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# Diffusive limit of asymmetric simple exclusion

R. Esposito [1]

Dipartimento di Matematica, Università di Roma Tor Vergata
Via della Ricerca Scientifica 00133 Roma, Italy. esposito@mat.utovrm.it

R. Marra [2]

Dipartimento di Fisica, Università di Roma Tor Vergata
Via della Ricerca Scientifica 00133 Roma, Italy. marra@vaxtov.infn.it

H. T. Yau [3]

Courant Institute of Mathematical Sciences, New York University
NY, NY, 10012. yau@math1.nyu.edu



*Dedicated to Elliott Lieb for his sixtieth birthday*

**Abstract**

We consider the asymmetric simple exclusion process on the lattice $Z^d \cap T^d$ with periodic b.c. for $d \geq 3$, in the diffusive space-time scaling with parameter $\varepsilon$. Assume the initial state is a product of Bernoulli measures with density of order $\varepsilon$, up to a fixed reference constant density $\theta$. We prove that the density at time $t$ is given to first order by $\theta - \varepsilon m(x - \varepsilon^{-1}vt, t)$ with $v$ a uniform velocity depending on $\theta$ and the dynamics and $m(x,t)$ satisfies the $d$-dimensional viscous Burgers equation. The diffusion matrix is given by a variational formula related to the Green-Kubo formula and it is strictly bigger than the diffusion matrix for the corresponding symmetric exclusion process.

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## 1 Introduction.

The macroscopic behavior of a large particle system is usually described in terms of the limiting behavior of some special local observables under a suitable space-time scaling. In the context of hydrodynamical limit, these observables are the locally conserved quantities of the dynamics. The system is assumed to be initially distributed according to a probability measure characterized by a slowly varying Gibbs states. Let $\varepsilon$ represent the scale separation between the micro and macro space scales. Looking at the time evolution of the conserved local observables of the system on times of order $\varepsilon^{-1}$ corresponds to the hyperbolic (Euler) scaling, while the diffusive (Navier-Stokes) scaling corresponds to times of order $\varepsilon^{-2}$. Rigorous results on the hydrodynamical limit are available for a wide range of models, though mostly for lattice systems. We refer to [1] and [2] for a review of the known results.

One of the simplest models of interacting particle systems is the simple exclusion process (SEP). It is a particle model on the lattice $Z^d$ with a hard core interaction which prevents the presence of more than one particle per site. Particles jump at random from a site to nearest neighbors according to the following rule: a particle in the site $x$, independently of the others, waits for an exponential time, then jumps with probability proportional to $p_e \geq 0$ to the site $x + e$ ($e$ being a vector of length 1 on the lattice) provided that the site $x + e$ is empty; otherwise stays in $x$ and waits a new exponential time. We denote by $\eta_x(t) = 0, 1$ the number of particles in the site $x$ at time $t$ and by $\mathcal{L}$ the generator of the process. The case $p_e = p_{-e}$ for all $e$ (symmetric simple exclusion process) is particularly simple and its diffusive limit is ruled by the linear heat equation. To certain extent, this model is exactly solvable. The more interesting case is the non symmetric one, $p_e \neq p_{-e}$, called asymmetric simple exclusion process (ASEP). For this system the product measure is still invariant, although the dynamics is not reversible with respect to it. To understand the macroscopic behavior of the system, we start with the hyperbolic scaling. Assume that the initial profile is a product measure with density $\rho_0(\varepsilon x) \in (0,1)$ slowly varying from site to site. Then the macroscopic particle density at time $\varepsilon^{-1}t$, $\rho^\varepsilon(x,t)$ (as a function of the macroscopic position and time), in the limit $\varepsilon \to 0$, satisfies (see [1] for references) the inviscid Burgers equation

$$\partial_t \rho + \delta \cdot \nabla_x [\rho(1-\rho)] = 0, \tag{1.1}$$

where $\delta$ is the vector with component $\delta_e = p_e - p_{-e}$ with $e$ being the unit vectors in the positive coordinate directions. To have an idea of the behavior in the diffusive scaling at heuristic level, it is convenient to look at the first order diffusive corrections in $\varepsilon$ to the eq. (1.1). One expects that in the diffusive scaling this term becomes of order 1. On the other hand, the macroscopic current $J = \delta[\rho(1-\rho)]$ in general has an extra factor $\varepsilon^{-1}$ in front in the diffusive scaling and it is difficult to make a conjecture on the scaling limit. If $\delta$ is of order $\varepsilon$, which correspond to the weakly asymmetric simple exclusion process (WASEP), the macroscopic current $J$ becomes of order $\varepsilon$ and the limiting equation can be proven to be just a diffusion with a (non linear) drift (see [3], [4]). This modification, however, does not manifest the asymmetric nature of the problem and the diffusive constant so obtained is the same as the symmetric one. In fact, both in [3] and in [4] the asymmetric part of the dynamics is treated as a small perturbation of the symmetric part and both methods rely heavily on this assumption.

In this paper we consider the ASEP with $\delta$ of order 1, but we analyze a different setting to make the macroscopic current $J$ of order $\varepsilon$. Suppose that the initial density $\rho_0(\varepsilon x)$ is of the form

$$\rho_0(\varepsilon x) = \theta - \varepsilon u_0(\varepsilon x)$$

with $\theta \in (0,1)$. Then, assuming that this form of the density persists at time $t$, the macroscopic current would be

$$J = \delta\theta(1-\theta) - \varepsilon v u - \varepsilon^2 \delta u^2, \quad v = (1-2\theta)\delta.$$

Therefore the equation for the fluctuation of density would become:

$$\partial_t u + \varepsilon^{-1} v \cdot \nabla_z u + \delta \cdot \nabla_z u^2 = \sum_{i,j=1}^{d} D_{i,j} \frac{\partial^2 u}{\partial z_i \partial z_j} \tag{1.2}$$

with $D_{i,j}$ some diffusion matrix. A global galileian transformation

$$m(z,t) = u(z + \varepsilon^{-1} v t, t)$$

allows to remove the diverging term and the equation for $m(z,t)$ is the viscous Burgers equation

$$\partial_t m + \delta \cdot \nabla_z m^2 = \sum_{i,j=1}^{d} D_{i,j} \frac{\partial^2 m}{\partial z_i \partial z_j} \tag{1.3}$$



Note that in the case $\theta = \frac{1}{2}$ no galileian transformation is necessary because $v = 0$.

Let $\nu_\varepsilon(z,t)$ be the empirical measure

$$\nu_\varepsilon(z,t) = \varepsilon^{d-1} \sum_x \delta(z - \varepsilon x)(\theta - \eta_x(\varepsilon^{-2} t)).$$

The rest of this paper is devoted to prove that $\nu_\varepsilon(z - \varepsilon^{-1} v t, t)$ converges weakly to the solution of (1.3) under quite general technical assumptions for dimension strictly bigger than 2, with periodic boundary condition. The periodic boundary condition is assumed for simplicity and we believe that it can be removed by using a Fritz's type argument [5, 6]. Our methods work for a general class of models, in particular the jump needs not to be nearest neighbor. In our proof the condition $d > 2$ is important for two reasons: our arguments strongly rely on a multiscale analysis which fails for $d = 1,2$; the factor $\varepsilon^{-1}$ in the definition of $\nu_\varepsilon$ prevents a "law of large numbers" for $d = 1,2$. The restriction on dimension, however, is intrinsic since in this scaling the diffusion coefficient is expected to be infinite for $d = 1$ or 2. It was conjectured that the correct scaling for $d = 1$ is $\varepsilon^{-3/2}$ while for $d = 2$ there is a logarithmic correction to $\varepsilon^{-2}$. See [1] for a review and references therein.

Compared with previous results [3, 4], our result differs in three major ways. First of all, our dynamics is fully asymmetric and non reversible. Furthermore, the diffusion coefficient, which is 1 in WASEP, in our case is given by a Green-Kubo formula which depends on the (global) density in a complicated manner. Strictly speaking, there is no proof that the typical expression of Green-Kubo formula is mathematically well-defined in this case, nor does our result establish this rigorously. We do, however, give a variational representation of the diffusion coefficient which is equivalent to the Green-Kubo formula heuristically. Finally, the WASEP has a non trivial behavior only on the diffusive time scale, while in ASEP the diffusive scale is beyond the usual hydrodynamical time scale (namely the Euler scale). In other words, we are looking at the long time behavior of a system with non trivial Euler limit. Admittedly, by restricting the initial data to be small perturbation of the global equilibrium, our setting is very special from this point of view. It nevertheless is the natural setting for the incompressible Navier-Stokes equation.

The last remark is indeed the key point of this paper and it relates to the important question of the Navier-Stokes correction to the Euler equations for the density, velocity and temperature of a fluid. In that case the analysis on the diffusive time scaling is made difficult by the presence of a transport term of order $\varepsilon^{-1}$, like the macroscopic current in the present situation. The incompressible approximation, which means to consider density, velocity and temperature constant up to terms of order $\varepsilon$ and look to the equations for the corrections, has a well defined limit as $\varepsilon$ goes to 0. It is actually this feature that makes possible to prove that the incompressible Navier-Stokes equations are the diffusive scaling limit of the Boltzmann equation (see [7], [8]). The same scaling argument can be used to obtain the incompressible Navier-Stokes equations formally from the Newton equations [9].

It should be emphasized that, as proved in Theorem 5.9, the diffusion coefficient (matrix) for the asymmetric process is strictly bigger than the corresponding one for the symmetric process, i.e. the process with jump intensities $\tilde{p}_{\pm e} = (p_e + p_{-e})/2$. Since the diffusion corresponding to the symmetric part is due to the stochasticity of the dynamics, to certain extent the diffusivity corresponding to the "deterministic" part is measured by the difference of the two diffusion coefficients. Therefore, it is important to prove that this difference is strictly positive. Our method is certainly far from dealing with purely deterministic systems. It nevertheless establishes the fact that the asymmetric part of the dynamics does enhance the diffusivity.

The strategy of the proof is the following. The main technical tools are a modification of the entropy method proposed in [10] (see also [11]), the method of Varadhan [12, 13, 14] for the non gradient systems and some multi-scale analysis. The general philosophy of the relative entropy method is to choose, as reference measure, the local equilibrium with parameter slowly varying in space and suitably chosen in such a way to reproduce the correct density in the limit. In our case the local equilibrium (for the system in the cube of size $\varepsilon^{-1}$ with periodic boundary conditions) is just a product measure with chemical potential $\theta + \varepsilon\lambda(\varepsilon x - \varepsilon^{-1}vt, t)$, $\theta(1-\theta)\lambda(z,t) = -m(z,t)$ and $m$ being the solution of (1.3) with diffusion matrix to be determined. The specific entropy of this measure, whose density with respect to global equilibrium, denoted by $\psi_t^\varepsilon$, is of order $\epsilon^2$. This fixes the typical size of the relative entropies. Our main result states that the specific entropy of the non equilibrium distribution $f_t^\varepsilon$ relative to $\psi_t^\varepsilon$ satisfies

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}s(f_t^\varepsilon|\psi_t^\varepsilon) = 0 \qquad (1.4)$$



provided that $\psi_0^\varepsilon$ is the initial measure. The limit (1.4) correctly pins down the chemical potential $\lambda$ and, together with the entropy inequality, implies a law of large numbers for the empirical density and its limit satisfies (1.3). As usual, the relative entropy method requires the smoothness of the solution of the hydrodynamic equation, which, in the case of the viscous Burgers equation, is assured globally in time.

The above outlines are standard procedures for applying the relative entropy method. In our case, it encounters significant difficulties because the system is in "wrong" scaling. To overcome these, we are lead to pick up the correct second order correction which can be achieved by adding to the local Gibbs state a suitable non product correction $\varepsilon^2\Phi$, with $\Phi$ to be determined. We emphasize that such a correction changes the entropy only to orders higher than $\varepsilon^2$. Such a correction is added purely for the purpose to prove (1.4).

The choice of $\Phi$ is dictated by the method of [12] for the non gradient systems, which, in its original form [12, 13], applies only to diffusive and reversible systems. Recently, it was extended to non reversible systems by [14] who considered the asymmetric simple exclusion process with a mean zero condition, i.e. the average (signed) distance for which a particle can jump is zero. This approach, however, can not be applied in our setting because, among other things, it is dimensionally independent. Our method relies strongly on the multiscale analysis and the logarithmic Sobolev inequality (for the corresponding symmetric process). It is far more general than [14] but is restricted to $d \geq 3$.

Finally we remark that one should be able to prove (1.4) via standard "one" and "two blocks" estimates and the "tightness" [12, 13, 14], using the multi-scale estimates in sect. 5 and 6. This would allow us to treat more general initial states. We choose the relative entropy method because it is somewhat shorter and it provides an example for which one can pin down the next order correction to the local Gibbs state. The rest of the paper is organized as follows. In sect. 2 we state our main result. Sect 3 is devoted to some simple large deviation bounds for local Gibbs states to be used later. The main part of relative entropy estimate is in sect. 4. In sect. 5 we identify the diffusion matrix via a variational formula. Finally the multiscale estimates are proved in sect. 6.

## 2 Statement of the problem and results.

Suppose $\Lambda_L \subset \mathbb{Z}^d$ is a cubic sublattice of size $(2L+1)$. Let $d\mu_{\beta,L}$ denote the product measure on $\Lambda_L$ with chemical potential $\beta$, i.e.

$$d\mu_{\beta,L} = Z^{-1}\exp[\beta \sum_{x\in\Lambda_L} \eta_x], \quad \eta_x \in \{0,1\}, \tag{2.1}$$

where $Z$ is the normalization. Let $\theta$ be the equilibrium density, namely

$$\theta = \frac{\sum_{\eta=0,1}\eta e^{\beta\eta}}{\sum_{\eta=0,1} e^{\beta\eta}} = \frac{e^\beta}{1+e^\beta}.$$

We assume $0 < \theta < 1$.

We define the generator $\mathcal{L}$ of simple exclusion process by $\mathcal{L}f = \sum_b \mathcal{L}_b f$ with the sum running on the set of all oriented bonds $b = (x,y)$ in $\mathbb{Z}^d$ such that $y - x = e$ and $e$ denotes any unitary vector on th lattice $\mathbb{Z}^d$. Here $\mathcal{L}_b$ is defined by

$$\mathcal{L}_b f = p_e \eta_x [f(\eta^b) - f(\eta)] \tag{2.2}$$

$$(\eta^b)_z = (\eta^{x,y})_z = \begin{cases} \eta_y, & z = x \\ \eta_x, & z = y \\ \eta_z, & \text{otherwise} \end{cases}$$

We shall require $p_e \geq 0$ and $p_{-e} + p_e = 2$ for all $e$ for convenience. We shall use the notation $\sum_e$ to denote the sum on all $e$ such that $|e| = 1$, while $\sum_{e>0}$ means sum for $e$ with non negative components.

