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Abstract

We discuss the behavior of a quantum 1/2-spin coupled to a time dependent magnetic field, which can be quasiperiodic or random. For a quasiperiodic field, the time evolution of the system on a coarse grained space has autocorrelation which does not decay and positive topological entropy. The information complexity diverges as a stretched exponential, although the Shannon entropy is zero and there is no divergence of nearby orbits. A simple random process is introduced in order to reproduce the main qualitative features of the observed weakly complex behavior.

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I. INTRODUCTION

For a classical system the meaning of deterministic chaos is clear: the largely used definition is in terms of sensitive dependence on the initial conditions, i.e., a positive maximum Lyapunov exponent λ [1]. It measures the exponential growth rate of the distance between initially close trajectories. For a quantum mechanical system this definition is not useful since $\lambda = 0$, because the evolution equation is linear.

However, systems with linear evolution laws can exhibit non trivial behaviors and some complex features. An important example is the passive advection of a scalar field $\Theta(\boldsymbol{x}, t)$ in a given velocity field, $\boldsymbol{u}(\boldsymbol{x}, t)$. The equation for $\Theta(\boldsymbol{x}, t)$ is:

$$\partial_t \Theta(\boldsymbol{x}, t) + (\boldsymbol{u}(\boldsymbol{x}, t) \cdot \nabla)\Theta(\boldsymbol{x}, t) = D\Delta\Theta(\boldsymbol{x}, t) \tag{1}$$

and it is not difficult to see that an initial uncertainty, $\delta\Theta(\boldsymbol{x}, 0)$, does not grow in time:

$$\int |\delta\Theta(\boldsymbol{x}, t)|^2 d\boldsymbol{x} \leq \int |\delta\Theta(\boldsymbol{x}, 0)|^2 d\boldsymbol{x}. \tag{2}$$

Nevertheless the solutions of (1) can appear "chaotic". This is easy to understand if $D = 0$, since in this case the evolution of $\Theta(\boldsymbol{x}, t)$ is

$$\Theta(\boldsymbol{x}, t) = \Theta(S^{-t}\boldsymbol{x}, 0), \tag{3}$$

where $S^t\boldsymbol{x}$ is the formal solution of the equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{u}(\boldsymbol{x}, t). \tag{4}$$

If the solution of equation (4) is chaotic, then a time record of $\Theta(\boldsymbol{x}, t)$ in a point $\boldsymbol{x}$, $F_{(\boldsymbol{x})}(t) = \Theta(\boldsymbol{x}, t)$, will appear "chaotic" although (1) is linear [2,3]. For instance, the Kolmogorov entropy of the signal $F_{(\boldsymbol{x})}(t)$, computed by means of the Grassberger-Procaccia method and the embedding technique, should have a positive value.

In this paper we shall consider a system which does not have the classical limit, since we want to concentrate on quantum effects with no relation with classical chaos.

To study the dynamical behavior of the quantum system we assume an "informatic" point of view: we study the rate of growing of the information complexities, as a function of the length n, of sequences of digits generated by the quantum dynamics in a coarse grained configuration space.

The model we consider in this paper, introduced in Sect. II, is a quantum 1/2-spin in a time dependent magnetic field. In Sect. III we study the case of a quasiperiodic magnetic field. We observed that the information complexity diverges as a stretched exponential of n while the Shannon entropy is zero. This case is between the standard chaos, where the divergence is exponential, and the regular case, where it is polynomial. This behaviour is called "sporadic chaos". In Sect. IV we introduce a simple random process which reproduces the sporadic chaotic behavior generated by the quantum system. In Sect. V we consider the case of random, in time, magnetic field. Finally in Sect. VI we report some conclusions.

II. THE MODEL

A simple quantum system which exhibits complex behavior is a quantum 1/2-spin in a time dependent magnetic field. This model has been widely studied in the literature to analyze chaotic oscillations in two level systems [4–8]. Given a single quantum 1/2-spin, its time evolution in a time dependent magnetic field $\mathcal{B}(t)$ is described by the Schrödinger equation with Hamiltonian

$$H = \mathcal{B}(t) \cdot \boldsymbol{\sigma} \tag{5}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices.

For example, choose $\mathcal{B}(t) = (B(t), 0, \omega)$, so that

$$H = \omega\sigma_z + B(t)\,\sigma_x, \tag{6}$$

then the Schrödinger equation for the spinor $\psi = (\psi_1,\, \psi_2)$ is

$$\begin{cases} i\,\partial_t\,\psi_1 = \ \omega\psi_1 + B(t)\,\psi_2 \\ \\ i\,\partial_t\,\psi_2 = \ -\omega\psi_2 + B(t)\,\psi_1. \end{cases} \tag{7}$$

Through all the paper we take $\hbar = 1$. Equation (7) corresponds to the systems studied by Pomeau et al. [4] for the special choice $\omega_1 = \omega$ and $\omega_2 = -\omega$.

By means of a straightforward transformation, one passes from (7) to the equations for the polarization vector $P_i = \langle\psi\,|\sigma_i|\,\psi\rangle$, $i = x, y, z$. One thus gets a three dimensional system

$$\frac{d\boldsymbol{P}}{dt} = D(t)\,\boldsymbol{P} \tag{8}$$

with the constraint $\sum_i P_i^2 = 1$, which reduces the number of degrees of freedom to two. In the case of (6) the evolution matrix reads

$$D(t) = \begin{pmatrix} 0 & -2\,\omega & 0 \\ 2\,\omega & 0 & -2\,B(t) \\ 0 & 2\,B(t) & 0 \end{pmatrix} \tag{9}$$

This matrix is the generator of an infinitesimal rotation with constant angular speed $2\,\omega$ around the z axis and with angular speed $2\,B(t)$ around the x axis. The linear system (8) describes the rotation on the unite sphere, and its solution is

$$\boldsymbol{P}(t) = \mathcal{T}\,\mathrm{e}^{\int_0^t D(t')\,dt'}\,\boldsymbol{P}(0) \tag{10}$$

where $\mathcal{T}$ is the time ordering operator.

