CERN-TH.6873/93
hep-th/9304149

A remark on the coupling-dependence in affine Toda field theories

Patrick Dorey

CERN TH, 1211 Geneva 23, Switzerland

dorey@surya11.cern.ch

The affine Toda field theories based on the non simply-laced Lie algebras are discussed. By rewriting the S-matrix formulae found by Delius *et al*, a universal form for the coupling-constant dependence of these models is obtained, and related to various general properties of the classical couplings. This is illustrated via the S-matrix associated with the dual pair of algebras $f_4^{(1)}$ and $e_6^{(2)}$.

CERN-TH.6873/93
April 1993

1. Introduction

For each affine Lie algebra $g^{(k)}$ of rank n, a classically integrable field theory in two

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)\,;$$

$$V(\phi) = \frac{m^2}{\beta^2}\sum_{i=0}^{n} n_i e^{\beta\alpha_i\cdot\phi}. \qquad (1.1)$$

sets the overall mass scale, while β is the coupling constant, taken to be real below. The relative values of the n_i can be changed by a shift in the field ϕ; it is convenient to impose $\sum n_i\alpha_i{=}0$ – so that $\phi{=}0$ minimises the potential – and conventional to choose $n_0{=}1$. The n_i then agree with the labelling of the affine diagrams in the book by Kac [1]. The resulting model is known as the affine Toda field theory based on $g^{(k)}$, and its integrability is reflected in the presence of infinitely many conserved quantities, at spins given by the exponents of the affine algebra. If these conservation laws survive quantisation, then (1.1) should also define a *quantum* integrable field theory, and its scattering amplitudes should factorise into products of two-particle S-matrices. As part of the general programme to understand two-dimensional quantum field theories, it is rather natural to ask what these two-particle S-matrix elements might be [2–8].

A straightforward way to approach this question starts by expanding the potential $V(\phi)$ in β, as follows:

$$V(\phi)=\frac{m^2}{\beta^2}\sum_{i=0}^{n}n_i+\frac{(M^2)^{ab}}{2}\phi^a\phi^b+\frac{C^{abc}}{3!}\phi^a\phi^b\phi^c+\dots \tag{1.2}$$

The leading non-trivial terms to appear are the classical (mass)^2 matrix, and a collection of classical three-point couplings:

$$(M^2)^{ab}=m^2\sum_{i=0}^{n}\alpha_i^a\alpha_i^b \quad ; \quad C^{abc}=m^2\beta\sum_{i=0}^{n}\alpha_i^a\alpha_i^b\alpha_i^c\,. \tag{1.3}$$

These objects have a number of remarkable 'universal' properties [3–5]. If $g^{(k)}$ is untwisted (that is, if $k{=}1$), then the set of classical masses $(m_1,m_2,\dots m_n)$ forms a right eigenvector of the Cartan matrix of the corresponding non-affine algebra G (taking $C_{ab}=2\alpha_a\cdot\alpha_b/\alpha_b^2$; $a,b{=}1\dots n$). For the couplings, there is a simple rule determining when they are non-vanishing, relying on the action of a Coxeter element in the relevant (finite) root system [9] (a twisted Coxeter element if $k{>}1$ [10]). The magnitudes of the non-vanishing couplings are given by the 'area law'

$$|C^{abc}|=\lambda^{abc}\frac{2\beta}{\sqrt{h^{(k)}}}m_am_b\sin U_{ab}^c\,, \tag{1.4}$$

where the fusing angle U_{ab}^c is defined via the relation

$$m_c^2=m_a^2+m_b^2+2m_am_b\cos U_{ab}^c\,. \tag{1.5}$$

The factor λ^{abc} is included both to take into account the normalisation of the roots α_i, and to allow for certain adjustments necessary in the untwisted non simply-laced cases; it will be reviewed in more detail later. The so-called 'k^{th} Coxeter number' $h^{(k)}$ is defined

$g^{(k)}$	k	$h^{(k)}$	$h^{(k)\vee}$	h	$h^{\vee}$
$b_n^{(1)}$	1	$2n$	$4n-2$	$2n$	$2n-1$
$a_{2n-1}^{(2)}$	2	$4n-2$	$2n$	$2n-1$	$2n$
$c_n^{(1)}$	1	$2n$	$2n+2$	$2n$	$n+1$
$d_{n+1}^{(2)}$	2	$2n+2$	$2n$	$n+1$	$2n$
$f_4^{(1)}$	1	12	18	12	9
$e_6^{(2)}$	2	18	12	9	12
$g_2^{(1)}$	1	6	12	6	4
$d_4^{(3)}$	3	12	6	4	6
$a_{2n}^{(2)}$	2	$4n+2$	$4n+2$	$2n+1$	$2n+1$

Table 1 : Lie algebra data

as follows [1]: if the usual Coxeter number of $g^{(k)}$ is $h=\sum_0^r n_i$, then $h^{(k)}=k.h$ (table 1 reproduces some of the relevant data from [1]). This quantity is relevant to the classical theory in one further way: all the fusing angles U_{ab}^c turn out to be integer multiples of $\pi/h^{(k)}$. General proofs of these classical results now exist [11].