Let $f_t$ be the density of the microscopic dynamics relative to $d\mu_{\beta,L}$. Then $f_t$ satisfies the forward equation

$$\partial_t f_t = \varepsilon^{-2}\mathcal{L}^* f_t$$

Here $\varepsilon = (2L+1)^{-1}$ and the adjoint $\mathcal{L}^*$ is defined by

$$\mathcal{L}^* f = \sum_b \mathcal{L}_b^* f, \quad \mathcal{L}_b^* f = p_{-e}\eta_x[f(\eta^b) - f(\eta)].$$

We shall assume periodic boundary condition on $\Lambda_L$ so that $\mathcal{L}^*$ is really the adjoint of $\mathcal{L}$ in $L_2(\mu_\beta)$. The initial data $f_{t=0} = \psi_0$ is chosen as

$$\psi_0 = \frac{\exp\{\varepsilon \sum_x \lambda(\varepsilon x)\eta_x\}}{Z_\lambda} \tag{2.3}$$



where $\lambda$ is a smooth periodic function and $Z_\lambda$ is the normalization. Recall the specific entropy of two densities $f$ and $g$ relative to $d\mu_\beta$ is defined by

$$s(f|g) = \varepsilon^d \int f \log\frac{f}{g} d\mu_\beta \tag{2.4}$$

and let $s(f) \equiv s(f|1)$. Let $v = (v_e)_{e>0}$, $v_e = (1-2\theta)\delta_e \equiv (1-2\theta)(p_e - p_{-e})$. Our main result is the following:

**Theorem 2.1.** *Let $f_t$ be the density at time $t$ relative to $d\mu_\beta$ with initial data $\psi_0$. Let $\psi_t$ be defined by*

$$\psi_t = \frac{1}{Z(t)}\exp\{\varepsilon \sum_x \lambda(\varepsilon x - \varepsilon^{-1}vt, t)\eta_x\} \tag{2.5}$$

*where $Z(t)$ is the normalization (w.r.t. $d\mu_\beta$) and $\lambda$ satisfies the equation*

$$\frac{\partial}{\partial t}\lambda(z,t) - \theta(1-\theta)\sum_{e>0}\delta_e(\partial_e\lambda^2)(z,t) = \sum_{e>0}\sum_{e'>0} D_{ee'}(\partial_e\partial_{e'}\lambda)(z,t). \tag{2.6}$$

*Here $\partial_e\lambda(z) = \partial\lambda(z)/\partial z_e$ and $0 < D < \infty$ ( in matrix sense) with $D$ given by Theorem 5.10. Then, for $d \geq 3$,*

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}s(f_t|\psi_t) = 0. \tag{2.7}$$

The proof of this theorem will be given in sect. 4.

<u>Remark.</u> The factor $\theta(1-\theta)$ is due to the relation between the density and the chemical potential, as explained in the following.

To understand the meaning of (2.7), let us first recall some definitions and elementary facts. Let $P(h)$ denote the pressure defined by

$$P(h) = \log \sum_{\eta=0,1} e^{h\eta} = \log(e^h + 1). \tag{2.8}$$

Define the entropy by the Legendre transformation

$$s(\rho) = \sup_{\alpha\in\mathbb{R}}[\alpha\rho - P(\alpha)].$$

The density $\rho$ and $h$ are related by

$$\rho = P'(\alpha) = e^\alpha(e^\alpha + 1)^{-1} = (1 + e^{-\alpha})^{-1}, \quad h = \log(\rho/(1-\rho)).$$

In particular, if $h = \beta + \varepsilon\alpha$ then

$$\rho = P'(h) = P'(\beta) + \varepsilon P''(\beta)\alpha + O(\varepsilon^2) = \theta + \varepsilon\theta(1-\theta)\alpha + O(\varepsilon^2). \tag{2.9}$$

Let $\psi$ be a local Gibbs state characterized by a smooth function $\xi$ as

$$\psi = \exp[\varepsilon \sum_x \xi(\varepsilon x)\eta_x] Z_\xi^{-1}, \tag{2.10}$$

$$\log Z_\xi = \log \int \exp[\varepsilon \sum_x \xi(\varepsilon x)\eta_x] d\mu_\beta = \sum_x [P(\beta + \varepsilon\xi(\varepsilon x)) - P(\beta)]. \tag{2.11}$$

Here and for the rest of this section, all summations are for $x \in \Lambda_L$. One can easily check that

$$s(\psi) = \varepsilon^d E^{\mu_\beta}[\psi \log \psi]$$
$$\leq \varepsilon^d \sum_x \{\varepsilon E^{\mu_\beta}[\xi(\varepsilon x)\eta_x] - P(\beta + \varepsilon\xi(\varepsilon x)) + P(\beta)\} \leq \text{const.}\ \varepsilon^2.$$

In order to pin down the function $\xi$, one needs to know the entropy beyond order $\varepsilon^2$. Indeed, one has the following Lemma where, as in the rest of this paper, $J$ denotes a smooth function of compact support in $\mathcal{R}^d$.

**Lemma 2.2.** *Let $f$ be a density satisfying*

$$\lim_{\varepsilon\to 0} \varepsilon^{-2} s(f|\psi) = 0$$

*Then for any $J$ and any bounded local function $F$ one has*

$$\lim_{\varepsilon\to 0} [E^f - E^\psi][\varepsilon^{d-1} \sum_x J(\varepsilon x)\tau_x F] = 0.$$

*If instead of $\varepsilon^{-2}s(f|\psi) \to 0$ one has $\varepsilon^{-2}s(f|\psi) \leq$ const. then the following bound holds*

$$|[E^f - E^\psi][\varepsilon^{d-1} \sum_x J(\varepsilon x)\tau_x F]| \leq \text{const.}$$

*In particular, $\psi$ can be the equilibrium measure $d\mu_\beta$.*

*Proof:* We use the entropy inequality:

$$E_f[X] \leq \gamma^{-1}\varepsilon^{-d}s(f|\psi) + \gamma^{-1} \log E^\psi[\exp\{\gamma X\}] \tag{2.12}$$

for any positive $\gamma$ and for any random variable $X$. Let $X = \varepsilon^{d-1}\sum_x J(\varepsilon x)\tau_x F$ and $\gamma = q\varepsilon^{2-d}$ We claim that, for such choices of $\gamma$ and $X$, one has, for any $q$ fixed,

$$\lim_{\varepsilon\to 0} \gamma^{-1} \log E^\psi \left[\exp\left\{\gamma[X - E^\psi\{X\}]\right\}\right] \leq \text{const}.q. \tag{2.13}$$



To prove this, suppose $F$ measurable w.r.t. the $\sigma$-algebra of a cube of size $\ell$. Let us label $x \in \Lambda_\ell$ by $(2\ell+1)\alpha + \sigma$ with $|\sigma| \leq \ell$ and $\alpha \in \Lambda_{L/2\ell+1}$. Hence we can write $X$ as

$$X = \text{Av}_\sigma (2\ell+1)^d \varepsilon^{d-1} \sum_\alpha J[\varepsilon\{(2\ell+1)\alpha + \sigma\}]\tau_{(2\ell+1)\alpha+\sigma}F, \tag{2.14}$$

where $\text{Av}_\sigma = (2\ell+1)^{-d}\sum_{|\sigma|\leq\ell}$. Without loss of generality one can assume $E^\psi[X] = 0$. Now, by the Jensen inequality (applied to $u \to \log E[e^u]$),

$$\gamma^{-1} \log E^\psi[\exp\{\gamma[X]\}] \leq \gamma^{-1}\text{Av}_\sigma (2\ell+1)^d \tag{2.15}$$
$$\log E^\psi \left[\exp\left\{q\varepsilon \sum_\alpha J[\varepsilon\{(2\ell+1)\alpha + \sigma\}]\tau_{(2\ell+1)\alpha+\sigma}F\right\}\right]. \tag{2.16}$$

Since $\psi$ is a product measure, the expectation factors into a product. Therefore, it suffices to prove that for any $\bar{q}$ fixed

$$\lim_{\varepsilon\to 0} \bar{q}\varepsilon^{-2} \log E^\psi[\exp\{\bar{q}\varepsilon F\} - E^\psi[F] \leq \text{const}.\bar{q}. \tag{2.17}$$

One can check (2.17) easily by expanding the exponential up to second order in $\varepsilon$.

To prove the first part of Lemma 2.2, one takes $\varepsilon \to 0$ and then $q \to 0$. This proves the upper bound. To obtain the lower bound, one simply switches $X$ to $-X$.

For the second part, one fixes $q$ to be some small constant instead of letting $q \to 0$. This concludes lemma 2.2. ∎

**Corollary 2.3** *Let $\nu_\varepsilon(z,t)$ be the empirical measure*

$$\nu_\varepsilon(z,t) = \varepsilon^{d-1} \sum_x \delta(z - \varepsilon x)(\theta - \eta_x(t)).$$

*Then, for $d \geq 3$, $\nu_\varepsilon(z - \varepsilon^{-1}vt, t)$ converges weakly in probability to $m(z,t)$ solving the equation*

$$\frac{\partial m(z,t)}{\partial t} + \sum_{e>0} \delta_e \partial_e (m(z,t)^2) = \sum_{e,e'>0} D_{ee'}(\partial_e\partial_{e'}m)(z,t). \tag{2.18}$$

*Proof:* From Theorem 2.1 and Lemma 2.2 it is enough to check the convergence with respect to the local equilibrium $\psi_t$. To prove this claim, recall the entropy bound [16, 11]

$$P^f[A] \leq \frac{\log 2 + \varepsilon^{-d}s(f|\psi)}{\log(1 + 1/P^\psi[A])}. \tag{2.19}$$

Let $A$ be the set $A = \{\eta | \varepsilon^{d-1}\sum_x J(\varepsilon x)(\eta_x - \theta + \varepsilon m(\varepsilon x - \varepsilon^{-1}vt, t)) > a\}$. From (2.13) and the Chebyshev inequality, $P^\psi[A]$ is bounded by $\exp\{-(a - \text{const}.q)\gamma\}$. Choose $q$ small enough so that $a - \text{const}.q > a/2$. From (2.7), (2.19) and this bound it follows $\lim_{\varepsilon\to 0} P^f[A] = 0$ for any choice of $a$ provided $d > 2$. This concludes Corollary 2.3. ∎

## 3 Large Deviations Estimate.

In this section we prove a simple result concerning large deviations of local Gibbs states. Most of the arguments are standard [10, 17] and we only sketch a few key steps. Let $\bar{\eta}_{x,k}$ be the average on a large block of size $2k+1$ centered at $x$

$$\bar{\eta}_{x,k} = (2k+1)^{-d} \sum_{|x-y|=k} \eta_y, \tag{3.1}$$

where $|x| = \max_{i=1,\dots,d} |x_i|$. For the rest of this section, $k = \ell \varepsilon^{-2/d}$ with $\ell$ being a large constant independent of $\varepsilon$.

**Lemma 3.1.** *Let $\psi$ be the density given by (2.10) and let*

$$\zeta_{x,k} = E^{\psi}[\bar{\eta}_{x,k}; \bar{\eta}_{x,k}] \equiv E^{\psi}[(\bar{\eta}_{x,k} - \overline{m}_{x,k})^2]$$

*where $E[A;B] = E[AB] - E[A]E[B]$ and $\overline{m}_{x,k}$ is defined by*

$$\overline{m}_{x,k} = E^{\psi}[\bar{\eta}_{x,k}]. \tag{3.2}$$

*Then, for $k = \ell\varepsilon^{-2/d}$ and for any $q < q_0$ with $q_0$ a small fixed constant,*

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0} \varepsilon^{d-2} \log E^{\psi}\left[\exp\left\{q\sum_x [(\bar{\eta}_{x,k} - \overline{m}_{x,k})^2 - \zeta_{x,k}]\right\}\right] = 0 \tag{3.3}$$

*In general, if $G_x$ is a family of bounded smooth functions such that*

$$G_x(\overline{m}_{x,k}) = \frac{\partial G_x}{\partial y}\Big|_{y=\overline{m}_{x,k}} = 0, \tag{3.4}$$

*then for any smooth function $J$*

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0} \varepsilon^{d-2} \log E^{\psi}\left[\exp\left\{\sum_x J(\varepsilon x)\{G_x(\bar{\eta}_{x,k}) - E^{\psi}[G_x(\bar{\eta}_{x,k})]\}\right\}\right] = 0. \tag{3.5}$$

*Proof.* Since $|G(u)| \le \text{const.}(u-m)^2$ for $G$ satisfying (3.4), we only have to prove the first part of the Lemma. This lemma was proved in [10, 17] in a slightly different context.

Divide $\Lambda_L$ into non overlapping cubes of size $2k+1$ and label them by $\sigma$. We shall use $\sigma$ to denote the center of the cube, too. Hence

$$\begin{aligned}\sum_x (\bar{\eta}_{x,k} - \overline{m}_{x,k})^2 &= \sum_{|j|\le k}\sum_\sigma (\bar{\eta}_{\sigma+j,k} - \overline{m}_{\sigma+j,k})^2 \\ &= \mathrm{Av}_{|j|\le k}(2k+1)^d \sum_\sigma (\bar{\eta}_{\sigma+j,k} - \overline{m}_{\sigma+j,k})^2\end{aligned}$$



By Jensen inequality for the function $\log E^{\psi}[\exp(u)]$, the left hand side of (3.3) is bounded by

$$\begin{aligned}&\varepsilon^{d-2}\mathrm{Av}_{|j|\le k} \log E^{\psi}\left\{\exp\left[q(2k+1)^d \sum_\sigma \{(\bar{\eta}_{\sigma+j,k} - \overline{m}_{\sigma+j,k})^2 - \zeta_{\sigma+j,k}\}\right]\right\} \\ &= \varepsilon^{d-2}\mathrm{Av}_{|j|\le k} \sum_\sigma \log E^{\psi}\left\{\exp\left[q(2k+1)^d \{(\bar{\eta}_{\sigma+j,k} - \overline{m}_{\sigma+j,k})^2 - \zeta_{\sigma+j,k}\}\right]\right\}\end{aligned} \tag{3.6}$$

Here we have used the product nature of $\psi$. It is not hard to estimate the expectation w.r.t. $\psi$, again since it is a product measure. For simplicity, let $\sigma = j = 0$ and suppress all indices $\sigma + j$. Note that $\bar{\eta}_k$ is a random variable with mean $\overline{m}_k$ and variance $\zeta_k$. Let $\xi_k = (\bar{\eta}_k - \overline{m}_k)/\zeta_k^{1/2}$. Since $\psi$ is a product measure, $\zeta_k$ is order $k^{-d}$. Together with last identity, (3.3) follows (for $d \ge 3$) if one establishes

$$\log E^{\psi}[\exp\{q(2k+1)^d \zeta_k \xi_k^2\}] \le \text{const.} \tag{3.7}$$

From the local central limit theorem and its expansion ([18], see also Lemma A.1 in [17]), $\xi_k$ has a density $\frac{1}{\sqrt{2\pi}}\exp[-\xi_k^2/2][1 + O(k^{-d/2})]$. Since $(2k+1)^d\zeta_k$ is uniformly bounded by a constant, say $\delta$, the integrand in (3.7) is bounded by $\exp[q\delta\xi_k^2]$. Clearly, for $q$ small enough, it is integrable w.r.t. $\frac{1}{\sqrt{2\pi}}\exp[-\xi_k^2/2]$. We have proved (3.7) and this concludes Lemma 3.1. ∎

We will be interested in local Gibbs states $\psi$ with small perturbation. More precisely, let $\hat{\psi}$ be defined by

$$\hat{\psi} = \hat{Z}^{-1}\exp\left\{\varepsilon \sum_x \xi(\varepsilon x)\eta_x + \varepsilon^2 \sum_x \zeta(\varepsilon x)\tau_x F\right\} \tag{3.8}$$

where $\hat{Z}$ is the normalization, $\xi$ and $\zeta$ are smooth function and $F$ is a local function. We shall assume for the rest of this paper that $E^{\mu_\theta}[F] = 0$. The following lemma allows us to replace $s(f|\psi)$ by $s(f|\hat{\psi})$.

