Spontaneously breaking non-Abelian gauge symmetry in non-Hermitian field theories

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We generalize our previous formulation of gauge-invariant $\mathcal{PT}$-symmetric field theories to include models with non-Abelian symmetries and discuss the extension to such models of the Englert-Brout-Higgs-Kibble mechanism for generating masses for vector bosons. As in the Abelian case, the non-Abelian gauge fields are coupled to nonconserved currents. We present a consistent scheme for gauge fixing, demonstrating Becchi-Rouet-Stora-Tyutin invariance, and show that the particle spectrum and interactions are gauge invariant. We exhibit the masses that gauge bosons in the simplest two-doublet $\mathrm{SU}(2) \times \mathrm{U}(1)$ model acquire when certain scalar fields develop vacuum expectation values: they and scalar masses depend quartically on the non-Hermitian mass parameter μ. The bosonic mass spectrum differs substantially from that in a Hermitian two-doublet model. This non-Hermitian extension of the Standard Model opens a new direction for particle model building, with distinctive predictions to be explored further.

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I. INTRODUCTION

Recent years have seen increasing interest in quantum-mechanical models with non-Hermitian, $\mathcal{PT}$-symmetric Hamiltonians [1–3], which have been shown to possess real energy spectra that are bounded below, and have extensive applications in photonics and other fields [4–6]. This interest has extended to $\mathcal{PT}$-symmetric quantum field theories with non-Hermitian Lagrangians, such as a scalar field theory with an $i\phi^3$ interaction [7–10], which has been shown to possess a physically meaningful effective potential, a $\mathcal{PT}$-symmetric $-\phi^4$ scalar field theory [11], and a fermionic model with a non-Hermitian mass term $\propto \bar{\psi}\gamma_5\psi$ that is unitary and has a conserved current [12,13]. Such non-Hermitian quantum field theories have been applied to describe neutrino masses and oscillations [14–17] (for a similar lattice fermion model, see Ref. [18]), and have also been considered in connection with dark matter [19] and decays of the Higgs boson [20]. We also note that effective non-Hermitian Hamiltonians can also be used to describe unstable systems with particle mixing [21].

The formulation of $\mathcal{PT}$-symmetric quantum field theories was extended in Refs. [15,22] to include an Abelian gauge symmetry. A particularity of this formulation is that the gauge field is coupled to a nonconserved current. The next step was to study spontaneous symmetry breaking and the Goldstone theorem [23–25] in a non-Hermitian, $\mathcal{PT}$-symmetric quantum field theory, which was done in Ref. [26] (cf. the alternative approach of Refs. [27,28]), where we exhibited a specific example with two complex scalar fields and a non-Hermitian bilinear scalar coupling μ^2, in which there is a massless boson at both the tree and one-loop levels.[1] We note that physical observables depend only on μ^4 and are therefore independent of the ambiguity in the sign of μ^2 that arises from the non-Hermiticity of the model. We subsequently explored in Ref. [29] the $\mathcal{PT}$-symmetric extension of the Englert-Brout-Higgs mechanism [30,31] for generating a mass for the Abelian gauge boson in a manner consistent with renormalizability of the quantum field theory. For a summary of these works, see Ref. [32].

In this paper, we further develop the formulation of $\mathcal{PT}$-symmetric gauge theories to include a non-Abelian gauge symmetry and Kibble's non-Abelian generalization [33] of the Englert-Brout-Higgs mechanism. We study a minimal

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[1]The behaviors of Goldstone modes in different phases of $\mathcal{PT}$ symmetry have also been studied in Ref. [28].

extension of the model considered in Refs. [26,29] that contains two complex scalar doublets and admits the same $SU(2) \times U(1)$ gauge symmetry as the Standard Model. We show how the gauge can be fixed in a consistent manner and demonstrate Becchi-Rouet-Stora-Tyutin (BRST) invariance [34]. We explore the scalar vacuum expectation values (VEVs) and tree-level spectra of the gauge and scalar boson masses in a simple version of the model with a single quartic coupling. They depend quartically on the non-Hermitian coupling μ^2 and differ significantly from the masses in the conventional Hermitian two-Higgs-doublet model (2HDM, see Ref. [35]). Thus, this non-Hermitian extension of the Standard Model offers prospects for distinctive experimental predictions that may be explored further in a systematic program of $\mathcal{PT}$-symmetric phenomenology.

II. SCALAR LAGRANGIAN

In this section, we extend the non-Hermitian model of Ref. [22] to include two complex scalar doublets, giving the non-Hermitian 2HDM on which we base the discussion of non-Abelian gauge symmetry and its breaking in the next section.

A. Lagrangian

Here we follow similar steps to those described in Ref. [22], starting with the Lagrangian

$$\mathcal{L} = \partial_\alpha \Phi_1^\dagger \partial^\alpha \Phi_1 + \partial_\alpha \Phi_2^\dagger \partial^\alpha \Phi_2 - m_1^2 |\Phi_1|^2 - m_2^2 |\Phi_2|^2 - \mu^2 (\Phi_1^\dagger \Phi_2 - \Phi_2^\dagger \Phi_1) - \frac{\kappa}{4} |\Phi_1|^4, \tag{1}$$

where Φ_i are complex doublets

$$\Phi_i = \begin{pmatrix} \phi_{ia} \\ \phi_{ib} \end{pmatrix}, \qquad i = 1, 2, \tag{2}$$

and μ is a non-Hermitian mass parameter. This system is invariant under $\mathcal{PT}$ symmetry, acting on the c-number fields as

$$\begin{aligned} \mathcal{PT}: \quad & \Phi_1(t,x) \to \Phi_1'(-t,-x) = \Phi_1^*(t,x), \\ & \Phi_2(t,x) \to \Phi_2'(-t,-x) = -\Phi_2^*(t,x), \end{aligned} \tag{3}$$

under which Φ_1 is a scalar doublet whereas Φ_2 is a pseudoscalar doublet. The eigenvalues of the squared mass matrix

$$M_\pm^2 = \frac{m_1^2 + m_2^2}{2} \pm \frac{1}{2} \sqrt{(m_1^2 - m_2^2)^2 - 4\mu^4} \tag{4}$$

are real provided the following inequality holds:

$$2|\mu^2| \leq |m_1^2 - m_2^2|, \tag{5}$$

which is assumed throughout the first two sections of this work. Note that the eigenvalues become degenerate at $|\mu^2| = |m_1^2 - m_2^2|/2$. This marks the *exceptional point*, which lies at the boundary between the regions of unbroken and broken $\mathcal{PT}$ symmetry. At this point, the squared mass matrix becomes defective and we lose an eigenvector. We discuss these exceptional points further in Sec. IV D.

Because of the non-Hermitian mass term proportional to μ^2, the equations of motion one obtains by varying the action with respect to Φ_i or to $\Phi_i^\dagger$ are not equivalent for nontrivial solutions, i.e.,

$$\frac{\delta S}{\delta \Phi_i^\dagger} \equiv \frac{\partial \mathcal{L}}{\partial \Phi_i^\dagger} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_i^\dagger)} = 0 \nLeftrightarrow \frac{\delta S}{\delta \Phi_i} \equiv \frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_i)} = 0. \tag{6}$$

These two sets of equations of motion are related by $\mathcal{PT}$ symmetry though or, equivalently, by a change in the sign of μ^2. As can be seen from the eigenvalues (4), observables depend on μ^4 only, so these two sets of equations of motion are physically equivalent. This is also valid at the quantum level (see Ref. [29]), as can be derived from the reality of the partition function, provided the sources for the scalar fields satisfy appropriate $\mathcal{PT}$ properties.

We choose here the equations of motion provided by the variation of the action with respect to $\Phi_i^\dagger$:

$$0 = \Box \Phi_1 + m_1^2 \Phi_1 + \mu^2 \Phi_2 + \frac{\kappa}{2} |\Phi_1|^2 \Phi_1, \tag{7a}$$

$$0 = \Box \Phi_2 + m_2^2 \Phi_2 - \mu^2 \Phi_1, \tag{7b}$$

together with their Hermitian conjugates

$$0 = \Box \Phi_1^\dagger + m_1^2 \Phi_1^\dagger + \mu^2 \Phi_2^\dagger + \frac{\kappa}{2} |\Phi_1|^2 \Phi_1^\dagger, \tag{8a}$$

$$0 = \Box \Phi_2^\dagger + m_2^2 \Phi_2^\dagger - \mu^2 \Phi_1^\dagger. \tag{8b}$$

We note that this formulation differs from that suggested in Ref. [27], where the author introduced a similarity transformation that transforms the non-Hermitian Lagrangian $\mathcal{L}$ to a Hermitian one $\mathcal{L}'$. The difference in approach is reflected in differences in the masses of the gauge fields, which we discuss in Sec. IV D.