We are not interested in a particular physical case, but rather in the general aspects of the problem. We thus study a version of this model which allows us to reduce the differential equation (8) into a map. To this end, we take a magnetic field of the following form: for t in the interval $[(i-1)\tau,\, i\tau]$, the magnetic field is

$$\mathcal{B}(t) = (\eta_i\,\omega_x,\, 0,\, (1-\eta_i)\,\omega_z) \tag{11}$$

where the variable η_i takes in each interval the value 0 or 1 with a rule to be specified, so that the string $(\eta_1, \cdots, \eta_n, \cdots)$ may be a periodic, or a quasi-periodic, or a random sequence of bits. The choice (11) corresponds to a series of square-like magnetic pulses of time length τ. A similar model was studied by Luck et al. [7] for a Fibonacci sequence of bits.

Inserting (11) into (8), the resulting matrix $D(t)$ has only two possible forms: the generator of a rotation around the z-axis

$$D(t) = 2\,\omega_z\, A = 2\,\omega_z \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{if } \eta_i = 0 \tag{12}$$

or of a rotation around the x-axis

$$D(t) = 2\,\omega_x\, B = 2\,\omega_x \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{if } \eta_i = 1. \tag{13}$$

The evolution equation (8) can be integrated exactly between two successive pulses. Therefore, if one observes the systems at the times τ, 2τ, 3τ, ... the time evolution (10) of the polarization vector is equivalent to the map

$$\boldsymbol{P}((i+1)\tau) = \mathcal{R}_i\, \boldsymbol{P}(i\tau) \tag{14}$$

describing a sequence of rotations of an angle $2\omega_x$ around x-axis or of an angle $2\omega_z$ around z-axis. In the following, without loosing in generality, we take $\omega_x = \omega_z = 1/2$ in order to simplify the notation. With this choice the rotation matrix takes the form

$$\mathcal{R}_i = \mathrm{e}^{A\tau} = \begin{pmatrix} \cos\tau & -\sin\tau & 0 \\ \sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{if } \eta_i = 0 \tag{15}$$

or

$$\mathcal{R}_i = \mathrm{e}^{B\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tau & -\sin\tau \\ 0 & \sin\tau & \cos\tau \end{pmatrix} \qquad \text{if } \eta_i = 1 \tag{16}$$

We expect that this map has the same qualitative features of the continuous-time model, although the numerical analysis becomes much simpler.

We note that in the class of models of Eq. (8) there is no growing of the errors, because of the conservation law $\sum_i P_i^2 = 1$. Therefore, besides the maximal Lyapunov exponent and the Kolmogorov entropy, the generalized Lyapunov exponents are also identically zero.

To proceed with the analysis of this model we make a partition of the unit sphere and consider the symbolic dynamics associated with the time evolution of the polarization vector. In practice, we define a sequence of bits b_i by using the following coding: the bit b_i is 1 when $P_z(i\tau) > 0$ and 0 otherwise. We shall call these bits the "output" bits to distinguish them from the "input" bits η_i.

We perform various kinds of analysis of the output signal of the model. In particular we shall study:

1. The time autocorrelation function $C(k) = \langle b_i\, b_{i+k} \rangle - \langle b_i \rangle \langle b_{i+k} \rangle$ where the average is a time average.

2. The topological entropy: we look at the number $N(n)$ of different strings of bits $\{b_1, b_2, \cdots b_n\}$ of length n. In a chaotic system this number is expected to diverge as $e^{n\, h_{top}}$ for large n.

3. The Shannon entropy [9]

$$K_1 = -\lim_{n\to\infty} \frac{1}{n} \sum_{(b_1,b_2,\cdots b_n)} \mathcal{P}(b_1, b_2, \ldots, b_n) \ln \mathcal{P}(b_1, b_2, \ldots, b_n) \tag{17}$$

   where $\mathcal{P}(b_1, b_2, \ldots, b_n)$ is the probability of the bit string $(b_1, \ldots, b_n)$.

4. The generalized information complexities

$$I_q(n) = \sum_{(b_1,b_2,\ldots,b_n)} [\mathcal{P}(b_1, b_2, \ldots, b_n)]^q . \tag{18}$$

In a "standard" chaotic system the information complexities diverge exponentially with the rates

$$K_q = -\frac{1}{q-1} \lim_{n\to\infty} \frac{1}{n} \ln I_q(n) \tag{19}$$

called generalized Renyi entropies [10]. Note that the topological entropy is K_0 and the Shannon entropy is obtained for $q \to 1$

$$K_1 = -\lim_{q\to 1} \lim_{n\to\infty} \frac{1}{q-1} \ln I_q(n). \tag{20}$$

Although a quantum system cannot have exponential divergence of nearby trajectories, and hence positive Lyapunov exponents, because the time evolution equation is linear, it could be chaotic in an informatic sense, since its information complexity may grow faster than any power of the time n. In the next section we shall investigate this possibility in our model.

III. SPORADIC CHAOS GENERATED BY A QUASIPERIODIC PERTURBATIONS

It is well known that in quantum systems with a time-periodic potential the physical observables are quasiperiodic because of the Floquet theorem. In the context of our model, a time periodic potential leads to a periodic input string η_i, which generates a quasiperiodic output string b_i. On the other hand, a quasiperiodic input breaks the invariance under time translation. In this case, the response of the system is not necessarily multiperiodic and one could expect a certain degree of randomness in the time evolution of the polarization vector.

In this section we study the model defined in Sect. II with a quasiperiodic input given by $\eta_i = \theta(\cos[2\pi\, i\, (\tau/T)])$, where θ is the Heaviside step-function. In order to have a quasiperiodic forcing, we have chosen the incommensurate ratio $T/\tau = 2\pi$ leading to $\eta_i = 0, 1$ if sign of $\cos(i) = \mp 1$, respectively.

The numerical study has been done by iterating the map (14) for $N = 10^7$ steps and recording the output bit b_i according the coding previously discussed.

In Fig. 1 we report the correlation function

$$C(k) = \frac{1}{N-k} \sum_{i=1}^{N-k} b_i\, b_{i+k} - \frac{1}{N} \left[\sum_{i=1}^{N} b_i \right]^2 . \tag{21}$$

One observes that the correlation does not decay, but exhibits rather modulated oscillations. The signal is, nevertheless, not fully correlated since it returns after periods of ≈ 250 steps to a value ≈ 0.7, and not back to 1.