**Lemma 3.2.** *Suppose that $f$ and $\mu_\theta$ are probability measures on $\Lambda_L$ with $\mu_\theta$ the equilibrium measure defined in (2.1). Moreover, $f$ satisfies $\varepsilon^{-2}s(f) \le$ const. Then for $\hat{\psi}$ and $\psi$ defined in (3.8) and (2.10), with $F$ such that $E^{\mu_\theta}[F] = 0$, one has*

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}[s(f|\hat{\psi}) - s(f|\psi)] = 0.$$

*Proof.* By definition,

$$\varepsilon^{-2}[s(f|\psi) - s(f|\dot\psi)] = \varepsilon^d E^f[\sum_x \zeta(\varepsilon x)\tau_x F] - \varepsilon^{d-2}\log(\dot Z/Z_\xi),$$

where

$$\begin{aligned}\varepsilon^{d-2}\log \dot Z/Z_\xi &= \varepsilon^{d-2}\log E^{\mu_\theta}\left[\exp\left\{\varepsilon\sum_x \xi(\varepsilon x)\eta_x + \varepsilon^2\sum_x \zeta(\varepsilon x)\tau_x F\right\}\right]\\ &- \varepsilon^{d-2}\log E^{\mu_\theta}[\exp\left\{\varepsilon\sum_x \xi(\varepsilon x)\eta_x\right\}]\\ &= \varepsilon^{d-2}\log E^{\nu}\left[\exp\left\{\varepsilon^2\sum_x \zeta(\varepsilon x)\tau_x F\right\}\right].\end{aligned}$$

To the leading order,

$$\varepsilon^{d-2}\log \dot Z/Z_\xi = \varepsilon^d E^\nu[\sum_x \zeta(\varepsilon x)\tau_x F] + O(\varepsilon^2)$$

By assumption on $F$ and Lemma 2.2, $\varepsilon^{d-2}\log \dot Z/Z_\xi \to 0$ as $\varepsilon \to 0$. The first term $\varepsilon^d E^f[\sum_x \zeta(\varepsilon x)\tau_x F]$ also tends to zero as $\varepsilon \to 0$ by assumption on $F$ and Lemma 2.2. This proves Lemma 3.2. ∎

# 4 Entropy estimate.

This section is devoted to the proof of Theorem 2.1. The aim is to get an integral inequality for the relative entropy in terms of a "large deviations" term and a "variance term". The first term will be bounded via Lemma 3.1. The second term is related to the fact that the current is non gradient. It will be handled with the method developed in [12, 13, 14] and in sect. 5 and 6. This method forces us to modify the local equilibrium by a second order non product term, which does not affect the relative entropy up to order $\varepsilon^2$ by Lemma 3.2, but it is crucial for the variance term to vanish.

We will replace $\psi_t$ with the density $\dot\psi_t$ defined by

$$\dot\psi_t = \dot Z_t^{-1}\exp\left\{\varepsilon\sum_x(\alpha * \dot\omega)(\varepsilon x, t)\eta_x + \varepsilon^2\Phi(\eta)\right\}, \tag{4.1}$$

where $\dot Z_t$ is the normalization, $*$ is the convolution product on $Z^d$, and $\alpha(z,t)$, $\dot\omega$ and $\Phi$ are chosen as follows.



(4.A) Let $\ell$ and $k$ be integers, $\tilde\ell = \ell^{d+2}$ and suppose that $\Lambda_k$ is divided in disjoint cubes of size $(2\tilde\ell+1)$, with centers $\sigma \in (2\tilde\ell+1)Z^d$, $|\sigma| \le k$. Let $\tilde\ell_1 = \tilde\ell - \ell^{1/d^2}$ and consider the cubes $\Lambda_{\tilde\ell_1,\sigma}$ and $\dot\Lambda_k = \bigcup_{|\sigma|\le k}\Lambda_{\tilde\ell_1,\sigma}$ is the region $\Lambda_k$ without corridors of width $2\ell^{1/d^2}$. Define $\dot\omega$ and $\omega$ to be the normalized characteristic functions $\dot\omega(x) = |\dot\Lambda_k|^{-1}1(x \in \dot\Lambda_k)$, $\omega(x) = (2k+1)^{-d}1(x \in \Lambda_k)$.

(4.B) $\alpha(\varepsilon x, t) = \lambda(\varepsilon x - \varepsilon^{-1}vt, t)$ with $\lambda$ and $v$ given by Theorem 2.1.

(4.C) Choice of $\Phi$. $\Phi(\eta) = -\sum_{x,e>0}(\partial_e\alpha)(\varepsilon x, t)(\dot\omega * \tau_y F)(x)$, where $F$ is a local function satisfying $\hat F(\theta) = 0 = \partial\hat F/\partial y|_{y=\theta}$, $\hat F(u) = E^{\mu_u}[F]$, with $u = P'(h)$.

The choice of $\dot\omega$ is particularly convenient to handle some boundary terms in sect. 5 and 6, although the use of $\ell^{1/d^2}$ is rather arbitrary. The choice of $\tilde\ell = \ell^{d+2}$ is arbitrary too and will be made clear in the proof of theorem 4.6 in Section 6. The choice of $\alpha(z,t)$ depends on the need of compensating a diverging transport term, according to what explained in sect. 1. Finally $\Phi$ is the second order corrections to $\psi$ and its choice will also be made clear below. We note that in the same conditions of Lemma 3.2, with $\dot\psi$ defined by (4.1), we have $\varepsilon^{-2}[(f|\psi) - s(f|\dot\psi)] \to 0$ from Lemma 3.2.

To state the main result of this section, we need the following definition.

Definition 4.1. *Let $G$ be a local function supported in $|x| \le s_0$ for some integer $s_0$. Suppose that $G$ satisfies $E^{\mu_\theta}[G] = 0$, where $\mu_\theta$ is the equilibrium state defined in (2.1). For any $y \in [0,1]$ we define the "variance" $V_\ell(G,y)$ by*

$$V_\ell(G,y) = (2\ell_1+1)^{-d}\left\langle\left[\sum_{|x|\le\ell_1}(\tau_x G - \alpha_\ell(G))\right](-\mathcal{L}_{s,\ell})^{-1}\left[\sum_{|x|\le\ell_1}(\tau_x G - \alpha_\ell(G))\right]\right\rangle_{\mu_{\ell,y}} \tag{4.2}$$

*where $\ell_1 = \ell - \ell^{1/d^2}$ and $\mu_{\ell,y}$ is the canonical Gibbs state of $(2\ell+1)^d$ sites with density $y$, namely,*

$$\mu_{\ell,y} = Q_{\ell,y}^{-1}\delta(\bar\eta_\ell - y)\mu_\ell.$$

*Here $\mu_\ell$ denotes the counting measure on the configurations in $\Lambda_\ell$ and $Q_{\ell,y}$ is the normalization. The generator $\mathcal{L}_{s,\ell}$ is defined by $\mathcal{L}_{s,\ell} = 1/2(\mathcal{L}_\ell + \mathcal{L}_\ell^*)$ with $\mathcal{L}_\ell = \sum_{|b|\le\ell}\mathcal{L}_b$ and $\mathcal{L}_\ell^*$ defined* *similarly. Here $|b| \le \ell$ means $b \in \Lambda_\ell$ and $\mathcal{L}_b$ is defined by (2.2). Finally, $\alpha_\ell(G)$ is a constant depending on $\bar\eta_\ell$ defined by*

$$\alpha_\ell(G) = E^{\mu_\beta}[G|\bar\eta_\ell]. \tag{4.3}$$

*We define also the "variance" of $G$ by*

$$V(G,\beta) = \limsup_{\ell\to\infty} E^{\mu_\beta}[V_\ell(G,\bar\eta_\ell)]. \tag{4.4}$$

We have chosen $\ell_1 < \ell$ to avoid boundary terms. The specific form of $\ell_1$ chosen above is not important.

Our main result of this section is the following theorem.

**Theorem 4.2** *Let $\hat\psi_t$ be defined as above and $f_t$ satisfying the assumptions of Theorem 2.1. Define the lattice gradient $\nabla_e f(x) \equiv f(x+e) - f(x)$. In particular, $\nabla_e \eta_0 = \eta_e - \eta_0$. Then, for $d \ge 3$,*

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0} \varepsilon^{-2} s(f_T|\hat\psi_T) \le C \lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\int_0^T \varepsilon^{-2} s(f_t|\hat\psi_t)dt + CT\sum_{e>0} V(\hat q_e,\beta) \tag{4.5}$$

*Here $C$ is a positive constant. $V$ is defined in (4.4) and*

$$\hat q_e = \delta_e(\eta_0 - \theta)(\eta_e - \theta) - \sum_{e'} \hat D_{ee'}\nabla_{e'}\eta_0 - \mathcal{L}_e^* \sum_x \tau_x F \tag{4.6}$$

*with $\hat D_{ee'} = \delta_{ee'} - D_{ee'}$ and $D_{ee'}$ is defined by (5.19).*

Let us recall a few basic facts from previous works.

**Lemma 4.3** *The density $f_t$ satisfies the following bounds*

*(i) $\frac{d}{dt}s(f_t) \le -\text{const.}\varepsilon^{-2+d}\sum_{b\in\Lambda_L} D_b(\sqrt{f_t})$, where $D_b(g) = \int[g(\eta^b) - g(\eta)]^2 d\mu_\beta(\eta)$ is the Dirichlet form.*

*(ii) For any $t \ge 0$, $\varepsilon^{-2}s(f_t) \le$ const.*

*(iii) For any local function $F$ and any $J$, $|[E^{f_t} - E^{\mu_\beta}][\varepsilon^{d-1}\sum_x J(\varepsilon x)\tau_x F]| \le$ const.*

*Proof.* The first two bounds (i) and (ii) were proved in [16]. The last bound (iii) follows from Lemma 2.2.

**Lemma 4.4.** *The relative entropy $s(f_t|\hat\psi_t)$ satisfies the bound*



*(i) $\frac{d}{dt}s(f_t|\hat\psi_t) \le \varepsilon^d \int f_t\hat\psi_t^{-1}(\varepsilon^{-2}\mathcal{L}^* - \partial_t)\hat\psi_t d\mu_\beta$.*

*(ii) There is a constant $c_t$ independent of $f_t$ such that*

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}\left[\frac{d}{dt}s(f_t|\hat\psi_t) - \varepsilon^d \int f_t(\varepsilon^{-2}\mathcal{L}^* - \partial_t)\log\hat\psi_t d\mu_\beta\right] - c_t \le 0.$$

*(iii) $\int \mathcal{L}^*\hat\psi_t d\mu_\beta = 0 = \int \partial_t\hat\psi_t d\mu_\beta$.*

*Proof.* The bound (i) is proved in [10, 11] while (iii) follows by inspection. Hence we only have to prove (ii). By comparing (i) and (ii), it suffices to prove that

$$\lim_{\varepsilon\to 0}\varepsilon^{d-4}\int f_t[\mathcal{L}^*\log\hat\psi_t - \hat\psi_t^{-1}\mathcal{L}^*\hat\psi_t]d\mu_\beta = c_t$$

By definition of $\mathcal{L}^*$, one has (omitting the subscript $t$)

$$\begin{aligned} W &= \mathcal{L}^*\log\hat\psi - \hat\psi^{-1}\mathcal{L}^*\hat\psi \\ &= \sum_{x,e}\left\{-\varepsilon p_{-e}\nabla_e\hat\alpha(\varepsilon x,t)\eta_x\nabla_e\eta_x + \varepsilon^2 p_{-e}\eta_x[\Phi(\eta^{x,x+e}) - \Phi(\eta)]\right\} \\ &\quad - \sum_{x,e}\eta_x p_{-e}\exp\left\{-\varepsilon[\nabla_e\hat\alpha(\varepsilon x,t)]\nabla_e\eta_x + \varepsilon^2[\Phi(\eta^{x,x+e}) - \Phi(\eta)] - 1\right\}, \end{aligned}$$

where $\hat\alpha = \alpha * \hat\omega$. Expanding the exponential to the second order,

$$W = \sum_{x,e} p_{-e}\varepsilon^4(\partial_e\hat\alpha(\varepsilon x,t))^2\eta_x(\nabla_e\eta_x)^2 + \varepsilon^4\sum_{x,e}p_{-e}\eta_x[\Phi(\eta^{x,x+e}) - \Phi(\eta)]^2 + O(\varepsilon^6).$$

If one choose $c_t = \lim_{\varepsilon\to 0}\varepsilon^{d-4}E^{\mu_\beta}[W]$, then (ii) follows from (iii) of Lemma 4.3. ∎

The generator $\mathcal{L}$ can be decomposed as $\mathcal{L} = \mathcal{L}_s + \mathcal{L}^{(a)}$ where

$$\mathcal{L}_s f = \frac{1}{2}(\mathcal{L} + \mathcal{L}^*)f = \sum_{b>0}\mathcal{L}_b^{(s)} f = \sum_{x,e>0}[f(\eta^{x,x+e}) - f(\eta)], \tag{4.7}$$

$$\mathcal{L}^{(a)} f = \frac{1}{2}(\mathcal{L} - \mathcal{L}^*)f = \sum_{b>0}\mathcal{L}_b^{(a)} f = \sum_{x,e>0}\delta_e(\eta_x - \eta_{x+e})[f(\eta^{x,x+e}) - f(\eta)]. \tag{4.8}$$

Here the sum over $b > 0$ means summing over <u>non oriented bonds</u> ($b = (x, x+e)$ with positive $e$). One can define the current and its antisymmetric part by

$$\mathcal{L}\eta_x = \sum_{e>0}\nabla_e^- w_{x,e}, \quad \mathcal{L}^{(a)}\eta_x = \sum_{e>0}\nabla_e^- w_{x,e}^{(a)} \tag{4.9}$$

with $\nabla_\epsilon^- g(x) \equiv g(x) - g(x-\epsilon)$. The currents $w_{x,\epsilon}$, $w^*_{x,\epsilon}$ and $w^{(a)}_{x,\epsilon}$ are given explicitly by

$$w_{x,\epsilon} = \nabla_\epsilon \eta_x + w^{(a)}_{x,\epsilon}, \quad w^*_{x,\epsilon} = \nabla_\epsilon \eta_x - w^{(a)}_{x,\epsilon}, \quad w^{(a)}_{x,\epsilon} = \delta_\epsilon u_{x,\epsilon} = \delta_\epsilon[\eta_x\eta_{x+\epsilon} - \frac{\eta_x + \eta_{x+\epsilon}}{2}]. \tag{4.10}$$

We shall now compute $\mathcal{L}^* \log \tilde\psi$. By definition of $\tilde\psi$

$$\begin{aligned} &\varepsilon^{d-4}\mathcal{L}^* \log \tilde\psi = \varepsilon^{d-4}\mathcal{L}_s \log \tilde\psi - \varepsilon^{d-4}\mathcal{L}^{(a)} \log \tilde\psi \\ = \ & \varepsilon^{d-1} \sum_{x,\epsilon>0} [\partial_\epsilon\partial_\epsilon \alpha(\varepsilon x, t)](\tilde\omega * \eta)_x - \varepsilon^{d-3} \sum_{x,\epsilon>0} \alpha(\varepsilon x, t)[\nabla_\epsilon^- w^{(a)}_{x,\epsilon} * \tilde\omega] \\ - \ & \varepsilon^{d-2} \sum_x \partial_\epsilon \alpha(\varepsilon x, t)\mathcal{L}^*(\tilde\omega * \tau_y F)(x) + o(1) = B_1 + B_2 + B_3 + o(1). \end{aligned} \tag{4.11}$$

Among the three terms in (4.11), $B_1$ is the simplest since it is already of order one (recall $\varepsilon^{-1}(\eta_x - \theta) \sim O(1)$). We now compute $B_2$. By Taylor expansion,

$$\begin{aligned} B_2 \ = \ & \varepsilon^{d-2} \sum_{x,\epsilon>0} [\varepsilon^{-1}\nabla_\epsilon \alpha(\varepsilon x, t)][\tilde\omega * w^{(a)}_{x,\epsilon}] \\ = \ & \varepsilon^{d-2} \sum_{x,\epsilon>0} (\partial_\epsilon \alpha)(\varepsilon x, t)[\tilde\omega * w^{(a)}_{x,\epsilon}] + \frac{1}{2}\varepsilon^{d-1} \sum_{x,\epsilon>0} (\partial_\epsilon\partial_\epsilon \alpha)(\varepsilon x, t)[\tilde\omega * w^{(a)}_{x,\epsilon}] \\ + \ & \varepsilon^d \sum_{x,\epsilon} a(\varepsilon x, t)(\tilde\omega * w^{(a)}_{x,\epsilon} - c_1) + \text{const}. \end{aligned} \tag{4.12}$$

Here $c_1$ is some constant and $a$ is proportional to the third derivatives of $\alpha(z,t)$. Here and below by *const.* we mean any quantity independent of the configuration and a cumulative constant will be determined at the end of this section using (iii) of Lemma 4.4. Note that from Lemma 4.3 the expectation of the third term goes to 0, by a suitable choice of $c_1$ so we only have to concentrate on the first two terms.

