QUARK MASSES FROM LATTICE QCD AT THE NEXT-TO-LEADING ORDER

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Abstract

Using the results of several quenched lattice simulations, we predict the value of the strange and charm quark masses in the continuum at the next-to-leading order, $m_s^{\overline{MS}}(\mu = 2\ GeV) = (127 \pm 18)\ MeV$ and $m_{ch}^{\overline{MS}}(\mu = 2\ GeV) = (1.47 \pm 0.28)\ GeV$. The errors quoted above have been estimated by taking into account the original statistical error of the lattice results and the uncertainties coming from the matching of the lattice to the continuum theory. A detailed presentation of the relevant formulae at the next-to-leading order and a discussion of the main sources of errors is also presented.

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1 Introduction

Quark masses are parameters of the QCD Lagrangian that cannot be determined within QCD by theoretical considerations only and cannot be measured directly since quarks do not appear as physical states. They are however very important for several reasons. In the framework of GUTs, quark and lepton masses at low energies are related to the pattern of symmetry breaking at the grand unification scale. The best-known example is that, within GUTs, one predicts $m_b \sim 3m_\tau$ as a consequence of the $SU(5)$ relation $m_b = m_\tau$ at the GUT scale and of strong-interaction effects [1]–[3]. Constraints on quark masses and relations between quark masses and matrix elements of the Cabibbo–Kobayashi–Maskawa mixing matrix are also found by using some ansatz on the form of the quark mass matrix [4]–[7]. In this framework, for example, the matrix element $|V_{cb}|$ depends, among other things, on the ratio of the strange to bottom quark mass. The mass of the strange quark is also a crucial factor appearing in the evaluation of matrix elements of penguin and electro-penguin operators, which enter in the $\Delta I = 1/2$ $K \to \pi\pi$ amplitude and in the CP violation parameter ϵ'/ϵ. The values of the light-quark masses are necessary also to estimate the quark-antiquark condensate $\langle 0|\bar{\psi}\psi|0\rangle$ which is the chiral order parameter of the QCD vacuum and finally the value of the up quark mass is fundamental for our understanding of the strong CP problem.

Lattice QCD is in principle able to predict the mass of any quark by fixing to its experimental value the mass of a hadron containing a quark with the same flavour. The quark mass that is directly determined in lattice simulations is the "bare" lattice quark mass $m(a)$, which can be converted to the continuum, renormalized mass $m(\mu)$ through a well-defined procedure [8]–[10]. The conversion factor relating $m(a)$ to $m(\mu)$ can be computed in perturbation theory. Since the typical scale is of order $1/a$ (or μ), where a is the lattice spacing, and in current numerical simulations $a^{-1} \sim 2$–4 GeV, we expect small non-perturbative effects. This would imply an accurate determination of quark masses at a scale larger than a^{-1}. In this respect lattice QCD is unique, since other techniques, like QCD sum rules [11]–[16], have to work at much smaller scales, where higher-order corrections [17] or non-perturbative effects [18] may be rather large. On the lattice, however, at values of the lattice strong coupling constant $\beta = 6/g_L^2(a)$ currently used

($\beta = 6.0 - 6.4$), perturbative corrections are often quite sizeable, due to the presence of "tadpole" diagrams, which are absent in continuum perturbation theory, and may give rise to large uncertainties. In order to reduce these uncertainties, two methods, based on "boosted" perturbation theory [19, 20] or on non-perturbative techniques [21], have recently been developed and will be used in the following.

In this paper, using values of quark masses determined in several numerical simulations [22]–[31], we predict the corresponding masses in the continuum and discuss the uncertainties entailed by the use of perturbation theory at the next-to-leading order. Following ref. [21], we also envisage a procedure to determine the continuum quark mass, which completely avoids the use of lattice perturbation theory. Our predictions are essentially limited to the strange and charm quark masses. Actually, simulations do not have direct access to very light (up and down) quarks, because of finite-volume effects. Moreover one expects that the "quenched" approximation can introduce uncontrolled systematic errors in this case. Thus we do not have much to say about the up and down quark masses. On the other hand, we cannot put the b quark on the lattice, because it is necessary to satisfy the condition $m_b a \ll 1$ in order to avoid huge discretization errors. However, m_b can be obtained from quarkonia spectroscopy [32, 33]. An alternative method, based on the heavy-quark effective theory on the lattice will be presented elsewhere [34].

In comparison with QCD sum rules, which is at present the only alternative approach to predicting quark masses, lattice QCD has advantages and disadvantages. The main advantage is that, on the lattice, one is able to work at scales large enough to avoid higher orders or non-perturbative effects which plague QCD sum rules calculations. Moreover, by making use of the proposals of refs.[19, 20] or [21], the error due to the large perturbative corrections encountered on the lattice can be reduced. Finally it is possible to reduce discretization errors, by using some "improved" lattice quark action, following the proposal of Symanzik [9, 35, 36]. The main disadvantage of the lattice approach is that essentially all the results have been obtained in the "quenched" approximation and it is difficult to quantify the size of the error introduced by this approximation.

We believe that the results of this work are useful as a presentation of the

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method, for the discussion of the uncertainties coming from the matching of the lattice to the continuum theory and for phenomenological applications. The plan of the paper is the following. In sec. 2 we give the formulae necessary to relate matrix elements of lattice operators and masses to the corresponding quantities in the continuum. The formulae introduced in sec. 2 are applied to quark masses in sec. 3, where lattice perturbation theory and non-perturbative methods based on Ward identities are reviewed. Numerical results and estimates of errors are given in sec. 4.

2   Bare and renormalized operators on the lattice

In this section we summarize the main formulae that relate lattice operators to the corresponding continuum renormalized operators[1]. These formulae will be used to give the continuum renormalized quark mass in terms of the lattice bare mass, as computed in Monte Carlo simulations.