B. Conserved currents

The Lagrangian (1) is invariant under the U(1) transformations

$$\Phi_1 \to e^{-i\frac{g'}{2}\beta_0} \Phi_1, \tag{9a}$$

$$\Phi_2 \to e^{-i\frac{g'}{2}\beta_0} \Phi_2, \tag{9b}$$

which correspond to the current

$$I_+^\alpha = i\frac{g'}{2}([\Phi_1^\dagger(\partial^\alpha\Phi_1) - (\partial^\alpha\Phi_1^\dagger)\Phi_1] + [\Phi_2^\dagger(\partial^\alpha\Phi_2) - (\partial^\alpha\Phi_2^\dagger)\Phi_2]), \tag{10}$$

and also invariant under the SU(2) transformations

$$\Phi_1 \to e^{-i\frac{g}{2}\vec{\beta}\cdot\vec{\tau}}\Phi_1, \tag{11a}$$

$$\Phi_2 \to e^{-i\frac{g}{2}\vec{\beta}\cdot\vec{\tau}}\Phi_2, \tag{11b}$$

which correspond to the current

$$\vec{J}_+^\alpha = i\frac{g}{2}([\Phi_1^\dagger\vec{\tau}(\partial^\alpha\Phi_1) - (\partial^\alpha\Phi_1^\dagger)\vec{\tau}\Phi_1] + [\Phi_2^\dagger\vec{\tau}(\partial^\alpha\Phi_2) - (\partial^\alpha\Phi_2^\dagger)\vec{\tau}\Phi_2]), \tag{12}$$

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ is composed of the Pauli matrices.

The equations of motion (7) show, however, that these currents are not conserved,

$$\partial_\alpha I_+^\alpha = ig'\mu^2(\Phi_2^\dagger\Phi_1 - \Phi_1^\dagger\Phi_2), \tag{13a}$$

$$\partial_\alpha \vec{J}_+^\alpha = ig\mu^2(\Phi_2^\dagger\vec{\tau}\Phi_1 - \Phi_1^\dagger\vec{\tau}\Phi_2), \tag{13b}$$

except at the Hermitian point $\mu^2 = 0$. The fact that symmetries of the Lagrangian do not correspond to conserved currents for non-Hermitian theories is a direct consequence of the fact that the two functional variations in Eq. (6) cannot vanish simultaneously for nontrivial solutions. Instead, a careful treatment of Noether's original derivation [36] shows that there still exist conserved currents for non-Hermitian theories, but these correspond to transformations that do not leave the Lagrangian invariant [22] (see also Ref. [37] for a summary).

In the present model, we find that the conserved currents are, in fact,

$$I_-^\alpha = i\frac{g'}{2}([\Phi_1^\dagger(\partial^\alpha\Phi_1) - (\partial^\alpha\Phi_1^\dagger)\Phi_1] - [\Phi_2^\dagger(\partial^\alpha\Phi_2) - (\partial^\alpha\Phi_2^\dagger)\Phi_2]), \tag{14a}$$

$$\vec{J}_-^\alpha = i\frac{g}{2}([\Phi_1^\dagger\vec{\tau}(\partial^\alpha\Phi_1) - (\partial^\alpha\Phi_1^\dagger)\vec{\tau}\Phi_1] - [\Phi_2^\dagger\vec{\tau}(\partial^\alpha\Phi_2) - (\partial^\alpha\Phi_2^\dagger)\vec{\tau}\Phi_2]), \tag{14b}$$

which correspond to the following transformations:

$$\Phi_1 \to e^{-i\frac{g'}{2}\beta_0}\Phi_1, \tag{15a}$$

$$\Phi_2 \to e^{+i\frac{g'}{2}\beta_0}\Phi_2, \tag{15b}$$

and

$$\Phi_1 \to e^{-i\frac{g}{2}\vec{\beta}\cdot\vec{\tau}}\Phi_1, \tag{16a}$$

$$\Phi_2 \to e^{+i\frac{g}{2}\vec{\beta}\cdot\vec{\tau}}\Phi_2. \tag{16b}$$

The relative sign between the charge assignments of the two fields reflects the usual interpretation of viable $\mathcal{PT}$-symmetric theories as systems with coupled gain and loss.

III. GAUGING THE SCALAR MODEL

Since the conserved currents do not correspond to the usual Noether currents, gauging the model (1) is nontrivial, as we describe in this section, generalizing the approach taken in Ref. [29] to the non-Abelian case. We refer to the *nonconserved* currents corresponding to symmetries of the Lagrangian as *Noether currents*, but note that the *conserved* currents are in fact those consistent with Noether's original derivation (see Ref. [22]).

A. Coupling to the Noether currents

We introduce an Abelian gauge field B^α and an SU(2) gauge field $\vec{W}^\alpha$, together with the $\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge transformations

$$\Phi_i \to e^{-i\frac{g'}{2}\beta_0}e^{-i\frac{g}{2}\vec{\beta}\cdot\vec{\tau}}\Phi_i, \tag{17a}$$

$$\vec{W}^\alpha \to \vec{W}^\alpha + g(\vec{\beta}\times\vec{W}^\alpha) + \partial^\alpha\vec{\beta} = \vec{W}^\alpha + \mathcal{D}^\alpha\vec{\beta}, \tag{17b}$$

$$B^\alpha \to B^\alpha + \partial^\alpha\beta_0, \tag{17c}$$

where $\mathcal{D}^\alpha\vec{\beta} = \partial^\alpha\vec{\beta} - g(\vec{W}^\alpha\times\vec{\beta})$. In order to write a gauge-invariant theory, one should couple the gauge fields to the Noether currents, such that the scalar kinetic terms are given by

$$\begin{aligned}\mathcal{L}_{\mathrm{kin}} &= [D_\alpha\Phi_1]^\dagger D^\alpha\Phi_1 + [D_\alpha\Phi_2]^\dagger D^\alpha\Phi_2 \\ &= \partial_\alpha\Phi_1^\dagger\partial^\alpha\Phi_1 + \partial_\alpha\Phi_2^\dagger\partial^\alpha\Phi_2 + \frac{i}{2}\partial_\alpha\Phi_1^\dagger(g'B^\alpha + g\vec{\tau}\cdot\vec{W}^\alpha)\Phi_1 - \frac{i}{2}\Phi_1^\dagger(g'B^\alpha + g\vec{\tau}\cdot\vec{W}^\alpha)\partial_\alpha\Phi_1 + \frac{i}{2}\partial_\alpha\Phi_2^\dagger(g'B^\alpha + g\vec{\tau}\cdot\vec{W}^\alpha)\Phi_2 \\ &\quad - \frac{i}{2}\Phi_2^\dagger(g'B^\alpha + g\vec{\tau}\cdot\vec{W}^\alpha)\partial_\alpha\Phi_2 + \frac{1}{4}\Phi_1^\dagger(g'B + g\vec{\tau}\cdot\vec{W})^2\Phi_1 + \frac{1}{4}\Phi_2^\dagger(g'B + g\vec{\tau}\cdot\vec{W})^2\Phi_2,\end{aligned} \tag{18}$$

where D_α is given by the usual minimal-coupling prescription, i.e.,

$$D^\alpha\Phi_i = \partial^\alpha\Phi_i + \frac{ig'}{2}B^\alpha\Phi_i + \frac{ig}{2}[\vec{\tau}\cdot\vec{W}^\alpha]\Phi_i. \tag{19}$$

As in the Standard Model, we rotate the gauge fields as

$$B^\alpha = \cos\theta_{\rm W}A^\alpha - \sin\theta_{\rm W}Z^\alpha, \tag{20a}$$

$$W_1^\alpha = \frac{W^\alpha + W^{\alpha\dagger}}{\sqrt{2}}, \qquad W_3^\alpha = \sin\theta_{\rm W}A^\alpha + \cos\theta_{\rm W}Z^\alpha, \qquad W_2^\alpha = i\frac{W^\alpha - W^{\alpha\dagger}}{\sqrt{2}} \tag{20b}$$

(where $\theta_{\rm W}$ is the weak mixing angle) to obtain

$$\begin{aligned}\mathcal{L}_{\rm kin} &= \partial_\alpha\Phi_1^\dagger\partial^\alpha\Phi_1 + \partial_\alpha\Phi_2^\dagger\partial^\alpha\Phi_2 - W_\alpha\left[\frac{J^\alpha_{+,1}+iJ^\alpha_{+,2}}{\sqrt{2}}\right] - W_\alpha^\dagger\left[\frac{J^\alpha_{+,1}-iJ^\alpha_{+,2}}{\sqrt{2}}\right] - Z_\alpha[J^\alpha_{+,3}\cos\theta_{\rm W} - I^\alpha_+\sin\theta_{\rm W}]\\
&\quad - A_\alpha[J^\alpha_{+,3}\sin\theta_{\rm W} + I^\alpha_+\cos\theta_{\rm W}] + \frac{g^2}{2}W_\alpha^\dagger W^\alpha(|\Phi_1|^2+|\Phi_2|^2)\\
&\quad + \frac{1}{4}Z_\alpha Z^\alpha\sum_i\Phi_i^\dagger([g'^2\sin^2\theta_{\rm W} + g^2\cos^2\theta_{\rm W}]\mathbb{I} - 2gg'\cos\theta_{\rm W}\sin\theta_{\rm W}\tau_3)\Phi_i\\
&\quad + \frac{1}{4}A_\alpha A^\alpha\sum_i\Phi_i^\dagger([g'^2\cos^2\theta_{\rm W} + g^2\sin^2\theta_{\rm W}]\mathbb{I} + 2gg'\cos\theta_{\rm W}\sin\theta_{\rm W}\tau_3)\Phi_i\\
&\quad + \frac{1}{4}Z_\alpha A^\alpha\sum_i\Phi_i^\dagger([(g^2-g'^2)\sin 2\theta_{\rm W}\mathbb{I} + 2gg'\cos 2\theta_{\rm W}\tau_3])\Phi_i\\
&\quad + \frac{1}{2}gg'(\cos\theta_{\rm W}A^\alpha - \sin\theta_{\rm W}Z^\alpha)\sum_i\Phi_i^\dagger\left(W_\alpha\left[\frac{\tau_1+i\tau_2}{\sqrt{2}}\right] + W_\alpha^\dagger\left[\frac{\tau_1-i\tau_2}{\sqrt{2}}\right]\right)\Phi_i.\end{aligned} \tag{21}$$

