LAGRANGIAN/HAMILTONIAN DERIVATION OF THE PHASE EQUATION

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ABSTRACT
Particle accelerator beams and celestial systems often share the common feature of making extremely large numbers of cycles. The long-term study of these systems requires a mathematical description that rigorously conserves phase-space density. This can be assured by using the Lagrangian/Hamiltonian formalism when deriving the equations of motion. In this paper, this technique is applied to the phase equation that describes the longitudinal, or energy oscillations, of a particle beam that is bunched by the action of an rf cavity. The use of a velocity-dependent potential to account for magnetic fields, the special case of replacing the magnetic guide field with an electric one, and various choices for the conjugate variables are discussed.

1. INTRODUCTION

The derivations given in the basic course for the phase equation (longitudinal motion) [1] and for the transverse motion [2, 3] appear very simple and easy to understand, which was the motivation for doing them in the ways presented. The final results are valid and are universally applied, but behind this economy of the truth there are some pitfalls. As often happens in applied physics, the final expressions are relatively simple, but only as a result of making some approximations, which are justified as having very small effects. Usually this is satisfactory compared to the desired accuracy, but in some cases, after very many oscillations of a system*, the consequences may no longer be negligible. In such cases the approximations are violating, however slightly, a fundamental principle of physics, which for accelerators would be the conservation of phase space. The same problem reappears in celestial mechanics where the equations of motion are required to accurately represent the motion of a planetary system through extremely large numbers of oscillations. In both cases, it is only approximations that do not violate the conservation of phase space that can be accepted. At present, there is a lot of research, motivated by the design of new accelerators, to determine the limit between stability and instability for very large numbers of oscillations in the presence of nonlinear fields. It is therefore imperative to be sure that the basic equations are phase-space conserving and to find a safe method for treating more complicated cases. This method is the Lagrangian/Hamiltonian formalism.

2. LAGRANGIAN/HAMILTONIAN FORMALISM

The motion of a single particle under an external force can be described by a Hamiltonian $H(\boldsymbol{q}, \boldsymbol{P}, t)$, where $\boldsymbol{q}$ is the position vector, $\boldsymbol{P}$ is known as the canonically conjugate momentum ($\boldsymbol{P}$ distinguishes the conjugate momentum from the more usual kinetic momentum $\boldsymbol{p}$) and t is an independent variable such as time. H is chosen so that,

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial P_i} \quad \text{and} \quad \frac{\mathrm{d}P_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i} . \qquad (1) \text{ and } (2)$$

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* In the CERN ISR, it was possible to have stable beams for 50h or more without cooling, which corresponds to about 4.6 x 10^{11} betatron oscillations

Thus in a system with n degrees of freedom the dynamics will be described by $2n$ first-order equations. The Hamiltonian can be found by expressing the system in whatever generalised set of coordinates is convenient and then constructing the Lagrangian denoted by L. The Lagrangian is the key to finding the conjugate momenta and the Hamiltonian via,

$$P_i = \frac{\partial L}{\partial \dot{q}_i} \quad \text{and} \quad H = \sum_i P_i \dot{q}_i - L\,. \qquad (3)\text{ and }(4)$$

In practice, the construction of the Lagrangian may not be easy, but the Lagrangian of a relativistic charged particle in an electromagnetic field is well-known and will be simply quoted and taken as the starting point, i.e.,

$$L = -\, m_0 c^2 \gamma^{-1} - e(\phi - \boldsymbol{A}.\boldsymbol{v}) \qquad (5)$$

where $\boldsymbol{A}(q)$ is the vector potential of the magnetic field such that,

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad (6)$$

and $\phi(q)$ is the scalar potential of the electrical field such that,

$$\boldsymbol{E} = -\, \nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}\,. \qquad (7)$$

m_0 is the particle's rest mass, c is the speed of light and $\gamma = (1\text{-}v^2/c^2)^{-1/2}$. This rather condensed explanation of Hamilton's equations should be supplemented by studying a standard work on mechanics, such as Ref. [4].

The development of the above theory in a curvilinear coordinate system for the transverse motion in a synchrotron is dealt with by Bell [5], Hagedoorn [6] and Montague [7] and for nonlinear resonances by Wilson [8]. In the present paper, the emphasis is on the longitudinal motion and the analysis is based on Refs. [9] and [10].

3. DERIVATION OF THE PHASE EQUATION

For the phase equation, it is convenient to use the cylindrical coordinates (R, Θ, z), where the azimuthal angle is defined as,

$$\Theta = s/R = 2\pi\, s/C \qquad (8)$$

where s is the distance along the central orbit, R is the average machine radius and C is the machine circumference. The latter is the preferred formulation since C is directly measurable. Partial differentiation of (5) with respect to $\dot{\Theta}$ gives the momentum, which is canonically conjugate to Θ. This is known as the <u>general angular momentum S</u>,

$$S = m_0 \gamma R^2 \dot{\Theta} + eRA_\Theta$$

$$S = \frac{C}{2\pi}\left(p_\Theta + eA_\Theta\right) \qquad (9)$$

where p_Θ is the tangential component of the kinetic momentum. Thus (Θ, S) are canonically conjugate variables for the azimuthal (longitudinal) motion. The most direct route to the motion equation is via the well-known Lagrangian equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}_i}\right]-\frac{\partial L}{\partial q_i}=0\ . \tag{10}$$

By virtue of (3) and (5), (10) can be rewritten as,

$$\frac{\mathrm{d}}{\mathrm{d}t}P_i=\frac{\partial}{\partial q_i}\left[-m_0c^2\gamma^{-1}-e(\phi-\boldsymbol{A}.\boldsymbol{v})\right].$$