Armed with this information, the quantum theory can be examined. A combination of perturbation theory, based on (1.2), and general (non-perturbative) principles [12,13], has led to conjectures for the *exact* S-matrices for all of the affine Toda models [2–8,14]. In the cases where $g^{(k)}$ is simply-laced, so that all of the α_a have the same length, the story seems to be particularly simple [2–7]. At least to one loop, the mass ratios do not renormalise, and so the classical fusing angles can be assumed to be directly relevant to the quantum theory as well. This allows the positions of both simple and higher poles in the S-matrix elements to be predicted, the latter via an analysis of the on-shell diagrams responsible for Landau singularities. Add in some zeroes (at coupling-constant dependent positions) to ensure that the S-matrix tends to the identity as $\beta\to 0$, demand agreement to lowest order in perturbation theory, and a surprisingly uniform collection of ansätze emerges, each S-matrix element being written as a product of elementary building blocks $\{x\}$,

$$\{x\}=\frac{(x-1)(x+1)}{(x-1+B)(x+1-B)}\quad;\quad (x)=\frac{\sinh\left(\theta/2+i\pi x/2h^{(k)}\right)}{\sinh\left(\theta/2-i\pi x/2h^{(k)}\right)}\ . \qquad (1.6a,b)$$

all the fusing angles are integer multiples of $\pi/h^{(k)}$, the parameter x for these blocks are always integers. Expressed in terms of such blocks, the β-dependence of the S-matrix elements is completely hidden: even inside the blocks it only appears via the function $B(\beta)$. Tree-level perturbation theory dictates that $B(\beta) = \beta^2/2\pi + \ldots$, and a natural conjecture, now backed up by higher-order perturbative calculations [15], has

$$B(\beta) = 2\beta^2/(\beta^2 + 4\pi)\,. \tag{1.7}$$

The combination of (1.6a) and (1.7) implies a strong-weak coupling duality for the simply-laced models, in that the S-matrix is unchanged on replacing β by $4\pi/\beta$: the operation simply sends B to $2{-}B$ and leaves all the blocks $\{x\}$ unchanged.

Excepting the theory based on $a_{2n}^{(2)}$ [4], the picture for the non simply-laced cases is more complicated. The mass ratios do not remain fixed; rather, one-loop corrections can already be seen to give them a non-trivial dependence on the coupling β. At first, this seemed to be an insurmountable obstacle to the construction of a consistent diagonal S-matrix for any of these models. However, more recent proposals by Delius *et al* have shown that this is not necessarily the case [8]. The physical-strip poles no longer have fixed positions, but move as the coupling constant varies. The price paid for this extra flexibility is that some of the expected bootstrap relations [13] are no longer obeyed, corresponding to certain poles in the S-matrix elements which are not at their expected positions even after the renormalisations of the mass ratios have been taken into account. Exactly in these situations, Delius *et al* were able to identify collections of Landau singularities which modify initial expectations in a delicate way (a similar phenomenon in the sine-Gordon model between the one- and two- breather thresholds had also been observed by Smirnov [16]). Furthermore, these poles can also be understood within a pure S-matrix context, by means of a particular generalisation of the Coleman-Thun mechanism [14]. While subtleties remain, enough supporting evidence now exists to leave little doubt that the S-matrices presented in [8] are correct.

One especially interesting feature is the way that the strong-weak coupling duality appears to extend [8,17]. On general grounds, one might expect that a fuller statement of the simply-laced duality is that the S-matrix should be unchanged under the simultaneous transformations $\beta \rightarrow 4\pi/\beta\,;\ \alpha_i \rightarrow \alpha_i^{\vee} = \frac{2}{\alpha_i^2}\alpha_i\,.$ Since the simply-laced roots were implicitly assumed to have had (length)^2 two, this duality reduces to the previous version in these cases. It groups the non simply-laced theories into the pairs indicated in table 1, corresponding to mutually dual affine Dynkin diagrams – with the exception of $a_{2n}^{(2)}$, whose diagram is self-dual (throughout, the addition of $^{\vee}$ to a symbol indicates the corresponding dual object). For example, the large-β mass spectrum of one member of such a pair should reproduce the small-β mass spectrum of its partner, a property that has now been checked numerically for the $g_2^{(1)}/d_4^{(3)}$ pair [18]. But beyond this, the S-matrices should also be pairwise equivalent under the replacement $\beta \rightarrow 4\pi/\beta$. The fact that the S-matrices proposed by Delius *et al* did indeed turn out to have this property – it had not been fed in at the start – provided further support for their conjectures.

