A tale of two exponentiations in $\mathcal{N} = 8$ supergravity

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Abstract

The structure of scattering amplitudes in supergravity theories continues to be of interest. Recently, the amplitude for $2 \to 2$ scattering in $\mathcal{N} = 8$ supergravity was presented at three-loop order for the first time. The result can be written in terms of an exponentiated one-loop contribution, modulo a *remainder function* which is free of infrared singularities, but contains leading terms in the high energy Regge limit. We explain the origin of these terms from a well-known, unitarity-restoring exponentiation of the high-energy gravitational S-matrix in impact-parameter space. Furthermore, we predict the existence of similar terms in the remainder function at all higher loop orders. Our results provide a non-trivial cross-check of the recent three-loop calculation, and a necessary consistency constraint for any future calculation at higher loops.

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1   Introduction

Scattering amplitudes in gauge and gravity theories continue to be intensively studied, due to a wide variety of both formal and phenomenological applications. Our focus in this paper is $\mathcal{N} = 8$ supergravity in four spacetime dimensions, which is of interest for a number of reasons. Firstly, it may prove to be an ultraviolet finite theory of perturbative quantum gravity [1–5], and in any case has a special status as its amplitudes arise in the low energy limit from type II superstring theory [6]. Secondly, calculations in maximally supersymmetric theories can be simpler than in less symmetric scenarios, making such theories the ideal frontier for developing new calculational techniques. Thirdly, there are a number of conjectures regarding the structure of amplitudes in maximally supersymmetric theories, which higher-order computations are able to shed light on.

One of the simplest amplitudes in terms of external multiplicity is that of four-graviton scattering, results for which have been previously calculated at one-loop [6–9] and two-loop [10–13] order. In the maximally supersymmetric theory, the tree-level result factors out, such that the amplitude may be written in the form

$$i\mathcal{M}_4 = i\mathcal{M}_4^{(0)} \left(1 + \sum_{L=1}^{\infty} M_4^{(L)} \right), \tag{1.1}$$

where $\mathcal{M}^{(0)}$ is the tree-level contribution, and $M^{(L)}$ an implicitly defined correction factor at L-loop order. The latter is infrared divergent, such that $M^{(L)}$ has a leading $1/\epsilon^L$ pole in $d = 4 - 2\epsilon$ spacetime dimensions. Additional structure arises, however, from the fact that infrared divergences in gravity theories are known to exponentiate [11, 14–19], where the logarithm of the soft (IR-divergent) part of the amplitude terminates at one-loop order, in marked contrast to (non-Abelian) gauge theories [20–29]. This motivates the following ansatz for the all-order amplitude:

$$i\mathcal{M}_4 = i\mathcal{M}_4^{(0)} \exp[M_4^{(1)}] \, \mathcal{F}_4, \tag{1.2}$$

where $M_4^{(1)}$ is the full one-loop correction factor, including also its infrared singular part, and $\mathcal{F}_4$ is an infrared finite *remainder function*, commencing at two-loop order. Indeed, results for the latter have been presented at two-loop order for a variety of supergravity theories in ref. [11–13], and their implications discussed further in refs. [19, 30].

Recently, the four-graviton scattering amplitude in $\mathcal{N} = 8$ supergravity has been obtained at an impressive three-loop order [31]. The authors compared their results with the form of eq. (1.2), confirming that the three-loop remainder function is infrared finite. This itself provided a highly non-trivial cross-check of their results. However, as in previous studies [13, 19, 30], they also examined the behaviour of the remainder function in the high energy *Regge limit*. This corresponds to highly forward high energy scattering, such that the centre of mass energy is much greater than the momentum transfer. The authors of ref. [31] noted in particular the curious property that the remainder function, although infrared finite, contains leading contributions in the high energy limit, suggesting that their structure can be explained using known results regarding high energy and / or soft limits. Indeed this is the case, as we will show in this paper.