In preparation to the use of the "one block" estimate (Theorem 4.6), we introduce the quantities

$$\Omega^{(1)}_{x,\epsilon} = E^\mu[u_{x,\epsilon}|\bar\eta_{x,k}], \quad \Omega^{(2)}_{x,\epsilon} = (2\theta - 1)[(\tilde\omega * \frac{\eta_y + \eta_{y+\epsilon}}{2})_x - \bar\eta_{x,k}], \tag{4.13}$$

where $\mu = \mu_\beta$ is the equilibrium measure. We can decompose $\tilde\omega * u$ as

$$(\tilde\omega * u)_{x,\epsilon} = \Omega^{(1)}_{x,\epsilon} + \Omega^{(2)}_{x,\epsilon} + \sum_{\epsilon'>0} \tilde D_{\epsilon\epsilon'}(\tilde\omega * \nabla_{\epsilon'}\eta)(x)\delta_\epsilon^{-1} + \tilde\omega * [g^{(x)}_{y,\epsilon} - \delta_\epsilon^{-1} \sum_{\epsilon'>0} \tilde D_{\epsilon\epsilon'}(\nabla_{\epsilon'}\eta_y)]. \tag{4.14}$$

where $g^{(x)}_{y,\epsilon}$ is defined by

$$g^{(x)}_{y,\epsilon} = u_{y,\epsilon} + (1 - 2\theta)(\frac{\eta_y + \eta_{y+\epsilon}}{2} - \theta) - \Omega^{(1)}_{x,\epsilon} - (1 - 2\theta)(\bar\eta_{x,k} - \theta \tag{4.15}$$



and the convolution in (4.14) is on the variable $y$. If we let $(b = (y, y + \epsilon))$

$$q_b \ = \ u_{y,\epsilon} + (1 - 2\theta)(\frac{\eta_y + \eta_{y+\epsilon}}{2} - \theta) - \theta(\theta - 1) = (\eta_y - \theta)(\eta_{y+\epsilon} - \theta) \tag{4.16}$$

then $g^{(x)}_{y,\epsilon}$ is nothing but

$$g^{(x)}_{y,\epsilon} = q_b - E^\mu[q_b|\bar\eta_{x,k}] \tag{4.17}$$

One can compute $\Omega^{(1)}_{x,\epsilon}$ easily. It is simply

$$E^\mu[u_{x,\epsilon}|\bar\eta_{x,k}] = [1 + ((2k+1)^d - 1)^{-1}]\left\{(\bar\eta_{x,k})^2 - \bar\eta_{x,k}\right\} \tag{4.18}$$

**Lemma 4.5** *Suppose $J$ is a smooth function. Then there is constant $C_2$ (independent of $f_t$) such that*

$$\lim_{\epsilon \to 0} \varepsilon^{d-2} E^{f_t}[\sum_x J(\varepsilon x)\Omega^{(2)}_{x,\epsilon} - \frac{\varepsilon}{2}(1 - 2\theta)\sum_x \partial_\epsilon J(\varepsilon x)\eta_x] - C_2 = 0$$

*Proof.* By definition of $\tilde\omega$ and $\omega$ in (4.A), up to a constant, the expression inside the expectation can be written as

$$\begin{aligned} &\varepsilon^{d-2}(2\theta - 1)\Big\{\sum_x [(\tilde\omega * J)(\varepsilon x)(\eta_x - \theta) - (\omega * J)(\varepsilon x)(\eta_x - \theta)] \\ - \ & \frac{1}{2}\sum_x [\nabla_\epsilon(\tilde\omega * J)(\varepsilon x) - \varepsilon\partial_\epsilon J(\varepsilon x)](\eta_x - \theta)\Big\}, \end{aligned} \tag{4.19}$$

The replacement of $\omega * J$ by $\tilde\omega * J$ produces an error

$$\begin{aligned} &\varepsilon^{d-2} E^{f_t}\{\sum_x (\eta_x - \theta)[\omega * J(\varepsilon x) - \tilde\omega * J(\varepsilon x)]\} \\ = \ & \varepsilon^{d-2} E^{f_t}\{\sum_x (\eta_x - \theta)Av_\sigma[\omega_\sigma * J(\varepsilon x) - \tilde\omega_\sigma * J(\varepsilon x)]\}, \end{aligned}$$

where $\tilde\omega_\sigma(x) = |\Lambda_{l_1,\sigma}|^{-1}1(x \in \Lambda_{l_1,\sigma})$ and $\omega_\sigma(x) = |\Lambda_{l,\sigma}|^{-1}1(x \in \Lambda_{l,\sigma})$ (recall $\sigma$ defined in (4.A)). Hence

$$\begin{aligned} &|\omega * J(\varepsilon x) - \tilde\omega * J(\varepsilon x)| = |Av_{y\in\Lambda_{l,\sigma}}[J(\epsilon(x - y)) - J(\epsilon(x - \sigma))] \\ - \ & Av_{y\in\Lambda_{l_1,\sigma}}[J(\epsilon(x - y)) - J(\epsilon(x - \sigma))]| \le \text{const}.\varepsilon^2 l^2|\partial_\epsilon\partial_\epsilon J|. \end{aligned}$$

Hence $|\text{error}| \le \varepsilon^d l^2 \sum_x E^{f_t}[\eta_x - \theta] \to 0$ because $E^{f_t}[\eta_x - \theta] = O(\varepsilon)$. Finally one can check the second term in (4.19) vanishes as $\varepsilon \to 0$. This concludes Lemma 4.5. ∎

Let us summarize what we have proved so far

$$\varepsilon^{-4+d}\mathcal{L}^*\log\hat{\psi}_t = B_1 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_7 + \text{const.} + o(1)$$

where $B_1$ is defined in (4.11) and $\Omega_i, i = 4,5,6,7$ are defined by ($\delta_e = p_e - p_{-e}$)

$$\begin{aligned}
\Omega_4 &= \varepsilon^{d-2}\sum_{x,e>0}(\partial_e\alpha)(\varepsilon x,t)\delta_e\Omega_x^{(1)},\\
\Omega_5 &= \varepsilon^{d-2}\sum_{x,e,e'>0}(\partial_e\alpha)(\varepsilon x,t)\hat{D}_{ee'}(\hat{\omega}*\nabla_{e'}\eta)(x),\\
\Omega_6 &= \frac{1}{2}\varepsilon^{d-1}\sum_{x,e>0}(\partial_e\partial_e)\alpha(\varepsilon x,t)\left\{[(\hat{\omega}*w_e^{(a)})(x)+(1-2\theta)\delta_e\eta_x\right\},\\
\Omega_7 &= \varepsilon^{d-2}\sum_{x,e>0}(\partial_e\alpha)(\varepsilon x,t)\Omega_{7,x,e}\\
\Omega_{7,x,e} &= \hat{\omega}*\left[\delta_e g_{y,e}^{(\tau)} - \sum_{e'>0}\hat{D}_{ee'}(\nabla_{e'}\eta)_y - \mathcal{L}^*_{y,y+e}\sum_z\tau_z F\right](x) \qquad (4.20)
\end{aligned}$$

The following theorem provides a key estimate replacing the usual "one block-two blocks" estimate in hydrodynamical limit. The traditional approach [16] fails here because the estimates are not strong enough to take into account fluctuations. Our method is based on a multi-scale analysis and logarithmic Sobolev inequality and it will be presented in sect. 6.

**Theorem 4.6** *For any $J$ and any constant $\gamma > 0$,*

$$\lim_{l\to\infty}\lim_{\varepsilon\to 0}\left\{\varepsilon^{d-2}\int f\sum_x J(\varepsilon x)\{\hat{\omega}*[\tau_y G - \alpha_k(G)](x)\}d\mu_\beta\right.$$
$$\left. -\ \text{const.}\ \frac{\varepsilon^d}{\gamma}\int f[\sum_x(J(\varepsilon x))^2 V_{\tilde{l}}(G,\bar{\eta}_{x,\tilde{l}})]d\mu_\beta - \gamma\varepsilon^{d-4}\sum_b D_b(\sqrt{f})\right\} \le 0. \qquad (4.21)$$

*Here $\tilde{l} = l^{d+2}$ and $k = l\varepsilon^{-2/d}$.*

**Corollary 4.7** *For any $\gamma > 0$ and $f$ such that $s(f) \le$ const. $\varepsilon^2$,*

$$\begin{aligned}
\lim_{l\to\infty}\lim_{\varepsilon\to 0}\ \varepsilon^{d-2} &\int\sum_x J(\varepsilon x)\{\hat{\omega}*(G-\alpha_k(G)\}(x)fd\mu_\beta\\
&\le \frac{1}{2\gamma}\int J^2(z)dz\ V(G,\beta) + \gamma\lim_{\varepsilon\to 0}\varepsilon^{d-4}\sum_b D_b(\sqrt{f}).
\end{aligned}$$

*In particular, if $f_t$ is the density in Theorem 2.1 then for any $\gamma > 0$,*

$$\lim_{l\to\infty}\lim_{\varepsilon\to 0}\int_0^T dt\left\{\varepsilon^{d-2}\int\sum_x\{J(\varepsilon x)\hat{\omega}*(G-\alpha_k(G))\}(x)f_t d\mu_\beta\right\} \le CTV(G,\beta) + \text{const.}\ \gamma$$

*where $C$ is some constant depending on $J$ and $\gamma$.*



*Proof*: By lemma 2.2 we can replace the density $f$ in the middle term of (4.21) by 1. The second part of the corollary is a consequence of (i) and (ii) of Lemma 4.3 ∎

**Corollary 4.8** *There is a constant $C_3$ independent of $f_t$ such that*

$$\lim_{l\to\infty}\lim_{\varepsilon\to 0}\int_0^T E^{f_t}[\Omega_6]dt - C_3 = 0$$

*Proof.* By Corollary 4.7, one can replace by $(\hat{\omega}*w_e^{(a)})(x)$ by $\hat{\omega}*[\bar{\eta}^2_{x,k} - \bar{\eta}_{x,k}]$. Corollary 4.8 then follows from Lemma 3.1 (with $\psi = \mu_\beta$ ) and the entropy inequality (2.12). ∎

**Corollary 4.9**

$$\lim_{l\to\infty}\lim_{\varepsilon\to 0}\int_0^T E^{f_t}[\Omega_7]dt \le CT\sum_{e>0}V(\hat{q}_e,\beta)\gamma^{-1} + \text{const.}\gamma$$

*where $\hat{q}_e$ is defined by*

$$\hat{q}_e(\eta) = \delta_e q_e - \sum_{e'}\hat{D}_{ee'}\nabla_{e'}\eta_0 - \mathcal{L}^*_e\sum_z\tau_z F \qquad (4.22)$$

*Proof.* Corollary 4.9 is just a reformulation of Corollary 4.7 to our setting. Notice that the term $\Omega_x^{(1)} + (1-2\theta)(\bar{\eta}_{x,k} - \theta)$ has been introduced to write (4.17), where $E^\mu[q_e|\bar{\eta}_{x,k}]$ is just the $\alpha_k(q_e)$ term in the l.h.s. of (4.21) . ∎

To summarize, we have proved

$$\varepsilon^{-4+d}\int_0^T E^{f_t}[\mathcal{L}^*\log\hat{\psi}_t]dt \le \int_0^T dt\int f_t\{\Omega_4 + \Omega_8 + \text{const.}\}d\mu_\beta + \Omega_9 + \text{const.}\gamma + o(1) \quad (4.23)$$

where $\Omega_8$ and $\Omega_9$ are defined by

$$\Omega_8 = B_1 + \Omega_5 = \frac{\varepsilon^{d-1}}{2}\sum_{x,e,e'>0}D_{ee'}(\partial_e\partial_{e'}\alpha)(\varepsilon x,t)(\hat{\omega}*\eta)(\varepsilon x), \qquad (4.24)$$

$$D_{ee'} = \delta_{ee'} - \hat{D}_{ee'}; \qquad \Omega_9 = CT\sum_{e>0}V(\hat{q}_e,\beta). \qquad (4.25)$$

We now compute $\partial_t\log\hat{\psi}_t$. By definition,

$$\begin{aligned}
&\varepsilon^{d-2}\partial_t\log\hat{\psi}_t = -\varepsilon^{d-2}\sum_{x,e>0}\partial_e\lambda(\varepsilon x - \varepsilon^{-1}vt,t)v_e(\hat{\omega}*\eta)(x) +\\
&\varepsilon^{d-1}\sum_x\frac{\partial\lambda}{\partial t}(\varepsilon x - \varepsilon^{-1}vt,t)(\hat{\omega}*\eta)(x) + \varepsilon^d\sum_x\frac{d}{dt}[\partial_e\lambda(\varepsilon x - \varepsilon^{-1}vt,t)](\hat{\omega}*\tau_y F)(x) + \text{const.}\\
&\equiv T_1 + T_2 + T_3 + \text{const.} \qquad (4.26)
\end{aligned}$$

The last term $T_3$ can be estimated as follows. By definition, the $t$-derivative has two terms, symbolically,

$$\frac{d}{dt} = -\varepsilon^{-1}\sum_{e>0} v_e\partial_e + \partial_t.$$

The contributions from the second term is negligible by (iii) of Lemma 4.3. The contribution from the first term is of the form

$$\varepsilon^{d-1}\sum_{x,e>0} J_e(\varepsilon x,t)\tau_x F \equiv X$$

If one uses the same argument used for the second term, it follows that $[E^{f_t} - E^{\mu_\beta}][X]$ is bounded. In order to have $[E^{f_t} - E^{\mu_\beta}][X]$ negligible, one needs Corollary 4.7 and suitable subtraction. By Theorem 4.6, one can replace $X$ by $\varepsilon^{d-1}\sum_{x,e>0} J_e(\varepsilon x,t)\hat{F}(\bar{\eta}_{x,k})$. By entropy inequality (2.12) and Lemma 3.1 ($\psi = \mu_\beta$), the second term of the estimate vanishes in the limit $\varepsilon \to 0$ and $\ell \to \infty$. Finally, letting $\gamma \to \infty$ we conclude that $T_3$ is negligible, provided that $F$ satisfies assumption (C) in the definition of $\hat{\psi}_t$.