Let us consider the renormalization of a generic lattice two-quark operator of the form $O_\Gamma(a) = \bar{\psi}\Gamma\psi$, where Γ is one of the Dirac matrices and ψ is the bare lattice fermion field. At the next-to-leading order (NLO) the generic, forward, two-point Green function, computed between quark states of virtuality $p^2 \ll 1/a^2$, has the form[2]:

$$
\begin{aligned}
\Gamma(pa) &= \Gamma_0 \Big[1 + \frac{g_L^2(a)}{16\pi^2}\big(\frac{\gamma^{(0)}}{2}\ln(\frac{pa}{\pi})^2 + C^L\big) \\
&+ \Big(\frac{g_L^2(a)}{16\pi^2}\Big)^2 \big(\frac{1}{8}\gamma^{(0)}(-2\beta_0 + \gamma^{(0)})\ln^2(\frac{pa}{\pi})^2 \\
&+ \frac{1}{2}\big(\bar{\gamma}^{(1)} + (-2\beta_0 + \gamma^{(0)})C^L + \beta_\lambda^0(\lambda\frac{\partial C^L}{\partial\lambda})\big)\ln(\frac{pa}{\pi})^2\big)\Big] + \dots \quad (1)
\end{aligned}
$$

where ... represent terms beyond the next-to-leading order and terms of $O(a)$, which we will assume negligible in the following; Γ_0 is the zeroth-order Green function, and $g_L^2(a) = 6/\beta$ is the bare lattice coupling constant; $\gamma^{(0)}$ and $\bar{\gamma}^{(1)}$ are the leading (regularization-independent) and next-

[1]A more detailed discussion on this point can be found in ref. [37].
[2]We work on a Euclidean lattice.

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to-leading (regularization-dependent) order anomalous dimensions, respectively; $\lambda = \lambda(a)$ is the lattice gauge parameter of the gluon propagator $\left(\Pi_{\mu\nu}(q^2) = -\delta_{\mu\nu} + (1 - \lambda)q_\mu q_\nu/q^2\right)$. It obeys the renormalization group equation:

$$\frac{1}{\lambda(a)}a\frac{\partial\lambda(a)}{\partial a} = -\beta_\lambda(g_L^2(a)) = -\beta_\lambda^0\frac{g_L^2(a)}{16\pi^2} + \cdots, \qquad \beta_\lambda^0 = \frac{5N - 2n_f}{6\pi}. \qquad (2)$$

We have introduced a scale-dependent gauge parameter in order to define a gauge-independent anomalous dimension and simplify the renormalization group equation for $\Gamma(pa)$, see eq. (5) below.

The lattice coupling constant obeys the equation:

$$a\frac{dg_L^2(a)}{da} = -\beta(g_L^2(a)) = 2\beta_0\frac{g_L^4(a)}{16\pi^2} + 2\beta_1\frac{g_L^6(a)}{(16\pi^2)^2}, \qquad (3)$$

where $\beta_{0,1}$ are given by:

$$\beta_0 = \frac{(11N - 2n_f)}{3}, \qquad \beta_1 = \frac{34}{3}N^2 - \frac{10}{3}Nn_f - \frac{(N^2 - 1)}{N}n_f \qquad (4)$$

and n_f is the number of flavours. Equation (1) guarantees that all the matrix elements can be made finite (as $a \to 0$) by multiplying the bare operator by a suitable renormalization constant, obtained by fixing the renormalization conditions for O_Γ.

From the above equations we find:

$$\left(a\frac{\partial}{\partial a} - \beta(g_L^2(a))\frac{\partial}{\partial g_L^2(a)} - \beta_\lambda(g_L^2(a))\lambda\frac{\partial}{\partial\lambda} - \bar{\gamma}(g_L^2(a))\right)\Gamma(pa) = 0 \qquad (5)$$

with

$$\bar{\gamma}(g_L^2(a)) = \frac{g_L^2(a)}{16\pi^2}\gamma^{(0)} + \frac{g_L^4(a)}{(16\pi^2)^2}\bar{\gamma}^{(1)}. \qquad (6)$$

In view of the comparison with some continuum regularization, it is convenient to expand the bare Green function $\Gamma(pa)$ of eq. (1) in terms of the continuum minimal subtraction $(\overline{MS})$ coupling constant, evaluated at the scale π/a. The continuum $\overline{MS}$ coupling $\alpha_s(\pi/a)$ is related to the lattice bare coupling $\alpha_s^L(a) = g_L^2(a)/4\pi$ by the equation:

$$\frac{1}{\alpha_s^L(a)} = \frac{1}{\alpha_s(\pi/a)}\left(1 + \frac{\alpha_s(\pi/a)}{4\pi}\Delta + \ldots\right), \qquad (7)$$

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where Δ is a numerical constant. With this substitution, eq. (5) becomes:

$$\left(a\frac{\partial}{\partial a} - \beta(\alpha_s)\frac{\partial}{\partial \alpha_s} - \beta_\lambda(\alpha_s)\lambda\frac{\partial}{\partial \lambda} - \gamma_L(\alpha_s)\right)\Gamma(pa) = 0 \tag{8}$$

and

$$\beta(\alpha_s) = -2\beta_0\frac{\alpha_s^2}{4\pi} - 2\beta_1\frac{\alpha_s^3}{(4\pi)^2}, \qquad \gamma_L(\alpha_s) = \frac{\alpha_s}{4\pi}\gamma^{(0)} + \frac{\alpha_s^2}{(4\pi)^2}\gamma_L^{(1)}, \tag{9}$$

$$\beta_\lambda(\alpha_s) = \beta_\lambda^0\frac{\alpha_s}{4\pi} + \cdots, \tag{10}$$

By changing the expansion parameter, one also has to change the two loop anomalous dimension [38]–[40], see also eq. (1):

$$\gamma_L^{(1)} = \bar{\gamma}^{(1)} - \Delta\gamma^{(0)} \tag{11}$$

Finally the running coupling constant α_s is given by:

$$\frac{\alpha_s(\mu^2)}{4\pi} = \frac{1}{\beta_0 ln(\mu^2/\Lambda_{QCD}^2)}\left(1 - \frac{\beta_1 ln[ln(\mu^2/\Lambda_{QCD}^2)]}{\beta_0^2 ln(\mu^2/\Lambda_{QCD}^2)}\right) + \cdots \tag{12}$$

The above equation defines the continuum $\overline{MS}$ scale parameter Λ_{QCD} at the NLO.

