divergence is removed over the interval $[\mu_R,\Lambda]$ only. The arbitrary separation scale $\mu_R \neq 0$ is needed to keep a-dimensional and finite the arguments of the logarithms appearing in the subtracted and finite parts. Thus

$$I_{\rm FDR} := I - \lim_{\Lambda\to\infty}\left(\int_0^\Lambda dx - \int_{\mu_R}^\Lambda dx\frac{M}{x}\right) = M\ln\frac{M}{\mu_R}. \tag{3.76}$$

The advantage of the definition in Eq. (3.76) is twofold:

– the UV cutoff Λ is traded for μ_R, which is interpreted, right away, as the renormalization scale;
– other than logarithmic UV divergences never contribute.

The use of Eq. (3.76) is inconvenient in practical calculations due to the explicit appearance of μ_R in the integration interval. An equivalent definition is obtained by adding an auxiliary unphysical scale μ to x,

$$x \to \bar{x} := x + \mu, \tag{3.77}$$

and introducing an integral operator $\int_0^\infty [dx]$ defined in such a way that it annihilates the FDR vacua before integration. Thus

$$I_{\rm FDR} = \int_0^\infty [dx]\frac{\bar{x}}{\bar{x}+M} = \int_0^\infty [dx]\left(1 - \frac{M}{\bar{x}} + \frac{M^2}{\bar{x}(\bar{x}+M)}\right) := M^2\lim_{\mu\to 0}\int_0^\infty dx\frac{1}{\bar{x}(\bar{x}+M)}\bigg|_{\mu=\mu_R} \tag{3.78}$$

where $\mu \to 0$ is an asymptotic limit. Note that, in order to keep the structure of the subtracted terms as in Eq. (3.75), the replacement $x \to \bar{x}$ must be performed in *both numerator and denominator* of the integrated function.

This strategy can be extended to more dimensions and to integrands which are rational functions of the integration variables, as is the case of multi-loop integrals. For instance, typical two-loop integrals contributing to $\sigma_V(\gamma^* \to jets)$ and $\sigma_V(H \to b\bar{b} + jets)$ are

$$K_1 := \int [d^4q_1][d^4q_2]\frac{1}{\bar{q}_1^2\bar{D}_1\bar{D}_2\bar{q}_2^2\bar{q}_{12}^2}, \qquad K_2^{\rho\sigma\alpha\beta} := \int [d^4q_1][d^4q_2]\frac{q_2^\rho q_2^\sigma q_1^\alpha q_1^\beta}{\bar{q}_1^4\bar{D}_1\bar{D}_2\bar{q}_2^2\bar{q}_{12}^2}, \tag{3.79}$$

where $q_{12} := q_1 + q_2$, $\bar{D}_{1,2} = \bar{q}_1^2 + 2(q_1\cdot p_{1,2})$, $p_{1,2}^2 = 0$, and $\bar{q}_i^2 := q_i^2 - \mu^2$ $(i = 1,2,12)$, in the same spirit of Eq. (3.77).

FDR integration keeps shift invariance in any of the loop integration variables and the possibility of cancelling reconstructed denominators, e.g.

$$\int [d^4q_1][d^4q_2]\frac{\bar{q}_1^2}{\bar{q}_1^4\bar{D}_1\bar{D}_2\bar{q}_2^2\bar{q}_{12}^2} = K_1. \tag{3.80}$$

Since, instead, $\int[d^4q_1][d^4q_2]\frac{q_1^2}{\bar{q}_1^4\bar{D}_1\bar{D}_2\bar{q}_2^2\bar{q}_{12}^2} \neq K_1$, this last property is maintained only if the replacement $q_i^2 \to \bar{q}_i^2$ is also performed in the numerator of the loop integrals whenever q_i^2 is generated by Feynman rules. This is called *global prescription* (GP), often denoted by $q_i^2 \stackrel{\text{GP}}{\to} \bar{q}_i^2$.

GP and shift invariance guarantee results which do not depend on the chosen gauge [12, 14]. Nevertheless, unitarity should also be maintained. This requires that any given UV divergent sub-diagram produce the same result when computed/manipulated separately or when embedded in the full diagram. Such a requirement is called *sub-integration consistency* (SIC) [15]. Enforcing SIC in the presence of IR divergent integrals, such as those in Eq. (3.79), needs extra care. In fact, the IR treatments of σ_V and σ_R should match with each other. In the next Sections I describe how this is achieved in the computation of the observable in Eq. (3.73).