High energy scattering in gauge and gravity theories has been studied for many decades. For example, generic scattering behaviour in the Regge limit formed a crucial ingredient in the S-matrix programme of strong interactions, which predated the discovery of QCD (see e.g. [32] for a review). Obtaining similar behaviour in perturbative quantum field theory has been pursued over many decades, with relevant work in (super-)gravity including [33–42]. Recently, methods from

gauge theory have been used to analyse gravitational physics, including clarifying the relationships between both theories in certain kinematic limits [15, 17, 19, 43–49]. Of particular relevance here is the outcome of the studies, started in the late 1980's [50–64], of high-energy (transplanckian) gravitational scattering in the Regge-asymptotics regime[7] in both string and field theories (see [67] for a recent review). Indeed, in order to explain the three-loop findings of ref. [31], we will use a very well-established property of gravitational scattering in the leading Regge limit, namely that the S-matrix has a certain exponential structure in transverse position (i.e. impact parameter) space, in terms of the so-called *eikonal phase*. This may be expanded in the gravitational coupling constant, before being Fourier transformed to momentum-transfer space order-by-order in perturbation theory. Given that a product in position space[8] is not a product (but rather a convolution) in momentum space, the exponentiated eikonal phase in the former does not directly lead to an exponential form in momentum space. The upshot of this is that by making the ansatz of eq. (1.2) in momentum space, a mismatch occurs, giving leading Regge contributions in the remainder function.

We will explicitly verify the form of the two- and three-loop remainder functions in the (leading) Regge limit. Furthermore, we will use our findings to predict additional terms at higher loops, before forming a conjecture for the leading Regge behaviour of the remainder function at arbitrary order in perturbation theory. Our results provide a cross-check of the three-loop calculation in ref. [31], whilst also setting consistency constraints on any future higher-loop calculations.

The structure of our paper is as follows. In section 2, we review previous results about fixed order results for supergravity amplitudes, and also the exponentiation of the position space amplitude in terms of the eikonal phase. In section 3, we verify the form of the remainder function up to three-loop order in the leading Regge limit. In section 4, we extend our analysis to arbitrary orders in perturbation theory. Finally, in section 5 we discuss our results and conclude.

2 Review of previous results

2.1 The remainder function up to three-loop order

As discussed above, the remainder function $\mathcal{F}_4$ of eq. (1.2) is defined after subtracting the one-loop contribution from the logarithm of the 4-graviton scattering amplitude. It thus begins at two-loop order, and we may then consider the perturbative expansion

$$\mathcal{F}_4 = 1 + \sum_{L=2}^{\infty} \mathcal{F}_4^{(L)}, \tag{2.1}$$

where $\mathcal{F}_4^{(L)}$ is the L-loop contribution, including coupling factors other than those associated with the tree-level amplitude. Explicit results for the two-loop contribution (in a variety of supergravity theories) have been presented in ref. [11–13]. To present results, we label 4-momenta as shown in figure 1, from which we may define the Mandelstam invariants

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2. \tag{2.2}$$

---

[7]A different high-energy regime, at fixed scattering angle, was also considered at about the same time within string theory [65, 66].

[8]In the following we will often refer to *transverse* space (momentum) as, simply, space (momentum), but it is important to stress that longitudinal momentum (energy) are never converted into the corresponding space (time) variables. This distinction is also important [50, 51] to recover classical General Relativity expectations from the eikonal approximation when the eikonal phase is parametrically large.
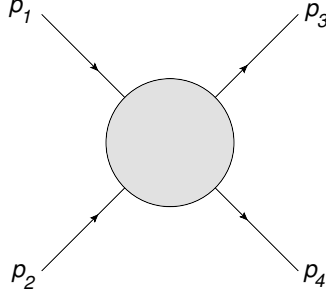
Figure 1: Labelling of 4-momenta for the four-graviton scattering process.