Also as in the Standard Model, the Lagrangian for the gauge fields is

$$\begin{aligned}\mathcal{L}_{\rm gauge} &= -\frac{1}{4}\vec{W}'_{\alpha\beta}\cdot\vec{W}'^{\alpha\beta} - \frac{1}{4}B_{\alpha\beta}B^{\alpha\beta}\\
&= -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{4}Z_{\alpha\beta}Z^{\alpha\beta} - \frac{1}{2}W^\dagger_{\alpha\beta}W^{\alpha\beta},\end{aligned} \tag{22}$$

with

$$\vec{W}'_{\alpha\beta} = \partial_\beta\vec{W}_\alpha - \partial_\alpha\vec{W}_\beta + g(\vec{W}_\alpha\times\vec{W}_\beta), \tag{23a}$$

$$B_{\alpha\beta} = \partial_\beta B_\alpha - \partial_\alpha B_\beta, \tag{23b}$$

$$\begin{aligned}W_{\alpha\beta} &= [\partial_\beta + ig(\sin\theta_{\rm W}A_\beta + \cos\theta_{\rm W}Z_\beta)]W_\alpha\\
&\quad - [\partial_\alpha + ig(\sin\theta_{\rm W}A_\alpha + \cos\theta_{\rm W}Z_\alpha)]W_\beta,\end{aligned} \tag{23c}$$

$$F_{\alpha\beta} = \partial_\beta A_\alpha - \partial_\alpha A_\beta + ig\sin\theta_{\rm W}[W^\dagger_\alpha W_\beta - W^\dagger_\beta W_\alpha], \tag{23d}$$

$$Z_{\alpha\beta} = \partial_\beta Z_\alpha - \partial_\alpha Z_\beta + ig\cos\theta_{\rm W}[W^\dagger_\alpha W_\beta - W^\dagger_\beta W_\alpha]. \tag{23e}$$

B. Consistent field equations

Since the gauge fields are coupled to currents that are not conserved, additional terms need to be added to the Lagrangian in order to have consistent field equations [29]. For this, it is enough to consider the usual gauge-fixing terms, which must be added to the *classical* equations of motion in the non-Hermitian case (not just at the quantum level in order to define the path integral, as in the Hermitian case). The gauge-fixing terms in the Lagrangian involve ghost fields $\vec{\eta}$ and $\vec{\bar{\eta}}$, taking the form

$$\begin{aligned}\mathcal{L}_{\rm GF} &= \partial_\alpha\vec{\bar{\eta}}\cdot[\mathcal{D}^\alpha\vec{\eta}] - \frac{1}{2\xi}[(\partial_\alpha B^\alpha)^2 + |\partial_\alpha\vec{W}^\alpha|^2]\\
&= \partial_\alpha\bar{\chi}^\dagger([\partial^\alpha + ig(\sin\theta_{\rm W}A^\alpha + \cos\theta_{\rm W}Z^\alpha)]\chi - igW^\alpha\eta_3)\\
&\quad + \partial_\alpha\bar{\chi}([\partial^\alpha - ig(\sin\theta_{\rm W}A^\alpha + \cos\theta_{\rm W}Z^\alpha)]\chi^\dagger + igW^{\alpha\dagger}\eta_3)\\
&\quad + \partial_\alpha\bar{\eta}_3(\partial^\alpha\eta_3 + ig[W^\alpha\chi^\dagger - W^{\alpha\dagger}\chi])\\
&\quad - \frac{1}{2\xi}[(\partial_\alpha A^\alpha)^2 + (\partial_\alpha Z^\alpha)^2 + 2|\partial_\alpha W^\alpha|^2],\end{aligned} \tag{24}$$

where

$$\bar{\chi} \equiv \frac{\bar{\eta}_1 - i\bar{\eta}_2}{\sqrt{2}}, \qquad \chi \equiv \frac{\eta_1 - i\eta_2}{\sqrt{2}}. \tag{25}$$

The equations of motion for the full Lagrangian are then given by

$$0 = D_\alpha D^\alpha \Phi_1 + m_1^2 \Phi_1 + \mu^2 \Phi_2 + \frac{\kappa}{2}|\Phi_1|^2 \Phi_1, \tag{26a}$$

$$0 = D_\alpha D^\alpha \Phi_2 + m_2^2 \Phi_2 - \mu^2 \Phi_1, \tag{26b}$$

$$0 = \mathcal{D}_\beta \vec{W}'^{\beta\alpha} + \vec{\mathcal{J}}_+^\alpha - \frac{1}{\xi}\partial^\alpha \partial^\beta \vec{W}_\beta - g(\partial^\alpha \vec{\eta} \times \vec{\eta}), \tag{26c}$$

$$0 = \partial_\beta B^{\beta\alpha} + \mathcal{I}_+^\alpha - \frac{1}{\xi}\partial^\alpha \partial^\beta B_\beta, \tag{26d}$$

$$0 = \partial_\alpha \mathcal{D}^\alpha \vec{\eta}, \tag{26e}$$

$$0 = \mathcal{D}_\alpha \partial^\alpha \vec{\eta}, \tag{26f}$$

together with their Hermitian conjugates, where

$$\mathcal{I}_+^\alpha \equiv i\frac{g'}{2}([\Phi_1^\dagger (D^\alpha \Phi_1) - (D^\alpha \Phi_1)^\dagger \Phi_1] + [\Phi_2^\dagger (D^\alpha \Phi_2) - (D^\alpha \Phi_2)^\dagger \Phi_2]), \tag{27a}$$

$$\vec{\mathcal{J}}_+^\alpha \equiv i\frac{g}{2}([\Phi_1^\dagger \vec{\tau} (D^\alpha \Phi_1) - (D^\alpha \Phi_1)^\dagger \vec{\tau} \Phi_1] + [\Phi_2^\dagger \vec{\tau} (D^\alpha \Phi_2) - (D^\alpha \Phi_2)^\dagger \vec{\tau} \Phi_2]). \tag{27b}$$

Taking into account the current divergences (13), the derivatives of the above equations of motion lead to the constraints

$$\frac{1}{\xi}\mathcal{D}_\alpha \partial^\alpha \partial^\beta \vec{W}_\beta = ig\mu^2(\Phi_2^\dagger \vec{\tau} \Phi_1 - \Phi_1^\dagger \vec{\tau} \Phi_2) - g\partial^\alpha \vec{\eta} \times \mathcal{D}_\alpha \vec{\eta}, \tag{28a}$$

$$\frac{1}{\xi}\Box \partial^\beta B_\beta = ig'\mu^2(\Phi_2^\dagger \Phi_1 - \Phi_1^\dagger \Phi_2), \tag{28b}$$

which must be satisfied in order for the field equations to be consistent. As explained in the next subsection, BRST symmetry allows one to write the latter constraints independently of the ghost fields as

$$\frac{1}{\xi}\mathcal{D}_\alpha \partial^\alpha \partial^\beta \vec{W}_\beta = \frac{ig\mu^2}{2}(\Phi_2^\dagger \vec{\tau} \Phi_1 - \Phi_1^\dagger \vec{\tau} \Phi_2), \tag{29a}$$

$$\frac{1}{\xi}\Box \partial^\beta B_\beta = ig'\mu^2(\Phi_2^\dagger \Phi_1 - \Phi_1^\dagger \Phi_2). \tag{29b}$$

We can summarize our approach as follows. In order to respect gauge invariance, we need to couple the gauge fields to the Noether currents. However, because these currents are not conserved, we need to introduce gauge-fixing terms, which restrict gauge invariance, but imply consistent field equations. The residual gauge invariance is enough to ensure that gauge fields remain massless in the absence of spontaneous symmetry breaking (SSB), and it is defined by the gauge functions β_0, $\vec{\beta}$ satisfying

$$\partial_\alpha \mathcal{D}^\alpha \vec{\beta} = 0, \tag{30a}$$

$$\Box \beta_0 = 0. \tag{30b}$$

We therefore obtain a consistent gauge theory with a non-Hermitian scalar sector, as in the Abelian case [29].