The lack of sensitive dependence on initial conditions and of correlation decay are not sufficient to assure that the dynamics is regular. We should introduce a more sophisticated definition of chaos by regarding at the possible number of output strings. In a quasiperiodic sequence of bits, the number $I_0(n)$ of different strings of length n grows as a power of n, e.g., in the previously defined quasiperiodic input sequence $I_0(n) = 2n$. On the contrary there are $I_0(n) = 2^n$ possible strings in a sequence generated by a Bernoulli extraction. We can speak of "topological chaos" in systems where

$$h_{\text{top}} = \lim_{n\to\infty} \frac{1}{n} \ln I_0(n) > 0 \tag{22}$$

We have computed $I_0(n)$ for the numerical output sequence of bits b_n of our model, as shown in Fig. 2. The result is compatible with $h_{\text{top}} = \ln(2)/4$, although $\ln I_0(n)$ seems to bend for large n. A direct inspection of the output signal shows the presence of "rigid" rules on short times. With very high probability, there are sub-sequences of 4 identical digits. On the contrary sub-sequences of 2 or 3 identical digits seems to be almost forbidden. A typical output string looks like

$$\cdots|1111|0000|1111|1111|1111|0000|1111|\cdots \tag{23}$$

This structure of the signal suggests a trick to increase the effective length of the analyzed strings. We have considered decimate sequences of bits obtained by taking one bit each 4 bits, i.e., by analyzing sequences $\tilde{b}_i = b_{K\cdot i}$ with $K = 4$.

The entropies of a Bernoulli shift are invariant under a decimation. In our case, we have found that the decimate signal has a topological entropy close to $\ln 2$, indicating that the original systems has strong short-range correlations which organizes the system in an almost deterministic way on the time-scale of 4 time-steps.

On the other hand, we have found that even if the Shannon entropy K_1 vanishes the information complexity $I_1(n)$ seems to diverge faster than any power of n. Figure 3 indicates that the data can be fitted by a stretched exponential of the form

$$I_1(n) \sim \exp[\gamma_1 \, n^{\alpha(1)}] \tag{24}$$

with $\alpha(1) = 0.64$. A similar form holds for all the generalized informatic complexities I_q with $q \geq 0$, which can be fitted by

$$I_q(n) \sim \exp[\gamma_q \, n^{\alpha(q)}] \tag{25}$$

with $\alpha(0) = 1$, $\gamma_0 = h_{\rm top}$ and $\alpha(q) < 1$ for $q > 0$. Figure 4 shows the numerical results for the scaling exponents $\alpha(q)$.

In a numerical calculation of the complexities $I_q(n)$ it is practically impossible to go beyond $n = 25$. On the other hand, the system exhibits a non-trivial behavior for times much larger than 25 time steps, e.g., the time correlation has very long return periods. Therefore, in order to obtain a stronger evidence of the complex nature of the output, we performed a systematic decimation of the signal. For a chaotic signal the decimation should increase, or leave unchanged, the value of $I_q(n)$. The quasi-periodic nature of a sequence, i.e., a polynomial increase of $I_q(n)$, is also invariant under decimation.

A clear numerical evidence of the presence of topological chaos is provided by a decimation with $K = 256$, the return-time for the correlation function. This leads again to $h_{\rm top} = \ln 2$ with high numerical accuracy. Similarly $I_q(n)$ show a stretched exponential behavior with exponents $\alpha(q)$ which are closer to 1 than those after the decimation with $K = 4$, see Fig. 4.

A similar behavior was called sporadic chaos by Gaspard and Wang [11], which found stretched exponential instabilities in intermittent maps of the type $x_{n+1} = x_n + c\,x_n^z$ (mod 1) when $z \geq 2$ and in some other systems. We stress that sporadic chaos is intermediate between multiperiodic and chaotic dynamics, that is between predictability and randomness. In Ref. [11], sporadicity was associated to very strong forms of dynamical intermittency which leads to temporal fluctuations of Levy's type rather than Gaussian. In the above quantum system the origin is not clear, however it is tempting to speculate that it is a generic feature of a quantum systems coupled with few external degrees of freedom. The quantum 1/2-spin system forced by the quasiperiodic pulse of the magnetic field, can be thought as an oversimplification of such a situation, which still retains all the basic ingredients. However, a deeper comprehension of this phenomenon requires a further step. Since we are just interested in an "informatic" point of view we introduce a random process which can reproduce the main feature of the output generated by the quantum system, and where the generalized complexities can be computed analytically.

Although the mechanism which produces this weakly complex behavior in the quantum system and in the random model could be different, this is not relevant in our approach which considers the informatic complexity of a sequence without any reference to its physical origin.

IV. A RANDOM PROCESS WITH SPORADIC CHAOS

The quantum system discussed in Sect. III gives an output signal with positive topological entropy although the autocorrelation function does not decay. At first glance, this feature may seem rather contradictory. In this section we introduce a random process which exhibits a similar behavior.

Consider a periodic signal, i.e., such that $z_{i+T} = z_i$, where T is the period, with zero average over a period. Next we allow for an error at time i with probability ϵ, and consider the new process

$$x_i = z_i \gamma_i \tag{26}$$

where γ_i are uncorrelated random variables which take the value $\gamma_i = 1$ with probability $1 - \epsilon$ and $\gamma_i = -1$ with probability ϵ.

By construction, it is immediate to verify that the generalized Renyi entropies of (26) are those of the γ_i process which is nothing but a Bernoulli shift. In particular, $h_{\rm top} = \ln 2$ and $K_1 = -[\epsilon \ln \epsilon + (1-\epsilon)\ln(1-\epsilon)]$.