Note that all 4-momenta in figure 1 are physical (e.g. rather than all outgoing), so that we are dealing with the physical scattering region

$$s \geq 0, \quad t, u \leq 0. \tag{2.3}$$

Furthermore, momentum conservation implies $s+t+u = 0$, so that only two Mandelstam invariants are independent. The Regge limit may then be formally defined as $s \gg -t$. Alternatively, defining the dimensionless ratio

$$x = \frac{-t}{s}, \tag{2.4}$$

the Regge limit corresponds simply to $x \to 0$. Until recently, only the $\mathcal{O}(\epsilon^0)$ contribution of the two-loop remainder function was known, whose leading behaviour in the Regge limit may be written as [19]

$$\mathcal{F}_4^{(2)} = x \left(\frac{\alpha_G s}{2}\right)^2 \left\{ -2\pi^2 \log^2 x - 4\pi^2 \log x + \pi^4 + 4\pi^2 \right. \tag{2.5}$$

$$\left. + i\pi \left[\frac{4}{3} \log^3 x + 4 \log^2 x + 8 \left(1 + \frac{\pi^2}{3}\right)(1 - \log x) + 16\zeta_3 \right] \right\} + \mathcal{O}(x^2) + \mathcal{O}(\epsilon),$$

where we introduced the parameter

$$\alpha_G \equiv \frac{G_N}{\pi}(4\pi)^\epsilon \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} = \frac{G_N}{\pi} + \mathcal{O}(\epsilon). \tag{2.6}$$

Given that $\mathcal{F}_4^{(2)}$ is $\mathcal{O}(x)$, it vanishes in the strict Regge limit. However, the results of ref. [31] have now demonstrated that this is not true at higher orders in the dimensional regularisation parameter ϵ, nor at higher-loop level. In fact, the result in eq. (6.5) of ref. [31] is[9]

$$\mathcal{F}_4^{(2)} = \alpha_G^2 s^2 \pi^2 \left[3\zeta_3 \epsilon + \left(\frac{\pi^4}{20} - 6\zeta_3 \log(-t)\right)\epsilon^2 + \mathcal{O}(x) + \mathcal{O}(\epsilon^3)\right],$$

$$\mathcal{F}_4^{(3)} = -\frac{2}{3} i \alpha_G^3 s^3 \pi^3 \zeta_3 + \mathcal{O}(x) + \mathcal{O}(\epsilon). \tag{2.7}$$

These contributions are non-vanishing as $x \to 0$; we will explain their origin in the following sections.

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[9]The ϵ^2 contribution to $\mathcal{F}_4^{(2)}$ is not explicitly written in (6.5) of [31], but can be deduced from the ancillary files attached to the arxiv version of [31]. The apparent sign discrepancy between $\mathcal{F}^{(3)}$ and eq. (6.5) results from our choosing $s > 0$ whereas they have $s < 0$.
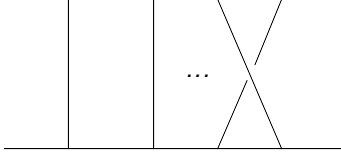
Figure 2: A representative (crossed) ladder graph, where all particles are gravitons. The sum of all such diagrams in the leading Regge limit builds up the exponentiated amplitude of eq. (2.8).

2.2 Impact-parameter exponentiation and the eikonal phase

The Regge limit of forward scattering consists of highly energetic particles that barely glance off each other. As such, any exchanged radiation must be soft (i.e. have an asymptotically small 4-momentum), and the emitting particles are then said to be in the *eikonal approximation*. One may then show [50, 51] that the dominant behaviour at arbitrary loop orders is given by the (crossed) horizontal ladder graphs of figure 2, in which all particles are gravitons. Furthermore, this situation does not depend on the amount of supersymmetry: in the leading Regge limit, the amplitude is dominated by the exchanged particle of highest spin, namely the graviton. It is then possible to sum such graphs to all perturbative orders by working at fixed impact parameter $\mathbf{x}_\perp$, a $(d-2)$-dimensional vector transverse to the incoming particle direction and which, at the leading eikonal level, can be thought as the transverse distance of closest approach between the two incoming hard gravitons. One may then write the complete *eikonal amplitude* as (see e.g. [68])

$$i\mathcal{M}_{\text{eik.}} = 2s \int d^{d-2}\mathbf{x}_\perp e^{-i\mathbf{q}_\perp \cdot \mathbf{x}_\perp} \left(e^{i\chi(\mathbf{x}_\perp)} - 1 \right), \tag{2.8}$$

where the quantity $i\chi(\mathbf{x}_\perp)$ is known as the *eikonal phase*, and is given in $d = 4 - 2\epsilon$ dimensions by[10] [50, 51]

$$i\chi(\mathbf{x}_\perp) = -iG_N s \Gamma(1-\epsilon) \frac{(\pi \mathbf{x}_\perp^2)^\epsilon}{\epsilon}. \tag{2.9}$$

