

Parametrized black hole quasinormal ringdown: Decoupled equations for nonrotating black holes

Vitor Cardoso,^{1,2} Masashi Kimura,^{1,3} Andrea Maselli,⁴ Emanuele Berti,⁵ Caio F. B. Macedo,⁶ and Ryan McManus⁵

¹*CENTRA, Departamento de Física, Instituto Superior Técnico—IST, Universidade de Lisboa—UL, Avenida Rovisco Pais 1, 1049 Lisboa, Portugal*

²*Theoretical Physics Department, CERN 1 Esplanade des Particules, Geneva 23, CH-1211, Switzerland*

³*Department of Physics, Rikkyo University, Tokyo 171-8501, Japan*

⁴*Dipartimento di Fisica, “Sapienza” Università di Roma & Sezione INFN Roma1, P.A. Moro 5, 00185 Roma, Italy*

⁵*Department of Physics and Astronomy, Johns Hopkins University, 3400 North Charles Street, Baltimore, Maryland 21218, USA*

⁶*Campus Salinópolis, Universidade Federal do Pará, Salinópolis, Pará, 68721-000, Brazil*



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Black hole solutions in general relativity are simple. The frequency spectrum of linear perturbations around these solutions (i.e., the quasinormal modes) is also simple, and therefore it is a prime target for fundamental tests of black hole spacetimes and of the underlying theory of gravity. The following technical calculations must be performed to understand the imprints of any modified gravity theory on the spectrum: 1. Identify a healthy theory; 2. Find black hole solutions within the theory; 3. Compute the equations governing linearized perturbations around the black hole spacetime; 4. Solve these equations to compute the characteristic quasinormal modes. In this work (the first of a series) we assume that the background spacetime has spherical symmetry, that the relevant physics is always close to general relativity, and that there is no coupling between the perturbation equations. Under these assumptions, we provide the general numerical solution to step 4. We provide publicly available data files such that the quasinormal modes of *any* spherically symmetric spacetime can be computed (in principle) to arbitrary precision once the linearized perturbation equations are known. We show that the isospectrality between the even- and odd-parity quasinormal mode spectra is fragile, and we identify the necessary conditions to preserve it. Finally, we point out that new modes can appear in the spectrum even in setups that are perturbatively close to general relativity.

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I. INTRODUCTION

The coalescence of two black holes (BHs) is a fascinating process from a theoretical perspective: two regions devoid of matter interact violently to produce enormous amounts of gravitational radiation, leaving behind a stationary, vacuum solution of the field equations. In general relativity (GR), this process is particularly interesting since both the initial BH spacetimes and the final state are characterized by only two parameters: their mass and angular momentum [1,2].

The transition from the initial to the final state is described by the so-called “ringdown” phase, during which the distorted remnant sheds away the nontrivial multipolar

structure through gravitational waves, relaxing to the final stationary solution. This exponential approach to equilibrium is a property of many physical systems. It can be mathematically described as a small deviation away from the *final* state, or as the time window in which the spacetime response to generic perturbations can be decomposed into a set of quasinormal modes (QNMs) characterized by complex frequencies [3,4]. The ringdown is of special relevance in the context of BH physics: if GR is correct, then QNM frequencies must depend only on the BH mass and spin, and therefore ringdown observations can provide simple and clean tests of GR [5–10].

It is generally accepted that GR does not offer the ultimate description of gravity. Rather, discrepancies in its strong-field or large-scale descriptions will eventually become apparent in a manner consistent with some of its known conceptual problems [8,11]. Thus, one expects that BH solutions will differ, if only slightly, from those of GR, and that the dynamics of BHs will also be described by slightly different equations. The correct description of gravity at all scales and energies is presently unknown.

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Traditionally, the strategy has been to identify modified theories of gravity which contain new features in their dynamics that could be smoking guns of new physics. Once the theory is identified, one computes its BH solutions, studies linearized perturbations around these BH solutions, and finally solves the relevant equations to compute QNM frequencies.

Recent efforts have focused instead on understanding the possible *functional form* of such corrections, using resummation methods. For example, effective field theory (EFT) arguments can produce the general form of the corrections in the absence of extra fundamental fields in the problem [12]. Such corrections can then be used to study BH solutions and their dynamics, most notably their QNM spectra [13]. Once new fundamental fields—for example scalars—are allowed, the dynamics become intrinsically more complicated and richer phenomenology arises from, for instance, the coupling between the scalar and the gravitational degrees of freedom (d.o.f.) [14–16].

The (technical) burden of computing the QNMs cannot be avoided. In the majority of the literature, tools yielding approximate answers, such as WKB techniques [17,18], have been used. The errors involved in the WKB approximation can be rather large even in GR (see e.g., Fig. 3 of Ref. [4]), and they are poorly known in modified theories of gravity [19]. Clearly, when testing GR precision physics will be involved. In addition, it quickly becomes apparent that one is repeatedly solving the same equation (or systems of equations) with the same boundary conditions and with results (the QNM frequencies) which are but small modifications to the GR predictions. This work is intended as a first step in the general program to address the repetitive task of finding the eigenvalues (i.e., the QNM frequencies) of these ordinary differential equations. Once the linearized equations of motion around a BH spacetime are known, our results can be used to find the corresponding QNM frequencies. In this first work, for simplicity, we assume that there are no couplings between the perturbation equations. These complications will be addressed in a forthcoming publication [20].

We also make the rather restrictive working hypothesis that the underlying equations are separable, and that the background GR solution is nonspinning. Many known modified theories of gravity indeed yield separable equations for nonspinning background spacetimes, but it is unclear whether this feature is generic. Certainly BHs in our Universe are spinning, generically breaking this hypothesis. Nevertheless, our results are interesting on their own and provide a better understanding of GR itself. Our formalism is theory agnostic and background agnostic. While in principle it only applies to nonspinning backgrounds, in Sec. V C we will show that, in fact, it can be used to accommodate (in a perturbative sense) the QNM frequencies of slowly spinning Kerr BHs. Therefore our

results seem to apply more broadly than these restrictions might imply.

