Statistical multi-step direct reaction models and the eikonal approximation

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Abstract
Nucleon-induced pre-equilibrium reactions are now recognized as consisting almost exclusively of direct reactions in which incident nucleons induce excitations over a wide range of energy in the target nuclei. At low energies, one step reactions dominate with more steps becoming important as the incident energy increases. The characterization of this multistep scattering process in terms of eikonal waves and an optical interaction potential could furnish an important simplification of the description of the collision process. In this preliminary work we perform an analysis of elastic angular distributions for different target nuclei and incident projectile energies, using the eikonal approximation and a $t\rho$ interaction potential.

1 Introduction

Nuclear reactions are of interest in a wide range of areas, from basic science to nuclear and atomic technological applications. Their measured cross sections are energy-dependent quantities, which allows the study of a variety of processes using different theories accordingly to the energy scales and experimental set up involved. Pre-equilibrium emissions are events that occur on an intermediate time scale when compared to the slower process of evaporation (CN compound nucleus) and the fast single interaction of a direct nuclear reaction. A pre-equlibrium particle is emitted after one or more collisions with the nucleons of the target nucleons, but leaves the target nucleus before the statistical equilibrium of the compound system is reached. The relevance of these reactions goes beyond fundamental studies, playing a key role in technical applications in applied areas, e.g., fast nuclear reactors and accelerator-driven system (ADS), radiation beam therapy and medical radioisotope production.

The quantum formalism describing the pre-equilibrium component relies on the multi-step reaction theory framework. The first model was proposed by Agassi, Weindenmüller and Mantzouranis [5], being more rigously deduced later [4], and denominated the multi-step compound theory. The direct (continuum) version of the theory were pioneered by Feshbach, Kerman and Koonin [1]; Tamura, Udagawa and Lenske [3] followed by Nishioka, Weindenmüller and Yoshida [2]. The multi-step direct models describe a collision in terms of a leading incident particle that interacts with the nucleons on its way throw the target nucleus. This processes can generate different particle hole excitations with energy values varying from a few to tens of MeVs. The number of projectile-target interactions is associated with number of steps in the multi-step formalism context. This is also directly related to the energy transferred in the reaction, where the one step is predominant for lower excitation energy events, while more and more steps are expected for reactions with higher energy transfer.

The transition matrix elements for each particle-hole excitations requires the description of the incoming and outgoing projectile waves. These functions represent scattering waves solutions and are usually taken as an expansion of distorted plane waves in the DWBA (distorted wave Born approximation). A sum of several terms representing waves with different angular momenta is required. For high energy scattering, the contribution of larger angular momenta are expected to be relevant, slowing down the convergence of this expansion. For these cases, the angular momentum sum can be conveniently substituted by an integral over the impact parameter. This is a semi-classical approximation, often called the Glauber approximation in a nuclear scattering context, but more often known as the eikonal approximation, since it was firstly applied in optics. The phase shift of the scattered particle is obtained by an integral of the optical potential along a straight line trajectory. The projectile-target interaction is taken as complex function to account for absorption of particles from the incident beam. For charged projectiles, the Coulomb part of the interaction is usually treated separately [6].

In this work we report our preliminary results on the description of elastic nucleon-induced reactions in which we use the eikonal distorted wave representation of the wave functions and the $t\rho$ approximation to the optical potential. This work is intended to be a test of both the eikonal approximation and the optical potential that we plan to use in the future in a description of multi-step reactions.

We organize the present work as follows. An introduction to the main concepts regarding the eikonal and the optical interactions used in this work are given in Sec. 2. In Section 3 we present the elastic cross sections computed for proton induced reactions on two different targets ^{90}Zr and ^{208}Pb. We summarize our results in Sec. 4. Sec. 4.1 contains the expression we plan to use to obtain the inelastic cross section corresponding to a particle-hole excitation in the first step of a reaction.