In eq. (2.8), $\mathbf{q}_\perp$ is the $(d-2)$-dimensional momentum transfer that is Fourier-conjugate to $\mathbf{x}_\perp$. In terms of the above Mandelstam invariants, one has $t \simeq -|\mathbf{q}_\perp|^2$ in the leading Regge limit. The exponentiation of the amplitude in terms of a large eikonal phase has the important consequence of restoring partial-wave unitarity, which is violated as $s \to \infty$ at each loop order due to graviton exchange [50,51]. Equation (2.8) has a well-defined physical interpretation [50–52], in which $i\chi$ represents the phase shift experienced by one of the incoming particles in the field set up by the other, thus forming a link between old-fashioned quantum mechanical scattering theory and perturbative QFT approaches (see e.g. ref. [69] for an excellent review). Importantly, the exponentiation occurs in position space. To obtain the momentum-space amplitude at a given order in perturbation theory, one must Taylor expand the exponential in the Newton constant G_N, before carrying out the Fourier transform:

$$i\mathcal{M}_{\text{eik.}} = 2s \sum_{n=1}^{\infty} \frac{1}{n!} \int d^{d-2}\mathbf{x}_\perp e^{-i\mathbf{q}_\perp \cdot \mathbf{x}_\perp} [i\chi(\mathbf{x}_\perp)]^n. \tag{2.10}$$

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[10]This result holds at finite ϵ and its validity is unrelated to the problem of infrared singularities originating in the $\epsilon \to 0$ limit discussed in this paper.

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In each term, the product of phase factors $[i\chi(\mathbf{x}_\perp)]^n$ becomes a convolution in momentum space, which may itself be given a direct physical interpretation. First, one may express the position-space eikonal phase as an inverse Fourier transform:

$$i\chi(\mathbf{x}_\perp) = -4\pi i G_N s \int \frac{d^{d-2}\mathbf{k}_\perp}{(2\pi)^{d-2}} \frac{e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp}}{(-\mathbf{k}_\perp^2 + i\varepsilon)}, \tag{2.11}$$

where $i\varepsilon$ denotes the usual Feynman prescription. This allows us to rewrite eq. (2.10) as

$$
\begin{aligned}
i\mathcal{M}_{\text{eik.}} &= 2s \sum_{n=1}^{\infty} \frac{(-4\pi i G_N s)^n}{n!} \int d^{d-2}\mathbf{x}_\perp e^{-i\mathbf{q}_\perp \cdot \mathbf{x}_\perp} \left(\prod_{i=1}^{n} \int \frac{d^{d-2}\mathbf{k}_{i\perp}}{(2\pi)^{d-2}} \frac{e^{i\mathbf{k}_i \cdot \mathbf{x}_\perp}}{(-\mathbf{k}_{i\perp}^2 + i\varepsilon)} \right) \\
&= 2s(2\pi)^{d-2} \sum_{n=1}^{\infty} \frac{(-4\pi i G_N s)^n}{n!} \prod_{i=1}^{n} \left(\int \frac{d^{d-2}\mathbf{k}_{i\perp}}{(2\pi)^{d-2}} \frac{1}{(-\mathbf{k}_{i\perp}^2 + i\varepsilon)} \right) \delta^{(d-2)} \left(\mathbf{q}_\perp - \sum_{i=1}^{n} \mathbf{k}_{i\perp} \right).
\end{aligned}
\tag{2.12}
$$

Each term in the second line consists of a momentum space Feynman integral, with n particles being exchanged, each described by a standard propagator in $(d-2)$-dimensions. It is the delta function that makes this a convolution in momenta rather than a simple product, and it simply corresponds to the fact that the sum of the exchanged momenta should be equal to $\mathbf{q}_\perp$, namely the total momentum transfer that is conjugate to the impact parameter. As we will see in the following section, it is precisely the lack of a simple product in momentum space that leads to the presence of the non-trivial remainder function of eq. (2.7).