The plan of the paper is as follows. In Sec. II we present the general perturbative framework underlying our analysis. In Sec. III we present our main results and their implications for isospectrality. In Sec. IV we discuss analytically and numerically the convergence of the perturbative expansion, and we present a simple example of a perturbation to the known GR potential that may lead to divergences. In Sec. V we verify our formalism in some interesting cases where QNM solutions are known: an EFT calculation of deviations from GR, the Reissner-Nordström (RN) solution, and scalar perturbations are slowly rotating BHs. In Sec. VI we present a preliminary estimate of the conditions under which perturbative deviations from the GR frequencies could be detectable. We conclude in Sec. VII with a discussion of the limitations of this analysis and directions for future work. In Appendices A and B we prove that our assumed form for the perturbative potentials is general enough for our present purposes, and in Appendix C we study the large- ℓ (eikonal) limit of our approximation.

II. FRAMEWORK

In GR, gravitational fluctuations have 2 d.o.f., which can be described by a linear combination Φ of the metric perturbations. In a nonspinning background, such modes are described by an “axial” (or odd, or Regge-Wheeler) and a “polar” (or even, or Zerilli) equation of the form [21,22]

$$f \frac{d}{dr} \left(f \frac{d\Phi}{dr} \right) + [\omega^2 - fV_{\pm}] \Phi = 0, \quad (1)$$

where $f = 1 - r_H/r$, and r_H is the horizon radius in standard Schwarzschild coordinates.

The effective potential V_{\pm} is

$$V_- = \frac{\ell(\ell+1)}{r^2} - \frac{3r_H}{r^3} \quad (2)$$

for odd perturbations, and

$$V_+ = \frac{9\lambda r_H^2 r + 3\lambda^2 r_H r^2 + \lambda^2(\lambda+2)r^3 + 9r_H^3}{r^3(\lambda r + 3r_H)^2} \quad (3)$$

for even perturbations, where $\lambda = \ell^2 + \ell - 2$, and ℓ is an angular harmonic index labeling the tensorial spherical harmonics used to separate the angular coordinates from the problem (the azimuthal index can be set to $m = 0$ for spherically symmetric backgrounds).

The dynamics of test scalar or vector fields in GR is also described by similar master equations, with potentials [4]

$$V_s = \frac{\ell(\ell+1)}{r^2} + (1-s^2) \frac{r_H}{r^3}, \quad (4)$$

where $s = 0, 1$ for scalar and vector perturbations, respectively.

Given the strong observational constraints on deviations from GR [23], it might be expected that a more fundamental description of gravity will add small corrections to the GR description of astrophysical BHs. Such corrections will slightly disturb both the equilibrium configurations (the Schwarzschild geometry) *and* the equations governing dynamical fluctuations [24]. Our guiding light is asymptotically flat spacetime and, therefore, we parametrize the modified master equations by adding power-law corrections to the effective potential:

$$V = V_{\pm} + \delta V_{\pm}, \quad (5)$$

$$\delta V_{\pm} = \frac{1}{r_H^2} \sum_{j=0}^{\infty} \alpha_j^{\pm} \left(\frac{r_H}{r} \right)^j \quad (6)$$

for gravitational fluctuations and

$$V = V_s + \delta V_s, \quad (7)$$

$$\delta V_s = \frac{1}{r_H^2} \sum_{j=0}^{\infty} \beta_j^s \left(\frac{r_H}{r} \right)^j \quad (8)$$

for scalars and vectors. Here, α_j^{\pm} , β_j^s are constant coefficients. Since the maximum value of the j th term $f(r)\alpha_j^{\pm}(r_H/r)^j$ is $\alpha_j^{\pm}(1+1/j)^{-j}/(j+1)$, the smallness of the corrections translates to the criterion:

$$(\alpha_j^{\pm}, \beta_j^s) \ll (1+1/j)^j(j+1). \quad (9)$$

Note that α_0^{\pm} and β_0^s in Eqs. (6) and (8) are effectively mass-square terms, hence they should be positive. Similar constraints may apply to the $j = 1$ terms, which dominate the potential at large distances and which are therefore constrained by weak-field physics. For example, a $1/r$ term appears when dealing with charged scalars in a *charged* background [see, e.g., Eq. (4) in Ref. [25] or Eq. (4.54) in Ref. [26]]. As we show in Sec. V, the leading-order coefficients (α_0 , β_0 and β_1 , in the notation of this section) are nonzero for charged (RN) or slowly rotating Kerr BHs. Although these terms are dominant in the power-law expansion—falling off even slower than the GR terms—they should still be considered small when compared to the GR potential $V_{\pm,s}$, due to the smallness of the parameters α_j^{\pm} , β_j^s .

In general, perturbation equations can be reduced to the form (1), but with $f \neq 1 - r_H/r$. In Appendix A we prove this for scalar and vector perturbations of static, spherically symmetric spacetimes, which indeed obey a generalized wave equation [Eq. (A4)]. However in Appendix B we show that, at linear order in perturbations, Eq. (A4) is in fact equivalent to Eq. (1). Gravitational perturbations

depend on the chosen modified gravity theory, so it is nontrivial to provide a general proof that they always reduce to Eq. (A4), but in Appendix A we provide arguments supporting this conclusion. Note that, as we show below, even perturbations of slowly rotating BHs can be accommodated within our formalism, so its regime of validity may be larger than anticipated.

The parametrization used here embodies both the change in the background spacetime and the change in the dynamical equations. It assumes no couplings to other fundamental d.o.f., and it also assumes no coupling between the odd and even modes; these issues will be addressed in a forthcoming study [20].

As predicted by perturbation theory [27,28], we find that the QNM frequencies for these potentials are described by small corrections to the GR modes [4,5,29]:

$$\omega_{\text{QNM}}^{\pm} = \omega_0^{\pm} + \sum_{j=0}^{\infty} \alpha_j^{\pm} e_j^{\pm}, \quad (10)$$

$$\omega_{\text{QNM}}^s = \omega_0^s + \sum_{j=0}^{\infty} \beta_j^s d_j^s. \quad (11)$$

The frequencies in GR ω_0^{\pm} , ω_0^s can be found online [4,5,29,30]. The corrections in the frequencies are linear in the corrections to the potentials, and the complex numbers (e_j^{\pm}, d_j^s) , which represent a “basis” set¹ for the perturbations, must be computed numerically. Below, when the context is clear, we omit the superscripts from the coefficients (e_j, d_j) for simplicity.