The aims of this note are twofold: on the one hand, to systematise previous results with a 'naturally dual' notation applicable to both simply- and non simply- laced cases, in the process observing the simple generalisation of (1.7); and on the other to illustrate this with some features of the $f_4^{(1)}/e_6^{(2)}$ S-matrix.

2. A general block notation for all affine Toda theories

In [8], the coupling dependence of the pole positions was incorporated in two ways: first, by allowing $h^{(k)}$ to wander away from its 'classical' (k^{th} Coxeter number) value, replacing it in (1.6b) by $H(\beta)$, the 'renormalised Coxeter number'; and second, by using parameters x in the blocks defined by (1.6a) that were no longer always integers, but rather could depend on β via $H(\beta)$, and sometimes also via $B(\beta)$. Most, though not in fact quite all, of the S-matrix elements were then written in terms of such blocks alone.

The alternative to be advocated here starts by dropping the idea that $B(\beta)$ should always be given by the expression (1.7). (With standard normalisations for the roots, this is in any case impossible to enforce: cf. [18] for the $g_2^{(1)}$ case.) Instead, for the theory based on $g^{(k)}$, define $B^{[g^{(k)}]}(\beta)$ to be that function of β which satisfies

(a) $B^{[g^{(k)}]}(0)=0\ ,\quad B^{[g^{(k)}]}(\infty)=2\,;$

(b) The positions of the zeroes and poles in the S-matrix elements of $g^{(k)}$ depend *linearly* on $B^{[g^{(k)}]}$.

Given the S-matrix, this specifies $B^{[g^{(k)}]}$ uniquely, and it reproduces (1.7) in the simply-laced cases. Strictly speaking, just a single mobile pole or zero is enough to pin $B^{[g^{(k)}]}$ down, so part (b) relies on the observation that all poles and zeroes can be linearised by a single function. If this seems surprising, it should be no more so than the fact that a single function $B(\beta)$ sufficed in each of the simply-laced models; the underlying reason is that the bootstrap relations, binding the different S-matrix elements together in a set of overdetermined equations, forces the coupling-dependence to enter in a coherent way.

An example: in the $a_{2n-1}^{(2)}$ theory, Delius *et al* found that $S_{nn}(\theta)$ had (amongst others) a pole at $2/(2n{-}1{+}B/2)$, when $B(\beta)$ is given by (1.7). So, this B fails on count (b). But

$$\frac{2}{2n-1+B/2}=\frac{2}{2n-1}+\frac{-B}{(2n-1)(2n-1+B/2)}\ , \tag{2.1}$$

and the second term on the RHS does satisfy (b) above, along with the first half of (a); multiplying it by $-2n(2n{-}1)$ to give it the desired limiting value of 2 as $\beta{\to}\infty$, and substituting for B using (1.7), yields $B^{[a^{(2)}_{2n-1}]}(\beta) = 2\beta^2/(\beta^2 + 4\pi\frac{2n-1}{2n})$. Comparing with the data in table 1 then suggests the following modification to (1.7):

$$B^{[g^{(k)}]}(\beta) = 2\beta^2/(\beta^2 + 4\pi\tfrac{h}{h^\vee}) . \tag{2.2}$$

This does indeed turn out to be the case, and will be discussed further below.

To continue, it is convenient to set

$$\langle x\rangle = \frac{\sinh\left(\theta/2 + i\pi x/2\right)}{\sinh\left(\theta/2 - i\pi x/2\right)} \;; \tag{2.3}$$

the same as (x), but without the $h^{(k)}$. Assuming duality, *two* k^{th} Coxeter numbers will be relevant to each non simply-laced theory, $h^{(k)}$ and $h^{(k)\vee}$. At weak coupling, pole positions should approach integer multiples of $i\pi/h^{(k)}$; at strong coupling (weak dual coupling), integer multiples of $i\pi/h^{(k)\vee}$. It is therefore natural to define an interpolating block $\langle x,y\rangle$, to replace (x), now depending on two integers rather than just one:

$$\langle x,y\rangle = \langle (2-B)x/2h^{(k)} + By/2h^{(k)\vee}\rangle \tag{2.4}$$

The first index will 'see' the classical data of $g^{(k)}$, the second that of $g^{(k)\vee}$. Now a generalisation of (1.6a) can be given:

$$\{x,y\} = \frac{\langle x-1,y-1\rangle\langle x+1,y+1\rangle}{\langle x-1,y+1\rangle\langle x+1,y-1\rangle} \;. \tag{2.5}$$