3 The three-loop remainder function in the Regge limit

Having seen how to describe the leading Regge limit of the four-graviton amplitude in supergravity to all orders via the eikonal phase, we now have everything we need to explain the results of ref. [31], presented here in eq. (2.7). To obtain the L-loop remainder function, we may start with eq. (2.10), and identify $n = L + 1$. Substituting eq. (2.9) then yields

$$
\begin{aligned}
i\mathcal{M}_{\text{eik.}} &= 2s \sum_{L=0}^{\infty} \frac{1}{(L+1)!} \left(\frac{-i G_N s \Gamma(1-\epsilon)\pi^\epsilon}{\epsilon} \right)^{L+1} \int d^{2-2\epsilon}\mathbf{x}_\perp e^{-i\mathbf{q}_\perp \cdot \mathbf{x}_\perp} \left(\mathbf{x}_\perp^2 \right)^{(L+1)\epsilon} \\
&= 2s \frac{4\pi i G_N s}{\mathbf{q}_\perp^2} \sum_{L=0}^{\infty} \frac{1}{L!} \left[-\frac{i G_N s \Gamma(1-\epsilon)}{\epsilon} \left(\frac{4\pi}{\mathbf{q}_\perp^2} \right)^\epsilon \right]^L \frac{\Gamma(1-\epsilon)\Gamma(1+L\epsilon)}{\Gamma(1-(L+1)\epsilon)}.
\end{aligned}
\tag{3.1}
$$

The second line allows us to identify the Regge limit of the tree-level amplitude from the $L = 0$ term:

$$i\mathcal{M}_4^{(0)} = \frac{8\pi i G_N s^2}{-t} + \mathcal{O}(x). \tag{3.2}$$

Examining the one-loop term then allows us to construct the correction factor entering eq. (1.2)

$$M_4^{(1)} = \frac{i\mathcal{M}_4^{(1)}}{i\mathcal{M}_4^{(0)}} = -\frac{i G_N s}{\epsilon} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{4\pi}{-t} \right)^\epsilon. \tag{3.3}$$

Let us now rewrite eq. (3.1) as

$$i\mathcal{M}_{\text{eik.}} = i\mathcal{M}_4^{(0)} \sum_{L=0}^{\infty} \frac{1}{L!} \left[-\frac{iG_N s}{\epsilon} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{4\pi}{-t}\right)^{\epsilon} \right]^L$$
$$\times \left\{ \frac{\Gamma^L(1-2\epsilon)\Gamma(1+L\epsilon)}{\Gamma^{L-1}(1-\epsilon)\Gamma^L(1+\epsilon)\Gamma(1-(L+1)\epsilon)} \right\}. \tag{3.4}$$

Were it not for the term in curly brackets, we would find that the full momentum-space amplitude is simply the tree-level amplitude multiplied by the *exponential of the one-loop correction* of eq. (3.3). By comparing eqs. (1.2) and (3.4), we thus find that the remainder function is given by

$$\mathcal{F}_4 = \exp\left[-M_4^{(1)}\right] \sum_{L=0}^{\infty} \frac{\left[M_4^{(1)}\right]^L}{L!} \left\{ \frac{\Gamma^L(1-2\epsilon)\Gamma(1+L\epsilon)}{\Gamma^{L-1}(1-\epsilon)\Gamma^L(1+\epsilon)\Gamma(1-(L+1)\epsilon)} \right\} + \mathcal{O}(x). \tag{3.5}$$

This is a complete all-orders expression in the leading Regge limit $x \to 0$, which may be systematically expanded in G_N to obtain the result at a given loop order. Performing such an expansion (also in the dimensional regularisation parameter ϵ), one finds

$$\mathcal{F}_4 = 1 + \alpha_G^2 s^2 \pi^2 \left[3\zeta_3 \epsilon + \left(\frac{\pi^4}{20} - 6\zeta_3 \log(-t)\right) \epsilon^2 + \mathcal{O}(\epsilon^3) \right]$$
$$+ \alpha_G^3 s^3 \pi^3 \left[-\frac{2}{3} i\zeta_3 + \mathcal{O}(\epsilon^2) \right] + \mathcal{O}(\alpha_G^4) + \mathcal{O}(x). \tag{3.6}$$

in agreement with eq. (2.7) and thus precisely confirming the results[11] of ref. [31]. We can now go further than this, however, and predict the structure of the remainder function in the leading Regge limit at higher orders in perturbation theory.