We should note that, generically, there are frequency-dependent terms in the corrections to the potential. In our approach, any ω -dependent term in the perturbing potential should be set to ω_0 , thereby corresponding to a redefinition of α_j^{\pm} . We have indeed verified numerically that the induced corrections are linear in the perturbations to the potential, and that ω -dependent terms can be handled by setting them equal to their GR value.

Corrections to the QNM frequencies scale *linearly* with corrections to the potential appearing in the master equation. In a forthcoming paper we will show that, when couplings between fluctuations are allowed, corrections in the frequency appear at *quadratic* order in the couplings. In some modified gravity theories both corrections are of the same order of magnitude, and therefore couplings cannot be neglected. These issues will be addressed in a separate study [20].

¹We use this terminology for convenience, but we make no claims on the completeness of this basis.

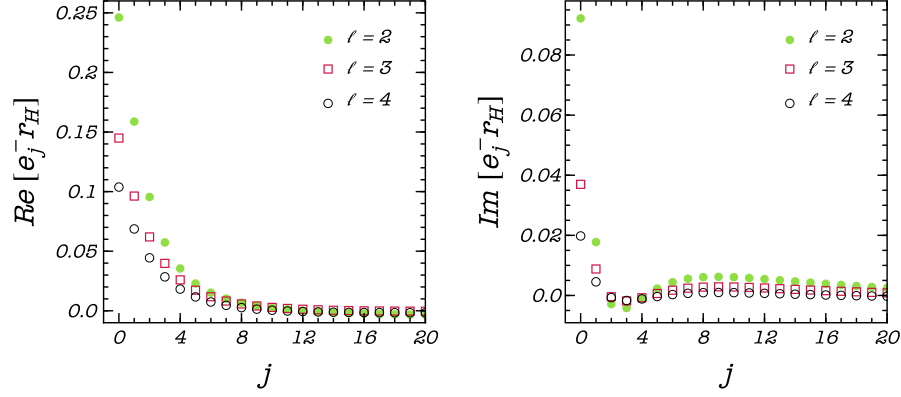


FIG. 1. Real and imaginary parts of the components e_j^- defined in (10) for $j = 1, \dots, 20$ and odd-parity gravitational perturbations.

III. RESULTS

The most important results of our analysis are the numerical values of the complex factors e_j^\pm and d_j^s , which were obtained via high-precision direct integration of the relevant equations. Figure 1 and Table I show representative results for the coefficients e_j^\pm and d_j^s in (10) and (11) for odd gravitational and scalar modes, respectively. Results for all fields and the lowest multipole numbers $\ell = |s|, |s+1|, \dots, 10$ are available online [30]. Our calculated e_j^\pm (and d_j^s) have a five-digit accuracy for $j \lesssim 30$.

A. Isospectrality

A remarkable result in GR concerns the isospectrality of potentials (2) and (3). These potentials are sometimes referred to as “superpartners,” since they can both be obtained from a “superpotential” W_0 [4,31]:

$$fV_\pm = W_0^2 \mp f \frac{dW_0}{dr} - \frac{\lambda^2(\lambda+2)^2}{36r_H^2}, \quad (12)$$

$$W_0 = \frac{3r_H(r_H - r)}{r^2(3r_H + \lambda r)} - \frac{\lambda(\lambda+2)}{6r_H}. \quad (13)$$

When the potentials are perturbed from their GR values as in Eq. (6), isospectrality will in general be broken, unless the coefficients α_j^+ are related in some way to α_j^- .

TABLE I. Real and imaginary part of a few of the first frequency components for odd-parity gravitational (e_j^-) and scalar field (d_j^0) perturbations with $\ell = 2$. The full set of frequencies up to $j = 100$ is provided online [30].

j	$r_H e_j^-$	$r_H d_j^0$
0	0.24725 + 0.092643i	0.15782 + 0.054078i
1	0.15985 + 0.018208i	0.11307 + 0.015119i
2	0.096632 - 0.0024155i	0.076570 + 0.00016782i
3	0.058491 - 0.0037179i	0.051121 - 0.0032973i
4	0.036679 - 0.00043870i	0.034527 - 0.0024724i
10	0.0036853 + 0.0065244i	0.0050350 + 0.0037363i

The linearized superpartner relations now yield

$$2 \frac{d\delta W}{dr} = \delta V_- - \delta V_+, \quad (14)$$

$$4 \frac{W_0}{f} \delta W = \delta V_+ + \delta V_-, \quad (15)$$

with δW the induced change in the superpotential W with respect to its GR value W_0 [Eq. (13)].

Let us admit that the theory predicts a set of α_j^- . The relations above give us a unique possibility for α_j^+ , namely,

$$\alpha_0^+ = \alpha_0^-, \quad \alpha_1^+ = \alpha_1^-, \quad (16)$$

$$\alpha_2^+ = \alpha_2^- + \frac{6(\alpha_0^- - \alpha_1^-)}{\lambda(\lambda+2)}, \dots \quad (17)$$

These relations show that isospectrality is very fragile, and that in general it will be broken.

IV. ASYMPTOTIC BEHAVIOR AND CONVERGENCE OF SERIES

Having defined the “basis vectors” e_j , it is crucial to investigate their behavior, and the overall convergence of the QNM frequency expansion (10). For large j , the contribution to the potential from the j th term can be described by a Gaussian profile depending on the tortoise coordinate² r_* , i.e.,

$$fr_H^2 \delta V_j = f \alpha_j^\pm \left(\frac{r_H}{r} \right)^j \simeq \frac{\alpha_j^\pm}{e_j} e^{-(r_* + r_H \ln j)^2 / (2r_H^2)}, \quad (18)$$

showing that the location of the peak is proportional to $-\ln j$ (in terms of the tortoise coordinate r_*). Moreover,

²The Gaussian fit (18) represents a good approximation for the peak of the potential’s modifications $fr_H^2 \delta V_j$. Although its precision is limited for large values r_* , far from the maximum of the function, the fit is accurate enough for the purposes of this section.

since the amplitude of $f\delta V_j$ is proportional to $1/j$ for large j , we can expect that $e_j \rightarrow 0$ for $j \rightarrow \infty$. Following [32], a perturbative expansion of the QNM frequency yields the following correction:

$$e_j \alpha_j^\pm \propto \int_{-\infty}^{\infty} dr_* f r_H^2 \delta V_j \Phi_0^2, \quad (19)$$

where Φ_0 is the unperturbed mode itself.