Even for $g^{(k)}$ simply-laced, this provides quite a succinct way to write the basic block, the correspondence being $\{x\} = \{x,x\}$ for these (self-dual) cases. However, one feature of the non simply-laced S-matrices has yet to be captured: 'extra' cancellations between physical-strip poles and zeroes. These never happen in the simply-laced cases – the poles and zeroes go through the bootstrap independently, allowing minimal parts of these S-matrices to be defined – but is crucial for $g^{(k)}$ non simply-laced (excepting the self-dual $a^{(2)}_{2n}$ theory, for which an – albeit not one-particle unitary – minimal S-matrix can indeed be defined [19]). The extra freedom given by the *pair* of indices entering (2.5) permits the product of two blocks to have fewer zeroes and poles than expected, owing to cross-cancellations. There are precisely two ways in which a product of two blocks (2.5) can have a such a 'partial' cancellation, one involving shifts of the left index, and one shifts of the right. This motivates the definitions

$$\begin{aligned}\{x,y\}_2 &= \{x,y-1\}\{x,y+1\}\\ {}_2\{x,y\} &= \{x-1,y\}\{x+1,y\}\end{aligned} \tag{2.6}$$

[illegible]

$${}_a\{x,y\}_b = \frac{\langle x-a,y-b\rangle\langle x+a,y+b\rangle}{\langle x-a,y+b\rangle\langle x+a,y-b\rangle} \;. \tag{2.7}$$

$$\begin{aligned}\{x,y\}_3 &= \{x,y-2\}\{x,y\}\{x,y+2\}\\ {}_2\{x,y\}_2 &= {}_2\{x,y-1\}\,{}_2\{x,y+1\} = \{x-1,y\}_2\{x+1,y\}_2 \,.\end{aligned} \tag{2.8}$$

after fusings, in terms of the more elaborate objects (2.8). The resulting notation may be viewed as a further refinement of the 'single index' scheme used in [14]: the correspondence between the two involves sending blocks of the form $\{,\}_b$ above into the blocks $\{\}_{(b-1)/2}$ of [14] in all cases but $b_n^{(1)}/a_{2n-1}^{(2)}$, where they become $\{\}_{(b-2)/4}$ instead.

To see how the duality works, note from (2.2) that $B^{[g^{(k)}]}(4\pi/\beta) = 2 - B^{[g^{(k)\vee}]}(\beta)$, and so from (2.4) it follows that expressions for the dual theory are obtained just by swapping over the integers x and y in each block, along with any subscripts present – that is, read the formulae from right to left instead of left to right. Some 'universal' features of the pole residues can also be derived. Consider a single block ${}_a\{x,y\}_b$, in the weak-coupling limit. This has two poles, which approach $i\pi(x \pm a)/h^{(k)}$ as $\beta \to 0$. To leading order in β (and with $B(\beta) = \kappa\beta^2/2\pi + \ldots$) their residues are

$$\mathcal{R}_{\pm}\left({}_a\{x,y\}_b\right) = \pm\frac{i\beta^2\kappa C}{2h^{(k)\vee}}\ , \quad \text{with } C = b\ . \tag{2.9}$$

(This is a little delicate, as the position of the pole may also depend on β; the residue given here corresponds to a simultaneous expansion, near the pole, in β and $\theta - \theta_0$, with θ_0 the (mobile) pole position. Following [8], this seems to be the appropriate prescription for comparison with perturbation theory based on the renormalised masses.) The new feature compared to the simply-laced cases is the correction factor C – the 'fusing' of blocks changes their residues. There is another mechanism by which a simple pole residue can be modified: other poles and zeroes, from other blocks, which also approach the location of the pole in question as $\beta \to 0$. While these do not change the order of the pole (for $\beta \neq 0$, they are not in the same place), they do change its residue, even to leading order in β. The general formula for this change is a little unwieldy; one example should suffice. Consider the product $\{x,y\}_b\{x{+}2,y{+}p\}_{b'}$. For small β, there are two simple poles near to $i\pi(x{+}1)/h^{(k)}$, one from each block. Without the presence of the other block, their residues would be given by (2.9) with $C = b, b'$. Including this extra effect then multiplies C by the additional factors of

$$\frac{p{+}b'{-}b}{p{-}b'{-}b}\quad,\quad\frac{p{+}b{-}b'}{p{-}b{-}b'} \tag{2.10}$$

respectively. These two new phenomena – the fusing and the interference of blocks – mesh with another extra feature of the non simply-laced theories: the already-mentioned occurrence of varying λ^{abc} factors correcting the area law (1.4). This is illustrated in the next section by means of the $f_4^{(1)}/e_6^{(2)}$ S-matrix.

to demonstrate the approach just outlined, and to highlight aspects of the comparison with