2 Formalism and Interaction

Particle scattering wave functions are usually taken as free plane-waves in the first order Born approximation to the scattering amplitude. A better representation, for higher energy particles, is to assume a straight line trajectory for the incident particle and to add a phase correction taking into account the effects of the potential on the scattered wave. This can be considered a semi-classical approximation due to its connection to the more generic WKB (Wentzel-Kramers-Brillouin) approach. The eikonal distorted wave in cylindrical coordinate system is given by [6]

$$\psi_k^{(+)}(z,\,\mathbf{b}) = \exp\left[i\mathbf{k}\cdot(\mathbf{b}+z\hat{\mathbf{z}}) - \frac{i}{\hbar v}\int_{-\infty}^{z} U\left(z',\,\mathbf{b}\right)dz'\right], \tag{1}$$

where b is the cylindrical radius variable. For an interaction potential U centered at the origin, the distance b can be interpreted as the classic impact parameter.
With this wave function, the elastic scattering amplitude becomes

$$\begin{aligned} f_{el}(\mathbf{k}_f,\,\mathbf{k}_i) &= -\frac{1}{4\pi}\int d^3r\, e^{-i\mathbf{k}_f\cdot\mathbf{r}}\,\frac{2\mu}{\hbar^2}U(z,\,\mathbf{b})\,\psi_{k_i}^{(+)}(z,\,\mathbf{b}) \\ &= \frac{k}{i}\int bdb\, J_0(qb)\left(e^{2i\delta(b)}-1\right) \end{aligned}$$

where the eikonal phase shift is defined as

$$\delta(b) = -\frac{1}{\hbar v}\int_0^{\infty} U(z,\,\mathbf{b})\,dz\,. \tag{2}$$

For the transferred momentum $\mathbf{q} = \mathbf{k}_i - \mathbf{k}_f$, since $|\mathbf{k}_f| = |\mathbf{k}_i| = k$, we take

$$q = |\mathbf{k}_i - \mathbf{k}_f| = 2k\sin(\theta/2)\ ,$$

where θ is the scattering angle.
The differential elastic cross section is given by the squared absolute value of the scattering amplitude as

$$\frac{d\sigma_{el}}{d\Omega} = |f_{el}(k,\,\theta)|^2\ .$$

Here, we approximate the projectile-target optical interaction present in the eikonal phase (2), using the $t\rho$ approximation [7,8]. The forward-angle nucleon-nucleon t-matrix is often parameterized as

$$t_{n_1n_2}(\mathbf{q}=0) = -\frac{2\pi\hbar^2}{\mu}f_{n_1n_2}(\mathbf{q}=0) = -\frac{\hbar v}{2}\sigma_{n_1n_2}^T(\alpha_{n_1n_2}+i)$$

where $f_{n_1 n_2}$ is the $n_1 - n_2$ scattering amplitude ($n_{1,2} = n,\, p$) and $\sigma_{pp}^T = \sigma_{nn}^T$ and σ_{pn}^T are the proton-proton, neutron-neutron and proton-neutron total cross sections. The quantity $\alpha_{n_1 n_2}$ represents the ratio between the imaginary and the real part of the proton-nucleon scattering amplitude. We take for the proton-target optical potential

$$U(\mathbf{r}) = -\frac{\hbar v}{2}\left[\sigma_{pp}^T (i + \alpha_{pp})\,\rho_p(\mathbf{r}) + \sigma_{pn}^T (i + \alpha_{pn})\,\rho_n(\mathbf{r})\right],$$

where the total cross sections $\sigma_{n_1 n_2}^T$ as well as the factors $\alpha_{n_1 n_2}$ are energy dependent. We take both parameters from [7,9] and interpolate between the point given there when necessary. We assume that the cross sections and $\alpha_{n_1 n_2}$ factors used in the optical potential also contain the effects of Pauli blocking in the nuclear medium. The position dependent quantities $\rho_p(\mathbf{r})$ and $\rho_n(\mathbf{r})$ are the target proton and neutron densities. These are often approximated as Z/A and N/A times the total nucleon density, $\rho_m(\mathbf{r})$, where Z and N are the proton and neutron number of the nucleus of mass number $A = Z + N$.