Let us assume that $f\delta V_j$ has support only in $-r_H \ln j - r_H \sigma \leq r_* \leq -r_H \ln j + r_H \sigma$, with a constant $\sigma = \mathcal{O}(1)$ (the particular choice of σ does not affect our results). Then the integral can be estimated as follows:

$$\begin{aligned} e_j \alpha_j^\pm &\propto \int_{-r_H \ln j - r_H \sigma}^{-r_H \ln j + r_H \sigma} dr_* f \delta V_j \Phi_0^2 \\ &= \int_{-r_H \ln j - r_H \sigma}^{-r_H \ln j + r_H \sigma} dr_* \frac{\alpha_j^\pm}{e_j} e^{-(r_* + r_H \ln j)^2 / (2r_H^2)} e^{-i2\omega_0 r_*} \\ &= \frac{\alpha_j^\pm}{j} j^{2ir_H \omega_0} \times \text{constant}. \end{aligned} \quad (20)$$

In conclusion we find

$$e_j \propto \frac{j^{2i\omega_0 r_H}}{j} = \frac{j^{2ir_H \omega_R - 2r_H \omega_I}}{j} = \frac{e^{2ir_H \omega_R \ln j}}{j^{1+2r_H \omega_I}}, \quad (21)$$

where $\omega_0 = \omega_R + i\omega_I$. Equation (21) shows that the basis frequency e_j decays as $1/j^{1+2r_H \omega_I}$, and oscillates as $\sin(2r_H \omega_R \ln j)$. For tensor perturbations³ with $\ell = 2$, we have $e_j \simeq \frac{e^{1.51 \ln j}}{j^{0.64}}$. This result is also confirmed by a numerical study of e_j for large j . We fitted our large- j numerical results to the following functional form:

$$e_j \sim \frac{\kappa}{j^\beta} \sin(\gamma \ln j + \zeta), \quad (22)$$

with $(\kappa, \beta, \gamma, \zeta)$ numerical coefficients to be determined. We find $\beta \simeq 0.66$ and $\gamma \simeq 1.5$, in very good agreement with the analytical estimate of Eq. (21). Figure 2 compares the fit with the actual data for e_j . The agreement is extremely good, and it is strong evidence in support of the asymptotic behavior (21).

Having assessed the asymptotic properties of the basis e_j , we can now study in detail the convergence of the frequencies ω_{QNM} . We assume that the effective potential of the specific theory under consideration is C^∞ , such that we can expand it for large distances as $\delta V = \sum_{j=0}^{\infty} \frac{a_j}{r^j}$, where the a_j 's are constant coefficients. Moreover, following our formalism developed in Sec. II, we expand the corrections with respect to the Schwarzschild term according to Eq. (6),

³Note that Eq. (19) holds regardless of the overtone number and of the multipole number of the mode, if j is sufficiently large.

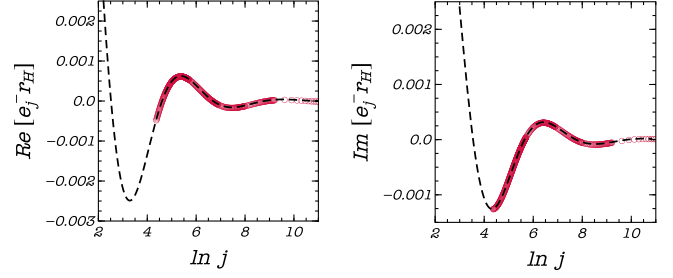


FIG. 2. Values of the frequency components e_j^- (red dots) for the odd tensor modes with $\ell = 2$, compared against the numerical fit of Eq. (22) (black dashed curve).

such that a mapping exists between the α_j^\pm and the coefficients of the theory a_j , i.e., $\alpha_j^\pm = r_H^2 \frac{a_j}{r_H^j}$. To study the convergence of the QNM expansion (10), we can compute the ratio

$$\Upsilon = \lim_{n \rightarrow \infty} \left| \frac{\alpha_{n+1}^\pm e_{n+1}}{\alpha_n^\pm e_n} \right|. \quad (23)$$

The series converges when $\Upsilon < 1$. Replacing the explicit form of the expansion coefficients α_j^\pm , and using the fact that the frequency basis vectors e_j behave as in Eq. (22)—or equivalently Eq. (21)—we find

$$\Upsilon = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \frac{n^\beta}{n^{\beta+1}} \frac{\sin[\gamma \ln(n+1) + \zeta]}{r_H \sin[\gamma \ln n + \zeta]} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

The previous equation implies that, in our approach, the convergence of the QNM frequencies depends on the behavior of the coefficients of the (expanded) potential for the specific theory of gravity.

As a nontrivial example to test the formalism, consider the case

$$r_H^2 \delta V = 10^{-2} \frac{\rho_0^k}{\rho_0^k + r^k} = 10^{-2} \sum_{j=1}^{\infty} (-1)^{j+1} \left(\frac{\rho_0}{r} \right)^{kj}, \quad (24)$$

where ρ_0 is a constant, and $\alpha_j^\pm = (-1)^{j+1} 10^{-2} (\rho_0/r_H)^{kj}$. For this potential the convergence criterion reduces to

$$\Upsilon = \lim_{n \rightarrow \infty} \left| \frac{\rho_0^{n+1}}{\rho_0^n} \right| = |\rho_0|, \quad (25)$$

and therefore the perturbative formalism is valid if $\rho_0 < 1$.

V. EXAMPLES

In this section we will provide some specific examples of theories of gravity whose gravitational perturbations can be described in terms of a master equation with the same functional form of Eq. (1). In order to determine the domain of validity and the accuracy of our semianalytical approach,

we will compare the results obtained by solving the exact equations of the theory on a case-by-case basis with the frequencies computed through Eq. (10).