3. An example: the S-matrix for $f_4^{(1)}$ and $e_6^{(2)}$

This S-matrix has been presented in [14], but will be developed from scratch here both perturbation theory. To start with, the classical Lagrangian will be taken to be that of the $f_4^{(1)}$ theory; however, an *a priori* assumption of duality allows data from both algebras of the pair to be used as input. The two classical theories both involve 4 non-degenerate masses, $m_1 \dots m_4$. (The $e_6^{(2)}$ masses are, via the folding idea [20,5], found as the masses of particles 2,4,5 and 7 of the $e_7^{(1)}$ model.) The full sets of couplings can be found in [5], but as an empirical rule the only 'good' three-point couplings, that survive to have bootstrap implications in the quantum theory, are those which are found in *both* classical theories of a dual pair. In terms of the generalised bootstrap principle put forward in [14], this means that the corresponding S-matrix poles will turn out to be 'positive definite', their residues being positive multiples of i thoughout the entire range of β. For the case in hand, this selects the couplings C^{111}, C^{222}, C^{333}, C^{444}, C^{112}, C^{113}, C^{224}, C^{123}, and C^{134}. (It is interesting to note that, apart from the four ϕ^3-type couplings, the couplings listed are those having depth [6] one in the $e_7^{(1)}$ theory.) Their classical fusing angles are:

$$
\begin{array}{lll}
U_{11}^1 = U_{22}^2 = U_{33}^3 = U_{44}^4 = (8,12) & & U_{11}^2 = (6,8)\ ; \quad U_{12}^1 = (9,14) \\
U_{11}^3 = (2,2)\ ; \qquad U_{13}^1 = (11,17) & & U_{22}^4 = (2,4)\ ; \quad U_{24}^2 = (11,16) \\
U_{23}^1 = (10,15)\ ; \quad U_{31}^2 = (9,13)\ ; \quad U_{12}^3 = (5,8) & & \\
U_{34}^1 = (11,17)\ ; \quad U_{41}^3 = (10,16)\ ; \quad U_{13}^4 = (3,3) & &
\end{array}
\qquad (3.1)
$$

In this list, the first number of each pair, multiplied by $\pi/12$, is the fusing angle for $f_4^{(1)}$ (that is, the angle that emerges from (1.5) when the classical $f_4^{(1)}$ masses are used for $m_1 \dots m_4$), while the second, multiplied by $\pi/18$, is that for $e_6^{(2)}$ (12 and 18 are the k^{th} Coxeter numbers for $f_4^{(1)}$ and $e_6^{(2)}$). These two fusing angles will be written as $U_{bc}^a(0)$ and $U_{bc}^a(2)$ respectively. If it is assumed that for the $f_4^{(1)}/e_6^{(2)}$ theory there is again a linearising B-function $B^{[f_4^{(1)}]} \equiv B$, then the general fusing angle must be

$$
U_{bc}^a(B) = \tfrac{2-B}{2} U_{bc}^a(0) + \tfrac{B}{2} U_{bc}^a(2)\,. \qquad (3.2)
$$

This is enough to postulate the ratios of the conserved charges, via the bootstrap relations. For example, using the couplings C^{123} and C^{224} leads to the quadruplet $(\sin 2\theta_s, \sin(3{+}B/6)\theta_s, \sin(7{-}B/6)\theta_s, \sin 2\theta_s{+}\sin(4{+}B/3)\theta_s)$, where $\theta_s{=}\pi s/12$. The spin s runs over all integers coprime to 6, the set of exponents of $f_4^{(1)}$. At $B{=}0$, these are the 'classical' conserved charges of $f_4^{(1)}$, forming eigenvectors of the Cartan matrix of the non-affine algebra. Next, combining the implications of (3.1) and (3.2) for pole structure with the hypothesis that the S-matrix can be written in terms of the blocks $\{x,y\}$ introduced in the last section selects S_{11} uniquely: $S_{11} = \{1,1\}\{5,7\}\{7,11\}\{11,17\}$. The bootstrap can now be followed through, a process which is simplified by the linear notation. Note first that if U_{ab}^c is specified by the pair of integers (m,n) in (3.1), then $\overline{U}_{ab}^c = \pi{-}U_{ab}^c$ is $(\overline{m},\overline{n})$ with $\overline{m} = 12{-}m$, $\overline{n} = 18{-}n$. Now to find $S_{db}(\theta + i\overline{U}_{ab}^c)$, say, first dismember the constituent blocks $\{x,y\}$ as $\{x,y\}_+/\{-x,-y\}_+$, where $\{x,y\}_+$ is defined as in the earlier sequence of definitions (2.3) – (2.5), but now starting from $\langle x\rangle_+ = \sinh(\theta/2 + i\pi x/2)$. Such blocks preserve their forms under shifts in θ (cf. the discussion in [9]), and for any B,