Combining with previous estimates, one has

$$\int_0^T dt E^{f_t}\left\{[\varepsilon^{d-4}\mathcal{L}^* - \varepsilon^{d-2}\partial_t]\log\hat{\psi}_t\right\} \le \int_0^T dt E^{f_t}[\Omega_4 - T_1 + \Omega_8 - T_2 + \text{const.}] + \Omega_9 + \text{const.}\gamma + o(1). \tag{4.27}$$

We now estimate $\Omega_4 - T_1$.

$$\Omega_4 - T_1 = \varepsilon^{d-2}\sum_{x,e>0}(\partial_e\alpha)(\varepsilon x,t)[\delta_e\Omega_x^{(1)} + v_e(\hat{\omega} * \eta)(x)]$$

By (4.18) we can write $\Omega_{x,e}^{(1)}$ as

$$\Omega_x^{(1)} = (\bar{\eta}_{x,k} - \theta)^2 + (2\theta - 1)(\bar{\eta}_{x,k} - \theta) + (\theta^2 - \theta) + O(k^{-d})$$

By definition, $v_e = (1-2\theta)\delta_e$. Hence $\Omega_4 - T_1 = \Omega_{10} + \Omega_{11} + \text{const.} + O(\ell^{-d})$ (recall $k = \ell\varepsilon^{-2/d}$), where

$$\begin{aligned}
\Omega_{10} &= \varepsilon^{d-2}\sum_{x,e>0}(\partial_e\alpha)(\varepsilon x,t)(2\theta - 1)\delta_e[\bar{\eta}_{x,k} - (\hat{\omega} * \eta)(x)], \\
\Omega_{11} &= \varepsilon^{d-2}\sum_{x,e>0}\delta_e(\partial_e\alpha)(\varepsilon x,t)(\bar{\eta}_{x,k} - \theta)^2
\end{aligned} \tag{4.28}$$

Note that $\Omega_{10}$ is of the same form as the first term of (4.19), hence $\lim_{\varepsilon\to 0}[E^{f_t} - E^{\mu_\beta}][\Omega_{10}] = 0$. Therefore

$$\int_0^T dt E^{f_t}\left\{[\varepsilon^{d-4}\mathcal{L}^* - \varepsilon^{d-2}\partial_t]\log\hat{\psi}_t\right\} \le \int_0^T dt E^{f_t}[\Omega_{12} + \Omega_9 + \text{const.}] + o(1)$$



where $\Omega_{12}$ is defined by

$$\Omega_{12} = \varepsilon^{d-1}\sum_{x,e,e'>0}\left\{[(D_{ee'}\partial_e\partial_{e'} - \frac{\partial}{\partial t})\lambda(\varepsilon x - \varepsilon^{-1}vt,t)](\hat{\omega} * \eta)(x)\right. \tag{4.29}$$

$$\left. + \ \varepsilon^{-1}\delta_e(\partial_e\lambda)(\varepsilon x - \varepsilon^{-1}vt,t)(\bar{\eta}_{x,k} - \theta)^2\right\} = \varepsilon^{d-1}\sum_{x,e>0}\Omega_{12,x,e}(\bar{\eta}_{x,k}) + \text{const.} + o(1)$$

with

$$\Omega_{12,x,e}(u) = [(D_{ee'}\partial_e\partial_{e'} - \frac{\partial}{\partial t})\lambda](u - \theta) + \varepsilon^{-1}\delta_e(\partial_e\lambda)(u - \theta)^2 \tag{4.30}$$

In the last step we have again replaced $\hat{\omega} * \eta$ by $\omega * \eta = \bar{\eta}_{x,k}$ and the error goes to 0 by the same argument used before. By assumption $\lambda$ satisfies equation (2.6) of Theorem 2.1. Let $m(\lambda) = -\theta(1-\theta)\lambda$. Then $\Omega_{12}$ can be simplified as

$$\begin{aligned}
\Omega_{12,x,e}(u) &= \delta_e(\partial_e\alpha)[2m(\alpha)(u - \theta) + \varepsilon^{-1}(u - \theta)^2] \\
&= \varepsilon^{-1}\delta_e(\partial_e\alpha)(u - \theta + \varepsilon m(\alpha))^2 + \text{const.}
\end{aligned} \tag{4.31}$$

Therefore we have the equation

$$\int_0^T dt E^{f_t}[(\varepsilon^{d-4}\mathcal{L}^* - \varepsilon^{d-2}\partial_t)\log\hat{\psi}] \le \varepsilon^{d-2}\int_0^T dt E^{f_t}[\sum_{x,e}\bar{\Gamma}_{x,e}(\bar{\eta}_{x,k}) + \text{const.}] + \Omega_9 + \text{const.}\gamma + o(1)$$

where

$$\bar{\Gamma}_{x,e}(u) = \delta_e(\partial_e\alpha)(\varepsilon x,t)(u - \theta + \varepsilon m(\alpha)(\varepsilon x,t))^2 \tag{4.32}$$

Note that $\theta - \varepsilon m(\alpha)$ is not exactly the mean $E^{\hat{\psi}_t}[\bar{\eta}_{x,k}] \equiv \theta - \varepsilon\bar{\rho}_k(\varepsilon x,t)$. One can prove easily that the replacement of $m$ by $\bar{\rho}_k$ produces only negligible errors.

To summarize, one has

$$\begin{aligned}
&\int_0^T dt E^{f_t}[\varepsilon^{d-4}\hat{\psi}^{-1}\mathcal{L}*\hat{\psi} - \varepsilon^{d-2}\partial_t\hat{\psi}]d\mu_\beta \\
\le &\int_0^T dt\int f_t[\varepsilon^d\sum_{x,e}\Gamma_{x,e}(\bar{\eta}_{x,k}) + \text{const.}]d\mu_\beta + \Omega_9 + \text{const.}\gamma + o(1)
\end{aligned} \tag{4.33}$$

Here $\Gamma$ is defined by (4.32) with $m$ replaced by $\bar{\rho}_k$. We now determine the constant by (iii) of Lemma 4.4. Note that (4.33) is an inequality but, if one traces the argument through, it is not hard to find that, in case $f_t = \hat{\psi}_t$, (4.33) is an identity with $\Omega_9$ and $\gamma$ set to zero. Hence the constant satisfies the equation

$$\int\hat{\psi}_t[\varepsilon^{d-2}\sum_{x,e}\Gamma_{x,e}(\bar{\eta}_{x,k}) + \text{const.}]d\mu_\beta = o(1)$$

By Lemma 3.2, one can replace $\hat{\psi}_t$ by $\psi_t$. Hence the constant is nothing but

$$\text{const.} = -\varepsilon^{-2}\delta_e(\partial_e\alpha)(\varepsilon x, t)\zeta_{x,k},$$

with $\zeta_{x,k}$ defined in Lemma 3.1. One can now apply the entropy inequality (2.12) and Lemma 3.1 and conclude that, with $k = \ell\varepsilon^{-2/d}$,

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\varepsilon^{-2}s(f_T|\hat{\psi}_T) \le C\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\int_0^T \varepsilon^{-2}s(f_t|\hat{\psi}_t)dt + \Omega_9 + T\gamma, \tag{4.34}$$

for any $\gamma > 0$. Therefore Theorem 4.2 follows since $\gamma$ is arbitrary. ∎

In next sections we prove that we can choose $D_{ee'}$ and $F$ so that $\Omega_9$=0. A replacement of $\hat{\psi}$ by $\psi$ produces negligible errors and Theorem 2.1 follows by Gronwall lemma.

## 5 The diffusion coefficient.

In this section we identify the diffusion coefficient and characterize the function $F$, introduced in 4.C. by a variational condition. This is done by a geometric argument which relies on the construction of the appropriate Hilbert space below.

Let $\mathcal{G}$ be the linear space $\mathcal{G} = \{g|g$ is a local function and $g$ satisfies (5.1)$\}$

$$\hat{g}(\theta) = 0 = \frac{\partial\hat{g}}{\partial y}|_{y=\theta}. \tag{5.1}$$

Here $\hat{g}$ is defined by $\hat{g}(y) = E^{\mu_\beta}[g]$, $y = P'(\beta)$. The condition (5.1) is equivalent to

$$E^{\mu_\theta}[g] = 0 = \sum_x E^{\mu_\theta}[g;\eta_x] \tag{5.2}$$

where $E^{\mu_\theta}[A;B] = E^{\mu_\theta}[AB] - E^{\mu_\theta}[A]E^{\mu_\theta}[B]$. *In this section* $\theta = P'(\beta)$, $< \cdot >= E^{\mu_\theta}[\cdot]$ *and we will drop the index* $\beta$ *in the measure* $\mu_\beta$. Recall the definition of $V(g,\beta)$ (Definition 4.1, eq. (4.4) ). We claim that the variance $V$ is a norm on $\mathcal{G}$ and it defines an inner product via polarization.

**Lemma 5.1** *For* $g$ *and* $h \in \mathcal{G}$ *define*

$$V(g,h) = \frac{1}{4}[V((g+h),\beta) - V((g-h),\beta)]. \tag{5.3}$$

*Then* $V(\cdot,\cdot)$ *is a inner product, that we denote by* $\ll \cdot,\cdot \gg$. *Furthermore, in the definition 4.2 of* $V(g,\beta)$ *the limit exists and hence* lim sup *can be replaced by* lim.



We shall prove this Lemma in the following Theorem 5.2 - Corollary 5.6. Theorem 5.2 is the key input and we shall prove it in sect. 6. Throughout this section $\ell$ will denote an integer (independent of $\varepsilon$).

**Theorem 5.2** *The variance* $V(g)$ *satisfies the variational principle*

$$\frac{1}{2}V(g) = \sup_{\alpha_e\in\mathcal{R},G\in\mathcal{G}}\left\{\sum_{e>0}\alpha_e t_e(g) + (g,G)_0 - \frac{1}{4}\sum_{e>0}E^\mu[\alpha_e\nabla_e\eta + \nabla_e\sum_x\tau_x G]^2\right\} \tag{5.4}$$

*Here*

$$(g,G)_0 = E^\mu[g\sum_x\tau_x G], \quad t_e(g) = \lim_{\ell\to\infty}E^\mu[g\sum_{x\in\Lambda_\ell}(e\cdot x)\eta_x]. \tag{5.5}$$

*Note the right side is independent of* $\ell$, *for* $\ell$ *large enough, since* $E^\mu[g\eta_x] = 0$ *if* $g$ *does not depend on* $\eta_x$.

**Lemma 5.3.** $\mathcal{L}\mathcal{G}\subset\mathcal{G}$, $\mathcal{L}^*\mathcal{G}\subset\mathcal{G}$ *and* $\mathcal{L}_s\mathcal{G}\subset\mathcal{G}$.

*Proof.* We shall only prove $\mathcal{L}\mathcal{G}\subset\mathcal{G}$. It suffices to check (5.2) for $\mathcal{L}g$. The first equality of (5.2) is trivial. The second one follows since our dynamics conserves the total particle number. ∎

We start with the definition of the inner product on a subspace of $\mathcal{G}$ and then extend it to $\mathcal{G}$. The following lemma is a simple consequence of Theorem 5.2

**Lemma 5.4.** *For* $g, G \subset \mathcal{G}$:

$$V(\mathcal{L}_s g) = \frac{1}{2}\sum_{e>0}E^\mu[(\nabla_e\sum_x\tau_x g)^2] \tag{5.6}$$

$$(\nabla_e\eta, G)_0 = 0 \tag{5.7}$$

$$t_{e'}(\nabla_e\eta) = \delta_{e,e'}V(\nabla_e\eta) = E^\mu[(\nabla_e\eta)\eta_e] = \frac{1}{2}E^\mu[(\nabla_e\eta)^2] \tag{5.8}$$

*where* $(\cdot,\cdot)_0$ *is defined in (5.5). Hence* $\ll \cdot,\cdot \gg = V(g,h)$ *defines an inner product on* $\mathcal{G}^{(1)} = \mathcal{G}^{(0)} + \mathcal{L}_s\mathcal{G}$ *where* $\mathcal{G}^{(0)} = \{\sum_{e>0}\alpha_e\nabla_e\eta\}$. *Furthermore* $\mathcal{G}^{(0)}\perp\mathcal{L}_s\mathcal{G}$ *and, for* $\xi\in\mathcal{G}^{(1)}$,

$$\ll \mathcal{L}_s g, \xi \gg = -(g,\xi)_0. \tag{5.9}$$

*Proof.* By (5.5), $t_e(\mathcal{L}_s g) = 0$. Hence the sup in (5.4) is attained for $\alpha_e = 0$ and this proves (5.6). (5.7) follows from the definition of $(\cdot,\cdot)_0$. By (5.7) the sup in (5.4) for $g = \nabla_e\eta$ is attained for $G = 0$ and this proves (5.8). From (5.6) and (5.3)

$$\ll \mathcal{L}_s G, \mathcal{L}_s g \gg = \frac{1}{2}\sum_{e>0}E^\mu[(\nabla_e\sum_x\tau_x g)(\nabla_e\sum_x\tau_x g)] = -(g,\mathcal{L}_s G)_0$$

Let $\xi = \sum_e a_e \nabla_e \eta + \mathcal{L}_s G$. Since $\ll \mathcal{L}_s g, \nabla_e \eta \gg = 0$, (5.9) is proven. ∎

**Lemma 5.5.** *Let $g, G \in \mathcal{G}$ and $\xi = \sum_{e>0} a_e \nabla_e \eta + \mathcal{L}_s G \in \mathcal{G}^{(1)}$. Define $S_g(\xi)$ by*

$$S_g(\xi) = \sum_{e>0} a_e t_e(g) + \langle g, G\rangle_0. \tag{5.10}$$

*Then $S_g(\xi)$ is a bounded linear functional on $\mathcal{G}^{(1)}$. Hence $S_g$ defines an element in $\overline{\mathcal{G}^{(1)}}$ via the Riesz representation theorem, which we still denote by $S_g$. The map $S : g \to S_g$ is a linear transformation from $\mathcal{G}$ to $\overline{\mathcal{G}^{(1)}}$. Furthermore, $S = I$ when restricted on $\mathcal{G}^{(1)}$, namely $S_g = g$ for $g \in \mathcal{G}^{(1)}$.*

*Proof.* The only thing we need to prove is the boundedness of $S_g(\xi)$. It suffices to prove that for any $g$ fixed,

$$|\langle g, G\rangle_0|^2 \le \text{const.(g)} \sum_{e>0} E^\mu[(\nabla_e \sum_x \tau_x G)^2].$$

By definition,

$$\langle g, G\rangle_0 = E^\mu[g \sum_x \tau_x G].$$

Since both $g$ and $G$ are local functions, we can truncate the summation of $x$ at, say, $|x| \le M$. By Lemma 6.1,

$$E^\mu[g \sum_{|x|\le M} \tau_x G]^2 \le \text{const.(g)} \sum_b E^\mu[(\nabla_b \sum_{|x|\le M} \tau_x G)^2 |b|^{-d-1/2}] \tag{5.11}$$

Now let $M \to \infty$ and use the fact that for all $b$ one has

$$E^\mu[(\nabla_b \sum_x \tau_x G)^2] = E^\mu[(\nabla_e \sum_x \tau_x G)^2]$$

for some $e > 0$. Hence the right side of (5.11) is bounded by const.(g)$E^\mu[\sum_{e>0}(\nabla_e \sum_x \tau_x G)^2]$. This proves the boundedness of $S_g$. ∎

Let $\mathcal{N}$ be the kernel of $S$ in $\mathcal{G}$. Then $\mathcal{G}/\mathcal{N}$ is isomorphic to $\overline{\mathcal{G}^{(1)}}$. Note that $g \in \mathcal{N}$ iff $V(g) = 0$. Hence one can identify $\overline{\mathcal{G}}$ with $\mathcal{G}/\mathcal{N}$.

**Corollary 5.6.** *$\overline{\mathcal{G}} = \overline{\mathcal{G}^{(1)}}$ and $\ll, \gg$ defines an inner product on $\overline{\mathcal{G}}$ so that $\ll g, h \gg = V(g, h)$.*

This proves Lemma 5.1 and also establishes the structure of $\mathcal{G}$. In consequence of above discussion we can think of the elements of $\mathcal{G}$, roughly speaking, as sums of a gradient term $\nabla_e \eta$ and a fluctuation term $\mathcal{L}_s G$.



We note that up to now we have only used the symmetric part of the generator. In our model the generator is not symmetric. To determine the diffusion coefficient, we need to decompose the elements of $\mathcal{G}$ in terms of a gradient and $\mathcal{L}G$. The way to do that is not straightforward, and the first step is the lemma 5.7 below.

Recall the definition (4.10) of the current $w_e = w_{0,e}$. Note that $w_e \notin \mathcal{G}$. We introduce the function

$$\sigma_e = (\nabla_e \eta) + \delta_e(\eta_0 - \theta)(\eta_e - \theta) \tag{5.12}$$

so that $\sigma_e \in \mathcal{G}$ and $\langle g, w_e\rangle_0 = \langle g, \sigma_e\rangle_0$ for $g \in \mathcal{G}$. Note that (5.2) is crucial for the last identity to hold. Similarly, one can define $\sigma_e^* = (\nabla_e \eta) - \delta_e(\eta_0 - \theta)(\eta_e - \theta)$ so that $\langle g, w_e^*\rangle_0 = \langle g, \sigma_e^*\rangle_0$.