In continuum dimensional regularizations, after subtraction of the poles in $1/\epsilon$, the renormalized Green function has a form similar to eq. (1):

$$\Gamma(\frac{p}{\mu}) = \Gamma_0\Big[1 + \frac{\alpha_s(\mu)}{4\pi}(\frac{1}{2}\gamma^{(0)}\ln(\frac{p}{\mu})^2 + C^C)$$

$$+ \Big(\frac{\alpha_s(\mu)}{4\pi}\Big)^2\Big(\frac{1}{8}\gamma^{(0)}(-2\beta_0 + \gamma^{(0)})\ln^2(\frac{p}{\mu})^2 + \frac{1}{2}\Big(\gamma^{(1)}$$

$$+ (-2\beta_0 + \gamma^{(0)})C^C + \beta_\lambda^0(\lambda\frac{\partial C^C}{\partial \lambda})\Big)\ln(\frac{p}{\mu})^2\Big] + \ldots \tag{13}$$

It obeys the renormalization group equation:

$$\left(\mu\frac{\partial}{\partial \mu} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s} + \beta_\lambda(\alpha_s)\lambda\frac{\partial}{\partial \lambda} + \gamma(\alpha_s)\right)\Gamma(\frac{p}{\mu}) = 0, \tag{14}$$

with

$$\gamma(\alpha_s) = \frac{\alpha_s}{4\pi}\gamma^{(0)} + \frac{\alpha_s^2}{(4\pi)^2}\gamma^{(1)} \tag{15}$$

and

$$\mu\frac{\partial\alpha_s(\mu)}{\partial\mu} = \beta(\alpha_s(\mu)), \qquad \frac{1}{\lambda(\mu)}\mu\frac{\partial\lambda(\mu)}{\partial\mu} = \beta_\lambda(\alpha_s). \tag{16}$$

By solving the renormalization group equations for $\Gamma(pa)$ and $\Gamma(p/\mu)$, and imposing the matching condition $\Gamma(p/\mu) = C(\mu a)\Gamma(pa)$, we find [37]:

$$
\begin{aligned}
O_\Gamma(\mu) &= C(\mu a)O_\Gamma(a) \\
&= \Big(\frac{\alpha_s(\mu)}{\alpha_s(\pi/a)}\Big)^{\gamma^{(0)}/2\beta_0}\Big[1 + \frac{\alpha_s(\mu) - \alpha_s(\pi/a)}{4\pi}\frac{\gamma^{(1)}\beta_0 - \gamma^{(0)}\beta_1}{2\beta_0^2} \\
&\quad + \frac{\alpha_s(\pi/a)}{4\pi}\big(C^C - C^L\big)\Big]O_\Gamma(a) \\
&\simeq \Big(\frac{\alpha_s(\mu)}{\alpha_s(\pi/a)}\Big)^{\gamma^{(0)}/2\beta_0}\Big(1 + \frac{\alpha_s(\mu) - \alpha_s(\pi/a)}{4\pi}\frac{\gamma^{(1)}\beta_0 - \gamma^{(0)}\beta_1}{2\beta_0^2}\Big) \\
&\quad \times \Big[1 + \frac{\alpha_s(\pi/a)}{4\pi}\big(C^C - C^L\big)\Big]O_\Gamma(a)
\end{aligned}
\tag{17}
$$

Equation (17) has been obtained using the universality of the combination[3]:

$$\frac{\gamma^{(1)}}{2\beta_0} - C^C = \frac{\gamma_L^{(1)}}{2\beta_0} - C^L. \tag{18}$$

Equation (17) is interpreted as follows: the lattice operator is matched into the continuum operator at the scale π/a through the factor $1 + \alpha_s(\pi/a)/4\pi$ $\big(C^C - C^L\big)$ and then evolved in the continuum, according to eq. (14), from the scale π/a up to μ. A few comments may be useful at this point:

- One can eliminate the coefficient C^L in eq. (17) by defining a new lattice operator

$$O'_\Gamma(a) = O_\Gamma(a)(1 - \frac{\alpha_s(\pi/a)}{4\pi}C^L). \tag{19}$$

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[3]This relation is true only when we use the same coupling constant and renormalized gauge parameter on the lattice and in the continuum.

This is equivalent to the regularization independent definition of the renormalized operators discussed in refs.[38]–[40]. This definition is such that $\Gamma(p = \pi/a)/\Gamma_0 = 1 + O(\alpha_s^2)$[4]. Notice however that this procedure requires a knowledge of the external states for which we have computed C^L and that it is in general gauge-dependent.

- We have decided to expand the lattice Green function in $\alpha_s(\pi/a)$. However we have the freedom to expand in $\alpha_s(1/a)$ or any other scale we like since the change will be completely compensated at the NLO. For the same reason we can expand the Green functions in a different coupling constant, for example the "boosted" coupling α_s^V defined in refs.[19, 20]. There, it was argued that the series in α_s^V may minimize $O(\alpha_s^2)$ NNLO corrections. If we expand in α_s^V we have accordingly to reorganize the expression used for the coefficient function. It is wise, however, to check the stability of the physical amplitudes under a change in the scale used for α_s and assume this as a theoretical uncertainty.

3 Quark masses

3.1 Standard perturbative approach

From lattice simulations, by fixing the value of the lattice spacing and the physical mass of a strange or charmed hadron, it is possible to evaluate the bare lattice quark mass $m(a) = m_s(a)$ or $m_{ch}(a)$ (strange and charmed quark masses). From $m(a)$ one can compute the quark mass $m^{\overline{MS}}(\mu)$, renormalized in the continuum minimal-subtraction dimensional scheme, at the NLO; $m^{\overline{MS}}(\mu)$ will be defined below. In the following we report the relevant formulae used to relate $m^{\overline{MS}}(\mu)$ to $m(a)$ and discuss the different sources of theoretical uncertainty. The final values and errors of $m^{\overline{MS}}(\mu)$ for different flavours have been obtained by using the results of refs.[22]–[31], as shown in table 1, and include the theoretical uncertainties discussed below.

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[4]This is not in contrast with the condition $p^2 \ll (\pi/a)^2$ that we have to impose in order to avoid discretization errors. The renormalization condition is simply a consequence of eq. (1), which is derived by expanding in $\alpha_s(\pi/a)$ for $p^2 \ll (\pi/a)^2$.

For definiteness, the formulae used in this section are valid for the lattice Wilson [43] and SW–Clover [35] quark actions. The case of staggered fermions will not be considered here.

In perturbation theory, the inverse quark propagator on the lattice can be written as:

$$S^{-1}(p, m_0) = i\!\!\not{p}\Big[1 - \Sigma_1^L\Big(pa, m_0 a\Big)\Big] + m_0\Big[1 - \Sigma_2^L(pa, m_0 a)\Big] + \frac{\Sigma_0^L}{a}, \qquad (20)$$

where a is the lattice spacing. The last term in eq. (20) diverges linearly as $a \to 0$. This term is induced by the explicit chiral symmetry breaking that is present in the lattice formulation of the quark action. The linear divergence is a mass term, which can be eliminated by a suitable redefinition of the bare quark mass [8]-[10], [44, 45]. At $O(\alpha_s)$, this is obtained by writing $S^{-1}(p, m_0)$ as follows:

$$S^{-1}(p, m(a)) = \Big[1 - \Sigma_1^L(pa, m(a)a)\Big]$$
$$\times \Big[i\!\!\not{p} + m(a)\Big(1 + \Sigma_1^L(pa, m(a)a) - \Sigma_2^L(pa, m(a)a)\Big)\Big], \qquad (21)$$