$$\{x,y\}_+(\theta{+}i\overline{U}^{c}_{ab}(B)) \equiv (\mathcal{T}_{\overline{m},\overline{n}}\{x,y\}_+)\,(\theta) = \{x{+}\overline{m},y{+}\overline{n}\}_+(\theta)\ , \tag{3.3}$$

owing to the linear natures of (3.2) and (2.5). The shift operator $\mathcal{T}_{\overline{m},\overline{n}}$ thus defined manages to hide away all θ- and B- dependence, making for more elegant calculations. With such tools to hand, the complete S-matrix emerges as follows:

$$\begin{aligned}
S_{11} &= \{1,1\}\{5,7\} \times \text{crossing}\\
S_{12} &= \{4,6\}_2 \times \text{crossing}\\
S_{13} &= \big(\{2,2\}\{4,6\} \times \text{crossing}\big)\{6,9\}_2\\
S_{14} &= \{3,4\}_2\{5,8\}_2 \times \text{crossing}\\
S_{22} &= \{1,2\}_2\{5,8\}_2 \times \text{crossing}\\
S_{23} &= \{3,5\}_2\{5,7\}_2 \times \text{crossing}\\
S_{24} &= \{2,4\}_2\{4,6\}_2\{6,8\}_2 \times \text{crossing}\\
S_{33} &= \{1,1\}\{3,4\}_2\{5,8\}_2\{5,7\} \times \text{crossing}\\
S_{34} &= \{2,3\}_2\{4,5\}_2\{4,7\}_2\{6,9\}_2 \times \text{crossing}\\
S_{44} &= \{1,2\}_2\{3,4\}_2({}_2\{4,6\}_2)(\{5,8\}_2)^2 \times \text{crossing}
\end{aligned} \tag{3.4}$$

All the S-matrix elements are crossing-symmetric: the omitted blocks can be restored using the general relation ${}_a\{x,y\}_b(i\pi{-}\theta) = {}_a\{h^{(k)}{-}x, h^{(k)\vee}{-}y\}_b(\theta)$.

The dual form of this S-matrix, describing the situation for which the classical theory is based on $e_6^{(2)}$, can be found by reading (3.4) backwards: for example, $S_{12} = {}_2\{6,4\} \times$ crossing, now with $h^{(k)}{=}18$ and $h^{(k)\vee}{=}12$. The dual conserved charges emerge on substituting $2-B$ for B in the previous expressions: this yields $(\sin 3\theta_s, \sin(5{-}B/4)\theta_s, \sin(10{-}B/4)\theta_s, \sin 3\theta_s{+}\sin(7{-}B/4)\theta_s)$, with $\theta_s{=}\pi s/18$. The spin s takes the same values as before, the exponents of $e_6^{(2)}$ being equal to those of $f_4^{(1)}$, but setting $B{=}0$ now reveals the classical $e_6^{(2)}$ charges, the subset $(q_2^{(s)}, q_4^{(s)}, q_5^{(s)}, q_7^{(s)})$ from the $e_7^{(1)}$ theory. While $e_7^{(1)}$ has extra exponents over $e_6^{(2)}$, at integers equal to 9 modulo 18, the corresponding conserved charges are identically zero for the particles 2, 4, 5 and 7 which survive the fold to $e_6^{(2)}$. This can be understood in the spirit of [9] via the alternative characterisation of twisted Coxeter elements described in [10]: within the E_7 root system, those roots annihilated by projection onto the spin-9 eigenplane of a Coxeter element w – the w-orbits for particles 2, 4, 5 and 7 – form a root system for E_6, within which w acts as a twisted Coxeter element of E_6.

There remains the question of the form of $B(\beta)$. This can be approached via the simple poles corresponding to the 'good' couplings. So long as attention is restricted to these poles, the situation is only a little more complicated than in the simply-laced cases. To leading order in a perturbative treatment based on the renormalised masses, the predicted residue of the simple pole in the $a\,b$ scattering amplitude due to a $\overline{c}$ bound state is $i(C^{abc})^2/8m_a^2m_b^2\sin^2 U_{ab}^c$. Using (1.4), this becomes

$$\mathcal{R}=\frac{i(\lambda^{abc})^2\beta^2}{2h^{(k)}}\ . \tag{3.5}$$

Consistency with (2.9) then demands $B(\beta)=\kappa\beta^2/2\pi+\ldots$, with

$$\kappa=\frac{(\lambda^{abc})^2}{C}\frac{h^{(k)\vee}}{h^{(k)}}\ . \tag{3.6}$$

To fix the numbers λ^{abc}, the normalisation convention for the roots $\alpha_0,\ldots\alpha_n$ must be decided. That which leads to the rule (2.2) for general $g^{(k)}$ starts by imposing $|\alpha_L|^2\!=\!2$ in all untwisted ($k\!=\!1$) cases, where α_L is a longest root. Via $\alpha_i^\vee=\frac{2}{\alpha_i^2}\alpha_i$, this fixes $|\alpha_L|^2\!=\!2k$ for all but $a_{2n}^{(2)}$; in this case, requiring that the roots are mapped into themselves under duality gives the same rule again. This change of normalisations over that used in [5] leads to the basic relation