The variables γ_i and z_i are uncorrelated, hence the time-correlation of the x_i can be factorized for all time delay $\tau \geq 1$ and becomes

$$C(\tau) = \langle x_{i+\tau}\, x_i \rangle = \langle z_i\, z_{i+\tau} \rangle \langle \gamma_i \rangle^2 = (1-2\epsilon)^2 C_0(\tau) \tag{27}$$

where $C_0(\tau) = \langle z_i\, z_{i+\tau} \rangle$ is the correlation function of the original periodic signal. To obtain the last equality we have used the fact that the γ_i are independent random variables with mean

$$\langle \gamma_i \rangle = 1 - 2\epsilon. \tag{28}$$

This model is still too simple to reproduce the sporadic form of chaos observed in the quantum system. In fact, here all the generalized complexities increase exponentially in time. To mimic sporadicity, we should lower the degree of randomness of the process. This can be achieved by considering a concentration of errors $\gamma_i = -1$ which depends on the knowledge of the past. We thus assume that the probability $P(x_{n+1}|x_n, \ldots, x_1)$ that $x_{n+1} = z_{n+1}$, i.e., no error, after a sequence of n symbols $x_n, \ldots, x_1$ is given by $1 - \epsilon n^{-w}$ with $0 < w \leq 1$. Note that $P(x_{n+1}|x_n, \ldots, x_1)$ depends only on the number n of known symbols and not on their value. In practice, this account for replacing the constant probability ϵ with ϵn^{-w}.

The correlation function $C(\tau)$ of the model with "memory" cannot be lower than that of the case without memory, and consequently does not decay. The topological entropy is unchanged, $h_{\text{top}} = \ln 2$, for $\epsilon \neq 0$ since it does not depend on the probability distribution. This is not the case for the information complexity and generalizations defined by

$$I_q(n) = \sum_{\{x_1, \ldots, x_n\}} \mathcal{P}(x_1, \ldots, x_n)^q. \tag{29}$$

In fact, since we can write

$$\mathcal{P}(x_1, \ldots, x_n) = \mathcal{P}(x_1, \ldots, x_{n-1})\, P(x_n|x_{n-1}, \ldots, x_1) \tag{30}$$

a simple calculation gives

$$I_q(n+1) = I_q(n) \times \left[(1-\epsilon n^{-w})^q + (\epsilon\, n^{-w})^q\right] \tag{31}$$

As a consequence, for large n, the complexity of order q diverges with a stretched exponential law

$$\frac{1}{n} \ln I_q(n) = \frac{q}{q-1}\, n^{-w}, \qquad q > 1 \tag{32}$$

$$\frac{1}{n} \ln I_q(n) = \frac{\text{const.}}{1-\text{q}}\, n^{-w\,q}, \qquad 0 < q < 1 \tag{33}$$

Moreover, for $q \rightarrow 1$, the information complexity scales as

$$\frac{1}{n} \ln I_1(n) = n^{-w} \ln n. \tag{34}$$

From equations (32) and (33) it follows that the exponent of the stretched exponential of the model with memory is $\alpha(q) = w$ for $q > 1$ and $\alpha(q) = w\,q$ for $0 < q \leq 1$. This result is compared with the scaling exponent of the quantum system in Fig. 4 which shows a qualitatively similar behavior. It is worth stressing that for large values of q the very rare chaotic events dominates the sum in (29), and $\alpha(q)$ becomes independent of q. This follows from the type of behavior exhibited by the model, where "chaotic bursts" are generated by the introduction of errors on a regular background. As consequence $\alpha(q)$ should coincide for large q with the only scaling exponent w which determines the degree of memory.

V. BEHAVIOR GENERATED BY A RANDOM PERTURBATION

A very different dynamical regime appears when the quantum 1/2-spin is driven by a random magnetic field. In this case, the driving perturbation can be regarded, e.g., as originated by an external environment on the quantum two level system. In the context of our model, a random magnetic field leads to a random sequence of input bits η_i. It is natural to expect that in this case the output is chaotic in the informatic sense, without the sporadic intermittency of the previous sections. In fact, now all the generalized informations diverge with an exponential rate given by the Renyi entropies K_q.

However, it is surprising that the quantum system reduces the degree of "topological" chaos of the input perturbation. This can be understood from the following argument:

given any initial condition $\boldsymbol{P}(0)$, from each input string of bits a deterministic rule leads to one output string. Under the hypothesis, well confirmed numerically, that $\boldsymbol{P}$ takes all the possible values on the unitary sphere, the output topological entropy cannot depend on the details of the input rules, but on the topological entropy of the input. Moreover, unlike, e.g., the Shannon entropy, the topological entropy is not sensitive to the probability of the strings, so that the topological entropy of the output takes a fixed value which is obviously less or equal than that of the topological entropy of the input.

For example, if we take as input the random process x_i, equation (26), with $z_i = (-1)^i$, then the topological entropy of the input is maximal, i.e., $h_{\text{top}} = \ln 2$. On the other hand, for the process without memory, we found that for any value of ϵ the topological entropy of the output is $h_{\text{top}} = 0.44$, see Figs. 5 and 6.

As a consequence the quantum system can amplify or reduce the complexity of the random input. In fact, the input has the entropies $h_{\text{top}} = \ln 2$ and $K_1 = -\epsilon \ln \epsilon - (1-\epsilon)\ln(1-\epsilon)$. On the other side, the output has a K_1 that is larger (smaller) than $-\epsilon \ln \epsilon - (1-\epsilon)\ln(1-\epsilon)$ for values of ϵ close to zero (or 1/2), see Figs. 5 and 6. This implies that the complexity is amplified for small ϵ and reduced for $\epsilon \simeq 1/2$.

VI. CONCLUSIONS

Chaos in classical systems is defined in terms of instability with respect to a perturbation of the initial conditions. One would be tempted to extend this definition to quantum mechanics but this is not possible since the evolution equations are linear. This is puzzling since classical mechanics is obtained as the limit of quantum mechanics for $\hbar \to 0$, so that one could expect to have some indications on chaos at least in the semiclassical region. This is true, but only for finite times. It is possible to show that the quantum version of a classically chaotic system, shares the same chaotic behavior of the classical mechanics up to a time $t^* \sim (1/\lambda)\ln(1/\hbar)$ [12,13,3]. When $t > t^*$ the system has a genuine quantum character, with features very different from the classical behavior, such as a suppression of the diffusion properties. Nevertheless, this does not imply the absence of complex behavior in quantum systems for large times, at least in an informatic sense. In fact, we have provided a numerical evidence that the time evolution of the wave function of a simple quantum 1/2-spin coupled with a time dependent magnetic field is complex in an "informatic" sense. In particular we have found that for quasiperiodic magnetic fields the information complexity of the time evolution diverges as a stretched exponential, implying a vanishing Shannon entropy, although the topological entropy is positive. A similar phenomenon is known as sporadic chaos in the theory of dynamical systems [11,14]. A second result, less surprising , is that a random magnetic field produces chaotic time evolution with positive Shannon entropy and positive topological entropy. The topological entropy has a non-trivial maximal value which has been computed numerically.