C. BRST transformation

In this subsection, we derive the gauge constraint (29) for $\vec{W}_\beta$ using the BRST transformation, which is a residual symmetry of the Lagrangian after gauge fixing. In order to define it, one can introduce an auxiliary field $\vec{T}$ to write the gauge-fixing Lagrangian (24) in the alternative form

$$\mathcal{L}_{\mathrm{GF}} = \partial_\alpha \vec{\eta} \cdot \mathcal{D}^\alpha \vec{\eta} + \frac{\xi}{2}|\vec{T}|^2 - \vec{T} \cdot \partial^\alpha \vec{W}_\alpha - \frac{1}{2\xi}(\partial_\alpha B^\alpha)^2, \tag{31}$$

and the original Lagrangian (24) can be recovered after integrating out $\vec{T}$. The BRST transformations are defined as

$$\delta\phi_i = -i\frac{g}{2}\theta(\vec{\tau} \cdot \vec{\eta})\phi_i, \tag{32a}$$

$$\delta \vec{W}^\alpha = \theta \mathcal{D}^\alpha \vec{\eta}, \tag{32b}$$

$$\delta B^\alpha = 0, \tag{32c}$$

$$\delta \vec{\eta} = -\theta \vec{T}, \tag{32d}$$

$$\delta \vec{\eta} = \frac{g}{2}\theta(\vec{\eta} \times \vec{\eta}), \tag{32e}$$

$$\delta \vec{T} = 0, \tag{32f}$$

where θ is an infinitesimal Grassmann parameter. The gauge-invariant terms (18) and (22) in the Lagrangian are invariant under the BRST transformation, and the gauge-fixing Lagrangian (31) transforms as a total derivative, so the action is invariant under this BRST transformation. Using the auxiliary field $\vec{T}$, the equation of motion (26c) for the gauge field $\vec{W}^\alpha$ can be written in the form

$$0 = \mathcal{D}_\beta \vec{W}'^{\beta\alpha} + \vec{\mathcal{J}}_+^\alpha - \partial^\alpha \vec{T} - g(\partial^\alpha \vec{\eta} \times \vec{\eta}), \tag{33}$$

and a covariant derivative leads to

$$\mathcal{D}_\alpha \partial^\alpha \vec{T} = ig\mu^2(\Phi_2^\dagger \vec{\tau}\Phi_1 - \Phi_1^\dagger \vec{\tau}\Phi_2) - g\partial^\alpha \vec{\eta} \times \mathcal{D}_\alpha \vec{\eta}. \quad (34)$$

A BRST transformation of Eq. (26f) leads to the relation

$$0 = \delta(\mathcal{D}_\alpha \partial^\alpha \vec{\eta}) = -\theta(\mathcal{D}_\alpha \partial^\alpha \vec{T} - g\partial_\alpha \vec{\eta} \times \mathcal{D}^\alpha \vec{\eta}), \quad (35)$$

so that

$$\mathcal{D}_\alpha \partial^\alpha \vec{T} = g\partial_\alpha \vec{\eta} \times \mathcal{D}^\alpha \vec{\eta}, \quad (36)$$

which, together with Eq. (34), leads to

$$\mathcal{D}_\alpha \partial^\alpha \vec{T} = \frac{ig\mu^2}{2}(\Phi_2^\dagger \vec{\tau}\Phi_1 - \Phi_1^\dagger \vec{\tau}\Phi_2). \quad (37)$$

Since, from the equations of motion for $\vec{T}$, one finds

$$\vec{T} = \frac{1}{\xi}\partial_\alpha \vec{W}^\alpha, \quad (38)$$

one finally obtains the expected constraint

$$\frac{1}{\xi}\mathcal{D}_\alpha \partial^\alpha \partial^\beta \vec{W}_\beta = \frac{ig\mu^2}{2}(\Phi_2^\dagger \vec{\tau}\Phi_1 - \Phi_1^\dagger \vec{\tau}\Phi_2), \quad (39)$$

which, unlike Eq. (28a), is independent of the ghost fields.

For further discussions of BRST (and anti-BRST) symmetries in the context of non-Hermitian field theories, see Ref. [38].

IV. SPONTANEOUS SYMMETRY BREAKING

SSB is possible if the sign of m_1^2 in the Lagrangian (1) is changed, and here we study the corresponding scalar vacuum expectation values and vector masses.

A. Vacuum expectation value

With this change of sign, the Lagrangian (1) has a symmetry-breaking vacuum that is given by

$$\frac{\kappa}{2}|\langle\Phi_1\rangle|^2 = m_1^2 - \frac{\mu^4}{m_2^2}, \quad (40a)$$

$$\langle\Phi_2\rangle = \frac{\mu^2}{m_2^2}\langle\Phi_1\rangle, \quad (40b)$$

which is physical as long as

$$m_1^2 m_2^2 > \mu^4. \quad (41)$$

The vacuum is defined up to a $\mathrm{SU}(2) \times \mathrm{U}(1)$ transformation, and it can be chosen so that

$$\langle\Phi_1\rangle = \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \equiv V_1, \qquad \langle\Phi_2\rangle = \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \equiv V_2, \quad (42)$$

with

$$v_1 = \sqrt{\frac{2}{\kappa}\left(m_1^2 - \frac{\mu^4}{m_2^2}\right)}, \qquad v_2 = \frac{\mu^2}{m_2^2}\sqrt{\frac{2}{\kappa}\left(m_1^2 - \frac{\mu^4}{m_2^2}\right)}. \quad (43)$$

With this choice, the vacuum expectation value is unbroken by the transformation

$$\langle\Phi_i\rangle \to e^{-i\frac{e}{2}\beta_0(\mathbb{I}+\tau_3)}\langle\Phi_i\rangle = \begin{pmatrix} e^{-ie\beta_0} & 0 \\ 0 & 1 \end{pmatrix}\langle\Phi_i\rangle = \langle\Phi_i\rangle, \quad (44)$$

such that the Abelian subgroup of $\mathrm{SU}(2) \times \mathrm{U}(1)$ generated by $\sigma \equiv \mathbb{I} + \tau_3$ remains unbroken. This subgroup corresponds to the electromagnetic interaction, with Noether current

$$\begin{aligned} Q^\alpha &= \frac{ie}{2}[\Phi_1^\dagger \sigma(\partial^\alpha \Phi_1) - (\partial^\alpha \Phi_1^\dagger)\sigma\Phi_1] \\ &\quad + \frac{ie}{2}[\Phi_2^\dagger \sigma(\partial^\alpha \Phi_2) - (\partial^\alpha \Phi_2^\dagger)\sigma\Phi_2] \\ &= \frac{e}{g'} I_+^\alpha + \frac{e}{g} J_{+,3}^\alpha. \end{aligned} \quad (45)$$

From Eq. (21), we see that the gauge field A^μ couples to the current $I_+^\alpha \cos\theta_\mathrm{W} + J_{+,3}^\alpha \sin\theta_\mathrm{W}$, which can be identified with the current (45) if

$$e = g' \cos\theta_\mathrm{W} = g \sin\theta_\mathrm{W}. \quad (46)$$

The $\mathrm{U}(1)_\mathrm{EM}$ charge is conserved at the tree level, although the Noether current is in general not conserved. An exploration of the possibility of charge nonconservation beyond the tree level lies beyond the scope of this paper. Its existence and observability would in principle depend upon the completion of the bosonic model considered here to include fermions, which is also a topic for future work.

We can then express the scalar Lagrangian in terms of fluctuations around the vacuum (42) as

$$\begin{aligned} \mathcal{L}_\mathrm{scal} &= \partial_\alpha \hat{\Phi}_1^\dagger \partial^\alpha \hat{\Phi}_1 + \partial_\alpha \hat{\Phi}_2^\dagger \partial^\alpha \hat{\Phi}_2 + \frac{2\mu^4}{m_2^2}(V_1^\dagger \hat{\Phi}_1) \\ &\quad - 2m_2^2(V_2^\dagger \hat{\Phi}_2) - m_2^2|\hat{\Phi}_2|^2 + \frac{\mu^4}{m_2^2}|\hat{\Phi}_1|^2 \\ &\quad - \frac{\kappa}{4}(V_1^\dagger \hat{\Phi}_1 + \hat{\Phi}_1^\dagger V_1)^2 - \mu^2(\hat{\Phi}_1^\dagger \hat{\Phi}_2 - \hat{\Phi}_2^\dagger \hat{\Phi}_1) \\ &\quad - \frac{\kappa}{2}(V_1^\dagger \hat{\Phi}_1 + \hat{\Phi}_1^\dagger V_1)|\hat{\Phi}_1|^2 - \frac{\kappa}{4}|\hat{\Phi}_1|^4, \end{aligned} \quad (47)$$

where

$$\Phi_i = \hat{\Phi}_i + V_i = \begin{pmatrix} \phi_i^+ \\ v_i + \rho_i + i\psi_i \end{pmatrix}, \tag{48a}$$

$$\Phi_i^* = \hat{\Phi}_i^* + V_i = \begin{pmatrix} \phi_i^- \\ v_i + \rho_i - i\psi_i \end{pmatrix}. \tag{48b}$$

We note that the terms linear in fluctuations are a consequence of the non-Hermitian nature of the system. However, they do not play a role in the equations of motion $\delta S/\delta\hat{\Phi}_i^\dagger \equiv 0$, since they depend on $\hat{\Phi}_i$ only. These equations of motion are

$$\begin{aligned} 0 = {} & \Box\hat{\Phi}_1 - \frac{\mu^4}{m_2^2}\hat{\Phi}_1 + \frac{\kappa}{2}(V_1^\dagger\hat{\Phi}_1 + \hat{\Phi}_1^\dagger V_1)V_1 + \mu^2\hat{\Phi}_2 \\ & + \frac{\kappa}{2}|\hat{\Phi}_1|^2 V_1 + \frac{\kappa}{2}(V_1^\dagger\hat{\Phi}_1 + \hat{\Phi}_1^\dagger V_1)\hat{\Phi}_1 + \frac{\kappa}{2}|\hat{\Phi}_1|^2\hat{\Phi}_1, \end{aligned} \tag{49a}$$

$$0 = \Box\hat{\Phi}_2 + m_2^2\hat{\Phi}_2 - \mu^2\hat{\Phi}_1. \tag{49b}$$

The massless Goldstone modes consist of charged and neutral fields:

$$G^\pm = \frac{1}{\sqrt{v_1^2 - v_2^2}}(v_1\phi_1^\pm - v_2\phi_2^\pm), \tag{50a}$$

$$G = \frac{1}{\sqrt{v_1^2 - v_2^2}}(v_1\psi_1 - v_2\psi_2). \tag{50b}$$

The remaining fields consist of a charged field and three neutral fields. The charged fields are given by

$$H^\pm = \frac{1}{\sqrt{v_1^2 - v_2^2}}(v_2\phi_1^\pm - v_1\phi_2^\pm), \tag{51}$$

and one neutral field is given by

$$D = \frac{1}{\sqrt{v_1^2 - v_2^2}}(v_2\psi_1 - v_1\psi_2), \tag{52}$$

with degenerate squared mass

$$M^2 = \frac{v_1^2 - v_2^2}{v_1 v_2}\mu^2 = m_2^2 - \frac{\mu^4}{m_2^2}. \tag{53}$$