4 The remainder function to all orders in the Regge limit

In the previous section, we obtained a general expression, eq. (3.5), for the remainder function $\mathcal{F}_4$ in the leading Regge limit, and confirmed the results of a recent three-loop calculation (which also necessarily included $\mathcal{O}(\epsilon)$ at $\mathcal{O}(G_N^2)$). However, the all-order nature of eq. (3.5), in both G_N and ϵ, means that we can expand this further. In doing so, we predict the existence of non-zero terms in the remainder function at four-loop order and beyond. This potentially provides a highly non-trivial cross-check of any future calculations in perturbative gravity.

We have expanded eq. (3.5) to 16 orders in G_N, finding that all poles in ϵ vanish. This is to be expected, given the aforementioned fact that all infrared singularities in gravity are generated by the exponentiation of the one-loop amplitude [11,14–19]. Turning to the $\mathcal{O}(\epsilon^0)$ terms of the leading energy remainder $\mathcal{F}_4 = \mathcal{F}_{4,0} + \mathcal{O}(\epsilon) + \mathcal{O}(x)$, we may write the L-loop contribution as

$$\mathcal{F}_{4,0}^{(L)} = (iG_N s)^L f^{(L)}, \tag{4.1}$$

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[11]See footnote 9.

where we find the explicit results

$$f^{(2)} = 0 \qquad f^{(7)} = \bar{\zeta}_7 \qquad f^{(12)} = \frac{1}{4!}\bar{\zeta}_3^4 + \bar{\zeta}_9\bar{\zeta}_3 + \bar{\zeta}_7\bar{\zeta}_5$$

$$f^{(3)} = \bar{\zeta}_3 \qquad f^{(8)} = \bar{\zeta}_5\bar{\zeta}_3 \qquad f^{(13)} = \frac{1}{2}\bar{\zeta}_7\bar{\zeta}_3^2 + \frac{1}{2}\bar{\zeta}_5^2\bar{\zeta}_3 + \bar{\zeta}_{13}$$

$$f^{(4)} = 0 \qquad f^{(9)} = \frac{1}{3!}\bar{\zeta}_3^3 + \bar{\zeta}_9 \qquad f^{(14)} = \frac{1}{3!}\bar{\zeta}_5\bar{\zeta}_3^3 + \bar{\zeta}_{11}\bar{\zeta}_3 + \frac{1}{2}\bar{\zeta}_7^2 + \bar{\zeta}_9\bar{\zeta}_5 \qquad (4.2)$$

$$f^{(5)} = \bar{\zeta}_5 \qquad f^{(10)} = \frac{1}{2}\bar{\zeta}_5^2 + \bar{\zeta}_7\bar{\zeta}_3 \qquad f^{(15)} = \frac{1}{5!}\bar{\zeta}_3^5 + \frac{1}{2}\bar{\zeta}_9\bar{\zeta}_3^2 + \bar{\zeta}_7\bar{\zeta}_5\bar{\zeta}_3 + \frac{1}{3!}\bar{\zeta}_5^3 + \bar{\zeta}_{15}$$

$$f^{(6)} = \frac{1}{2}\bar{\zeta}_3^2 \qquad f^{(11)} = \frac{1}{2}\bar{\zeta}_5\bar{\zeta}_3^2 + \bar{\zeta}_{11} \qquad f^{(16)} = \frac{1}{3!}\bar{\zeta}_7\bar{\zeta}_3^3 + \frac{1}{4}\bar{\zeta}_5^2\bar{\zeta}_3^2 + \bar{\zeta}_{13}\bar{\zeta}_3 + \bar{\zeta}_9\bar{\zeta}_7 + \bar{\zeta}_{11}\bar{\zeta}_5$$

with $\bar{\zeta}_n = 2\zeta_n/n$. Despite the rather formidable nature of eq. (3.5), we see that the results for the $\mathcal{O}(\epsilon^0)$ contributions have a simple form. It is apparent that the arguments of the zeta functions in each term in the sums are such that they form a partition of L into a sum of odd integers greater than one. The generating function for the number of such partitions is