3.3 Keeping unitarity in the virtual component

Any integral contributing to σ_V has the form

$$I_V = \int[d^4q_1][d^4q_2]\frac{N_V}{\bar{D}\bar{q}_2^2\bar{q}_{12}^2}, \tag{3.81}$$

where $\bar{D}$ collects all q_2-independent propagators and N_V is the numerator of the corresponding Feynman diagram. I_V can be sub-divergent or globally divergent for large values of the integration momenta. For example, K_1 in Eq. (3.79) only diverges when $q_2 \to \infty$, while K_2 also when $q_{1,2} \to \infty$. This means that FDR prescribes the subtraction of a *global vacuum* (GV) involving both integration variables in K_2, while the *sub-vacuum* (SV) developed when $q_2 \to \infty$ should be removed from both K_1 and K_2. In addition, IR infinities are generated by the on-shell conditions $p_{1,2}^2 = 0$. Even though IR divergences are automatically regulated when barring the loop denominators, a careful SIC preserving treatment is necessary in order not to spoil unitarity. Since the only possible UV sub-divergence is produced by the quark loop in Fig. C.2-(b), this is accomplished as follows [10]:

- one does not apply GP to the contractions $g_{\rho\sigma}q_2^\rho q_2^\sigma$ when $g_{\rho\sigma}$ refers to indices external to the UV divergent sub-diagram;
- one replaces back everywhere $\bar{q}_1^2 \to q_1^2$ *after* GV subtraction.

The external indices entering the calculation of σ_V in Eq. (3.73) are denoted by $\hat{\rho}$ and $\hat{\sigma}$ in Fig. C.3-(a,b). Using this convention, one can rephrase the first rule as follows: $g_{\rho\sigma}q_2^\rho q_2^\sigma =$

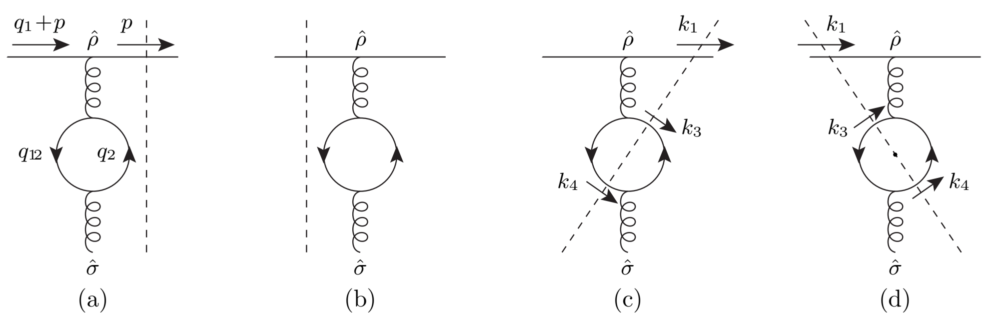

Fig. C.3: Virtual and real cuts contributing to the IR divergent parts of σ_V (a,b) and σ_R (c,d).

$q_2^2 \overset{\rm GP}{\to} \bar q_2^2$, but $g_{\hat\rho\hat\sigma} q_2^\rho q_2^\sigma := \hat q_2^2 \overset{\rm GP}{\to} q_2^2$, which gives, for instance,

$$\begin{aligned} g_{\rho\sigma} K_2^{\rho\sigma\alpha\beta} &\overset{\rm GP}{\to} \bar K_2^{\alpha\beta} = \int [d^4 q_1] \frac{q_1^\alpha q_1^\beta}{\bar q_1^4 \bar D_1 \bar D_2} \int [d^4 q_2] \frac{1}{\bar q_{12}^2} = 0, \text{ but} \\ g_{\hat\rho\hat\sigma} K_2^{\rho\sigma\alpha\beta} &\overset{\rm GP}{\to} \hat K_2^{\alpha\beta} = \int [d^4 q_1][d^4 q_2] \frac{q_2^2 q_1^\alpha q_1^\beta}{\bar q_1^4 \bar D_1 \bar D_2 \bar q_2^2 \bar q_{12}^2} \neq 0, \end{aligned} \tag{3.82}$$

where $\bar K_2^{\alpha\beta}$ vanishes because the shift $q_2 \to q_2 - q_1$ makes it proportional to the sub-vacuum $1/\bar q_2^2$, which is annihilated by the $\int [d^4 q_2]$ operator. It can be shown [10, 15] that integrals such as $\hat K_2^{\alpha\beta}$ generate the unitarity restoring logarithms missed by $\bar K_2^{\alpha\beta}$.