Finally, we can express the last two neutral fields as

$$H = \rho_1\cosh\alpha - \rho_2\sinh\alpha, \tag{54a}$$

$$h = \rho_1\sinh\alpha - \rho_2\cosh\alpha, \tag{54b}$$

with masses

$$\begin{aligned} M_h^2 &= \frac{1}{2}\left(m_2^2 + 2m_1^2 - 3\mu^4/m_2^2 \right. \\ &\quad \left. - \sqrt{(2m_1^2 - m_2^2 - 3\mu^4/m_2^2)^2 - 4\mu^4}\right) \\ &= (v_1^2 - v_2^2)\left[\lambda - \frac{\hat{\lambda}\cosh(\beta-\alpha)}{\sinh(\beta-\alpha)}\right], \end{aligned} \tag{55a}$$

$$\begin{aligned} M_H^2 &= \frac{1}{2}\left(m_2^2 + 2m_1^2 - 3\mu^4/m_2^2 \right. \\ &\quad \left. + \sqrt{(2m_1^2 - m_2^2 - 3\mu^4/m_2^2)^2 - 4\mu^4}\right) \\ &= (v_1^2 - v_2^2)\left[\lambda - \frac{\hat{\lambda}\sinh(\beta-\alpha)}{\cosh(\beta-\alpha)}\right], \end{aligned} \tag{55b}$$

where

$$\tanh\alpha = \frac{-\mu^2}{(M_H^2 - m_2^2)}, \tag{56a}$$

$$\tanh\beta = \frac{v_2}{v_1}, \tag{56b}$$

and

$$\lambda = \kappa\cosh^4\beta, \tag{57a}$$

$$\hat{\lambda} = \frac{\kappa}{2}\sinh 2\beta\cosh^2\beta. \tag{57b}$$

It is not obvious that M^2 is positive or that M_H^2 and M_h^2 are real, and we derive the corresponding conditions on μ^2 in the next section.

The eigenvectors of non-Hermitian matrices are not orthogonal with respect to the Hermitian inner product

$$\langle\phi,\varphi\rangle = \int_x \phi^\dagger\varphi. \tag{58}$$

In the case of $\mathcal{PT}$-symmetric theories, however, the eigenmodes of the non-Hermitian Hamiltonian are orthogonal with respect to the $\mathcal{PT}$ inner product

$$\langle\phi,\varphi\rangle_{\mathcal{PT}} = \int_x (\phi^{\mathcal{PT}})^{\mathsf{T}}\varphi, \tag{59}$$

and we have normalized the fields $G^\pm$, G, $H^\pm$, D, H, and h accordingly. These eigenmodes are nontrivial linear combinations of the scalar components of Φ_1 and the pseudoscalar components of Φ_2 and, as such, they cannot be eigenstates of $\mathcal{P}$. Instead, the $\mathcal{P}$ transformation relates the left and right eigenmodes, which are distinct for a non-Hermitian Hamiltonian.

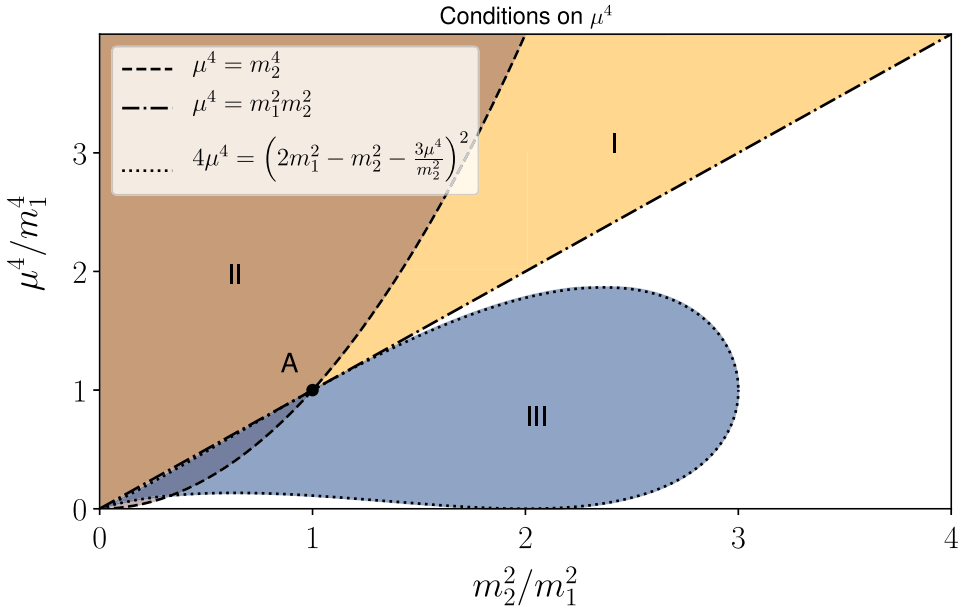

FIG. 1. The excluded regions for the parameter μ^4, corresponding to the constraints I, II, and III, plotted as functions of m_2^2/m_1^2. Region I corresponds to the symmetric phase of the $\mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry [see Eq. (61)], region II corresponds to the broken phase of $\mathcal{PT}$ symmetry [see Eq. (62)] in which M^2 is negative, and region III corresponds to the broken phase of $\mathcal{PT}$ symmetry in which M_h^2 and M_H^2 [see Eq. (63)] are complex. The unshaded region corresponds to a physical SSB phase for the $\mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry. For $m_2^2/m_1^2 < 1/3$, the allowed region is determined only by Condition II. For $m_1^2/3 < m_2^2 < 3m_1^2$, the allowed region is determined by Conditions I and III. Last, in the region $m_2^2 > 3m_1^2$, the allowed region is determined only by Condition III. At the point A, all the conditions become equivalent.

We remark that the $\mathcal{PT}$ norm used for the modes G, $G^\pm$, D, and $H^\pm$ in Eqs. (50), (51), and (52) diverges when $\mu^4 = m_2^4$ ($v_1^2 = v_2^2$). At this point—the *zero exceptional point* described in Ref. [28]—we lose three eigendirections: $D \propto G$ and $H^\pm \propto G^\pm$. On the other hand, when $\mu^4 = T_{H(h)}$, where

$$T_{H(h)} = \frac{m_2^2}{9}\left(6m_1^2 - m_2^2 + (-)2\sqrt{2m_2^2(3m_1^2 - m_2^2)}\right), \tag{60}$$

$|\alpha| \to \infty$ and the $\mathcal{PT}$ norm of h and H in Eq. (54) diverges. In this case, we lose one eigendirection: $H \propto h$. We discuss these exceptional points further in Sec. IV D.

B. Conditions on μ^2

Ensuring that we are in a physical regime of spontaneous symmetry breaking leads to a number of constraints on the parameter μ^2:

(I) In order for the symmetry to be broken [see Eqs. (40a)–(40b)], we require that

$$\mu^4 < m_1^2 m_2^2. \tag{61}$$

(II) In order to ensure that the squared mass M^2 [defined in Eq. (53)] remains positive, we require that

$$\mu^4 < m_2^4. \tag{62}$$

(III) In order for the squared masses M_h^2 and M_H^2 [defined in Eqs. (55a)–(55b)] to be real, we require that

$$4\mu^4 \leq \left(2m_1^2 - m_2^2 - \frac{3\mu^4}{m_2^2}\right)^2. \tag{63}$$

We remark that in the region $4\mu^4 \geq (2m_1^2 - m_2^2 - \frac{3\mu^4}{m_2^2})^2$ the squared mass matrix cannot be brought to a Hermitian form by a similarity transformation [27].

These constraints on the parameter μ^4 are plotted in Fig. 1. The unshaded regions correspond to values of μ^4 consistent with a physical spontaneous symmetry-breaking phase, satisfying all of the previously mentioned conditions. The various constraints on μ^4 can be summarized as follows:

(1) If $m_2^2 < \frac{m_1^2}{3}$, then $\mu^4 < m_2^4$ (Condition II).
(2) If $\frac{m_1^2}{3} < m_2^2 < m_1^2$, then $\mu^4 < T_h$ (Condition III).
(3) If $m_1^2 < m_2^2 < 3m_1^2$, then $\mu^4 < T_h$ (Condition III) or $T_H < \mu^4 < m_1^2 m_2^2$ (Conditions I and III).
(4) If $3m_1^2 < m_2^2$, then $\mu^4 < m_1^2 m_2^2$ (Condition I).