$$\prod_{j=1}^{\infty} \frac{1}{1 - z^{2j+1}} = 1 + z^3 + z^5 + z^6 + z^7 + z^8 + 2z^9 + 2z^{10} + 2z^{11} + 3z^{12} + 3z^{13} + 4z^{14} + 5z^{15} + 5z^{16} + O\left(z^{17}\right)$$

$$(4.3)$$

so the coefficient of z^L on the right-hand side of eq. (4.3) tells us the number of individual terms in each $f^{(L)}$ of eq. (4.2). We then find that we can summarise all of eq. (4.2) as the compact formula

$$\mathcal{F}_{4,0}^{(L)} = (iG_N s)^L \sum_{p_r(L)} \prod_j \frac{1}{n_j!}\left(\frac{2\zeta_{L_j}}{L_j}\right)^{n_j}, \qquad (4.4)$$

where the sum is over all restricted partitions of L, as mentioned above, the L_j's are the distinct odd integers entering in the partition and n_j is the number of times each L_j appears, so we have

$$L = \sum_j L_j n_j. \qquad (4.5)$$

In fact, one may observe[12] that eqs. (4.1), (4.2) and (4.4) may be compactly summarized by

$$\mathcal{F}_{4,0} = 1 + \sum_{L=2}^{\infty} \mathcal{F}_{4,0}^{(L)} = \exp\left[\sum_{j=1}^{\infty}(iG_N s)^{2j+1}\bar{\zeta}_{2j+1}\right]$$

$$= \frac{e^{-2iG_N s\gamma}}{\Gamma^2(1 + iG_N s)}\exp\left[\log\left(\frac{\pi iG_N s}{\sin(\pi iG_N s)}\right)\right] = e^{-2iG_N s\gamma}\frac{\Gamma(1 - iG_N s)}{\Gamma(1 + iG_N s)}. \qquad (4.6)$$

The same result can be obtained from the $\epsilon \to 0$ limit of (2.8) without expanding the exponential of $\chi(\mathbf{x}_\perp)$ (see ref. [19] for a similar observation). Denoting the ϵ^0 terms of eqs. (2.9) and (3.3) by χ_0 and $M_{4,0}^{(1)}$ respectively, we may write

$$\chi_0 = -G_N s\left(\log(\pi\mathbf{x}_\perp^2) + \gamma\right), \qquad e^{M_{4,0}^{(1)}} = e^{iG_N s\gamma}\left(\frac{4\pi}{\mathbf{q}_\perp^2}\right)^{-iG_N s}. \qquad (4.7)$$

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[12]We would like to thank Henrik Johansson for this observation.

We can then perform the Fourier transform of eq. (2.8) in $d = 4$ (i.e. restricting to the ϵ-independent part)

$$\int d^2\mathbf{x}_\perp e^{-i\mathbf{q}_\perp \cdot \mathbf{x}_\perp} e^{i\chi_0(\mathbf{x}_\perp)} = \frac{4\pi i G_N s}{\mathbf{q}_\perp^2} e^{-iG_N s \gamma} \left(\frac{4\pi}{\mathbf{q}_\perp^2}\right)^{-iG_N s} \frac{\Gamma(1 - iG_N s)}{\Gamma(1 + iG_N s)} = \frac{4\pi i G_N s}{\mathbf{q}_\perp^2} e^{M_{4,0}^{(1)}} \mathcal{F}_{4,0}, \quad (4.8)$$

and check explicitly that the last step is consistent with the result of eq. (4.6)[13]. This derivation can be seen as a proof of the result (4.4) for the ϵ^0 contribution, but we stress in any case that a complete all-order expression for the remainder function (which is more powerful than a finite-order ϵ expansion) has already been given in eq. (3.5).