where $m(a) = m_0 + \Sigma_0^L/a$. By comparing eq. (21) to the corresponding expression in the continuum:

$$S^{-1}(p, m^{\overline{MS}}(\mu)) = \Big[1 - \Sigma_1^C(p/\mu, m^{\overline{MS}}(\mu)/\mu)\Big]$$
$$\times \Big[i\!\!\not{p} + m^{\overline{MS}}(\mu)\Big(1 + \Sigma_1^C(p/\mu, m^{\overline{MS}}(\mu)/\mu) - \Sigma_2^C(p/\mu, m^{\overline{MS}}(\mu)/\mu)\Big)\Big], \quad (22)$$

we obtain:

$$m^{\overline{MS}}(\mu) = m(a)\Big[1 + \Sigma_1^L(pa, m(a)a) - \Sigma_2^L(pa, m(a)a)$$
$$-\Sigma_1^C(p/\mu, m^{\overline{MS}}(\mu)/\mu) + \Sigma_2^C(p/\mu, m^{\overline{MS}}(\mu)/\mu)\Big]$$
$$\simeq m(a)\Big[1 + \frac{\alpha_s}{4\pi}\Big(\gamma^{(0)}\ln(\frac{\pi}{\mu a}) + K_m\Big)\Big]; \qquad (23)$$

$\gamma^{(0)}$ and K_m are numerical constants, which can be computed from the general expressions of $\Sigma_{1,2}$:

$$\Sigma_1^{L,C}(p/\mu, m/\mu) = \frac{\alpha_s}{4\pi}\frac{N^2-1}{2N}\Big[\sigma_1^{L,C} + \int_0^1 dx\Big(2(1-x)\ln((p^2 x(1-x) + m^2 x)/\mu^2)\Big)\Big]$$

$$\Sigma_2^{L,C}(p/\mu, m/\mu) = \frac{\alpha_s}{4\pi}\frac{N^2-1}{2N}\Big[\sigma_2^{L,C} + \int_0^1 dx\Big(4\ln((p^2 x(1-x) + m^2 x)/\mu^2)\Big)\Big] \quad (24)$$

Action	β	a^{-1}	K_c	K_s	$m_s(a)$	$\bar{m}_s(a)$
Wilson [22]	6.0	$2.23(5)^a$	0.15704(3)	0.1548(1)	0.104(2)	0.127(3)
Wilson [22]	6.0	$2.21(8)^b$	0.15704(3)	0.1547(2)	0.105(4)	0.128(4)
Wilson [23]	6.0	$2.29(37)^b$	0.1570(1)	0.1553(8)	0.080(40)	0.098(49)
Clover [22]	6.0	$2.05(6)^a$	0.14551(3)	0.1437(1)	0.090(3)	0.103(3)
Clover [22]	6.0	$2.11(11)^b$	0.14551(4)	0.1438(2)	0.088(4)	0.100(5)
Clover [24]	6.0	$1.95(7)^a$	0.14548(2)	0.1434(1)	0.096(3)	0.108(4)
Clover [24]	6.0	$1.78(9)^b$	0.14548(2)	0.1430(2)	0.104(5)	0.117(6)
Clover [25]	6.0	$2.0^{+3}_{-2}{}^{a}$	0.14556(6)	0.1437^{+4}_{-5}	0.089(28)	0.101(31)
Wilson [26]	6.2	$2.9(2)^a$	0.15330(2)	0.1517(3)	0.101(9)	0.121(10)
Wilson [26]	6.2	$3.0(2)^b$	0.15330(2)	0.1518(2)	0.098(8)	0.118(9)
Wilson [27]	6.2	$2.77^{+9}_{-23}{}^{a}$	0.15328^{+7}_{-4}	0.1517^{+1}_{-3}	0.094(20)	0.113(24)
Clover [28]	6.2	$2.7(1)^a$	0.14315(2)	0.1419(1)	0.082(8)	0.093(8)
Clover [29]	6.2	$3.05(19)^a$	0.14313(2)	0.1421(1)	0.079(5)	0.090(5)
Clover [29]	6.2	$2.73(17)^b$	0.14313(2)	0.1418(2)	0.089(6)	0.100(7)
Wilson [23]	6.3	$3.01(26)^b$	0.15163(3)	0.1503(3)	0.088(21)	0.105(25)
Wilson [30]	6.4	$3.7(1)^a$	0.1506(1)	0.1495(1)	0.089(3)	0.106(3)
Wilson [30]	6.4	$4.0(6)^b$	0.1506(1)	0.1497(2)	0.082(13)	0.098(15)
Action	β	a^{-1}	K_c	K_{ch}	$m_{ch}(a)$	$\bar{m}_{ch}(a)$
Wilson [26]	6.0	$2.18(9)^a$	0.15702(4)	0.1313(16)	1.36(4)	1.26(2)
Wilson [26]	6.0	$1.97(8)^b$	0.15702(4)	0.1271(17)	1.48(5)	1.31(2)
Clover [26]	6.0	$1.92(11)^a$	0.14544(2)	0.1212(24)	1.32(8)	1.13(3)
Clover [26]	6.0	$1.94(5)^b$	0.14544(2)	0.1218(11)	1.30(4)	1.12(2)
Wilson [26]	6.2	$2.9(2)^a$	0.15330(2)	0.1354(24)	1.24(8)	1.22(5)
Wilson [26]	6.2	$3.0(2)^b$	0.15330(2)	0.1362(22)	1.22(8)	1.21(5)
Wilson [31]	6.2	$2.77^{+9}_{-23}{}^{a}$	0.15328^{+7}_{-4}	$0.1350(-)$	1.22(10)	1.20(10)
Clover [31]	6.2	$2.6(2)^a$	0.14313^{+7}_{-4}	$0.1290(-)$	0.99(8)	0.94(7)
Wilson [30]	6.4	$3.7(1)^a$	0.1506(1)	0.1386(7)	1.06(2)	1.10(2)
Wilson [30]	6.4	$4.0(6)^b$	0.1506(1)	0.1402(24)	1.00(11)	1.05(9)

Table 1: *Lattice results for the strange and charm quark masses from lattice QCD. In the table $\beta = 6.0/g_L^2(a)$, K_q (q = s, ch) is the lattice hopping parameter for the quark q, $am_q(a) = 1/2\left(1/K_q - 1/K_c\right)$ and $a\bar{m}_q(a) = \ln(4K_c/K_q - 3)$. a^{-1}, $m(a)$ and $\bar{m}(a)$ are given in GeV. The symbol ^{a} denotes the scale taken from M_ρ and ^{b} from f_π. In the second case, we used boosted perturbation theory for the renormalization constant Z_A in the Wilson case ($Z_A = 0.78, 0.79, 0.81$ at $\beta = 6.0, 6.2, 6.4$), and the non-perturbative values in the SW–Clover case ($Z_A = 1.09, 1.04$ at $\beta = 6.0, 6.2$ [41, 42]). Ref. [23] used $Z_A = 0.77, 0.79$ at $\beta = 6.0, 6.3$, respectively.*