One can give an interpretation to these behaviors assuming that the external quasiperiodic magnetic field represents an interaction of the spin with few degrees of freedom of a macroscopic object whose motion is very weakly affected by the spin. The random field represents an interaction with the infinite degrees of freedom of the environment. It is an open problem to understand whether a genuine chaotic behavior in quantum mechanics can arise in large system with infinite degrees of freedom (in our case spin plus environment) while for complexity, in the sense of sporadic chaos, it is sufficient that a small quantum system (the spin, in our case) interacts with some classical object. On these lines is the work of Jona-Lasinio et al. [15].

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FIGURES

FIG. 1. Quasiperiodic input: correlation function $C(k) = \langle b_{i+k}\, b_i \rangle - \langle b_i \rangle^2$ as a function of k. In the figure only one point each 256 is drawn.

FIG. 2. Quasiperiodic input: $\ln I_0(n)$ as a function of n. $I_0(n)$ is computed from the decimated strings b_{Kn} with $K = 1$ (cross), $K = 4$ (plus) and $K = 256$ (diamond). The full line has the slope equal to the topological entropy of the Bernoulli shift $h_{\rm top} = \ln 2$.

FIG. 3. Quasiperiodic input: $\ln I_1(n)$ as a function of n. $I_1(n)$ is computed from the decimated strings b_{Kn} with $K = 4$ (plus) and $K = 256$ (diamond). The dashed lines correspond to the stretched exponential $I_1(n) \sim \exp[\gamma_1\, n^{\alpha(1)}]$ with $\alpha(1) = 0.64$ for $K = 4$ and $\alpha(1) = 0.87$ for $K = 256$. The full line has slope $\ln 2$.

FIG. 4. Scaling exponent $\alpha(q)$ as a function of q for the stretched exponential of the generalized complexities I_q for $K = 4$ (diamond) and $K = 256$ (plus). The dashed line is $\alpha(q)$ of the random model with memory.

FIG. 5. Random input with $\epsilon = 0.03$: $\ln I_0(n)$ (square) and $\ln I_1(n)$ (cross) as a function of n. The slopes of the lines are: $h_{\rm top} = 0.44$ (full), $h_1 = 0.22$ (dashed) and $K_1 = 0.13$ (dot-dashed).

FIG. 6. Random input with $\epsilon = 0.5$: $\ln I_0(n)$ (square) and $\ln I_1(n)$ (cross) as a function of n. The slopes of the lines are: $h_{\rm top} = 0.44$ (full), $h_1 = 0.40$ (dashed) and $K_1 = \ln 2$ (dot-dashed).