The next unknown order in the four-graviton amplitude is four loops. It is easily checked from eq. (3.5) that, as at two loops, the $\mathcal{O}(\epsilon^0)$ contribution to the remainder function (in the leading Regge limit) vanishes. However, there is a nonzero contribution beyond this, given by

$$\mathcal{F}_4^{(4)} = -5(G_N s)^4 \zeta_5 \epsilon + \mathcal{O}(\epsilon^2) + \mathcal{O}(x). \quad (4.9)$$

We do not expect this result to be explicitly confirmed in the near future: calculating the $\mathcal{O}(\epsilon)$ part of the four-loop amplitude would presumably be first carried out as part of a five-loop calculation!

An interesting observation is that the above results respect the conjectured *uniform transcendentality* property of amplitudes in theories with maximal supersymmetry. That is, we can associate a transcendental weight n with the zeta value ζ_n, where all rational coefficients are taken to have weight zero. The sum of weights at $\mathcal{O}(\epsilon^m)$ and L-loop order is then

$$w = L + m. \quad (4.10)$$

Beyond the leading order, the Regge limit breaks this uniform transcendentality property, as, for instance, one approximates $\ln(-u/s) \sim x$ losing the transcendental contribution of the logarithm. Since the leading eikonal does not depend on the number of supersymmetries, the uniform weight property for $\mathcal{N} = 8$ supergravity manifest in (4.10) is inherited by the lower supersymmetric cases. We stress that this property of the leading term is exact to all orders, not just the ϵ^0 order considered above. For the amplitude itself at a given loop order, there is a dominant pole

$$\sim \frac{1}{\epsilon^L}$$

coming from the exponentiated IR singularity in the one-loop contribution. All subleading terms in ϵ (in the leading Regge limit) come from expanding Euler gamma functions, and the coefficients of all such expansions have increasing uniform weight as the power of ϵ increases. Thus, this accounts precisely for the dependence of eq. (4.10).

5   Discussion

In this paper, we examined the form of the four-graviton scattering amplitude in $\mathcal{N} = 8$ supergravity, which was recently calculated at three-loop order [31]. It is conventional to define a remainder function for this amplitude, constituting what is left upon factoring out the tree-level amplitude,

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[13]Note that, for $\epsilon \to 0$, the whole effect of $\mathcal{F}_{4,0}$ boils down to a renormalization of an unobservable (and not explicitly written) infinite Coulomb phase originating from the leading eikonal resummation.

and the one-loop correction factor [13]. The three-loop calculation, which includes an evaluation of the $\mathcal{O}(\epsilon)$ part of the two-loop result, revealed the existence of leading terms in the remainder function in Regge's high energy limit, at non-negative powers of the dimensional regularisation parameter ϵ.

In this paper, we have shown that these contributions follow precisely from the known exponentiation of the four-graviton amplitude in position space, in terms of the so-called *eikonal phase*. At a given order in perturbation theory, a product of one-loop amplitudes occurs, which becomes a convolution in momentum space, whose physical interpretation is that the transverse momentum transfer (conjugate to the impact parameter) must be democratically shared between the exchanged gravitons at that order. This in turn means that the amplitude does not straightforwardly exponentiate in momentum space, and we have derived an all-orders expression – in both the gravitational coupling G_N and dimensional regularisation parameter ϵ – for the remainder function in the Regge limit. As well as confirming the results of ref. [31], we also predict explicit contributions at higher loop orders. We obtained a particularly convenient combinatorial form for the $\mathcal{O}(\epsilon^0)$ contributions, which we showed can be directly obtained from the leading eikonal expression in $d = 4$. The higher loop remainder function respects maximal transcendentality to all orders.

There are a number of possible extensions of our analysis. Firstly, one could look at predicting the structure of subleading terms in the Regge limit (see e.g. refs. [30, 44, 54, 57, 61–64, 70] for previous work in this area). Secondly, it would be interesting to extend the analysis discussed in this paper to higher loops by starting from the integral expressions for the four- and five-loop amplitudes of refs. [71, 72]. Finally one can study the remainder function in theories with less than maximal supersymmetry. This is not independent of the exploration of subleading eikonal contributions. Indeed, the leading Regge behaviour would be expected to be the same for less supersymmetric gravity theories, given that this kinematic regime is dominated by the exchange only of leading soft particles of highest spin (i.e. the graviton). Three-loop calculations in non-maximal supergravity theories do not yet exist, thus our results already provide a highly useful constraint.

Acknowledgments

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