**Lemma 5.7.** *Suppose $g \in \mathcal{G}$. Then*

$$t_e(\mathcal{L}g) = \lim_{\ell\to\infty} E^\mu[\mathcal{L}g \sum_{|x|\le\ell} (x\cdot e)\eta_x] = -\langle g, w_e^*\rangle_0 = -\langle g, \sigma_e^*\rangle_0, \tag{5.13}$$

$$t_e(\mathcal{L}^* g) = -\langle g, w_e\rangle_0 = -\langle g, \sigma_e\rangle_0 \tag{5.14}$$

*Proof.* Since $\mathcal{L}g$ is a local function, one can fix $M$ much larger than the support of $g$ and replace $\mathcal{L}g$ by $\mathcal{L}_M g$ with $M \ll \ell$. One can now perform integration by parts and

$$E^\mu[\mathcal{L}_M g \sum_{|x|\le\ell} (x\cdot e)\eta_x] = \langle g \mathcal{L}_M^* \sum_{|x|\le\ell} (x\cdot e)\eta_x\rangle + E^\mu[g \left\{ \sum_{y\in\partial\Lambda_\ell} \eta_y \text{sign}(y) \right\} \sum_{|x|\le\ell} (x\cdot e)\eta_x],$$

where $\text{sign}(y) = \pm 1$ depending on which boundary $y$ lies. The choice of sign is not important; the crucial point is that opposite boundaries have opposite signs. The first term on the right is just $-\langle g, w_e^*\rangle_0$. Hence it remains to prove the last term is zero. We claim that the contribution from opposite boundaries cancel exactly and sum up to zero. To see this let $g$ depend only on the configuration in $\Lambda_s$ for some integer $s$ and $z \notin \Lambda_s$. Let $z'$ be the reflection of $z$ along the axis $e$. Note that for any $x$

$$E^\mu[g\eta_z\eta_x] = E^\mu[g\eta_{z'}\eta_x] \tag{5.15}$$

unless $x = z$ or $x = z'$. Hence

$$E^\mu[g(\eta_z - \eta_{z'}) \sum_x (x\cdot e)\eta_x] = E^\mu[g(\eta_z - \eta_{z'})\{(z\cdot e)\eta_z + (z'\cdot e)\eta_{z'}\}] = E^\mu[g; F(\eta_z, \eta_{z'})],$$

where $F(\eta_z,\eta_{z'}) = (\eta_z - \eta_{z'})\{(z\cdot\epsilon)\eta_z + (z'\cdot\epsilon)\eta_{z'}\}$. Since $\mu$ is a product measure and $g$ is independent of $\eta_z$ and $\eta_{z'}$, the correlation of $g$ and $F$ is zero. We have thus proved (5.15) and the Lemma. ∎

**Lemma 5.8**

$$\ll \sigma_\epsilon, \nabla_{e'}\eta \gg = t_{e'}(\sigma_e) = V(\nabla_e\eta)\delta_{e,e'} = t_e(\sigma^*_{e'})$$

*Proof.* By definition

$$t_{e'}(\sigma_e) = \lim_{\ell\to\infty} E^\mu[\sigma_e \sum_{|x|\le\ell}(x\cdot e')\eta_x]$$

Clearly, if $e' \neq e$ then $t_{e'}(\sigma_e) = 0$. Hence we assume $e = e'$. Since $\mu$ is a product measure, $E^\mu[\sigma_e\eta_x] = 0$ unless $x = 0$, or $e$. Hence

$$t_e(\sigma_e) = E^\mu[\sigma_e\eta_e] = E^\mu[(\nabla_e\eta)\eta_e] + \delta_e E^\mu[(\eta_0 - \theta)(\eta_e - \theta)\eta_e].$$

Clearly, the last term is zero. Thus we prove Lemma 5.8 using Lemma 5.4 ∎

**Theorem 5.9.**

*(i)* $\overline{\mathcal{L}\mathcal{G} + \mathcal{G}^{(0)}} = \overline{\mathcal{G}} = \overline{\mathcal{L}^*\mathcal{G} + \mathcal{G}^{(0)}}$

*(ii) Let* $\mathcal{G}_w = \{\sum_{e>0}\alpha_e\sigma_e\}$. *Then* $\overline{\mathcal{G}_w + \mathcal{L}\mathcal{G}} = \overline{\mathcal{G}} = \overline{\mathcal{G}_w + \mathcal{L}_s\mathcal{G}}$. *Furthermore,*

$$V(\sum_{e>0}\alpha_e\sigma_e + \mathcal{L}g) \ge V(\sum_{e>0}\alpha_e\nabla_e\eta + \mathcal{L}_s g) \tag{5.16}$$

*Similarly,* $\overline{\mathcal{G}_{w^*} + \mathcal{L}^*\mathcal{G}} = \overline{\mathcal{G}}$ *and*

$$V(\sum_{e>0}\alpha_e\sigma^*_e + \mathcal{L}^* g) \ge V(\sum_{e>0}\alpha_e\nabla_e\eta + \mathcal{L}_s g) \tag{5.17}$$

*(iii) Let $T$ be the linear transformation from $\mathcal{G}$ to $\mathcal{G}$ s.t.*

$$T(\sum_{e>0}\alpha_e\sigma_e + \mathcal{L}g) = \sum_{e>0}\alpha_e\nabla_e\eta + \mathcal{L}_s g. \tag{5.18}$$

*Then $T$ is bounded above by 1 and*

(A) $\quad T\nabla_e\eta \perp \mathcal{L}^*\mathcal{G} \quad \forall e$

(B) $\quad \ll T\nabla_e\eta, \sigma^*_{e'} \gg = \delta_{ee'}V(\nabla_e\eta)$



*(iv) Let $Q$ be the matrix* $Q_{ee'} = \ll \nabla_e\eta, T\nabla_{e'}\eta \gg$. *The diffusion matrix $D$ is given by*

$$D = V(\nabla_e\eta)Q^{-1} > 1. \tag{5.19}$$

*(v) Let $R$ be the linear transformation from $\mathcal{G}$ to $\mathcal{G}$ s.t.*

$$R(\sum_{e>0}\alpha_e\nabla_e\eta + \mathcal{L}_s g) = \sum_{e>0}\alpha_e\sigma_e + \mathcal{L}g \tag{5.20}$$

*Then $R$ is a bounded linear transformation, $R = T^{-1}$ and in consequence $D$ is bounded above.*

*Remark:* At formal level one can think to the maps $T$ and $R$ as $T = \mathcal{L}_s\mathcal{L}^{-1}$ and $R = \mathcal{L}\mathcal{L}_s^{-1}$. With this in mind, it is easy to understand the statements of Theorems 5.9 and 5.10 below and the calculus of the diffusion matrix. Unfortunately, we do not know how to give sense to $\mathcal{L}^{-1}$, so the arguments based on it are only formal and we had to use the above maps to make the computation rigorous.

*Proof of (i).* Clearly, one only has to check that the orthogonal projection from $\overline{\mathcal{L}\mathcal{G}}$ to $\overline{\mathcal{L}_s\mathcal{G}}$ is onto. Suppose that, for some $h \in \mathcal{G}$, $\ll \mathcal{L}g, \mathcal{L}_s h \gg = 0$ for all $g \in \mathcal{G}$. In particular, $\ll \mathcal{L}h, \mathcal{L}_s h \gg = 0$. By Lemma 5.4,

$$\ll \mathcal{L}h, \mathcal{L}_s h \gg = -(\mathcal{L}h, h)_0 = -(\mathcal{L}_s h, h)_0 = \ll \mathcal{L}_s h, \mathcal{L}_s h \gg$$

This proves $\mathcal{L}_s h = 0$ in $\mathcal{G}$ and hence (i).

*Proof of (ii).* We shall only prove the bound of $V$, namely (5.16). By Theorem 5.2 the left side of (5.16) is equal to

$$\sup_{\gamma,G}\Big\{\sum_{e'>0}\gamma_{e'}t_{e'}(\sum_{e>0}\alpha_e\sigma_e + \mathcal{L}g) + (\sum_{e>0}\alpha_e\sigma_e + \mathcal{L}g, G)_0 - \frac{1}{4}E^\mu[\sum_{e'>0}[\gamma_{e'}\nabla_{e'}\eta + \nabla_{e'}\sum_x \tau_x G]^2]\Big\}$$

In particular, let $G = -g$ and $\gamma = \alpha$. Hence from Lemma 5.4 we can bound the last expression from below by

$$\sum_{e'}\alpha_{e'}t_{e'}(\sum_{e>0}\alpha_e\sigma_e + \mathcal{L}g) - (\sum_{e>0}\alpha_e\sigma_e + \mathcal{L}g, g)_0 - \frac{1}{2}V(\sum_{e>0}\alpha_e\nabla_e\eta + \mathcal{L}_s g) \tag{5.21}$$

By definition of $\mathcal{L}$, $(\mathcal{L}g, g)_0 = (\mathcal{L}_s g, g)_0$. Also, from (5.13) $t_{e'}(\mathcal{L}g) = -(g, w^*_{e'})_0$. To summarize, we have

$$\begin{aligned}\frac{1}{2}V(\sum_{e>0}\alpha_e\sigma_e + \mathcal{L}g) \;\ge\; & \sum_{e'>0}[\alpha_{e'}t_{e'}(\sum_{e>0}\alpha_e\sigma_e) - \alpha_{e'}(g, w^*_{e'})_0] - \sum_{e>0}\alpha_e(\sigma_e, g)_0 \\ & - (\mathcal{L}_s g, g)_0 - \frac{1}{2}V(\sum_{e>0}\alpha_e\nabla_e\eta + \mathcal{L}_s g).\end{aligned} \tag{5.22}$$

Since $g \in \mathcal{G}$, $\langle \sigma_\epsilon, g\rangle_0 = \langle w_\epsilon, g\rangle_0$ by (5.14). By definition $w_\epsilon + w_\epsilon^* = 2\nabla_\epsilon\eta$ and hence $\langle g, w_\epsilon^* + w_\epsilon\rangle_0 = 0$. By Lemma 5.8, $t_{\epsilon'}(\sigma_\epsilon) = \delta_{\epsilon,\epsilon'}V(\nabla_\epsilon\eta)$. Therefore

$$\begin{aligned}\frac{1}{2}V(\sum_{\epsilon>0}\alpha_\epsilon\sigma_\epsilon + \mathcal{L}g) &\geq V(\sum_{\epsilon>0}\alpha_\epsilon\nabla_\epsilon\eta) - \langle\mathcal{L}_s g, g\rangle_0 \\ &- \frac{1}{2}V(\sum_{\epsilon>0}\alpha_\epsilon\nabla_\epsilon\eta + \mathcal{L}_s g) = \frac{1}{2}V(\sum_{\epsilon>0}\alpha_\epsilon\nabla_\epsilon\eta + \mathcal{L}_s g).\end{aligned}$$

Note that this inequality is precisely the statement that $T$ in (iii) is bounded by 1.

*Proof of (A) of (iii).* Suppose that $\nabla_\epsilon\eta = \sum_{\epsilon'} a_{\epsilon\epsilon'}\sigma_{\epsilon'} + \mathcal{L}F$ with $F \in \mathcal{G}$. In general, we may not have such a decomposition and some approximation is needed. Then

$$\ll T\nabla_\epsilon\eta, \mathcal{L}^*g \gg = \ll \sum_{\epsilon'>0} a_{\epsilon\epsilon'}\nabla_{\epsilon'}\eta, \mathcal{L}^*g \gg + \ll \mathcal{L}_s F, \mathcal{L}^*g \gg$$

By (5.9)

$$\ll \mathcal{L}_s F, \mathcal{L}^*g \gg = -\langle F, \mathcal{L}^*g\rangle_0 = -\langle \mathcal{L}F, g\rangle_0$$

Note that no boundary term arise when performing integrations by parts because $g$ and $F$ are local functions. Also, by Lemma 5.7, $t_\epsilon(\mathcal{L}^*g) = \ll \nabla_\epsilon\eta, \mathcal{L}^*g \gg = -\langle\sigma_\epsilon, g\rangle_0$. Hence

$$\ll T\nabla_\epsilon\eta, \mathcal{L}^*g \gg = -\langle \sum_{\epsilon'>0} a_{\epsilon\epsilon'}\sigma_{\epsilon'} + \mathcal{L}F, g\rangle_0 = \langle\nabla_\epsilon\eta, g\rangle_0 = 0$$

*Proof of (B) of (iii).* By definition of $T$

$$\ll T\nabla_\epsilon\eta, \sigma_{\bar\epsilon}^* \gg = \ll \sum_{\epsilon'>0} a_{\epsilon\epsilon'}\nabla_{\epsilon'}\eta, \sigma_{\bar\epsilon}^* \gg + \ll \mathcal{L}_s F, \sigma_{\bar\epsilon}^* \gg$$

For Lemma 5.4 and (5.13), for all $e$, $\ll \mathcal{L}_s F, \sigma_\epsilon^* \gg = -\langle F, \sigma_\epsilon^*\rangle_0 = t_\epsilon(\mathcal{L}F) = \ll \mathcal{L}F, \nabla_\epsilon\eta \gg$. Here the last identity follows from the isomorphism of $\overline{\mathcal{G}}$ and $\overline{\mathcal{G}^{(1)}}$ (Cor. 5.6) and the structure of $\mathcal{G}^{(1)}$ (Lemma 5.4). By Lemma 5.8, $\ll \nabla_{\epsilon'}\eta, \sigma_\epsilon^* \gg = \delta_{\epsilon,\epsilon'}V(\nabla_\epsilon\eta) = \ll \sigma_{\epsilon'}, \nabla_\epsilon\eta \gg$. Hence

$$\ll T\nabla_\epsilon\eta, \sigma_{\bar\epsilon}^* \gg = \ll \sum_{\epsilon'>0} a_{\epsilon\epsilon'}\sigma_{\epsilon'}^* + \mathcal{L}F, \nabla_{\bar\epsilon}\eta \gg = \ll \nabla_\epsilon\eta, \nabla_{\bar\epsilon}\eta \gg = \delta_{\epsilon\bar\epsilon}V(\nabla_\epsilon\eta).$$

*Proof of (iv).* The diffusion coefficient is such that $\sigma_\epsilon^* - \sum_{\epsilon'>0} D_{\epsilon\epsilon'}\nabla_{\epsilon'}\eta \in \mathcal{L}^*\mathcal{G}$; hence by (A) of (iii), it is characterized by

$$\ll \sigma_{\bar\epsilon}^* - \sum_{\epsilon'>0} D_{\epsilon\epsilon'}\nabla_{\epsilon'}\eta, T\nabla_{\bar\epsilon}\eta \gg = 0 \tag{5.23}$$



for all $\epsilon$ and $\bar\epsilon$. Solving (5.23), one has

$$D_{\epsilon\epsilon'} = V(\nabla_\epsilon\eta)(Q^{-1})_{\epsilon\epsilon'}. \tag{5.24}$$

Since $T$ is bounded above by 1, so is $Q$ (as a quadratic form). Hence $D \geq 1$. Finally, one has to prove the strict inequality holds. Note that $D = 1$ iff $\ll T\nabla_\epsilon\eta, \nabla_{\epsilon'}\eta \gg = \delta_{\epsilon\epsilon'}V(\nabla_\epsilon\eta)$. Since $T$ is bounded by 1, this implies $T\nabla_\epsilon\eta = \nabla_\epsilon\eta$. Hence by (A) of (iii), $\nabla_\epsilon\eta \perp \mathcal{L}^*\mathcal{G}$ and this leads to contradiction.