A. Effective field theory

An interesting scenario to test against our formalism is given by the BH solutions derived within an EFT approach in Refs. [12,13]. Deviations from GR in this theory are characterized by three coupling parameters, which define the scale of the strong curvature modifications. For the sake of simplicity, we focus on a single coupling parameter, namely $\tilde{\Lambda}$ (ϵ_2 in the notation of Ref. [13], which we adopt). In this setup, nonspinning BHs are described by the Schwarzschild geometry, and its axial perturbations lead to the following wave equation:

$$\frac{d^2\Psi_-}{dr_*^2} + [\omega^2 - f(V_- + \delta V_-)]\Psi_- = 0, \quad (26)$$

where V_- is given by Eq. (2), while the extra term in the effective potential reads

$$\delta V_- = \epsilon_2 \frac{18(\ell+2)(\ell+1)(\ell-1)r_H^8}{r^{10}}. \quad (27)$$

Note that in this case $f = 1 - r_H/r$ and $dr/dr_* = f$. According the formalism developed in Sec. II, the only coefficient of the expansion (10) is then

$$\alpha_{10}^- = 18(\ell+2)(\ell+1)(\ell-1)\epsilon_2. \quad (28)$$

From Table I we read off $r_H\omega = r_H\omega_0 + (0.0663354 + 0.117439i)\epsilon_2(\ell-1)(\ell+1)(\ell+2)$. For $\ell = 2$ this leads to a fractional deviation from GR of $\epsilon_2(2.13, -15.84)$ in the real and imaginary components, respectively, in excellent agreement with published results: cf. Eq. (20) of Ref. [13]. We find a similar level of agreement for higher- ℓ modes.

B. Reissner-Nordström black holes

The odd-mode gravitational perturbations of RN BHs are driven by the master equation

$$f_{\text{RN}} \frac{d}{dr} \left(f_{\text{RN}} \frac{d\Phi}{dr} \right) + (\omega^2 - f_{\text{RN}} V_{\text{RN}})\Phi = 0, \quad (29)$$

with

$$f_{\text{RN}} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \left(1 - \frac{r_H}{r}\right) \left(1 - \frac{r_-}{r}\right), \quad (30)$$

$$V_{\text{RN}} = \frac{\ell(\ell+1)}{r^2} + \frac{4r_H r_-}{r^4} - \frac{3(r_H + r_-)}{2r^3} - \frac{\sqrt{4(\ell-1)(\ell+2)r_H r_- + \frac{9}{4}(r_H + r_-)^2}}{r^3}. \quad (31)$$

We will consider the charge Q —or, equivalently, the Cauchy horizon radius $r_- \sim Q^2/(2M)$ —to be a small perturbation. Note, however, that r_H is *not* to be assumed to be equal to $2M$, as it would introduce an error of the same order of the leading order correction. According to the procedure in Appendix B, introducing $\phi = \sqrt{Z}\Phi = (1 - r_-/r)^{1/2}\Phi$, we can write the master equation in the form

$$f \frac{d}{dr} \left(f \frac{d\phi}{dr} \right) + \left[\left(1 - \frac{r_-}{r_H}\right)^{-2} \omega^2 - f(V_- + \delta V) \right] \phi = 0, \quad (32)$$

with

$$\delta V = 2 \frac{r_-}{r_H} \omega_0^2 + \frac{r_-}{r_H^3} \left(\frac{r_H}{r} \right)^3 \left[\frac{5r_H}{2r} - \frac{\lambda+6}{3} \right], \quad (33)$$

where $f = 1 - r_H/r$ and we ignored $\mathcal{O}(r_-^2)$ terms. We can now apply our formalism to RN perturbations. The coefficients of the effective potential expansion (10) read

$$\alpha_0^- = \frac{2r_-}{r_H} (\omega_0 r_H)^2, \quad \alpha_3^- = -\frac{(\lambda+6)r_-}{3r_H}, \quad \alpha_4^- = \frac{5r_-}{2r_H}.$$

Thus, the QNMs for $\ell = 2$ can be written as

$$\begin{aligned} \omega_{\text{QNM}} &= \left(1 - \frac{r_-}{r_H}\right) \left(\frac{2\Omega_0}{r_H} + e_0\alpha_0^- + e_3\alpha_3^- + e_4\alpha_4^- \right) \\ &= \frac{\Omega_0}{M} + \frac{(0.0258177 - 0.002824i)Q^2}{M^3}, \end{aligned} \quad (34)$$

where $r_H = M + \sqrt{M^2 - Q^2}$, $r_- = M - \sqrt{M^2 - Q^2}$, and in the last line we have expanded the equation up to order $\mathcal{O}(Q^2)$. Here, Ω_0 is the dimensionless Schwarzschild QNM frequency, so $\Omega_0 = 0.3736716844180418... - i0.0889623156889357...$ for $\ell = 2$.⁴ Using the notation of Ref. [28], these results imply $(\alpha_R^-, \alpha_I^-) = (-6.9, -3.2) \times 10^{-8} [Q/(10^{-3}M)]$, in excellent agreement with reported values in the literature [28,33,34].

We can quantify the accuracy of our approach and estimate the uncertainty on the non-GR modifications of the QNM frequencies by defining

$$\Delta_R = |\text{Re}(\omega_{\text{QNM}} - \omega_{\text{GR}}) / \text{Re}(\omega_{\text{full}} - \omega_{\text{GR}}) - 1|, \quad (35)$$

⁴Because ω_{QNM} in our formula is written as a function of r_H and α_j , our formalism requires that one first writes down ω_{QNM} as a function of r_H and α_j . We note that we should formally use ω_0 in Eq. (10) as $2\Omega_0/r_H$.

TABLE II. Relative percentage errors on the real and imaginary parts of the QNMs for RN BHs, as a function of the charge-to-mass ratio Q/M .

Q/M	Δ_R	Δ_I
0.00	0%	0%
0.05	0.11%	0.042%
0.10	0.43%	0.17%
0.20	1.7%	0.66%
0.30	3.8%	1.5%
0.40	6.8%	2.6%
0.50	11%	4.2%

$$\Delta_I = |\text{Im}(\omega_{\text{QNM}} - \omega_{\text{GR}}) / \text{Im}(\omega_{\text{full}} - \omega_{\text{GR}}) - 1|, \quad (36)$$

where ω_{full} is computed by numerically solving the *exact* master equation without any approximation, and ω_{QNM} is given by Eq. (34). For $Q/M = 0.2$, for example, the QNMs frequencies derived using Eq. (34) agree to within 0.05% with those in Ref. [33]. The errors obtained for different values of Q/M are listed in Table II.