The proton elastic scattering phase shift in the $t\rho$ approximation is then given by

$$\begin{aligned}\delta(b) &= \frac{1}{2}\sigma_{pp}^T (i + \alpha_{pp}) \int_0^\infty \rho_p(z,\, b)\, dz + \frac{1}{2}\sigma_{pn}^T (i + \alpha_{pn}) \int_0^\infty \rho_n\left(\sqrt{z^2 + b^2}\right) dz \\ &\approx \frac{1}{2}\left[\sigma_{pp}^T (i + \alpha_{pp})\frac{Z}{A} + \sigma_{pn}^T (i + \alpha_{pn})\frac{N}{A}\right] \int_0^\infty \rho\left(\sqrt{z^2 + b^2}\right) dz\,,\end{aligned}$$

where ρ is the nucleon density. For proton-induced reactions, we peform the usual separation into a pure Coulomb amplitude and a Coulomb-mdoified nuclear amplitude [6]. We also take into consideration the nuclear recoil correction [10].

In the next section we present the results obtained for proton induced reactions considering two different target nuclei, ^{90}Zr and ^{208}Pb. We performed this analysis for a large range of incident energy beams. Non-relativistic kinematics were employed. The experimental data sets used to compare with our calculations are given in Table 1.

Table 1: Data sets for the proton elastic cross sections.

Reaction	Proton Incident Energy E (MeV)
^{90}Zr(p,p)	61.4 [11], 80, 135, 160 [12], 156 [13], 185 [14], 400 [15]
^{208}Pb(p,p)	30.3 [16], 200, 400 [15], 318, 800 [17]

3 Results

We show in Figure 1 the normalized (to Rutherford) elastic proton angular distribution as a function of the scattering angle. We performed the calculations of p+^{90}Zr elastic scattering for several different incident energies. The theoretical calculations are in good agreement with the data, especially for higher energies and small angle deflections, as would be expected. At energies below 160 MeV, we overestimate the distribution at backward angles $\theta > 50°$. For lower incident energies, 80 and 61.4 MeV, the calculated distributions are slightly shifted in angle when compared to the experimental data. This is an effect caused by the geometry of the target nucleus and can be adjusted with the use of a more precise target radius.
Fig. 2 presents the cross section dependence on two different quantities for the same ^{208}Pb(p,p) reaction at different incident proton energies. In the left panel, the angular distribution is compared to data for three cases, two higher and one lower incident energies. The results for theses cases, just as in the previous figure, also follow the experimental data more closely for forward angular deflections. For

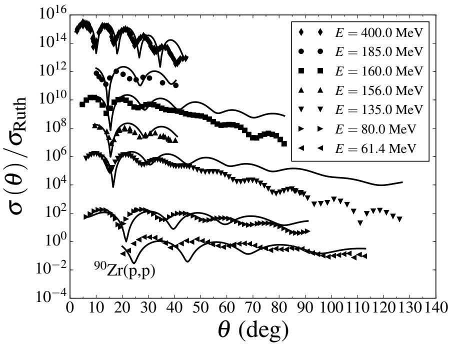

Fig. 1: Differential proton elastic scattering angular distribution (normalized to Rutherford) ^{90}Zr(p,p) for different incident projectile energies. Black solid curves are obtained with the eikonal and $t\rho$ approximations. The experimental data are shown as symbols (see Table 1). The cross sections are numerically shifted starting from the bottom.

the most extreme case at 30.3 MeV of incident energy, the oscillations in the calculation become very different from the experimental ones at the backward scattering angles, although the magnitude of the distribution is reproduced. In the right panel of Fig. 2, the dependence of the angular distribution on the transferred momentum is given. The curves at both 318 and 800 MeV show excellent agreement with the data. Small differences are seen for larger transferred momenta, corresponding to particle emission at backward angles. The results above are also aimed to test both the eikonal and $t\rho$ approaches close to