C. Equations of motion after SSB

After expressing the full Lagrangian in terms of fluctuations around the VEVs as done in Eqs. (48a)–(48b), we can express the equations of motion after symmetry breaking in terms of the gauge fields Z^α, W^α, and A^α. Introducing the notations

$$C_+^\alpha \equiv \frac{J_{+,1}^\alpha - iJ_{+,2}^\alpha}{\sqrt{2}}, \qquad K_+^\alpha \equiv J_{+,3}^\alpha \cos\theta_W - I_+^\alpha \sin\theta_W, \tag{64}$$

and

$$\sigma \equiv \mathbb{I} + \tau_3, \qquad \omega \equiv \frac{\tau_3 \cos^2\theta_W - \sin^2\theta_W}{\cos\theta_W}, \qquad \tau_+ \equiv \frac{\tau_1 - i\tau_2}{\sqrt{2}}, \qquad \tau_- \equiv \frac{\tau_1 + i\tau_2}{\sqrt{2}}, \tag{65}$$

the equations of motion read as follows.

Scalar fields:

$$0 = D_\alpha D^\alpha \hat{\Phi}_1 + D_\alpha\left(\frac{ig}{2} Z^\alpha \omega + \frac{ig}{2} W^\alpha \tau_-\right) V_1 - \frac{\mu^4}{m_2^2}\hat{\Phi}_1 + \frac{\kappa}{2}(V_1^\dagger \hat{\Phi}_1 + \hat{\Phi}_1^\dagger V_1) V_1 + \mu^2 \hat{\Phi}_2 + \frac{\kappa}{2}|\hat{\Phi}_1|^2 V_1 + \frac{\kappa}{2}(V_1^\dagger \hat{\Phi}_1 + \hat{\Phi}_1^\dagger V_1)\hat{\Phi}_1 + \frac{\kappa}{2}|\hat{\Phi}_1|^2 \hat{\Phi}_1, \tag{66a}$$

$$0 = D_\alpha D^\alpha \hat{\Phi}_2 + D_\alpha\left(\frac{ig}{2} Z^\alpha \omega + \frac{ig}{2} W^\alpha \tau_-\right) V_2 + m_2^2 \hat{\Phi}_2 - \mu^2 \hat{\Phi}_1. \tag{66b}$$

Z^α *gauge field*:

$$0 = \partial_\beta Z^{\alpha\beta} + ig\cos\theta_W (W_\beta^\dagger W^{\beta\alpha} - W^{\dagger\beta\alpha} W_\beta) + \frac{1}{\xi}\partial^\alpha \partial^\beta Z_\beta + \frac{g^2}{2\cos^2\theta_W}(|V_1|^2 + |V_2|^2) Z^\alpha - K_+^\alpha + ig\cos\theta_W(\partial^\alpha \bar{\chi}^\dagger \chi - \partial^\alpha \bar{\chi}\chi^\dagger) + \frac{g^2}{2} Z^\alpha \sum_i (\hat{\Phi}_i^\dagger \omega^2 \hat{\Phi}_i + [V_i^\dagger \hat{\Phi}_i + \hat{\Phi}_i^\dagger V_i]) + \frac{eg}{2} A^\alpha \sum_i \Phi_i^\dagger (\omega\sigma)\hat{\Phi}_i - \frac{gg'}{2}\sin\theta_W \sum_i ([\hat{\Phi}_i^\dagger \tau_- V_i] W^\alpha + [V_i^\dagger \tau_+ \hat{\Phi}_i] W^{\alpha\dagger}) - \frac{gg'}{2}\sin\theta_W \sum_i ([\hat{\Phi}_i^\dagger \tau_- \hat{\Phi}_i] W^\alpha + [\hat{\Phi}_i^\dagger \tau_+ \hat{\Phi}_i] W^{\alpha\dagger}). \tag{67}$$

A^α *gauge field*:

$$0 = \partial_\beta F^{\alpha\beta} + ig\sin\theta_W (W_\beta^\dagger W^{\beta\alpha} - W^{\dagger\beta\alpha} W_\beta) + \frac{1}{\xi}\partial^\alpha \partial^\beta A_\beta - Q^\alpha + ig\sin\theta_W(\partial^\alpha \bar{\chi}^\dagger \chi - \partial^\alpha \bar{\chi}\chi^\dagger) + e^2 A^\alpha \sum_i \hat{\Phi}_i^\dagger \sigma \hat{\Phi}_i + \frac{eg}{2} Z^\alpha \sum_i \hat{\Phi}_i^\dagger (\omega\sigma)\hat{\Phi}_i + \frac{eg}{2}\sum_i ([\hat{\Phi}_i^\dagger \tau_- \hat{\Phi}_i] W^\alpha + [\hat{\Phi}_i^\dagger \tau_+ \hat{\Phi}_i] W^{\alpha\dagger} + [\hat{\Phi}_i^\dagger \tau_- V_i] W^\alpha + [V_i^\dagger \tau_+ \hat{\Phi}_i] W^{\alpha\dagger}). \tag{68}$$

W^α *gauge fields*:

$$0 = \partial_\beta W^{\alpha\beta} + igW_\beta(\sin\theta_W F^{\beta\alpha} + \cos\theta_W Z^{\beta\alpha}) + \frac{1}{\xi}\partial^\alpha \partial^\beta W_\beta - ig(\sin\theta_W A_\beta + \cos\theta_W Z_\beta) W^{\beta\alpha} + \frac{g^2}{2} W^\alpha (|V_1|^2 + |V_2|^2) - C_+^\alpha + ig(\partial^\alpha \bar{\chi}\eta_3 - \partial^\alpha \bar{\eta}_3 \chi) + \frac{g^2}{2} W^\alpha \sum_i (V_i \hat{\Phi}_i + \hat{\Phi}_i^\dagger V_i + |\hat{\Phi}_i|^2) + \frac{g}{2}(eA^\alpha - g'\sin\theta_W Z^\alpha)\sum_i (\hat{\Phi}_i \tau_+ \hat{\Phi}_i + V_i^\dagger \tau_+ \hat{\Phi}_i). \tag{69}$$

From these equations, we can see that the gauge field masses are

$$M_W = g\sqrt{\frac{v_1^2+v_2^2}{2}} = \cos\theta_{\rm W} M_Z \quad \text{and} \quad M_A = 0, \tag{70}$$

as in the Hermitian 2HDM.

D. Comments on the exceptional points

At the zero exceptional points $\mu^2 = \pm m_2^2$, the VEVs become

$$v_1^2 = v_2^2 \equiv v^2 = \frac{2}{\kappa}(m_1^2 - m_2^2), \tag{71}$$

which vanish in the degenerate limit $m_1^2 = m_2^2$. For $m_1^2 \neq m_2^2$, though, the gauge boson masses at the exceptional points are

$$M_W^2 = g^2 v^2 = \cos^2\theta_{\rm W} M_Z^2 \neq 0, \tag{72}$$

remaining physical and nonzero.

In order to make sense of this, in spite of the divergence of the $\mathcal{PT}$ norm and the apparent nonnormalizability of the Goldstone modes (see Sec. IV A), it is helpful to reconsider the behavior of the non-Hermitian theory at the exceptional point. As an example, let us consider the following 2×2 squared mass matrix of the noninteracting theory [22]:

$$\mathbf{M}^2 = \begin{pmatrix} m_1^2 & \mu^2 \\ -\mu^2 & m_2^2 \end{pmatrix}. \tag{73}$$

For $m_1^2 > m_2^2$, the eigenvectors of this mass matrix are

$$\mathbf{e}_+ = N\begin{pmatrix} \eta \\ \sqrt{1-\eta^2}-1 \end{pmatrix} \text{ and } \mathbf{e}_- = N\begin{pmatrix} 1-\sqrt{1-\eta^2} \\ -\eta \end{pmatrix}, \tag{74}$$

where

$$\eta = \frac{2\mu^2}{m_1^2 - m_2^2} \tag{75}$$

(not to be confused with the ghost field appearing earlier).

The eigenvectors are not orthogonal with respect to the usual Hermitian inner product,

$$\mathbf{e}_+^* \cdot \mathbf{e}_- = 2N^2\eta\left(1-\sqrt{1-\eta^2}\right), \tag{76}$$

except in the Hermitian limit $\mu \to 0$ ($\eta \to 0$). They are, however, orthogonal with respect to the $\mathcal{PT}$ inner product, and orthonormality fixes

$$N = \left(2\eta^2 - 2 + 2\sqrt{1-\eta^2}\right)^{-1/2}. \tag{77}$$

The exceptional point of this mass matrix occurs when $\eta \to 1$, at which point the normalization of the eigenvectors diverges. This signals that the mass matrix has become defective, having the Jordan normal form

$$M^2|_{\eta\to 1} = \begin{pmatrix} (m_1^2+m_2^2)/2 & 1 \\ 0 & (m_1^2+m_2^2)/2 \end{pmatrix}, \tag{78}$$

and we lose an eigenvector. In fact, we see that in the limit $\eta \to 1$ the eigenvectors $\mathbf{e}_+$ and $\mathbf{e}_-$ become parallel to each other. However, the issue of the nonorthogonality of these eigenvectors is then moot, and we can normalize them with respect to the Hermitian inner product, fixing

$$N|_{\eta=1} = \frac{1}{\sqrt{2}}. \tag{79}$$

In other words, at the exceptional point the system behaves like a Hermitian theory with one fewer degree of freedom.

Returning to the case of spontaneously broken gauge symmetries at the zero exceptional point, the explanation for the nonvanishing masses of the gauge bosons is that the Goldstone modes must be normalized with respect to Hermitian conjugation and not $\mathcal{PT}$ conjugation (which has become ill defined). The discontinuity in the behavior of the system as we approach such exceptional points means that we must treat these particular points in parameter space separately.