As for the second rule, it states that a GV subtraction is needed first. In the case of $\hat K_2^{\alpha\beta}$, this is achieved by rewriting $\frac{1}{\bar D_1} = \frac{1}{\bar q_1^2} - \frac{2(q_1\cdot p_1)}{\bar D_1 \bar q_1^2}$. The first term gives a scaleless integral, annihilated by $\int [d^4 q_1][d^4 q_2]$, so that

$$\hat K_2^{\alpha\beta} = -2 \int [d^4 q_1][d^4 q_2] \frac{(q_1\cdot p_1) q_2^2 q_1^\alpha q_1^\beta}{\bar q_1^6 \bar D_1 \bar D_2 \bar q_2^2 \bar q_{12}^2}, \tag{3.83}$$

which is now only sub-divergent when $q_2 \to \infty$, as is K_1 in Eq. (3.79). After that, the replacement $\bar q_1^2 \to q_1^2$ produces

$$K_1 \to \tilde K_1 = \int d^4 q_1 [d^4 q_2] \frac{1}{q_1^2 D_1 D_2 \bar q_2^2 \bar q_{12}^2}, \quad \hat K_2^{\alpha\beta} \to \tilde K_2^{\alpha\beta} = -2 \int d^4 q_1 [d^4 q_2] \frac{(q_1\cdot p_1) q_2^2 q_1^\alpha q_1^\beta}{q_1^6 D_1 D_2 \bar q_2^2 \bar q_{12}^2}. \tag{3.84}$$

All two-loop integrals I_V in Eq. (3.81) should be treated in this way. In the case of the N_F part of $\sigma_V(\gamma^* \to jets)$ and $\sigma_V(H \to b\bar b + jets)$ this produces three master integrals, which can be computed as described in Appendix D of [10].

After loop integration, σ_V contains logarithms of μ^2 of both UV and IR origin. The former should be replaced by logarithms of μ_R^2, as dictated by Eq. (3.78), while the latter compensate the IR behavior of σ_R. To disentangle the two cases, it is convenient to renormalize σ_V first. This means expressing the bare strong coupling constant $a^0 := \alpha_S^0/4\pi$ and the bare bottom Yukawa coupling y_b^0 in terms of $a := \alpha_S^{\overline{\rm MS}}(s)/4\pi$ and y_b extracted from the the bottom pole mass m_b. The relevant relations in terms of $L := \ln \mu^2/(p_1 - p_2)^2$ and $L'' := \ln \mu^2/m_b^2$ are as follows [10]

$$a^0 = a\left(1 + a\delta_a^{(1)}\right), \quad y_b^0 = y_b\left(1 + a\delta_y^{(1)} + a^2\left(\delta_y^{(2)} + \delta_a^{(1)}\delta_y^{(1)}\right)\right), \tag{3.85}$$

with

$$\delta_a^{(1)} = \frac{2}{3} N_F L, \quad \delta_y^{(1)} = -C_F\left(3L'' + 5\right), \quad \delta_y^{(2)} = C_F N_F\left(L''^2 + \frac{13}{3}L'' + \frac{2}{3}\pi^2 + \frac{151}{18}\right). \tag{3.86}$$

After renormalization, the remaining μ^2s are the IR ones.

3.4 Keeping unitarity in the real component

The integrands in σ_R of Eq. (3.73) are represented in Fig. C.3-(c,d). They are of the form

$$J_R = \frac{N_R}{S s_{34}^\alpha s_{134}^\beta}, \quad s_{i\ldots j} := (k_i + \ldots + k_j)^2, \ 0 \le \alpha, \beta \le 2, \tag{3.87}$$

where N_R is the numerator of the amplitude squared and S collects the remaining propagators. Depending on the value of α and β, J_R becomes infrared divergent when integrated over Φ_{n+2}. These IR singularities must be regulated consistently with the SIC preserving treatment of σ_V described in Sec. 3.3.