Regularization	σ_1	$\bar{\sigma}_1$	σ_2	Σ_0^L
$\overline{MS}$	1	–	2	–
Lattice-Wilson	13.85	0.99	1.90	51.43
Lattice-Clover	10.11	2.12	−8.07	31.98

Table 2: *Quantities entering in the definition of K_m, eq. (26); Σ_0^L, defined in eq. (20), is given in units of $(\alpha_s/4\pi)(N^2-1)/2N$. They have been computed in refs.[8]–[10],[46],[47].*

where the constants $\sigma_{1,2}^{L,C}$ are reported in table 2, together with Σ_0^L, which was defined in eq. (20). One finds:

$$\gamma^{(0)} = 6\frac{N^2-1}{2N} \tag{25}$$

and

$$K_m = \frac{N^2-1}{2N}\Big(\sigma_1^L - \sigma_2^L - \sigma_1^C + \sigma_2^C - 6\ln(\pi)\Big) = C_m^C - C_m^L. \tag{26}$$

From eq. (23) we can derive the relation between $m^{\overline{MS}}(\mu)$ and $m(a)$ at the next-to-leading order, cf. eq. (17) in sec. 2:

$$m^{\overline{MS}}(\mu) = \Big(\frac{\alpha_s(\mu)}{\alpha_s(\pi/a)}\Big)^{\gamma^{(0)}/2\beta_0}\Big[1 + \frac{\alpha_s(\mu) - \alpha_s(\pi/a)}{4\pi}\Big(\frac{\gamma^{(1)}}{2\beta_0} - \frac{\gamma^{(0)}\beta_1}{2\beta_0^2}\Big)$$
$$+ \frac{\alpha_s(\pi/a)}{4\pi}K_m\Big]m(a); \tag{27}$$

$m(a)$, $m^{\overline{MS}}(\mu)$, $\gamma^{(1)}$ and K_m are regularization-dependent, gauge-invariant quantities; $\gamma^{(0)}$ is the LO anomalous dimension and $\gamma^{(1)}$ the NLO one, computed in the $\overline{MS}$ dimensional scheme; $\gamma^{(1)}$ is connected to the lattice anomalous dimension $\gamma_L^{(1)}$ through the relation, see also eq. (18):

$$\gamma^{(1)} = \gamma_L^{(1)} + 2\beta_0 K_m. \tag{28}$$

In ref. [48] they found:

$$\gamma^{(1)} = \frac{77}{3}N\frac{N^2-1}{2N} + 3\Big(\frac{N^2-1}{2N}\Big)^2 - \frac{10n_f}{3}\frac{N^2-1}{2N} \tag{29}$$

In eq. (27), α_s is the running coupling constant in the continuum minimal-subtraction scheme. It is related to the lattice bare coupling $\alpha_s^L(a) = g_L^2(a)/4\pi$ by the equation, see (7):

$$\frac{1}{\alpha_s^L(a)} = \frac{1}{\alpha_s(q)}\Big(1 + \frac{\alpha_s(q)}{4\pi}(\beta_0 \ln(\frac{\pi}{qa})^2 + 48.76)\Big). \tag{30}$$

At $\beta = 6.0$ one finds:

$$\alpha_s(q = \frac{\pi}{a}) \simeq 1.45 \ \alpha_s^L(1/a). \tag{31}$$

The above definition gives values of α_s close to the values of the "optimized perturbative lattice expansion" couplings α_s^V introduced in refs.[19, 20]. One typically finds $\alpha_s^V(\pi/a) \simeq$ 1.6–1.8 $\alpha_s^L(a)$. Among the definitions used in refs.[19, 20], we have:

$$\frac{1}{\alpha_s^L(a)} = \frac{1}{\alpha_s^{V_1}(q)}\Big(1 + \frac{\alpha_s^{V_1}(q)}{4\pi}(\beta_0 \ln(\frac{\pi}{qa})^2 + 59.09)\Big), \tag{32}$$

where $\alpha_s^{V_1}$ is defined from the Q–$\bar{Q}$ heavy-quark static potential. Using the perturbative expansion of the plaquette, one finds:

$$\frac{1}{\alpha_s^L(a)} = \frac{1}{\alpha_s^{V_1}(q = \pi/a)\langle\frac{1}{3}TrU_{plaq}\rangle}\Big(1 + \frac{\alpha_s^{V_1}(q = \pi/a)}{4\pi}6.45\Big). \tag{33}$$

One can expand in $\alpha_s^{V_1}$ by redefining $\gamma^{(1)}$ according to eqs.(7) and (11), and use the value of $\alpha_s^{V_1}$, found by using the plaquette computed in numerical simulations. Another commonly used definition of α_s^V is:

$$\frac{1}{\alpha_s^L(a)} = \frac{(8K_c)^4}{\alpha_s^{V_2}} \simeq \frac{1}{\alpha_s^{V_2}}(1 + \Sigma_0^L), \tag{34}$$

We will discuss the uncertainties in the quark mass, coming from different choices of the expansion parameter in sec. 4.

We can give an alternative definition of the renormalized mass, which is regularization-independent. One can for example define the renormalization constant of the mass Z_m as the inverse of the renormalization constant of the scalar density $S = \bar{\psi}\psi$:

$$Z_m = Z_S^{-1}, \tag{35}$$

where Z_S is defined by the renormalization condition:

$$Z_S(\mu a)\Gamma_S(\mu a)/\Gamma_S^0 = 1 \qquad (36)$$

imposed to the amputated Green function on off-shell quark states with $p^2 = \mu^2$. Since Z_S in eq. (36) is gauge-dependent, we will denote it as $Z_S^{gauge=lan,fey,\cdots}$ in the following. The renormalization scheme introduced in eq. (36), which we call in the following RI for regularization-independent, is particularly suitable for the implementation of the relevant Ward identities of the theory, see sec. 3.3, and for the non-perturbative renormalization of lattice operators [21]. In perturbation theory, in the Feynman or Landau gauge, one finds:

$$
\begin{aligned}
m^{fey,lan}(\mu) &= \left(\frac{\alpha_s(\mu)}{\alpha_s(\pi/a)}\right)^{\gamma^{(0)}/2\beta_0}\left[1 + \frac{\alpha_s(\mu) - \alpha_s(\pi/a)}{4\pi}\left(\frac{\gamma^{(1)}}{2\beta_0} - \frac{\gamma^{(0)}\beta_1}{2\beta_0^2}\right)\right.\\
&\quad + \left.\frac{\alpha_s(\pi/a)}{4\pi}K_m - \frac{\alpha_s(\mu)}{4\pi}C_m^{fey,lan}\right]m(a)\\
&\simeq \left[1 - \frac{\alpha_s(\mu)}{4\pi}C_m^{fey,lan}\right]m^{\overline{MS}}(\mu)\\
C_m^{fey} &= -\frac{N^2-1}{2N}5\\
C_m^{lan} &= -\frac{N^2-1}{2N}4 \qquad (37)
\end{aligned}
$$