The relative uncertainties $\Delta_{R,I}$ grow with Q/M up to $\sim 10\%$ (4%) for the real (imaginary) component of the mode's frequency when $Q/M = 0.5$. This behavior is consistent with the assumptions made to obtain Eq. (34), in which we neglected terms that are order $\mathcal{O}((\alpha_j^\pm)^2)$, the latter corresponding to ignoring $\mathcal{O}(Q^4)$ corrections to the RN perturbations. Therefore, for large values of Q , a better accuracy would require to compute ω_{QNM} by including second-order terms in both α_j^+ and α_j^- .

C. Scalars around a slowly spinning black hole

The massless Klein-Gordon equation $\square\Psi = 0$ around a slowly rotating Kerr BH can be written as

$$f \frac{d}{dr} \left(f \frac{d}{dr} \right) \Phi + \left(\omega^2 - fV_0 - \frac{4amM\omega}{r^3} \right) \Phi = 0, \quad (37)$$

at $\mathcal{O}(a)$, where $f = 1 - 2M/r$, a/M is the BH angular momentum, and we assumed $\Psi = e^{-i\omega t} Y_{\ell m}(\theta, \phi) \Phi(r)/r$. We can rewrite this equation in the following form:

$$f \frac{d}{dr} \left(f \frac{d}{dr} \right) \Phi + \left[\left(\omega - \frac{am}{r_H^2} \right)^2 - f \left(V_0 - \frac{2am\omega}{r_H^2} - \frac{2am\omega r_H}{r^2} - \frac{2am\omega}{r_H^2} \left(\frac{r_H}{r} \right)^2 \right) \right] \Phi = 0, \quad (38)$$

where we used $r_H = 2M + \mathcal{O}(a^2)$. Thus

$$\beta_0^0 = \beta_1^0 = \beta_2^0 = -2am\omega_0^0, \quad (39)$$

and we find

TABLE III. Relative percentage errors in the real and imaginary parts of the QNM frequencies for scalar perturbations around a slowly spinning black hole, as a function of the BH angular momentum a/M .

a/M	Δ_R	Δ_I
0	0%	0%
10^{-4}	0.0050%	0.83%
10^{-3}	0.049%	5.1%
10^{-2}	0.49%	34%

$$\omega_{\text{QNM}} = \omega_0^0 + \frac{am}{r_H^2} - 2am\omega_0^0(d_0^0 + d_1^0 + d_2^0). \quad (40)$$

Comparing these results with numerical data for $\ell = m = 2$ [4,30], we find the relative percentage errors (Δ_R, Δ_I) listed in Table III. Our formula is a good approximation for very small $a/M \sim 10^{-4}$. For $a/M \sim 10^{-2}$ the agreement gets worse, especially in the imaginary component. We find that $a/M \ll 10^{-2}$ is a necessary condition for our approximation to be valid.

VI. STATISTICAL ERRORS

In this section we analyze the detectability of the modifications of the QNM spectrum by space and terrestrial GW interferometers. We follow the approach described in Ref. [5], and we refer the reader to this paper and references therein for further details. Since we are interested only in an order-of-magnitude estimate of the associated observational errors, we *assume* that the mass of the BH is known (and therefore that the fundamental GR frequencies are known). For a more sophisticated analysis, we refer the reader to Refs. [7,35].

The gravitational waveform measured by the interferometers is a linear superposition of two polarization states of the form $h = h_+ F_+ + h_\times F_\times$, where F_+ and F_\times denotes the standard pattern functions (which depend on the source orientation with respect to the detector and on a ‘‘polarization angle’’). In the frequency domain, the two GW components are simply given by

$$\begin{aligned} \tilde{h}_+(f) &= \frac{A_{\ell mn}^+}{\sqrt{2}} [e^{i\phi_{\ell mn}^+} S_{\ell mn} b_+(f) + e^{-i\phi_{\ell mn}^+} S_{\ell mn}^* b_-(f)], \\ \tilde{h}_\times(f) &= -\frac{iA_{\ell mn}^\times}{\sqrt{2}} [e^{i\phi_{\ell mn}^\times} S_{\ell mn} b_+(f) + e^{-i\phi_{\ell mn}^\times} S_{\ell mn}^* b_-(f)], \end{aligned}$$

where the amplitude coefficients $A_{\ell mn}^{+,\times}$ and the phase coefficients $\phi_{\ell mn}^{+,\times}$ are real, $S_{\ell mn}$ represent the (complex) spin-weighted spheroidal harmonics of spin weight 2, which depend on the polar and azimuthal angles, and $b_\pm(f)$ are the Breit-Wigner functions:

$$b_{\pm}(f) = \frac{-\text{Im}[\omega_{\ell mn}]}{\text{Im}[\omega_{\ell mn}]^2 + (\omega \pm \text{Re}[\omega_{\ell mn}])^2}. \quad (41)$$

For simplicity, we consider $A_{\ell mn}^{\times} = N_{\times} A_{\ell mn}^{+}$ and $\phi_{\ell mn}^{\times} = \phi_{\ell mn}^{+} + \phi_0$, where N_{\times} and ϕ_0 are a scale factor and a phase shift, which are assumed to be known.

Moreover, consistently with the approach introduced in Sec. II, we express the modes' frequency and damping time as (small) deviations of the corresponding Schwarzschild values: $\omega_{\ell mn} = \omega_{\ell mn}^0 + \delta\omega$, where $\delta\omega = \delta\omega_1 + i\delta\omega_2$. The GW signal is then completely specified by four parameters, $(A_{\ell mn}^{+}, \phi_{\ell mn}^{+}, \delta\omega_1, \delta\omega_2)$. In order to estimate the errors on the coefficients $\delta\omega_{1,2}$, we consider a Fisher matrix approach, which is a reliable approximation of a full Bayesian analysis for signals with high signal-to-noise ratios [36]. For this reason we focus on possible detections by the space interferometer LISA [37], an ideal tool to exploit the full potential of QNM spectroscopy [38–40]. Finally, we set $N_{\times} = 1$, and $\phi_{\ell mn}^{+} = \phi_0 = 0$. We compute the errors on the fundamental $\ell = 2$ mode of nonspinning BHs, averaging over sky orientation through the following identities:

$$\langle F_{+}^2 \rangle = \langle F_{\times}^2 \rangle = \frac{1}{5}, \quad \langle F_{+} F_{\times} \rangle = 0, \quad \langle |S_{lmn}|^2 \rangle = \frac{1}{4\pi}. \quad (42)$$

Note that the amplitude of the signal depends in general on the mass, the source distance, and the energy released during the ringdown [5]. For simplicity, here we fix $A_{\ell mn}^{+}$ by requiring that for $\delta\omega_1 = \delta\omega_2 = 0$ the signal-to-noise ratio is equal to 10^4 (a typical value for LISA sources [40]).