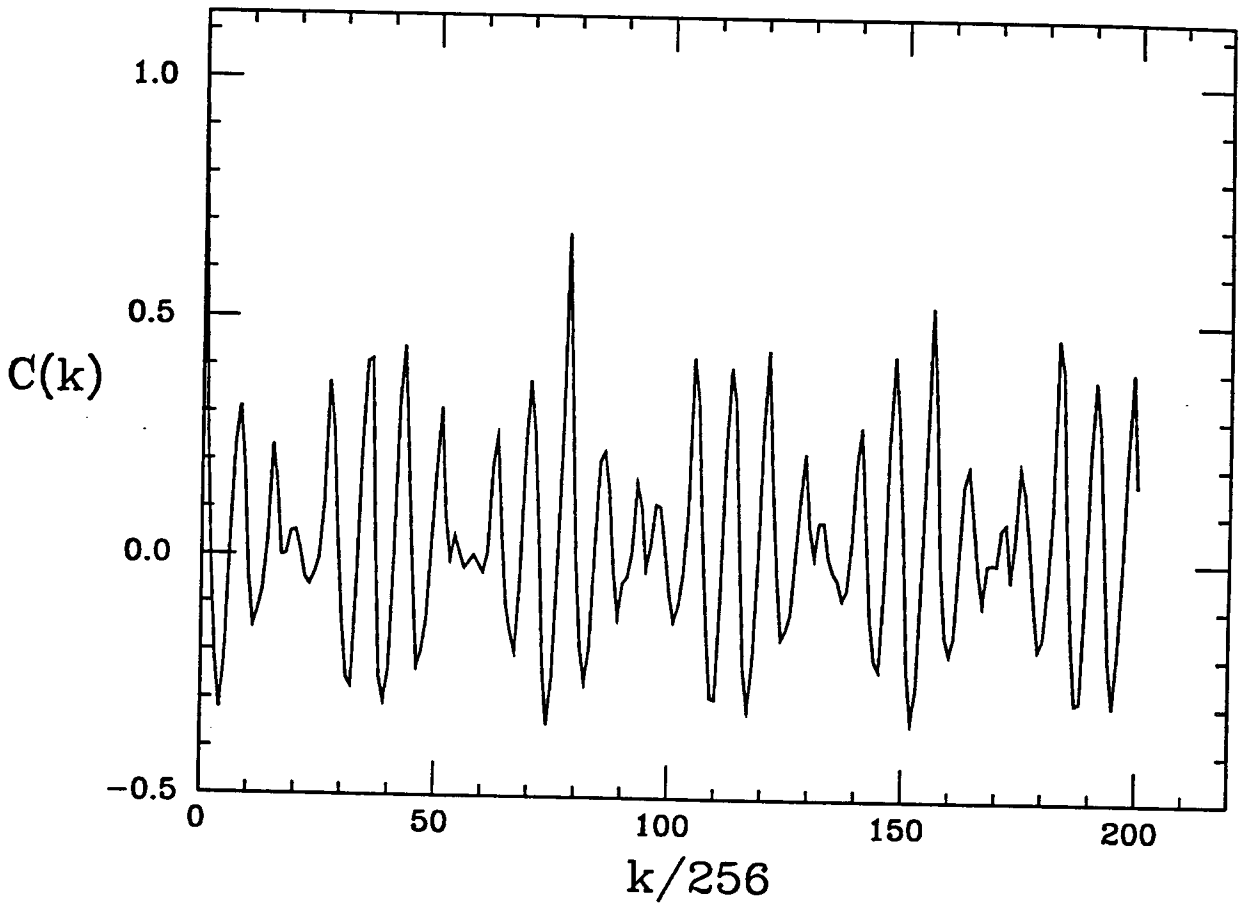

Fig. 1

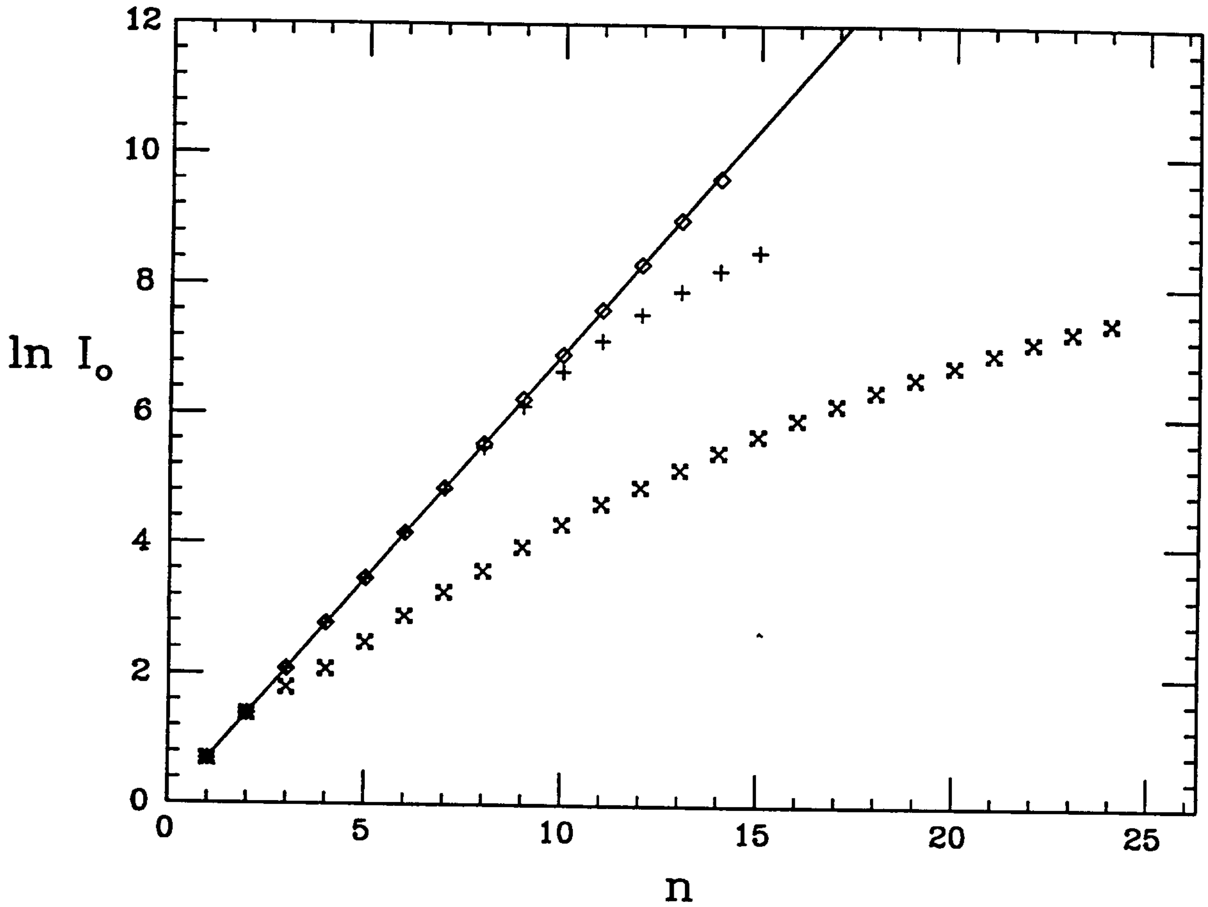

Fig. 2