The changes $q_2^2 \overset{\text{GP}}{\to} \bar{q}_2^2$ and $q_{12}^2 \overset{\text{GP}}{\to} \bar{q}_{12}^2$ in the virtual cuts of Fig. C.3-(a,b) imply the Cutkosky relation

$$\frac{1}{(\bar{q}_2^2 + i0^+)(\bar{q}_{12}^2 + i0^+)} \leftrightarrow \left(\frac{2\pi}{i}\right)^2 \delta_+(\bar{k}_3^2)\delta_+(\bar{k}_4^2), \tag{3.88}$$

with $\bar{k}_{3,4}^2 := k_{3,4}^2 - \mu^2$. Hence, one replaces in Eq. (3.73) $\Phi_{n+2} \to \tilde{\Phi}_{n+2}$, where the phase-space $\tilde{\Phi}_{n+2}$ is such that $k_3^2 = k_4^2 = \mu^2$ and $k_i^2 = 0$ when $i \neq 3, 4$. In [10] it is proven that SV subtraction in σ_V does not alter Eq. (3.88). Analogously, the correspondence between cuts (a) and (d)

$$\frac{1}{(q_1+p)^2 + i0^+} \leftrightarrow \frac{2\pi}{i}\delta_+(k_1^2) \tag{3.89}$$

is not altered by GV subtraction. Finally, k_3^2, k_4^2 and $(k_3+k_4)^2 = s_{34}$ in N_R of Eq. (3.87) should be treated using the same prescriptions imposed on q_2^2, q_{12}^2 and q_1^2 in N_V of Eq. (3.81), respectively. This means replacing

$$k_{3,4}^2 \to \bar{k}_{3,4}^2 = 0, \quad (k_3 \cdot k_4) = \frac{1}{2}\left(s_{34} - k_3^2 - k_4^2\right) \to \frac{1}{2}(s_{34} - \bar{k}_3^2 - \bar{k}_4^2) = \frac{1}{2}s_{34}, \tag{3.90}$$

where the last equalities are induced by the delta functions in Eq. (3.88). These changes should be performed everywhere in N_R except in contractions induced by the external indices $\hat{\rho}$ and $\hat{\sigma}$ in cuts (c,d). In this case

$$g_{\hat{\rho}\hat{\sigma}} k_{3,4}^{\rho} k_{3,4}^{\sigma} \to k_{3,4}^2 = \mu^2, \quad g_{\hat{\rho}\hat{\sigma}} k_3^{\rho} k_4^{\sigma} \to (k_3 \cdot k_4) = \frac{s_{34} - 2\mu^2}{2}. \tag{3.91}$$

In the case of the N_F part of $\sigma_R(\gamma^* \to jets)$ and $\sigma_R(H \to b\bar{b} + jets)$, integrating J_R over $\tilde{\Phi}_4$ and taking the asymptotic $\mu \to 0$ limit produces the phase-space integrals reported in Appendix E of [10].

3.5 Results and conclusions

Using the approach outlined in Sec. 3.3 and Sec. 3.4 one reproduces the known $\overline{\text{MS}}$ results for the N_F components of $\sigma^{\text{NNLO}}(H \to b\bar{b} + jets)$ and $\sigma^{\text{NNLO}}(\gamma^* \to jets)$ [10]

$$\begin{aligned} \sigma^{\text{NNLO}}(H \to b\bar{b} + jets) &= \Gamma_{\text{BORN}}(y_b^{\overline{\text{MS}}}(M_H))\left\{1 + a^2 C_F N_F \left(8\zeta_3 + \frac{2}{3}\pi^2 - \frac{65}{2}\right)\right\}, \\ \sigma^{\text{NNLO}}(\gamma^* \to jets) &= \sigma_{\text{BORN}}\left\{1 + a^2 C_F N_F\left(8\zeta_3 - 11\right)\right\}. \end{aligned} \tag{3.92}$$

This shows, for the first time, that a fully four-dimensional framework to compute NNLO quark-pair corrections can be constructed based on the requirement of preserving gauge invariance and unitarity. The basic principles leading to a consistent treatment of all the pieces contributing to the NNLO results in Eq. (3.92) are expected to remain valid also when considering more complicated environments. A general four-dimensional NNLO procedure including initial state IR singularities is currently under investigation.

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