The contour plot of Fig. 3 shows the relative errors $\sigma_{\delta\omega_1}/\delta\omega_1$ as a function of various combinations of $(\delta\omega_1, \delta\omega_2)$. Dashed white curves identify all the possible GR modifications that lead to uncertainties of 20% and 1%. We immediately note from the plot that the values of $\sigma_{\delta\omega_1}/\delta\omega_1$ are almost completely degenerate with respect to $\delta\omega_2$. More interestingly, all the parameter space sampled seems to indicate that LISA would be able to provide high-precision measurements of the frequency shifts, with a relative accuracy better than 1%. This is extremely promising for some of the GR modifications which are currently under investigation. For the EFT described in Sec. VA, and identified by a red star in Fig. 3, the frequency change $\delta\omega_1$ could be constrained with an error of $\sim 20\%$.

We obtain similar results for the second parameter $\delta\omega_2$, with relative uncertainties being always below $\sim 50\%$ for all the configurations analyzed. Note finally that a different choice of the waveform parameters, like the quality factor $Q_{\ell mn} = \text{Re}[\omega_{\ell mn}]/\text{Im}[\omega_{\ell mn}]/2$, may provide better results. We defer more detailed investigations of the statistical errors, as well a multiband analysis including third-generation detectors [6,41], to future work.

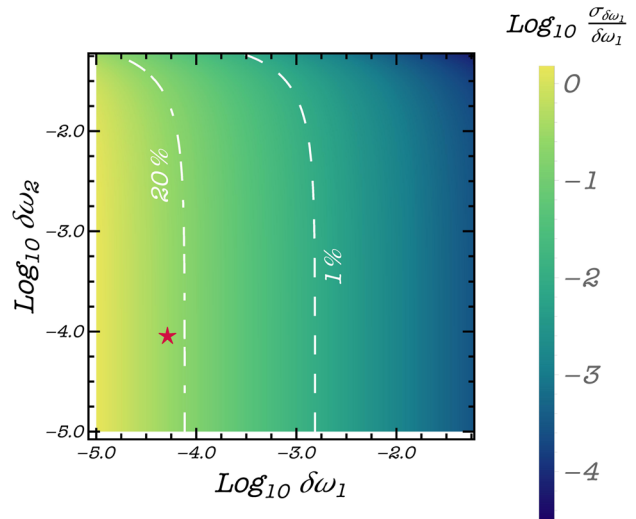


FIG. 3. Contour plot for the relative errors on the parameters $\delta\omega_1$ which modifies the frequency of the $\ell = 2$ fundamental BH mode. Vertical dashed lines represent configurations that lead to 20% and 1% fixed uncertainty. The red star identifies the values on $(\delta\omega_1, \delta\omega_2)$ for the EFT model described in Sec. VA.

VII. DISCUSSION

Our results yield (modulo an algebraic multiplication and sum) the QNM frequencies of any BH spacetime, provided that it describes a small deviation away from the Schwarzschild geometry. Thus, step 4 in the abstract can be avoided altogether by using our results, which are available online.

There are some important open questions, in particular the extension to coupled master equations [20]. The expansion (6) describes many known theories, and is “reasonable.” However, our formalism requires Υ in Eq. (23) to be smaller than unity. This restricts the possible space of theories, as we discussed in the example of Eq. (24). In fact, we suspect that potentials localized far away from the horizon may be hard to accommodate: perturbation theory projects the perturbing potential onto the undisturbed QNMs, but these diverge exponentially at large distances. Thus, there are known, exponentially sensitive corrections when the potentials have structure at large distances [28,32,42].

Alternative expansions are possible, and have been used. Since the GR QNMs are strongly connected to the light ring [4,43,44], a tempting alternative is to expand the perturbing potential around the light ring [16,19]. Such an approach is adequate for approximate, WKB-like expansions, but it may not be suitable for accurate numerical results.

It is intrinsic to our approach (and to many others) that the calculations are performed in a very special coordinate system, and the gauge invariance of the final results is not obvious. On the other hand, background solutions in generic theories are also usually written in very special coordinate systems. This weakness comes hand in hand with the power of the present approach: it handles any potential, but it is blind

to the underlying theory or background. This blindness prevents us from understanding the connection between the expansion parameters in the potentials and the metric multipoles, or to explore the possible connection between geodesic motion and QNMs [44].

Finally, we should highlight the fact that our results apply only to those modes which differ slightly from those of GR. The addition of a small perturbation to a known potential causes the appearance of both a perturbatively small correction to the modes, and new nonperturbative solutions. Take for instance the term β_0 for scalar perturbations, which is equivalent to a mass parameter $\mu^2 = \beta_0/r_H^2$. It is well known that for small μ , the GR QNMs are slightly displaced, and a new quasibound state with frequency $\omega^2 \sim \mu^2 - r_H^2 \mu^4 / (4(\ell + 1)^2)$ branches off from a zero-mode in the zero-mass limit [26,45,46]. We find, numerically and analytically, that a similar mode exists when α_0 is turned on. Such phenomena occur in electromagnetic waveguides as well [47,48]. Our analysis is unable to cover systematically the emergence of these new families of modes.