Thus, our conclusion is that it is also possible to give masses to gauge bosons in a gauge-invariant way through SSB for non-Hermitian theories, even at the exceptional points. At these points, however, the counting of eigendirections must allow for the fact that the Hamiltonian has become defective.

We note that different results were derived in Ref. [27], which is based on an alternative interpretation of a similar (Abelian) non-Hermitian theory, and where the gauge boson masses are zero at the zero exceptional point. The difference in our results can be traced back to differing interpretations of the complex conjugate: we take complex conjugation to act linearly on the fields, whereas in Ref. [27] it was taken to act antilinearly on one of the fields (as motivated by a similarity transformation to a Hermitian theory). This has the effect of interchanging $v_2^2 \to -v_2^2$ in the expression for the gauge boson masses, such that they then vanish at the zero exceptional point, when $v_1^2 = v_2^2$. It was then argued that this is consistent with the fact that the Goldstone modes cannot be normalized with respect to the $\mathcal{PT}$ norm, which diverges at exceptional points, and these modes therefore cannot be "eaten" by the gauge field. This then led Ref. [27] to conclude that it is possible to spontaneously break the gauge symmetry of a non-Hermitian model without giving a mass to the gauge bosons. Our conclusion is different: the gauge boson remains massive in the symmetry-broken phase, even at the zero exceptional point.

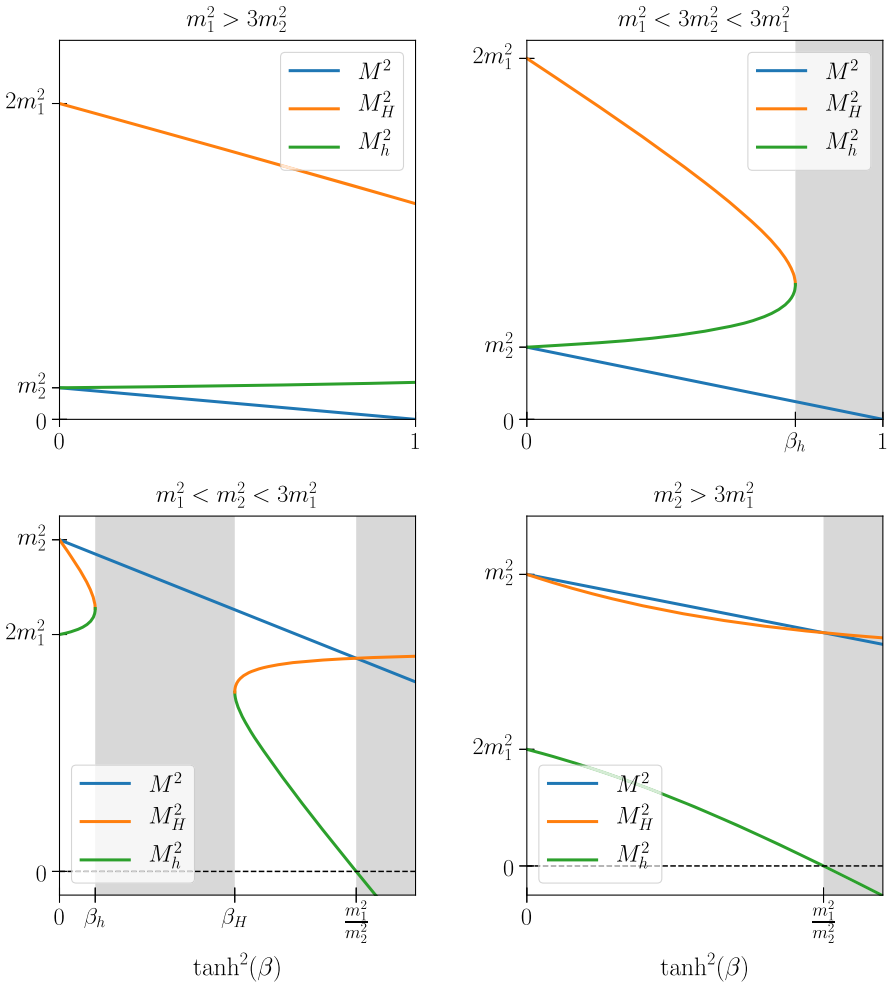

FIG. 2. The masses of the physical scalar bosons as functions of $\tanh^2\beta$ in different parameter regions. Unphysical parameter regions are shaded grey. The upper left panel shows the region where $m_1^2 > 3m_2^2$, the upper right panel shows the region where $m_1^2 < 3m_2^2 < 3m_1^2$, the lower left panel shows the region where $m_1^2 < m_2^2 < 3m_1^2$, and the lower right panel shows the region where $m_2^2 > 3m_1^2$.

V. MASSES IN THE NON-HERMITIAN MODEL COMPARED WITH THE HERMITIAN MODEL

In this section, we discuss the dependences of the scalar and vector masses in the non-Hermitian 2HDM on the non-Hermitian mixing parameter μ^2. These dependences are shown in Figs. 2 and 3 for the scalar and vector bosons, respectively, wherein we have introduced the notation $\beta_{H(h)} \equiv T_{H(h)}/m_2^4$. In addition, we make a comparison with the dependence of the scalar and vector masses on a Hermitian mixing parameter in the corresponding Hermitian 2HDM.

We note the following features from each panel of Fig. 2:

(1) In the region $m_1^2 > 3m_2^2$, the mass M^2 goes to zero at the exceptional point $\mu^2 = m_2^2$. If μ^2 were to become larger then m_2^2, then M^2 would become negative and we would enter the phase of broken $\mathcal{PT}$ symmetry.

(2) In the region $m_1^2/3 < m_2^2 < m_1^2$, the masses M_H^2 and M_h^2 become equal at the point $\tanh^2\beta = \beta_h$. For larger values of μ^2, both M_H^2 and M_h^2 would become complex.

(3) For $m_1^2 < m_2^2 < 3m_1^2$, the masses M_H^2 and M_h^2 become equal at the point $\tanh^2\beta = \beta_h$ or $\tanh^2\beta = \beta_H$. Between these points, M_H^2 and M_h^2 become complex. When $\tanh^2\beta > m_1^2/m_2^2$, the mass M_H^2 becomes negative. The unshaded regions correspond to physical masses.

(4) For $m_2^2 > 3m_1^2$, the masses are all real and positive as long as $\tanh^2\beta < m_1^2/m_2^2$.

We note in the lower right panel of Fig. 3 that the gauge-boson masses vanish at the point $\mu^4 = m_1^2 m_2^2$, where the symmetry is restored, as we would expect.

It is interesting to compare the masses in this $\mathcal{PT}$-symmetric non-Hermitian model with those in a similar Hermitian 2HDM with the following Lagrangian, involving a Hermitian mass-mixing term:

$$\mathcal{L} = \partial_\alpha \Phi_1^\dagger \partial^\alpha \Phi_1 + \partial_\alpha \Phi_2^\dagger \partial^\alpha \Phi_2 + m_1^2 |\Phi_1|^2 - m_2^2 |\Phi_2|^2 + m_{12}^2 (\Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1) - \frac{\kappa}{4} |\Phi_1|^4. \tag{80}$$

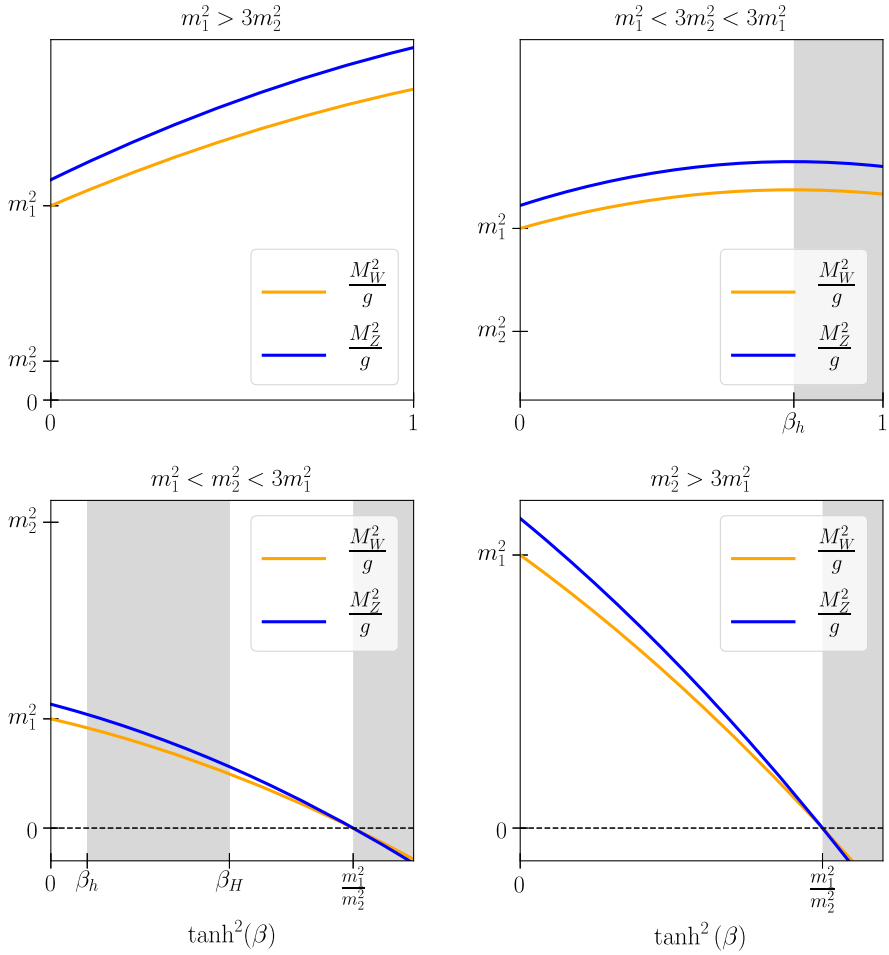

FIG. 3. The masses of the charged and neutral gauge bosons as functions of $\tanh^2\beta$ in the same parameter regions as in Fig. 2. Unphysical parameter regions are shaded grey.