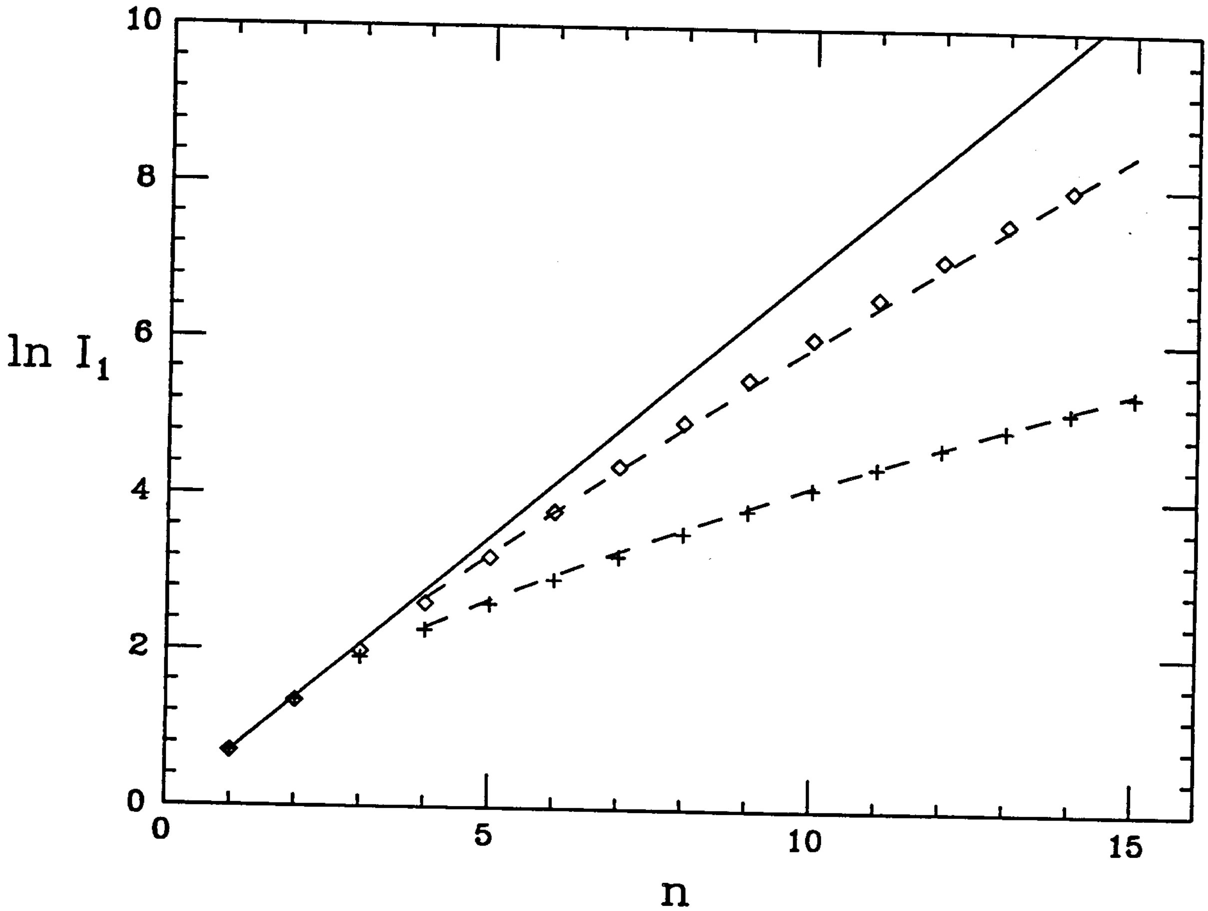

Fig. 3

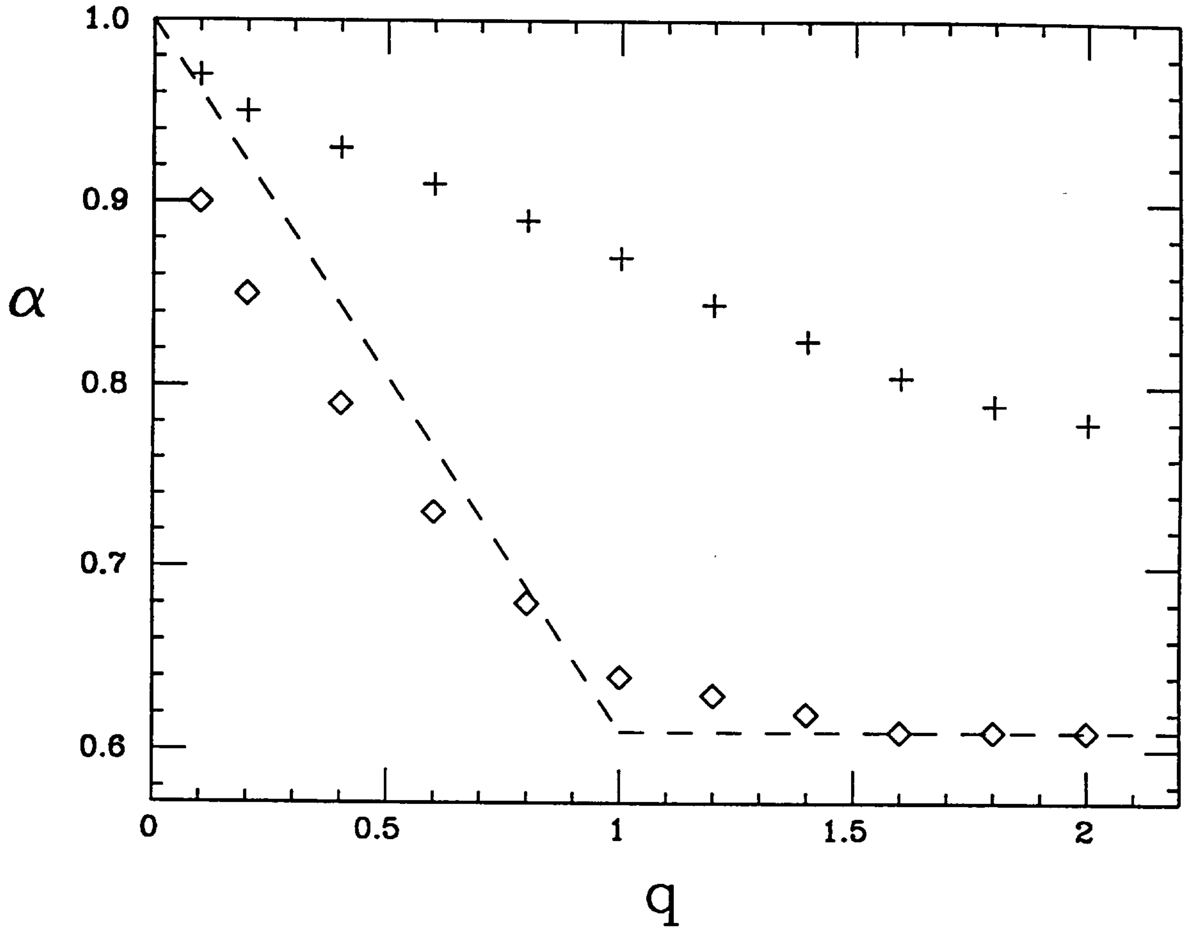

Fig. 4