As explained in ref. [40], $m^{fey,lan}(\mu)$ are regularization-independent. In fact they are obtained by imposing the same renormalization conditions in all the regularizations. The price is that this definition can be gauge-dependent and one has to specify the external states at which the renormalization conditions have been imposed.

3.2   Tadpole resummation

In the Wilson or SW–Clover lattice formulation of the quark action, $m(a)$ is expressed in terms of the hopping parameter as:

$$m(a) = \ln\left(1 + \frac{1}{2}\left(\frac{1}{K} - \frac{1}{K_c^0}\right)\right) \simeq \frac{1}{2}\left(\frac{1}{K} - \frac{1}{K_c^0}\right), \qquad (38)$$

12

where the critical value of K, corresponding to $m(a) = 0$, is $K_c^0 = 1/8$. At first order in perturbation theory, K_c is given by:

$$\frac{1}{K_c} = 8 - 2\Sigma_0^L = 8u_0. \tag{39}$$

It has been argued that lattice perturbation theory is dominated by tadpole diagrams, which have no correspondence in the continuum, and that the hopping parameter should be replaced by an "effective" one [19, 20]:

$$\bar{K} = Ku_0 \sim \frac{K}{8K_c}, \tag{40}$$

where K_c is defined non-perturbatively as the value of K at which the pseudoscalar meson mass vanishes. The same tadpole diagrams contributing to Σ_0^L also enter the calculation of Σ_1. At first order in perturbation theory, we can write:

$$1 - \Sigma_1^L = 1 - \frac{\Sigma_0^L}{4} - (\Sigma_1^L - \frac{\Sigma_0^L}{4})$$

$$\simeq (1 - \frac{\Sigma_0^L}{4})(1 - \bar{\Sigma}_1^L) = \frac{1}{8K_c}(1 - \bar{\Sigma}_1^L). \tag{41}$$

Using eq. (41), the lattice inverse quark propagator becomes:

$$S^{-1}(p, m(a)) = \frac{\left[1 - \bar{\Sigma}_1^L(pa, \bar{m}(a)a)\right]}{8K_c}$$

$$\times \left[i\not{p} + \bar{m}(a)\left(1 + \bar{\Sigma}_1^L(pa, \bar{m}(a)a) - \Sigma_2^L(pa, \bar{m}(a)a)\right)\right] \tag{42}$$

where $\bar{m}(a) = 4K_c/K - 4 \simeq \ln(4K_c/K - 3)$. The relation between the lattice quark mass $\bar{m}(a)$ and $m^{\overline{MS}}(\mu)$ is then given by the same expression as in eq. (27), with $\bar{m}(a)$ instead of $m(a)$ and $\bar{\sigma}_1 = \sigma_1 - \Sigma_0^L/4$ instead of σ_1 (see table 2).

3.3 Renormalization conditions and Ward identities

In this section we discuss the definition of the quark mass via the Ward identities of the regularized theory and its relation with the quark mass defined in the previous section[5].

---

[5]To avoid problems connected with anomalous terms, we will only consider Ward indentities for non-singlet currents.

In the continuum, the renormalization of the quark mass and of the scalar density are related by the vector current Ward identity:

$$\langle\alpha|\partial^\mu V_\mu^a|\beta\rangle = \langle\alpha|\partial^\mu\bar{\psi}\gamma_\mu\frac{\lambda^a}{2}\psi|\beta\rangle = \langle\alpha|\bar{\psi}\left[M,\frac{\lambda^a}{2}\right]\psi|\beta\rangle \tag{43}$$

where the ψ's are bare fields, M is the bare mass matrix and $|\alpha,\beta\rangle$ are arbitrary on-shell physical states. Equation (43) guarantees that the product $m\bar{\psi}\psi$ is unrenormalized by strong interactions:

$$m_R\bar{\psi}_R\psi_R = mZ_mZ_s\bar{\psi}\psi, \tag{44}$$

i.e. that $Z_m = Z_S^{-1}$ as in eq. (35), which remains valid on the lattice, where it is also possible to define a conserved vector current in the limit of degenerate quark masses. It ensures that the mass and scalar density renormalization constants are the inverse of one another to all orders in perturbation theory and beyond. Still we have the freedom to decide which definition of the renormalized mass we wish to use: renormalized on the mass-shell, in the $\overline{MS}$ (gauge-invariant) or in the RI (gauge-dependent) schemes, as explained in sec. 3. Of course we could decide to renormalize the mass in the minimal-subtraction scheme and the scalar density in RI. Such an exotic choice would only obscure the understanding of the Ward identities and will not be considered here. Differences between different definitions of the quark mass are related by terms computable in perturbation theory. This is true for heavy quarks with the renormalization on the mass shell and for heavy or light quarks with the $\overline{MS}$ or RI renormalizations, provided that μ and π/a are much larger than Λ_{QCD}.

The quark mass can also be defined through the Ward identity of the axial current:

$$\langle\alpha|\partial^\mu A_\mu^a|\beta\rangle = \langle\alpha|\partial^\mu\bar{\psi}\gamma_\mu\gamma_5\frac{\lambda^a}{2}\psi|\beta\rangle = \langle\alpha|\bar{\psi}\left\{M,\frac{\lambda^a}{2}\right\}\gamma_5\psi|\beta\rangle. \tag{45}$$

In a regularization that preserves chirality[6], as for example naïve $\overline{MS}$, the scalar and pseudoscalar ($P^a = \bar{\psi}\lambda^a/2\gamma_5\psi$) densities have the same renormalization constants, $Z_S = Z_P$, so that the quark mass defined from eq. (43)

---

[6]This regularization strictly speaking does not exist.

or (45) coincide. This is not true in continuum regularizations, which break chirality, like 't Hooft–Veltman or dimensional reduction in the continuum, or in the Wilson (SW–Clover) formulation of QCD on the lattice, where an explicit term that violates chirality is present in the action. We now explain how the quark mass can be defined through the axial Ward identity in the presence of an explicit breaking of chiral invariance. The argument is made with the lattice regularization, but is general.