*Proof of (v).* First of all, we claim that $T$ has a bounded inverse. This follows from general abstract arguments as follows. Since $T^*$ ( the adjoint of $T$ w.r.t. $\ll \cdot, \cdot \gg$) is surjective from (ii), $T$ is a bijection from $\mathcal{G}$ to $\mathcal{G}$. The inverse mapping theorem [20] then assures that $T^{-1}$ is bounded. It is now trivial to check that $R = T^{-1}$. ∎

**Theorem 5.10**: *For the map $T$ defined in (iii) of Theorem 5.9 the following variational formula holds ($\xi = \sum_{\epsilon>0}\alpha_\epsilon\nabla_\epsilon\eta + \mathcal{L}_s g$):*

$$\begin{aligned}\ll \xi, T\xi \gg = \sup_{\beta_\epsilon, h}\inf_{\gamma_\epsilon, k}\Big\{&\sum_{\epsilon>0}\beta_\epsilon(\alpha_\epsilon - \gamma_\epsilon)E^{\mu_\rho}[(\nabla_\epsilon\eta)^2] - 2\langle g, \mathcal{L}_s h\rangle_0 \\ - 2&\sum_{\epsilon>0}[\beta_\epsilon\langle\sigma_\epsilon, k\rangle_0 + \gamma_\epsilon\langle\sigma_\epsilon^*, h\rangle_0] + 2\langle\mathcal{L}h, k\rangle_0 + \sum_{\epsilon>0}E^{\mu_\rho}[(\gamma_\epsilon\nabla_\epsilon\eta + \nabla_\epsilon\sum_x \tau_x k)^2]\Big\}\end{aligned} \tag{5.25}$$

*where $h$ and $k$ are local functions. In particular, if $\xi = \sum_{\epsilon>0}\alpha_\epsilon\nabla_\epsilon\eta$, the second term on the right side of (5.25) is zero. Hence we have the following representation of the diffusion matrix:*

$$\begin{aligned}\sum_{\epsilon,\epsilon'>0}(D^{-1})_{\epsilon\epsilon'}\alpha_\epsilon\alpha_{\epsilon'} = V(\nabla_\epsilon\eta)^{-1}\sup_{\beta_\epsilon, h}\inf_{\gamma_\epsilon, k}\Big\{&\sum_{\epsilon>0}\beta_\epsilon(\alpha_\epsilon - \gamma_\epsilon)E^{\mu_\rho}[(\nabla_\epsilon\eta)^2] \\ - 2\sum_{\epsilon>0}[\beta_\epsilon\langle\sigma_\epsilon, k\rangle_0 + \gamma_\epsilon\langle\sigma_\epsilon^*, h\rangle_0] + 2\langle\mathcal{L}h, k\rangle_0 + &\sum_{\epsilon>0}E^{\mu_\rho}[(\gamma_\epsilon\nabla_\epsilon\eta + \nabla_\epsilon\sum_x \tau_x k)^2]\Big\}\end{aligned} \tag{5.26}$$

*Remark*: By the formal relation $T = \mathcal{L}_s\mathcal{L}^{-1}$, we have $(D^{-1})_{\epsilon\epsilon'} = V(\nabla_\epsilon\eta)^{-1}\langle\nabla_\epsilon\eta, \mathcal{L}^{-1}\nabla_{\epsilon'}\eta\rangle_0$. Hence (5.26) is a way to give a rigorous sense to this expression.

*Proof.* By definition of $R$, we have the identity $\ll \xi, T\xi \gg = \ll \xi, R^{-1}\xi \gg = \ll \xi, (R^{-1})_s\xi \gg$, with $(R^{-1})_s = [R^{-1} + (R^{-1})^*]/2$ and $(R^{-1})^*$ is the adjoint of $R^{-1}$ w.r.t. $\ll, \gg$. Hence we have the variational formula

$$\ll \xi, (R^{-1})_s\xi \gg = \sup_{u\in\mathcal{G}}\{2 \ll \xi, u \gg - \ll u, [(R^{-1})_s]^{-1}u \gg\}. \tag{5.27}$$

It is a general fact that

$$[(R^{-1})_s]^{-1} = R^* R_s^{-1} R. \tag{5.28}$$

One can check (5.28) straightforwardly by simply computing

$$[R^{-1} + (R^{-1})^*] R^* R_s^{-1} R = [R^{-1} R^* + 1] R_s^{-1} R = R^{-1}[R^* + R] R_s^{-1} R = 2R^{-1} R = 2.$$

We can now rewrite $\ll u, [(R^{-1})_s]^{-1} u \gg$ by

$$\ll u, [(R^{-1})_s]^{-1} u \gg = \sup_{\zeta \in \mathcal{G}} \{2 \ll Ru, \zeta \gg - \ll \zeta, R_s \zeta \gg \}. \tag{5.29}$$

By the structure theorem on $\mathcal{G}$, we can assume $u = \sum_{e>0} \beta_e \nabla_e \eta + \mathcal{L}_s h$ and $\zeta = \sum_{e>0} \gamma_e \nabla_e \eta + \mathcal{L}_s k$. Hence, by definition of $R$

$$\ll \zeta, R_s \zeta \gg = \ll \zeta, R\zeta \gg = \sum_{e,e'>0} \gamma_e \gamma_{e'} \ll \nabla_e \eta, \sigma_{e'} \gg +$$

$$\sum_{e>0} \gamma_e [\ll \nabla_e \eta, \mathcal{L} k \gg + \ll \mathcal{L}_s k, \sigma_e \gg] + \ll \mathcal{L}_s k, \mathcal{L} k \gg .$$

From Lemma 5.7, $\ll \nabla_e \eta, \mathcal{L} k \gg = -\langle k, \sigma_e^* \rangle_0$. Together with (5.9)

$$\ll \nabla_e \eta, \mathcal{L} k \gg + \ll \mathcal{L}_s k, \sigma_e \gg = -[\langle k, \sigma_e^* \rangle_0 + \langle k, \sigma_e \rangle_0] = -\langle k, \nabla_e \eta \rangle_0 = 0.$$

Combining that with Lemma 5.8 and the fact that $\ll \mathcal{L}_s k, \mathcal{L} k \gg = \ll \mathcal{L}_s k, \mathcal{L}_s k \gg = \sum_{e>0} E^{\mu_\beta} [\nabla_e \sum_x \tau_x k]^2$ we conclude that

$$\ll \zeta, R_s \zeta \gg = \sum_{e>0} E[\gamma_e \nabla_e \eta + \nabla_e \sum_x \tau_x k]^2. \tag{5.30}$$

To get the variational formula for $\ll \xi, T\xi \gg$, it remains to compute $\ll Ru, \zeta \gg$. From the definition of $R$, (5.20), Lemma 5.7 and Lemma 5.8,

$$\begin{aligned} \ll Ru, \zeta \gg &= \ll \sum_{e>0} \beta_e \sigma_e + \mathcal{L} h, \sum_{e'>0} \gamma_{e'} \nabla_{e'} \eta + \mathcal{L}_s k \gg \\ &= \sum_{e>0} \beta_e \gamma_e V(\nabla_e \eta) - \sum_{e>0} \beta_e \langle \sigma_e, k \rangle_0 - \sum_{e'>0} \gamma_{e'} \langle h, \sigma_{e'}^* \rangle_0 - \langle \mathcal{L} h, k \rangle_0. \end{aligned} \tag{5.31}$$

Combining (5.27)-(5.31) and recalling that $\ll \xi, u \gg = \sum_{e>0} \beta_e \alpha_e V(\nabla_e \eta) - \langle g, \mathcal{L}_s h \rangle_0$ (Lemma 5.4), we conclude the proof. ∎



# 6 Multiscale analysis.

We are interested in estimating $E^{\mu_\beta}[V_{\bar\ell}(g,y)]$ for $g \in \mathcal{G}$ with $\bar\ell = \ell^{d+2}$ as in Theorem 4.6. We will use $\bar\ell_1 = \bar\ell - \ell^{1/d^2}$. First of all, one can rewrite $E^{\mu_\beta}[V_{\bar\ell}(g,y)]$ by a variational principle as

$$\frac{1}{2} E^{\mu_\beta}[V_{\bar\ell}(g,y)] = (2\bar\ell_1 + 1)^{-d} \left\{ \sup_u \sum_{|x| \le \bar\ell_1} E^{\mu_\beta}[(\tau_x g - \alpha_{\bar\ell}(g)) u] - \frac{1}{4} E^{\mu_\beta}[\sum_{|b| \le \bar\ell}' (\nabla_b u)^2] \right\} \tag{6.1}$$

Here $u$ is any function on $\Lambda_{\bar\ell}$. Note that the expectation is w.r.t. $\mu = \mu_\beta$ which is the infinite volume Gibbs state. One can drop $\alpha_{\bar\ell}(g)$ if one assumes instead $E^{\mu_\beta}[g|\bar\eta_{\bar\ell}] = 0$ for any choice of $\bar\eta_{\bar\ell}$.

**Lemma 6.1** (Integration by parts). *Suppose $g$ is a local function satisfying*

*(i)* $E^{\mu_\beta}[g] = 0$

*(ii)* $\frac{\partial \hat g}{\partial m}|_{m=\theta} = 0$, *where* $\hat g(m) = E^{\mu_\beta}[g]$ *with* $m = P'(\beta)$.

*Then for $d \ge 3$ there exist $\hat\Phi_b(g)$ s.t. for any $u$ in $L^2(\mu_\beta)$*

$$E^{\mu_\beta}[gu] = \sum_b E^{\mu_\beta}[\hat\Phi_b(g) \cdot \nabla_b u]. \tag{6.2}$$

*Furthermore, $\hat\Phi_b$ satisfies*

$$\sum_b E^{\mu_\beta}[\hat\Phi_b^2 |b|^{d+1/2}] < \infty, \tag{6.3}$$

*where* $|b| \equiv \mathrm{dist}(b, 0)$

*Proof.*

*Step 1.* Fix an integer $q$. Define $g_n$ by

$$g_n = E^{\mu_\beta}[g|\mathcal{F}_n], \tag{6.4}$$

where $\mathcal{F}_n = \mathcal{F}_{\Lambda^{(n)}}$ is the $\sigma$-field generated by $\eta_x$ for $x \in \Lambda - \Lambda^{(n)}$ and $\bar\eta_{q^n} = \mathrm{Av}_{|y| \le q^n} \eta_y$. Then

$$E^{\mu_\beta}[gu] = \sum_n E^{\mu_\beta}[(g_n - g_{n+1}) u]$$

Define $\mathcal{L}_{s,n+1} = \sum_{b \in \Lambda^{(n+1)}} \mathcal{L}_{s,b}$ with $\Lambda^{(n)} = \Lambda_{q^n}$. In this section, all bonds are unoriented. Note that $(\mathcal{L}_{s,n+1})^{-1}(g_n - g_{n+1})$ makes sense since $E^{\mu_\beta}[g_n - g_{n+1}|\bar\eta_{q^{n+1}}] = 0$. Therefore one can write

$$E^{\mu_\beta}[u(g_n - g_{n+1})] = E^{\mu_\beta}[u(-\mathcal{L}_{s,n+1})(-\mathcal{L}_{s,n+1})^{-1}(g_n - g_{n+1})]$$

Hence

$$E^{\mu_\beta}[u(g_n - g_{n+1})] = \sum_{b\in\Lambda^{(n+1)}} E^{\mu_\beta}[T_{n,b}\cdot(\nabla_b u)]$$

Here

$$T_{n,b} = \nabla_b(-\mathcal{L}_{s,n+1})^{-1}(g_n - g_{n+1}). \tag{6.5}$$

By definition, $T_{n,b}$ is bounded by

$$\sum_{b\in\Lambda^{(n+1)}} E^{\mu_\beta}[(T_{n,b})^2] = E^{\mu_\beta}[(g_n - g_{n+1})(-\mathcal{L}_{s,n+1})^{-1}(g_n - g_{n+1})] \le \text{const}.q^{2(n+1)}E^{\mu_\beta}[(g_n - g_{n+1})^2]$$

We have used the spectral gap of the symmetric simple exclusion in the last inequality.

*Step 2.* By definition of $g_n$ and $g_{n+1}$, $E^{\mu_\beta}[(g_n - g_{n+1})^2] = E^{\mu_\beta}[E^{\mu_\beta}[g_n; g_n|\mathcal{F}_{n+1}]]$. Here $E^{\mu_\beta}[A; B|\mathcal{F}_{n+1}] = E^{\mu_\beta}[AB|\mathcal{F}_{n+1}] - E^{\mu_\beta}[A|\mathcal{F}_{n+1}]E^{\mu_\beta}[B|\mathcal{F}_{n+1}]$. Clearly,

$$E^{\mu_\beta}[E^{\mu_\beta}[g_n; g_n|\mathcal{F}_{n+1}]] \le E^{\mu_\beta}[g_n; g_n].$$

From the local limit theorem and its expansion ((3.11) of [17]) one has $g_n = \hat{g} + O(q^{-nd})$. Hence

$$E^{\mu_\beta}[g_n; g_n] \le O(q^{-2nd}) + E^{\mu_\beta}[\hat{g}(\bar{\eta}_{q^n}); \hat{g}(\bar{\eta}_{q^n})]$$

Note that our assumptions (i) and (ii) are the same as (3.4) with $\bar{m}_{x,k}$ replaced by $m$. From (3.5) one has $\log E^{\mu_\beta}[\exp\{q^n\hat{g}\}] \le$ const. With the same argument, $E^{\mu_\beta}[\hat{g}(\bar{\eta}_{q^n}); \hat{g}(\bar{\eta}_{q^n})] \le \text{const}.q^{-2nd}$. Hence

$$\sum_{b\in\Lambda^{(n+1)}} E^{\mu_\beta}[(T_{n,b})^2] \le \text{const}.q^{2n-2nd}. \tag{6.6}$$

*Step 3.* Finally, let $\dot{\Phi}_b = \sum_n T_{n,b}$. One can check from Step 2 that the bound (6.3) holds for $d \ge 3$. This concludes Lemma 6.1. ∎

Once Lemma 6.1 is established, Theorem 5.2 can be proved as in [12, 13]. The only difference is that the function $\dot{\Phi}_b$ in [13] vanishes for all but finitely many $b$. We have, however, the decay bound (6.3) at our disposal. Hence we can follow exactly the same argument given in [12, 13]. In the following, we sketch a proof which is essentially the same as in [12, 13].

*Proof of Theorem 5.2.* From the variational principle, there exists a function $u$ such that

$$\frac{1}{2}E^{\mu_\beta}[V_l(g,y)] \le \text{Av}_{|x|\le\bar{l}_1}E^{\mu_\beta}[(\tau_x g)u] - \frac{1}{4}\text{Av}_{|b|\le l}E^{\mu_\beta}(\nabla_b u)^2] + \delta. \tag{6.7}$$



where $\delta$ is a small positive constant and $\bar{l}_1 = l - l^{1/d^2}$. By Lemma 6.1,

$$\sum_x E^{\mu_\beta}[(\tau_x g)u] = \sum_b E^{\mu_\beta}[\sum_x \dot{\Phi}_b(\tau_x g)\nabla_b u] \tag{6.8}$$

Since $E^{\mu_\beta}[\dot{\Phi}_b(\tau_x g)^2] = E^{\mu_\beta}[\{\dot{\Phi}_{\tau_x b}(g)\}^2], \sum_x \dot{\Phi}_b(\tau_x g)$ converges *strongly* to an element $\Phi_b$ in $L_2(\mu_\beta)$ and $\Phi_b$ is translationally covariant in $b$, in the sense that $\Phi_b(\tau_x\eta) = \Phi_{\tau_x b}(\eta)$. Therefore, the tail of the summation on $x$ in (6.8) is not important. Furthermore, it follows from the Schwartz inequality, (6.7) and the translational covariance of $\Phi_b$ that

$$\frac{1}{2}E^{\mu_\beta}[V_l(g,y)] \le -Av_{|b|\le l}E^{\mu_\beta}[(\frac{1}{2}\nabla_b u - \Phi_b)^2] + \sum_{e>0}E^{\mu_\beta}[\Phi_e^2] + \delta \le \sum_{e>0}E^{\mu_\beta}[\Phi_e^2] + \delta. \tag{6.9}$$

As a corollary, we also have

$$Av_{|b|\le l}E^{\mu_\beta}[(\nabla_b u)^2] \le \text{const}.\sum_{e>0}E^{\mu_\beta}[\Phi_e^2]. \tag{6.10}$$

Suppose for the moment that $\Phi_e$ is measurable w.r.t. the $\sigma$-algebra $\mathcal{F}^{(s)}$ generated by $\{\eta_y \| y| \le s\}$ with $s$ a fixed integer independent on $l$. Then, from the translational covariance of $\Phi_b$ one has the identity

$$E^{\mu_\beta}[\Phi_{\tau_y e}(\eta)\nabla_{\tau_y e}u(\eta)] = E^{\mu_\beta}\{\Phi_e(\eta)\nabla_e(\tau_y u)(\eta)\}. \tag{6.11}$$

Averaging (6.11) over $|y| \le \bar{l}_1$ and summing over $e > 0$,

$$\text{Av}_{|b|\le\bar{l}_1}E^{\mu_\beta}[\Phi_b(\eta)\nabla_b u(\eta)] = \sum_{e>0}E^{\mu_\beta}[\Phi_e(\eta)\nabla_e\bar{u}(\eta)].$$

Here $\bar{u} = \text{Av}_{|y|\le\bar{l}_1}\tau_y u$. Let $\xi_e^{(l)} = \nabla_e\bar{u}$. One can check that

$$E^{\mu_\beta}[(\xi_e^{(l)})^2] \le \text{Av}_{|b|\le l}E^{\mu_\beta}[(\nabla_b u)^2] \tag{6.12}$$

Hence $E^{\mu_\beta}[(\xi_e^{(l)})^2]$ is uniformly bounded and it converges (up to subsequences) weakly to an element $\xi_e$ in $L_2(\mu_\beta)$ such that

$$\lim_{l\to\infty}\text{Av}_{|b|\le\bar{l}_1}E^{\mu_\beta}[\Phi_b(\eta)\nabla_b u(\eta)] = \sum_{e>0}E^{\mu_\beta}[\Phi_b(\eta)\xi_e],$$

$$\liminf_{l\to\infty}\text{Av}_{|b|\le l}E^{\mu_\beta}[(\nabla_b u)^2] \ge \sum_{e>0}E^{\mu_\beta}[(\xi_e)^2].$$

We have thus proved that for any $\delta > 0$,

$$\lim_{\ell\to\infty} \frac{1}{2} E^{\mu_\rho}[V_\ell(g,\bar\eta_\ell)] \le E^{\mu_\rho}[\sum_{e>0} \Phi_e(g)\xi_e] - \frac{1}{4} E^{\mu_\rho}[\sum_{e>0}(\xi_e)^2] + \delta.$$

Finally we can let $\delta \to 0$. Because a weak limit is taken, we do not known whether $\xi_e$ is of the form $\nabla_e \sum_x \tau_x u$. The key remark from [12, 13] is that $\xi_e$ can be well approximated by function of the form $\alpha_e \nabla_e \eta + \nabla_e \sum_x \tau_x G$. More precisely, define $\xi_b = \tau_x \xi_e$ for $b = (x, x+e)$. Then $\xi_b$ satisfies the compatibility conditions $\nabla_e \xi_b = \nabla_b \xi_e$. From [13], Theorem 9.2, $\xi_e$ can be approximated by functions of the form just described.