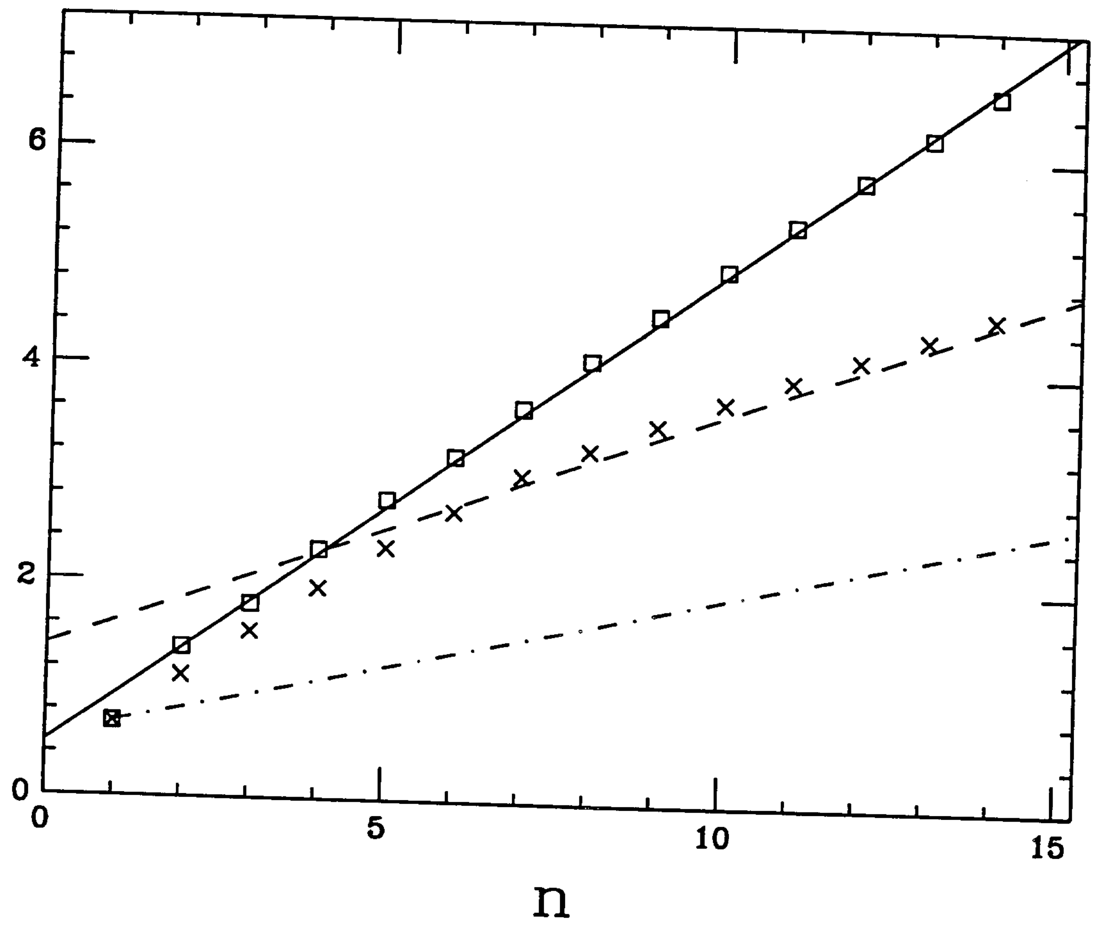

Fig. 5

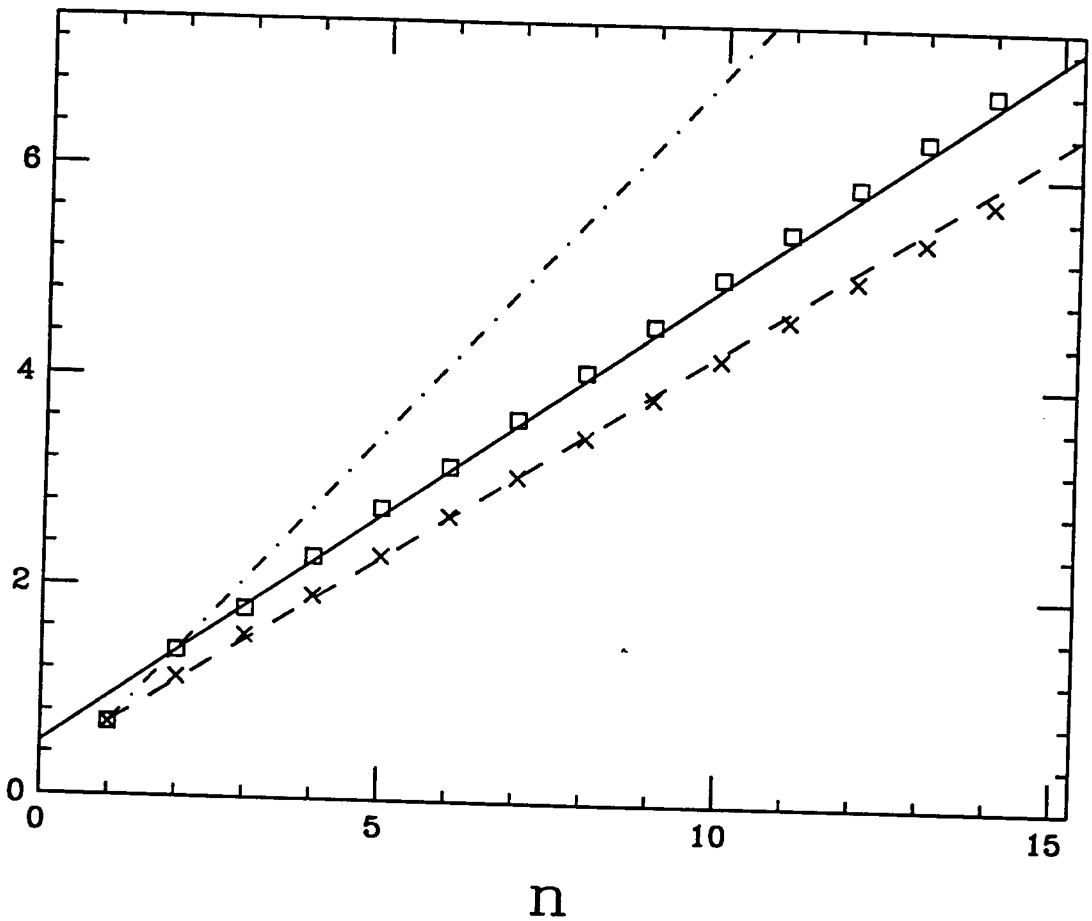

Fig. 6