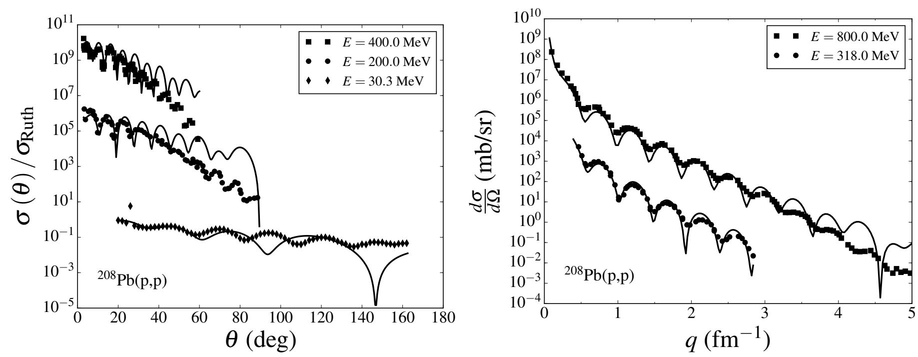

Fig. 2: Normalized differential proton elastic scattering cross sections is shown in the left panel for different incident proton energies. The dependence of the angular distribution on the transferred momentum is presented in the right panel. For these cases the ^{208}Pb(p,p) reaction is studied. Black solid curves are obtained with the eikonal and $t\rho$ approximations. The experimental data are shown as symbols (see Table 1). The cross sections are numerically shifted starting from the bottom.

their limits of validity. The $t\rho$ optical potential is expected to work for higher energy processes, where the medium effects present at lower energies can be neglected. The straight-line trajectory assumption of the eikonal scattering is also questionable in the low energy region. These are the main reasons for the observed discrepancy at lower energies and larger deflection angles. Apart from these limiting cases, our results are in fairly good agreement with the data for proton-induced reactions.

4 Conclusions

Calculations of elastic angular distributions for proton-induced reaction were performed at different incident energies for different targets. The eikonal distorted wave approximation together with a $t\rho$ approximation to the optical potential form the basis of the description we used to compare the theoretical cross sections to the experimental data. The results for ^{90}Zr and ^{208}Pb provided a good description of the experimental angular distributions. The agreement with the data is rather good for small angles and small transferred momenta, i.e. in the forward part of the angular distributions.

Although the eikonal approach was developed for high-energy processes, we have shown that the model can still provide accurate results for incoming particle energies of around 60 MeV. More extreme cases, such as at 30 MeV, can also be studied in this formalism with some restriction. The application of the present model for inelastic scattering is now in progress. Our objective is to represent emitted particles in the eikonal plus $t\rho$ approximation. The simplified picture of this approach provides important guidance for a better description of the inelastic scattering present in the multi-step direct formalism. Below we give the expression for a particle-hole excitation resulting from a collision between the incident particle and the nucleons in the target.

4.1 Perspectives

Here we consider the extension of the use of eikonal waves to represent the distorted waves of the DWBA. The integral over impact parameter reduces the complicated angular momentum coupling drastically and provides a simpler formula for implementations. The one-step DWBA amplitude of a nucleon-induced reaction has the general form

$$T_{DWBA} = \int d^3r \, \psi_{k_f}^{(-)*}(\mathbf{r}) \left\langle B \left| \sum_{j=1}^{A} V(\mathbf{r}-\mathbf{r}_j) \right| A \right\rangle \psi_{k_i}^{(+)}(\mathbf{r}) \,,$$

where we have written the nucleon-nucleus interaction as a sum of nucleon-nucleon interactions $V(\mathbf{r}-\mathbf{r}')$. Any individual interaction can be written as

$$\langle \mathbf{k}_f; ph \,|T|\, \mathbf{k}_i \rangle = \int d^3r \, d^3r' \, \psi_{k_f}^{(-)*}(\mathbf{r}) \, \psi_p^*(\mathbf{r}') \, V(\mathbf{r}-\mathbf{r}') \, \psi_h(\mathbf{r}') \, \psi_{k_i}^{(+)}(\mathbf{r}) \,,$$

where ψ_h is an occupied orbital in the initial nucleus (a hole state after the collision) and $\psi_p$is an unoccupied orbital or continuum state of the initial nucleus.

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