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APPENDIX A: KLEIN-GORDON AND MAXWELL EQUATIONS IN STATIC, SPHERICALLY SYMMETRIC SPACETIMES

Consider a static, spherically symmetric spacetime

$$ds^2 = -f_t dt^2 + \frac{1}{f_r} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (\text{A1})$$

with $f_t = f_t(r)$, $f_r = f_r(r)$. The Klein-Gordon equation $\square\Psi = 0$ on this background spacetime reads

$$\sqrt{f_t f_r} [\sqrt{f_t f_r} (\Phi^0)']' + \left[\omega^2 - f_t \frac{\ell(\ell+1)}{r^2} - \frac{(f_t f_r)'}{2r} \right] \Phi^0 = 0,$$

where primes stand for radial derivatives, and the scalar $\Psi = e^{-i\omega t} Y_{\ell m} \Phi^0(r)/r$. Similarly, the Maxwell equations $\nabla_\mu F^{\mu\nu} = 0$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ reduce to a single master equation

$$\sqrt{f_t f_r} [\sqrt{f_t f_r} (\Phi^{1,\pm})']' + \left[\omega^2 - f_t \frac{\ell(\ell+1)}{r^2} \right] \Phi^{1,\pm} = 0,$$

where we assumed

$$\begin{aligned} A_\mu dx^\mu &= a_t(r) e^{-i\omega t} Y_{\ell m} dt + a_r(r) e^{-i\omega t} Y_{\ell m} dr \\ &+ a_+(r) e^{-i\omega t} (\partial_\theta Y_{\ell m} d\theta + \partial_\phi Y_{\ell m} d\phi) \\ &+ a_-(r) e^{-i\omega t} \left(\frac{\partial_\phi Y_{\ell m}}{\sin\theta} d\theta - \partial_\theta Y_{\ell m} d\phi \right), \end{aligned} \quad (\text{A2})$$

and

$$\Phi^{1,+} = \frac{\sqrt{f_t f_r} a_r}{i\omega}, \quad \Phi^{1,-} = a_-. \quad (\text{A3})$$

When the spacetime is close to the Schwarzschild spacetime, we can assume $f_t = (1 - r_H/r)[1 + \delta_t(r)]$, $f_r = (1 - r_H/r)[1 + \delta_r(r)]$, with $\delta_{t,r}$ small quantities. Therefore the perturbations induced by scalar and vector fields on a spherically symmetric background can always be cast in the form

$$F \frac{d}{dr} \left[F \frac{d\Phi}{dr} \right] + (\omega^2 - F\bar{V})\Phi = 0, \quad (\text{A4})$$

with

$$F(r) \equiv f(r)Z(r) = \left(1 - \frac{r_H}{r} \right) Z(r). \quad (\text{A5})$$

The description of tensor perturbations requires a knowledge of the underlying gravitational theory. While we do not have a rigorous proof, we expect that under the assumptions spelled out in the Introduction it should be

possible to reduce tensor perturbations to the study of Eq. (A4). As shown in Ref. [13], even when the equations of motion are of higher order, the relations between metric quantities in GR at zeroth order can be used to eliminate higher-order derivatives. Then, algorithms similar to that introduced within GR in Ref. [22] can be used to cast the equations in the form (A4).

APPENDIX B: TRANSFORMATION TO EQ. (1)

As shown in Appendix A, when working out the perturbation equations the “natural” form of the master equation is not our Eq. (1), but rather Eq. (A4). Defining $\phi \equiv \sqrt{Z}\Phi$, Eq. (A4) reads

$$f \frac{d}{dr} \left[f \frac{d\phi}{dr} \right] + \left[\frac{\omega^2}{Z^2} - fV \right] \phi = 0, \quad (\text{B1})$$

$$V = \frac{\bar{V}}{Z} - \frac{f(Z')^2 - 2Z(fZ)'}{4Z^2}. \quad (\text{B2})$$

If \bar{V} and Z are written in terms of small perturbations $\delta\bar{V}$ and δZ as

$$\bar{V} = V_{\text{GR}} + \delta\bar{V}, \quad (\text{B3})$$

$$Z = 1 + \delta Z, \quad (\text{B4})$$

then the potential V becomes (to leading order in the perturbations)

$$V = V_{\text{GR}} + \delta\bar{V} - V_{\text{GR}}\delta Z + \frac{1}{2}(f\delta Z)', \quad (\text{B5})$$

and the frequency-dependent term in Eq. (B1) reads

$$\frac{\omega^2}{Z^2} = \omega^2[1 - 2\delta Z(r_H)] - 2\omega^2[\delta Z(r) - \delta Z(r_H)]. \quad (\text{B6})$$

The factor $[1 - 2\delta Z(r_H)]$ in the first term on the right-hand side amounts to a rescaling of the frequencies. The second

term on the right-hand side vanishes at $r = r_H$, so one can factor out f in this term and absorb it in the perturbed potential. Therefore Eq. (A4) is equivalent to the starting point of our analysis, i.e., Eq. (1).

APPENDIX C: e_3^\pm AND e_4^\pm FOR LARGE ℓ

We can regard the deviation between V_+ in Eq. (3) as V_- in Eq. (2) as a small correction for large ℓ , because $V_+ - V_- = \mathcal{O}(\ell^{-2})$. More explicitly, we have

$$V_+ = V_- + \frac{1}{r_H^2} \left[-\frac{12}{\ell^2} \left(\frac{r_H}{r} \right)^3 + \frac{18}{\ell^2} \left(\frac{r_H}{r} \right)^4 \right] + \frac{1}{r_H^2} \left[\frac{12}{\ell^3} \left(\frac{r_H}{r} \right)^3 - \frac{18}{\ell^3} \left(\frac{r_H}{r} \right)^4 \right] + \mathcal{O}(\ell^{-4}). \quad (\text{C1})$$

Since $V_+ - V_-$ can be expanded in powers of r_H/r and the series is convergent in $r_H \leq r < \infty$, we can apply our formalism to this case. From our QNM formula, the relation,

$$\omega_{\text{QNM}}^+ = \omega_{\text{QNM}}^- + 6(-2e_3^- + 3e_4^-) \left(\frac{1}{\ell^2} - \frac{1}{\ell^3} \right) + \mathcal{O}(\ell^{-4}),$$

should hold. On the other hand, because QNM frequencies for even and odd modes are the same, we can obtain the large- ℓ relations

$$e_4^- = 2e_3^-/3 + \mathcal{O}(\ell^{-2}), \quad (\text{C2})$$

$$e_4^+ = 2e_3^+/3 + \mathcal{O}(\ell^{-2}). \quad (\text{C3})$$

A numerical calculation shows that $|1 - 2e_3^-/(3e_4^-)|$ is 0.00309 for $\ell = 10$, and 0.000798 for $\ell = 20$. This is consistent with the fact that the error is of order $\mathcal{O}(\ell^{-2})$. We should note that since our QNM formula contains an error of order $\mathcal{O}((\alpha_2)^2) = \mathcal{O}(\ell^{-4})$, we cannot obtain relations for e_j with $j \geq 5$.

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