The vacuum expectation values for this Lagrangian are

$$\langle\Phi_1\rangle = \begin{pmatrix} 0 \\ v_1^H \end{pmatrix} = V_1^H, \qquad \langle\Phi_2\rangle = \begin{pmatrix} 0 \\ v_2^H \end{pmatrix} = V_2^H, \tag{81}$$

with

$$v_1^H = \sqrt{\frac{2}{\kappa}\left(m_1^2 + \frac{m_{12}^4}{m_2^2}\right)}, \qquad v_2^H = \frac{m_{12}^2}{m_2^2}\sqrt{\frac{2}{\kappa}\left(m_1^2 + \frac{m_{12}^4}{m_2^2}\right)}. \tag{82}$$

After expressing the Lagrangian in terms of the shifted field $\hat{\Phi}_i$, where

$$\Phi_i = \hat{\Phi}_i + V_i^H = \begin{pmatrix} \phi_i^+ \\ v_i^H + \rho_i + i\psi_i \end{pmatrix}, \tag{83a}$$

$$\Phi_i^* = \hat{\Phi}_i^* + V_i^H = \begin{pmatrix} \phi_i^- \\ v_i^H + \rho_i - i\psi_i \end{pmatrix}, \tag{83b}$$

we can calculate the eigenvalues. As in the non-Hermitian model, the massless states consist of massless charged scalar and pseudoscalar Goldstone fields

$$G^\pm = \frac{1}{\sqrt{(v_1^H)^2 + (v_2^H)^2}}(v_1^H\phi_1^\pm + v_2^H\phi_2^\pm), \tag{84a}$$

$$G = \frac{1}{\sqrt{(v_1^H)^2 + (v_2^H)^2}}(v_1^H\psi_1 + v_2^H\psi_2). \tag{84b}$$

The normalizations of the eigenmodes should be compared with those in Sec. IV A. We remark that this Hermitian model is not $\mathcal{PT}$ symmetric if Φ_1 and Φ_2 transform as a scalar and a pseudoscalar, respectively. It is, however, $\mathcal{PT}$ symmetric if both Φ_1 and Φ_2 transform as scalars or pseudoscalars, and the Hermitian and $\mathcal{PT}$ norms coincide, as is expected for a Hermitian, $\mathcal{PT}$-symmetric theory.

The remaining massive fields include a charged scalar, a neutral pseudoscalar, and two neutral scalar fields. The charged scalars are

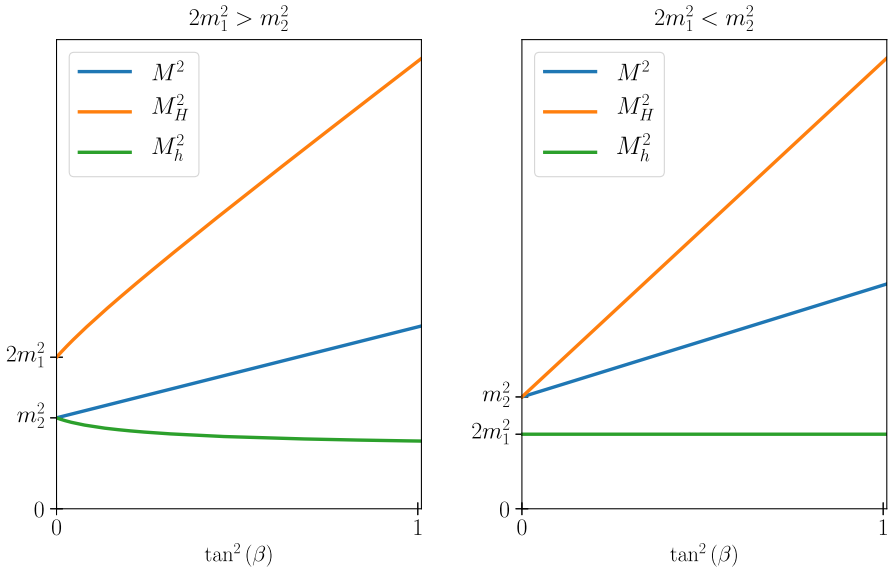

FIG. 4. The masses of the scalar fields in the Hermitian 2HDM as functions of $\tan^2\beta$ in the parameter ranges $2m_1^2 > m_2^2$ (left panel) and $2m_1^2 < m_2^2$ (right panel).

$$H^{\pm} = \frac{1}{\sqrt{(v_1^H)^2 + (v_2^H)^2}}(v_2^H \phi_1^{\pm} - v_1^H \phi_2^{\pm}), \tag{85}$$

and the pseudoscalar is

$$D = \frac{1}{\sqrt{(v_1^H)^2 + (v_2^H)^2}}(v_2^H \psi_1 - v_1^H \psi_2), \tag{86}$$

with degenerate squared mass

$$M^2 = \frac{(v_1^H)^2 + (v_2^H)^2}{v_1^H v_2^H} m_{12}^2. \tag{87}$$

Last, we can express the neutral scalar boson fields as

$$H = -\rho_1 \cos\alpha - \rho_2 \sin\alpha, \tag{88a}$$

$$h = \rho_1 \sin\alpha - \rho_2 \cos\alpha, \tag{88b}$$

with squared masses

$$M_h^2 = ((v_1^H)^2 + (v_2^H)^2)\left[\lambda - \frac{\hat{\lambda}\cos(\beta-\alpha)}{\sin(\beta-\alpha)}\right], \tag{89a}$$

$$M_H^2 = ((v_1^H)^2 + (v_2^H)^2)\left[\lambda + \frac{\hat{\lambda}\sin(\beta-\alpha)}{\cos(\beta-\alpha)}\right], \tag{89b}$$

where

$$\tan\alpha = \frac{-m_{12}^2}{(M_H^2 - m_2^2)}, \tag{90a}$$

$$\tan\beta = \frac{v_2^H}{v_1^H}, \tag{90b}$$

and

$$\lambda = \kappa \cos^4\beta, \tag{91a}$$

$$\hat{\lambda} = \frac{\kappa}{2}\sin 2\beta \cos^2\beta. \tag{91b}$$

The squared masses for this Hermitian model are plotted in Fig. 4 in the parameter ranges $2m_1^2 > m_2^2$ (left panel) and $2m_1^2 < m_2^2$ (right panel). We see that the mass spectra are completely different from the non-Hermitian, $\mathcal{PT}$-symmetric case, offering distinctive phenomenological possibilities.

Before concluding, we remark that, by comparing the expressions above with those in Sec. IV A, we can see that the non-Hermitian 2HDM that we have considered in this work is an analytic continuation of the Hermitian 2HDM, obtained by taking $m_{12}^4 \to -\mu^4$. In other words, the Hermitian 2HDM lies in the fourth quadrant of the $(m_2^2/m_1^2, \mu^4/m_1^4)$ plane (not shown in Fig. 1).

VI. CONCLUSION

In this paper, we have exhibited a consistent description of a non-Abelian two-Higgs-doublet model with a non-Hermitian scalar mass-mixing term, which generalizes the non-Hermitian extension of the Abelian Higgs model given in Refs. [26,29]. As in Ref. [29], the main point that led to a consistent model in the present article consists of restricting gauge invariance to a subclass of gauge field configurations. The corresponding constraint plays the role of a conventional gauge-fixing condition, but which must be taken into account at the classical level already, in order to find consistent field equations. Within this framework,

we have described the realization of SSB and compared its features with the Hermitian case.

An interesting question is the significance of the exceptional points. As explained in this paper, the number of eigendirections is reduced there, so that this limit is not continuous. It is indeed easy to see that, in the noninteracting model, one can write a unique equation of motion for $\Phi_1 + \Phi_2$ only, with mass $(m_1^2 + m_2^2)/2$, when taking the exceptional limit $|\mu^2| \to |m_1^2 - m_2^2|/2$, cf. Eq. (78). The introduction of gauge or self-interactions does not allow this though, and one can therefore question the stability of the exceptional points under quantum corrections, which appear as soon as interactions are switched on. However, the treatment of radiative corrections and further study of the exceptional points goes beyond the scope of the present paper.

We have noted that physical observables depend on μ^4, and thus not on the set of equations of motion we choose. This can also be checked with the masses of scalar excitations and gauge bosons: the transformation $\mu^2 \to -\mu^2$ leads to changes in the signs of α and β [see Eq. (56)], such that the masses obtained after SSB are not modified. It was shown in Ref. [29] that the quantum theory also depends on μ^4 only, and we expect the same to be valid here, since this feature is based on the scalar sector properties of the partition function, which is very similar here.

Finally, we note that the scalar boson mass spectrum in the non-Abelian non-Hermitian model differs significantly from that in the Hermitian version. This shows that the non-Hermitian model opens up new phenomenological perspectives, which merit a subsequent, more detailed discussion.

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