$$\lambda^{abc}=\sqrt{k}\,. \tag{3.7}$$

Modifications are found in the untwisted non simply-laced theories. Via the eigenvector property of the masses, their particles are associated with the spots on the non-affine Dynkin diagram of the corresponding finite algebra. They can thus be called 'short' or 'long', depending on whether the corresponding (non-affine) root is short or long. The change to (3.7) occurs when a, b and c are all short, and is [4,5,11]

$$a,\ b,\ c\ \text{all short:}\quad \lambda^{abc}=\begin{cases}\frac{1}{\sqrt{2}} & \text{for } c_n^{(1)},\ f_4^{(1)};\\ \frac{2}{\sqrt{3}} & \text{for } g_2^{(1)}.\end{cases} \tag{3.8}$$

The other couplings continue to obey (3.7). (For $b_n^{(1)}$ there are no short-short-short couplings at all, and so (3.8) is not needed.) The normalisations leading to (3.7) and (3.8) only disagree with [8] for the $c_n^{(1)}/d_{n+1}^{(2)}$ pair, where $|\alpha_L|^2$=4 was used for $c_n^{(1)}$, and $|\alpha_L|^2$=2 for $d_{n+1}^{(2)}$.

Of the four particle types in the $f_4^{(1)}$ theory, 1 and 3 are short. Consider first $S_{11}(\theta)$. In the notation of (3.1), this has forward-channel poles at $(2,2)$, $(6,8)$ and $(8,12)$, associated with particles 3, 2 and 1 respectively. The S-matrix predicts residues given by (2.9) with $C=1,2,1$, the 2 for the $(6,8)$ pole coming via the mechanism (2.10). On the other hand, (3.8) gives the λ factors for the 113, 112 and 111 couplings to be $1/\sqrt{2}$, 1, and $1/\sqrt{2}$

respectively. Thus in all three cases, (3.6) predicts $\kappa = \frac{1}{2} h^{(k)\vee}/h^{(k)} = 9/12 = h^\vee/h$. The other mechanism by which simple S-matrix residues change, namely the fusing of blocks, comes into play when S_{12} is examined. This has forward-channel poles at $(5, 8)$ and $(9, 14)$, due to particles 3 and 1. Whilst 1 is short, 2 is long and so the λ factors are 1 for both the 123 and 121 couplings. However, the relevant blocks enter (2.9) with $b{=}2$, so $C{=}2$ in (3.6), and $\kappa{=}9/12$ is again confirmed. The rest of (3.4) can be checked in a similar fashion, and in all cases the combination of (2.9), (2.10) and (3.8) conspires to produce $(\lambda^{abc})^2/C = 1/2$, exactly as required in (3.6) to convert $h^{(k)\vee}/h^{(k)}$ into $h^\vee/h$.

On the other hand, the dual form of (3.4) can be checked against the classical $e_6^{(2)}$ data. This is even more straightforward, at least if attention is again restricted to the 'good' three-point couplings: read right-to-left, all the relevant blocks enter (2.9) with $b{=}1$, and never feel the effect (2.10) of neighbouring blocks, so $C{=}1$. Furthermore, the λ factors behave in a uniform way for twisted algebras, being equal in this case to $\sqrt{2}$. Thus $(\lambda^{abc})^2/C = 2$ throughout, and equation (3.6) yields $\kappa = 2 \times 12/18 = 12/9$, in line with the $e_6^{(2)}$ entry of table 1.

4. Conclusions

The analysis of the last section can be repeated for the block forms of all the other non simply-laced Toda S-matrices, and at every 'good' simple pole, the mechanisms listed above – the fusing of blocks, the influence of neighbouring blocks, and the corrections to the basic area law – combine to give the linearising B-function as defined earlier the leading behaviour $B^{[g^{(k)}]}(\beta) = \frac{h^\vee}{2\pi h}\beta^2 + \dots$. The expression (2.2) is then the natural extension of this with the desired duality properties. Of course, for all the other cases the full β-dependence has already given in [8] (for $g_2^{(1)}/d_4^{(3)}$, see also [18]), and one can also check that the manipulations which lead to (2.1) in the $a_{2n-1}^{(2)}$ case also manage to confirm (2.2) for the others. However, in doing this the already-noted change in root normalisations for $c_n^{(1)}/d_{n+1}^{(2)}$ should be taken into account.