Clearly, by adjusting $\alpha_e$, one can replace $G$ by $G - a - b\eta_0$ without changing $\xi$. Hence we can assume $G \in \mathcal{G}$ and

$$\limsup_{\ell\to\infty} \frac{1}{2} E^{\mu_\rho}[V_\ell(g,\bar\eta_\ell)] \le$$
$$\sup_{\alpha, G\in\mathcal{G}} \left\{ E^{\mu_\rho}[\sum_{e>0} \Phi_e(g)\{\alpha_e \nabla_e \eta + \nabla_e \sum_x \tau_x G\}] - \frac{1}{4} E^{\mu_\rho}[\sum_e (\alpha_e \nabla_e \eta + \nabla_e \sum_x \tau_x G)^2] \right\}.$$

One can check that

$$E^{\mu_\rho}[\Phi_e(g)\nabla_e \eta] = t_e(g), \quad \sum_{e>0} E^{\mu_\rho}[\Phi_e(g)\nabla_e \sum_x \tau_x G] = \langle g, G\rangle_0$$

This concludes the upper bound for Theorem 5.2 provided that $\Phi_b$ is measurable w.r.t. $\mathcal{F}^{(s)}$. For the general case, let $\Phi_b = \Phi_b^{(s)} + \Psi_b^{(s)}$ where $\Phi_b^{(s)}$ is measurable in $\mathcal{F}^{(s)}$. Hence

$$\frac{1}{2} E^{\mu_\rho}[V_\ell(g,\bar\eta_\ell)] \le \mathrm{Av}_{|x|\le \ell_1} E^{\mu_\rho}[\Phi_b^{(s)} \nabla_b u] - \frac{1-\gamma}{4} \mathrm{Av}_{|b|\le \ell} E^{\mu_\rho}[(\nabla_b u)^2] + \Omega + \delta$$

$$\Omega = \mathrm{Av}_{|b|\le \ell_1} E^{\mu_\rho}[\Psi_b^{(s)} \nabla_b u] - \frac{\gamma}{4} \mathrm{Av}_{|b|\le \ell} E^{\mu_\rho}[(\nabla_b u)^2].$$

From the Schwartz inequality, $\Omega \to 0$ as $s \to \infty$ for any $\gamma > 0$ fixed. This provides the cut-off we need to complete our proof of the upper bound.

The lower bound is much easier and can be proved by plugging in a test function $u$ of the form $\sum_{x\in\Lambda_M} \alpha_e(e\cdot x)\eta_x + \sum_x \tau_x G$. ∎

*Proof of Theorem 4.6.* Consider first the eigenvalue problem

$$\mathcal{U}(f) = q \int \hat\omega * [\tau_y G - \alpha_k(G)] f d\mu_k - \varepsilon^{-2} \mathrm{Av}_{|b|\le k} D_b(\sqrt{f}) \tag{6.13}$$

where $d\mu_k$ is the *canonical* Gibbs state with $\bar\eta_k$ fixed and $G$ is a local function. Recall that $k = \ell\varepsilon^{-2/d}$. For each $s = 0, \dots, n$, divide the cube $\Lambda_k$ into disjoint subcubes of size $2\ell^{d+2+s}+1$



and label each subcube by $(\sigma, s)$, $s = 0, 1, \dots, n$ with $2\ell^{d+2+n} + 1 = 2k+1$. The choice of the exponent $d+2+s$ is made for convenience and will be made clear later. Let $\Lambda_\sigma^{(s)}$ be the cube labeled by $(\sigma, s)$ and denote its density by $\bar\eta_\sigma^{(s)}$, namely $\bar\eta_\sigma^{(s)} = \mathrm{Av}_{x\in\Lambda_\sigma^{(s)}} \eta_x$. Let us require that these decompositions are compatible in the sense that, for each $(\sigma, s)$, there is a $(\hat\sigma, s+1)$ such that $\Lambda_\sigma^{(s)} \subset \Lambda_{\hat\sigma}^{(s+1)}$. Denote by $G_\sigma^{(s)}$ the function

$$G_\sigma^{(s)}(u) = E^u[G|\bar\eta_\sigma^{(s)} = u] \tag{6.14}$$

where $\bar\eta_\sigma^{(s)} = \mathrm{Av}_{|x|\le \ell^{d+2+s}} \eta_x$. We can now decompose the average over $y$ as

$$\sum_y \hat\omega(y)\tau_y G = \sum_{s=0}^{\infty} \mathrm{Av}_{\hat\sigma} \left\{ \mathrm{Av}_{(\sigma,s)\in(\hat\sigma,s+1)} G_\sigma^{(s)}(\bar\eta_\sigma^{(s)}) - G_{\hat\sigma}^{(s+1)}(\bar\eta_{\hat\sigma}^{(s+1)}) \right\}. \tag{6.15}$$

By definition, $G_y^{(0)} = \tau_y G$ and $G_\sigma^{(s)} = 0$ for $s > n$.

In the r.h.s. of (6.15) the average over $\sigma$ for the first level, $s = 0$, is taken in $\Lambda_\sigma^{(1)}$ such that the distance from the boundary, $|x - \partial\Lambda_\sigma^{(1)}|$, is greater than $\ell^{1/d^2}$. Since $G$ is a local function, this will ensure the support of $\tau_x G$ lies strictly inside $\Lambda_\sigma^{(1)}$ for $\ell$ large and it is consistent with the definition of $\hat\omega$ given in Section 4. Actually, $\hat\omega$ has been chosen in this way to avoid boundary terms arising from the corridors.

Taking into account the decomposition (6.15) and the analogous for the Dirichlet form, we can bound $\mathcal{U}$ from above by taking the sup on each cube conditioned on $\bar\eta_{\hat\sigma(s+1)} = \rho$. Then $\mathcal{U}$ is bounded by

$$\sum_{s=0}^{\infty} \mathcal{U}^{(s+1)} = \int \sum_{s=0}^{\infty} \mathrm{Av}_{\hat\sigma} \mathcal{U}_{\hat\sigma}^{(s+1)}(\bar\eta_{\hat\sigma(s+1)}) f d\mu_k.$$

$$\begin{aligned}\mathcal{U}_{\hat\sigma}^{(s+1)}(\rho) \;=\; & \sup_{\int h^2 d\nu_\rho = 1} q \int \mathrm{Av}_{(\sigma,s)\in(\hat\sigma,s+1)} \left\{ G_\sigma^{(s)}(\bar\eta_\sigma^{(s)}) - G_{\hat\sigma}^{(s+1)}(\rho) \right\} h^2 d\nu_\rho \\ & - \frac{1}{4(s+1)^2} \varepsilon^{-2} \mathrm{Av}_{|b|\le \ell^{d+3+s}} D_b(h).\end{aligned}$$

Here $d\nu_\rho$ is the canonical Gibbs state in $\Lambda_{\hat\sigma}^{(s+1)}$ with density $\rho$. The factor $(s+1)^{-2}$ makes the sum over $s$ finite. Note that $G_{\hat\sigma}^{(s+1)}(\rho)$ is a constant w.r.t. $d\nu_\rho$. Hence we can redefine $G_\sigma^{(s)}$ so that $G_\sigma^{(s)}(\rho) = 0$. Furthermore, by subtracting a linear term $c(\rho)(\bar\eta_{\sigma,s} - \rho)$ from $G_\sigma^{(s)}$, we can assume that $G_\sigma^{(s)}$ satisfies (3.4) with $\overline{m}_k$ replaced by $\rho$. Such a subtraction is justified since the error vanishes after sum over $\sigma$.

*Case 1, $s \geq 1$.* From the logarithmic Sobolev inequality for *symmetric* simple exclusion [19],

$$\mathrm{Av}_{|b|\leq \ell\hat{\ell}} D_b(h) \geq \mathrm{const.}(\ell\hat{\ell})^{-2} s(h^2|\nu_\rho), \tag{6.16}$$

where $\hat{\ell} = \ell^{d+s+2}$. Hence by the entropy inequality (2.12),

$$\mathcal{U}^{(s+1)} \leq \mathrm{const.}\left\{\frac{1}{(s+1)^2}\varepsilon^{-2}(2\ell\hat{\ell}+1)^{-(d+2)}\log\int d\nu_\rho \exp[Q] - qG_{\hat{\sigma}}^{(s+1)}\right\}. \tag{6.17}$$

$$Q = q(s+1)^2\varepsilon^2(2\ell\hat{\ell}+1)^{(d+2)}\mathrm{Av}_{(\sigma,s)}G_\sigma^{(s)}(\bar{\eta}_\sigma^{(s)}) \tag{6.18}$$

From the Jensen inequality, the average over $\sigma$ can be moved to the front of the log in (6.17). Therefore, $\mathcal{U}^{(s+1)}$ is bounded by

$$\mathcal{U}^{(s+1)} \leq (2\hat{\ell}+1)^{-d}\bar{\varepsilon}^{\,-1}\mathrm{Av}_\sigma \log E^{\nu_\rho}[\exp\{q\bar{\varepsilon}(2\hat{\ell}+1)^d G_\sigma^{(s)}(\bar{\eta}_\sigma^{(s)})\}] - qG_{\hat{\sigma}}^{(s+1)} \tag{6.19}$$

where $\bar{\varepsilon} = (s+1)^2\varepsilon^2(2\ell\hat{\ell}+1)^{(d+2)}(2\hat{\ell}+1)^{-d} \leq 4(s+1)^2\varepsilon^2\ell^{2+d}\hat{\ell}^2$.

We claim that (6.19) is bounded above by

$$\mathcal{U}^{(s+1)} \leq \mathrm{const.}q^2\bar{\varepsilon}\hat{\ell}^{-d} \leq \mathrm{const.}(s+1)^2q^2\varepsilon^2\ell^{2+d}(\hat{\ell})^{2-d}. \tag{6.20}$$

This implies the bound

$$\sum_{s=1}^{\infty}\mathcal{U}^{(s+1)} \leq \mathrm{const.}\sum_{s=1}^{\infty}(s+1)^2q^2\varepsilon^2\ell^{d+2}(\hat{\ell})^{2-d} \leq \mathrm{const.}\varepsilon^2\ell^{-1}, \tag{6.21}$$

provided that $d \geq 3$. Note that it is the fluctuation $\ell^{-d}$ that helps the convergence. Note also the choice of $\hat{\ell} = \ell^{d+2}$ to have the last factor $\ell^{-1}$.

We now prove (6.20). Our strategy is as follows. Since $\bar{\varepsilon}$ is small, in fact $\bar{\varepsilon} < \varepsilon^{2-4/d}$, one can expand the exponential of (6.19) provided $(2\hat{\ell}+1)^d G_\sigma^{(s)}(\bar{\eta}_\sigma^{(s)})$ is of order 1. Assuming this, one gets (6.20) by expanding the exponential up to the second order in $\bar{\varepsilon}$. Note that the first order term cancels due to the subtraction of $qG_{\hat{\sigma}}^{(s+1)}(\rho)$. We now prove that $\xi^2 = (2\hat{\ell}+1)^d G_\sigma^{(s)}(\bar{\eta}_\sigma^{(s)})$ is of order 1. As in the proof of Lemma 3.1 it is enough to consider the special case $G_\sigma^{(s)}(\bar{\eta}_\sigma^{(s)}) = (\bar{\eta}_\sigma^{(s)} - \rho)^2$ since $G$ satisfies (3.4). If one replaces the canonical Gibbs state $d\nu_\rho$ by the grand canonical Gibbs state (namely the product state), it follows from the local central limit theorem that $\exp[q\bar{\varepsilon}\xi^2]$ is integrable and $\xi$ is of order 1 (see the proof of Lemma 3.1). To extend this result to $d\nu_\rho$, let

$$X = (2\hat{\ell}+1)^{d/2}(\bar{\eta}_\sigma^{(s)} - \rho), \quad Y = [2(\ell-1)\hat{\ell}]^{-d/2}\sum_{x\in\Lambda_b^{(s+1)}-\Lambda_\sigma^{(s)}}(\eta_x - \rho).$$



Clearly, $(2\ell\hat{\ell}+1)^{d/2}(\bar{\eta}_{\hat{\sigma}}^{(s+1)} - \rho) = \alpha X + \beta Y$ with $\alpha = (2\hat{\ell}+1)^{d/2}(2\ell\hat{\ell}+1)^{-d/2}$ and $\beta = [2(\ell-1)\hat{\ell}]^{d/2}(2\ell\hat{\ell}+1)^{-d/2}$. Note that $X$ and $Y$ are independent random variables (w.r.t. $d\mu_\beta$) and the densities $g(x)$ and $h(y)$ of $X$ and $Y$ respectively are given by the local central limit theorem [19, 18] as

$$g(x) = \frac{1}{\sqrt{2\pi\sigma}}\exp\{-\frac{x^2}{2\sigma^2}\}[1+O(\hat{\ell}^{-d/2})]; \quad h(y) = \frac{1}{\sqrt{2\pi\sigma}}\exp\{-\frac{y^2}{2\sigma^2}\}[1+O(\hat{\ell}^{-d/2})]. \tag{6.22}$$

where $\sigma$ is the variance of $\eta - \rho$ w.r.t. $d\mu_\beta$. Therefore, the density of $X$ w.r.t. $d\nu_\rho$ is simply the marginal density of $X$ conditioned on $\alpha X + \beta Y = 0$. Since $\alpha$ and $\beta$ are bounded by a constant depending only on $\ell$ (and independent of $\hat{\ell}$ and $\varepsilon$), one can prove easily from (6.22) that $\exp[\mathrm{const.}(\ell)X^2]$ is integrable w.r.t. $d\nu_\rho$ for some const$(\ell)$ depending on $\ell$ only. This provides the bound one needs to complete the proof of (6.20).

*Case 2, s=0.* From the regular perturbation theory [20], $\mathcal{U}_\sigma^{(1)}$ is bounded by

$$\mathcal{U}_\sigma^{(1)} \leq \varepsilon^2q^2\int\int V_\ell(G,\bar{\eta}_{\sigma,\ell})d\mu_k + \mathrm{const.}(\ell)O(\varepsilon^4).$$

We can now average over $\sigma$ to have an upper bound on $\mathcal{U}^{(1)}$.

Together with the bound on $\mathcal{U}^{(s+1)}$ for $s \geq 1$ in (6.21), we conclude Theorem 4.6 by averaging w.r.t. $J$. ∎

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