At zero order in α_s the axial current Ward identity on the lattice can be written as [10, 44]:

$$\langle \alpha | \partial^\mu A_\mu^a | \beta \rangle = 2m \langle \alpha | P^a | \beta \rangle + \langle \alpha | \xi_A^a | \beta \rangle \tag{46}$$

in terms of bare fields and masses. We have simplified the Ward identity by taking degenerate quark masses; ξ_A^a is the chiral rotation of the Wilson term, which is of $O(a)$ only in the free-field case ($\sim a\bar{\psi}D^2\psi$). Because of interaction, ξ_A^a develops terms of $O(g_L^2/a)$ and g_L^2 [10], which renormalize the mass term, the pseudoscalar density and the axial current. Equation (46) becomes then:

$$Z_A \langle \alpha | \partial^\mu A_\mu^a | \beta \rangle = 2\Big(m - \bar{m}(m)\Big) \langle \alpha | P^a | \beta \rangle + \langle \alpha | \bar{\xi}_A^a | \beta \rangle, \tag{47}$$

where $\bar{\xi}_A^a$ is of $O(a)$, including strong-interaction effects. Close to the chiral limit, and neglecting terms of $O(a)$, eq. (47) becomes:

$$
\begin{aligned}
Z_A \langle \alpha | \partial^\mu A_\mu^a | \beta \rangle &= 2\Big(m - m_c\Big)\Big(1 - \frac{\partial \bar{m}}{\partial m_c}\Big)\langle \alpha | P^a | \beta \rangle \\
&= 2\Big(m - m_c\Big)\frac{Z_P}{Z_S}\langle \alpha | P^a | \beta \rangle, \tag{48}
\end{aligned}
$$

where the chiral value of the mass is defined by the equation $m_c = \bar{m}(m_c)$ and corresponds to $K = K_c$ in the notation used in sec. 3, $m - m_c \simeq 1/2(1/K - 1/K_c)$. The factor Z_P/Z_S appears since S and $P' = Z_P/Z_S P$ (and not S and P [10]) belong to the same chiral multiplet, because of the symmetry breaking present on the lattice. The same would be true with the 't Hooft–Veltman regularization.

Even though Z_A and Z_P/Z_S can be both computed in perturbation theory, an alternative, fully non-perturbative, definition of the quark mass can

be given. Using the Ward identities relative to the axial vector and scalar–pseudoscalar densities we can determine both Z_A and Z_P/Z_S [21, 41, 49, 50] in a non-perturbative way. Then we can measure the ratio 2ρ of the matrix elements of $\partial^\mu A_\mu^a$ and P^a on a physical state; typically one uses:

$$2\rho = \frac{\langle 0|\partial^0 A_0^a|\pi(\vec{p}=0)\rangle}{\langle 0|P^a|\pi(\vec{p}=0)\rangle} \tag{49}$$

and then, by saturating the Ward identity in eq. (48) we can find the lattice bare quark mass. This avoids ambiguities on the mass definition of the kind $m(a) = 4(K_c/K - 1) \sim \ln(4K_c/K - 3)$, see sec. 3.2 [7]. In practice we define $m = m(a)$ as:

$$m(a) = Z_A \times \frac{Z_S}{Z_P} \times \rho, \tag{50}$$

To find the continuum, renormalized quark mass, we can proceed at this point in two different ways. We can work in perturbation theory, and we essentially recover the results found in eq. (27) of sec. 3.2. Alternatively we can renormalize non-perturbatively the pseudoscalar density by imposing, on quark states of momentum $p^2 = \mu^2$ and in a fixed (Landau) gauge, the renormalization conditions [21]:

$$Z_S^{lan}(\mu a)\Gamma_S(\mu a)/\Gamma_S^0 = Z_S^{lan}(\mu a)\Gamma_{P'}(\mu a)/\Gamma_P^0 = Z_P^{lan}(\mu a)\Gamma_P(\mu a)/\Gamma_P^0 = 1. \tag{51}$$

We have used the fact that the renormalization conditions imposed on S and P may be chosen in such a way as to satisfy the continuum Ward identities, i.e. $Z_P/Z_S = Z_P^{lan}/Z_S^{lan}$. The continuum mass is then given by:

$$m^{lan}(\mu) = Z_m^{lan}(\mu a)m(a) = Z_A \times \left(Z_P^{lan}(\mu a)\right)^{-1} \times \rho, \tag{52}$$

We can then use eq. (37) to obtain $m^{\overline{MS}}(\mu)$ from $m^{lan}(\mu)$. The advantage of this procedure is that perturbation theory is used only to relate quark masses in the continuum, thus avoiding the large perturbative corrections present on the lattice [21]. The possible disadvantage is that μ has to satisfy the condition $\Lambda_{QCD} \ll \mu \ll 1/a$ to avoid large higher-order corrections

---

[7]There remains another ambiguity of $O(a)$ in the definition of the time derivative in eq. (49). This ambiguity can be made of $O(\alpha_s a)$ with improved actions.

Figure 1: *Histograms of the strange and charm quark mass distributions in arbitrary units, computed in the $\overline{MS}$ scheme at the scale $\mu = 2$ GeV. They have been obtained by considering all possible uncertainties discussed in the text. In the case of the strange quark all the results of table 1 have been included. In the charm case, in order to reduce discretization errors, we have only considered the results obtained at $\beta \geq 6.2$.*

and discretization errors[8]. It turns out that, using the SW–Clover action at $\beta = 6.0$, it is possible to choose a value of $\mu \sim 1/a \sim 2$ GeV, at which good agreement can be found between the determination of Z_P^{lan}/Z_S^{lan} on quark states [21] and Z_P/Z_S as computed using the Ward identities [41], with a relatively small discretization error. The systematic error due to discretization can be estimated to be of the order of $\sim$ 10–15%. Since at present Z_P^{lan} has been computed only at $\beta = 6.0$ and only with the SW–Clover action, the results of the non-perturbative method will be reported in the next section, but they will not be included in the final evaluation of m_s.

---

[8]One could imagine determining $Z_{S,P}^{lan}$ non-perturbatively on the lattice, at very large values of β, thus avoiding large higher-order effects. This however would not solve the problem because, in order to get a physical quantity from a matrix element computed in current simulations, one has in any case to evolve the operators down to a scale μ smaller than the inverse lattice spacing used in the numerical calculation of the matrix element. Thus large higher-order effects would be present anyway.