The main object of this note, apart from establishing the β-dependence of the $f_4^{(1)}/e_6^{(2)}$ S-matrix, has been to point out that the non simply-laced S-matrices share rather more of the universal features of the simply-laced cases than might have been thought. This reinforces the idea that there may ultimately be some geometrical structure underlying these theories, as described for the simply-laced cases in [9,10]. It is hoped that the alternative ways of viewing the S-matrices outlined here will lead to some insight into this question.

Acknowledgements

I would like to thank Gérard Watts, Robert Weston, Gustav Delius and Marc Grisaru for discussions, and Ed Corrigan and Ryu Sasaki both for discussions and for helpful comments on the manuscript. This work was supported by a grant under the Science Programme of the European Community.

References

[1] V. Kac, *Infinite dimensional Lie algebras* (Cambridge University Press 1990).

[2] A. E. Arinshtein, V. A. Fateev and A. B. Zamolodchikov, *Qunatum S-matrix of the 1+1 dimensional Toda chain*, *Phys. Lett.* **B87** (1979) 389.

[3] P. Christe and G. Mussardo, *Integrable systems away from criticality: the Toda field theory and S matrix of the tricritical Ising model*, *Nucl. Phys.* **B330** (1990) 465.

[4] P. Christe and G. Mussardo, *Elastic S-matrices in (1+1) dimensions and Toda field theories*, *Int. J. Mod. Phys.* **A5** (1990) 4581.

[5] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, *Affine Toda field theory and exact S-matrices*, *Nucl. Phys.* **B338** (1990) 689.

[6] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, *Multiple poles and other features of affine Toda field theory*, *Nucl. Phys.* **B356** (1991) 469.

[7] C. Destri and H. J. de Vega, *The exact S-matrix of the affine E_8 Toda field theory*, *Phys. Lett.* **B233** (1989) 336.

[8] G. W. Delius, M. T. Grisaru and D. Zanon, *Exact S-matrices for non simply-laced affine Toda theories*, *Nucl. Phys.* **B282** (1992) 365.

[9] P. E. Dorey, *Root systems and purely elastic S-matrices, I & II*, *Nucl. Phys.* **B358** (1991) 654; *Nucl. Phys.* **B374** (1992) 741.

[10] P. E. Dorey, *Hidden geometrical structures in integrable models*, preprint NI 92018, hep-th/9212143, CERN-TH.6772/93, to appear in the proceedings of the conference "Integrable Quantum Field Theories", Como, Italy, 13-19 September 1992.

[11] M. D. Freeman, *On the mass spectrum of affine Toda field theory*, *Phys. Lett.* **B261** (1991) 57;
A. Fring, H. C. Liao and D. I. Olive, *The mass spectrum and coupling in affine Toda theories*, *Phys. Lett.* **B266** (1991) 82.

[12] A. B. Zamolodchikov and Al. B. Zamolodchikov, *Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models*, *Ann. Phys.* **120** (1979) 253.

[13] A. B. Zamolodchikov, *Integrable Field Theory from Conformal Field Theory*, Proceedings of the Taniguchi Symposium, Kyoto (1988);
A. B. Zamolodchikov, *Integrals of motion and S-matrix of the (scaled) $T{=}T_c$ Ising model with magnetic field*, *Int. J. Mod. Phys.* **A4** (1989) 4235.

[14] E. Corrigan, P. E. Dorey and R. Sasaki, *On a generalised bootstrap principle*, preprint DTP-93/19, YITP/U-93-09, CERN-TH.6870/93; hep-th/9304065.

[15] H. W. Braden and R. Sasaki, *The S-matrix coupling dependence for a, d and e affine Toda field theory*, *Phys. Lett.* **B255** (1991) 343; *Affine Toda perturbation theory*, *Nucl. Phys.* **B379** (1992) 377.

[16] F. Smirnov, *Reductions of the sine-Gordon model as a perturbation of minimal models of conformal field theory*, *Nucl. Phys.* **B337** (1990) 156.

[17] H. G. Kausch and G. M. T. Watts, *Duality in quantum Toda theory and W-algebras*, *Nucl. Phys.* **B386** (1992) 166.

[18] G. M. T. Watts and R. A. Weston, $G_2^{(1)}$ *affine Toda field theory. A numerical test of exact S-matrix results*, *Phys. Lett.* **B289** (1992) 61.

[19] P. G. O. Freund, T. Klassen and E. Melzer, *S-matrices for perturbations of certain conformal field theories*, *Phys. Lett.* **B229** (1989) 243.

[20] D. I. Olive and N. Turok, *The symmetries of Dynkin diagrams and the reduction of Toda field equations*, *Nucl. Phys.* **B215** (1983) 470.