Figure 2: *Strange and charm $\overline{MS}$ masses and errors at $\mu = 2$ GeV from the results of table 1.*

4 Numerical estimates of quark masses from the lattice

4.1 Uncertainties of the lattice calculations

We are now ready to give the results for the continuum quark masses. These results have been obtained by combining the values of the bare lattice masses given in table 1 with the conversion factor in eq. (27), computed with all its possible allowed variations, value of the scale, α_s, etc. The theoretical predictions are subject to the uncertainties listed below:

- **Statistical error on** $m(a)a$: this error is usually in the range 5–20% depending on the value of $\beta = 6/g_L^2(a)$ and on the statistical accuracy of the simulation.

- **Calibration of the lattice spacing**: the uncertainty on the value of a^{-1} in physical units enters in two ways:

i) when we convert the dimensionless quark mass $m(a)a$, which is directly accessible in numerical simulations, to $m(a)$;

ii) in the scale used to evaluate $\alpha_s(\pi/a)$ or $\alpha_s(1/a)$, etc.

Only the first effect is important, since the second one gives a mild, logarithmic dependence in a. The value of a enters not only in the conversion of $m(a)a$ to $m(a)$, but also in the determination of K_s (K_{ch}) from the mass of a strange (charmed) hadron. There are several methods to evaluate the lattice spacing. The most popular ones are from the string tension, from the mass of the ρ, and from f_π. They usually differ from one another by $\sim 10\%$.

- **Choice of the expansion parameter**: we can use α_s, $\alpha_s^{V_1}$ or $\alpha_s^{V_2}$, or any other coupling constant that one believes makes the perturbative series converge rapidly. The differences are of $O(\alpha_s^2)$, but may be important with the lattice regularization [19, 20]. We have evaluated the conversion factor with all the three possibilities listed above. In particular, with the $\overline{MS}$ choice of α_s, we have evaluated α_s either by using eq. (30) or by using the "unquenched" LEP results [51, 52], i.e. we have computed α_s from eq. (12) with $\Lambda_{QCD}^{n_f=4} = (340 \pm 120)$ MeV.

- **Effects of $O(\alpha_s^2)$**: Values of the mass that differ by terms of order α_s^2 are obtained by using the two equivalent versions of eq. (17), but with different scales as arguments of α_s. We have allowed the scale in α_s to vary between π/a and $1/a$, which seems to us sufficient to cover a large range of possibilities.

- **Tadpole-improved definition of the mass**: We have evaluated the continuum mass from $m(a)$ defined as $1/2(1/K - 1/K_c)$ and from its tadpole improved expression $\ln(4K_c/K - 3)$ with the conversion factor changed accordingly.

- **Discretization errors**: In the final estimate of the mass we combine the results obtained with the Wilson and SW–Clover "improved" action, which suffer from different $O(a)$ effects. The comparison is useful because in the SW–Clover case, discretization errors are of $O(\alpha_s a)$ and have been found in some cases as small as 5%, to be compared to 30% in the Wilson case.

- **Quenching**: It is not possible to estimate and correct the error due to the quenched approximation. We have only applied the following procedure. From $m(a)$ one can get $m^{\overline{MS}}(\pi/a)$, which is subsequently evolved to $\mu = 2$ GeV. It is clear that $m^{\overline{MS}}(\pi/a)$ scales in a (and for consistency in μ) with a quenched anomalous dimension and β-function. However, by imposing $\alpha_s(\pi/a)^{quenched} = \alpha_s(\pi/a)^{unquenched}$, $m^{\overline{MS}}(\pi/a)$ has been evolved also with the unquenched formulae and the difference has been taken into account in the final error. This procedure does not pretend to correct for the systematic error entailed by the quenched approximation, but it is consistent with the observation that most of the quenched and unquenched results look very similar after a suitable rescaling of the lattice coupling constant.

4.2 Results for the strange and charmed quark masses

By changing the value of the lattice spacing, the expansion parameter α_s, the scale at which α_s is evaluated, etc., and taking into account the statistical errors of the lattice simulations, we obtain a pseudo-Gaussian distribution of the value of the mass from which it is possible to evaluate the theoretical error, following the method explained in ref. [37], see fig. 1. The continuum determination, at the scale $\mu = 2$ GeV, of masses and relative errors from different lattice simulations are reported in figs. 1 and 2. One notices a nice agreement between different simulations performed at different values of the lattice spacing and with different lattice quark actions. No systematic trend in μ or dependence on the lattice action appears for the strange quark. In the case of the charm quark, a slight reduction of the value of the mass seems to appear at larger values of β. This could be due to terms of $O(m_{ch}a)$. In order to reduce the systematic error due to these corrections, in the final estimate of the value of the mass, we have taken only the determinations of m_{ch} at $\beta \geq 6.2$. Combining all the results together, our best estimates, which is also reported in the abstract, are given by:

$$
\begin{aligned}
m_s^{\overline{MS}}(\mu = 2~GeV) &= (127 \pm 18)~MeV \\
m_{ch}^{\overline{MS}}(\mu = 2~GeV) &= (1.47 \pm 0.28)~GeV.
\end{aligned}
\tag{53}
$$

For completeness we also try to estimate m_s, with the SW–Clover action,

using the non-perturbative method envisaged in sec. 3.3. From a recent study at $\beta = 6.0$ they found [24]:

$$2\rho(K = K_s) = 0.071(5), \quad \beta = 6.0. \tag{54}$$

From the non-perturbative lattice renormalization of P^a, at $\beta = 6.0$ and $(pa)^2 \sim 1$, they estimated [21]:

$$Z_P^{lan}(\mu a \sim 1) = 0.49(1) \tag{55}$$

from which, using eq. (52), one finds:

$$m_s^{lan}(\mu \sim 2GeV) = (150 \pm 18) \; MeV, \tag{56}$$

or, from relation (37):

$$m_s^{\overline{MS}}(\mu = 2GeV) = (141 \pm 17) \; MeV, \tag{57}$$

in reasonable agreement with the results of eq. (53). In eqs. (56) and (57) only the statistical errors are reported. A further uncertainty of $\sim 10 - 15\%$, due to discretization errors, must be taken into account. A more accurate analysis can be found in ref. [21].

5 Conclusion

We have shown that lattice simulations of QCD can give accurate and reliable determinations of quark masses, using well-defined procedures. At present, it is possible to evaluate most of the uncertainties with the only exception of the systematic error introduced by the quenched approximation. The values of the strange and charm quark masses found in this work can readily be used for phenomenological applications.

Acknowledgements

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