The first term on the right-hand side is not an explicit function of position, so that,

$$\frac{\mathrm{d}P_i}{\mathrm{d}t}=-\frac{\partial}{\partial q_i}\left[e(\phi-\boldsymbol{A}.\boldsymbol{v})\right] \quad \text{or} \quad \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t}=-e\,\boldsymbol{\nabla}\,U \tag{11}$$

where $U = \phi-\boldsymbol{A}.\boldsymbol{v}$ and is called the generalised potential. It is this term that makes the conjugate momentum different from the kinetic momentum in the presence of a magnetic field. Equation (11) shows that the time rate of change of the conjugate momentum can be equated to the gradient of the generalised potential. This is a more general formulation of the simpler expression that equates the rate of change of kinetic momentum to the gradient of a scalar potential when magnetic fields are not present. The generalised potential is further discussed and shown to be the potential for the Lorentz force in Appendix A.

The application of (11) to the azimuthal motion in a synchrotron with the conjugate variables (Θ, S) gives,

$$\frac{\mathrm{d}S}{\mathrm{d}t}=-e\frac{\partial}{\partial\Theta}(\phi-\boldsymbol{A}.\boldsymbol{v})\ .$$

With the use of (9), this can be expanded into

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{C}{2\pi}(p_\theta+eA_\theta)\right]=-e\frac{\partial}{\partial\Theta}(\phi-\boldsymbol{A}.\boldsymbol{v}). \tag{12}$$

So far all equations are exact, but certain assumptions and approximations are now needed:

(i) Inter-particle forces will be neglected, so that there are only two sources of field to be considered in (12), i.e.,

$$\boldsymbol{A} = \boldsymbol{A}_{\text{guide field}} + \boldsymbol{A}_{\text{cavity}}$$
$$\phi = \phi_{\text{guide field}} + \phi_{\text{cavity}}.$$

(ii) The magnetic guide field will be taken as purely two-dimensional and will therefore be derivable from A_Θ alone.

(iii) It will be assumed that the guide field is constant and therefore $\partial A_\Theta/\partial\Theta = 0$.

(iv) Only the paraxial region of the cavity, where the beam passes, will be considered. In this region the rf magnetic field is essentially zero and the electric field can be represented by a time-varying potential, $\phi(t)$. The representation of electromagnetic fields by potential functions is discussed further in Appendix B.

Approximations (ii) and (iii) rely on the transitions between magnets being very short compared to the synchrotron wavelength, which is generally the case. The term $\phi_{\text{guide field}}$ could contain electric bending and focusing forces, but this is not very usual and this term will be put to zero. The special case of a radial electric guide field will, however, be discussed later, but note that such a field would still not have an azimuthal component. Thus the terms in (12) are assigned as shown below,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{C}{2\pi}(p_\theta + eA_\theta)\right] = -e\frac{\partial}{\partial\Theta}(\ \phi \qquad - \qquad A.v).$$

⇓	⇓	⇓	(12)'
magnetic guide field	rf cavity	magnetic guide field	

With the approximations made concerning A_Θ and since v does not explicitly depend on Θ, this equation reduces to

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -e\frac{\partial\phi}{\partial\Theta} \,. \qquad (13)$$

On the left-hand side, S contains both actions of the guide field, i.e. deflection and betatron acceleration. The right-hand side contains only the action of the rf cavity and can be rewritten with the help of the active wave component of the electric field derived in Appendix C.

$$\frac{\mathrm{d}}{\mathrm{d}t}S = \frac{e\hat{u}}{2\pi}\cos(h\Theta - \int \omega dt) \,. \qquad (14)$$

Equation (14) describes the azimuthal motion of a single particle using the conjugate variables (Θ, S). In order to demonstrate that a beam will be focused it is necessary to show how a given particle behaves with respect to a reference particle called the <u>synchronous particle</u>*. This is achieved by applying (14) to an arbitrary particle and a synchronous particle and forming the difference equation to give,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta S = \frac{e\hat{u}}{2\pi}(\cos\theta - \cos\theta_0) \qquad (15)$$

where the argument of the cosine in (14) has been replaced by θ, so that

$$\theta = h\Theta - \int \omega \mathrm{d}t \,. \qquad (16)$$

The subscript 0 denotes the synchronous particle and Δ is used to denote the deviation of a variable from that of the synchronous particle. The term ΔS on the left-hand side of (15) can be expanded as,

$$\Delta S = S - S_0 = [(p_\Theta + eA_\Theta)C - (p_0 + eA_0)C_0]/(2\pi). \qquad (17)$$

The absolute value of $A_\Theta C$ has no meaning and it can be chosen to be zero on the synchronous orbit. Further, $A_\Theta C$ can be expanded in a series of ΔC in the median plane. The expression (17) is then linearised by retaining only the first-order terms to give,

$$\Delta S = \left\{1 + \left(1 + \frac{e}{p_0}\left[\frac{\partial(A_\Theta C)}{\partial C}\right]_0\right)\alpha\right\}\frac{C_0}{2\pi}\Delta p \qquad (18)$$

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* The synchronous particle is an ideal particle whose revolution frequency is synchronised with the rf and whose phase θ with respect to the rf is such that its energy gain matches the change in the guide field and its closed orbit therefore stays constant

where $\alpha = \frac{p_0}{C_0}\frac{\Delta C}{\Delta p}$, the momentum compaction. Since it has already been assumed that A_Θ is uniform, or close enough that average values can be used, and that the field is derived from A_Θ only,

$$\left[\frac{\partial(A_\Theta C)}{\partial C}\right]_0 = \frac{C_0 B_0}{2\pi} . \tag{19}$$

The substitution of (19) into (18) gives,

$$\Delta S = \left\{1 + \left(1 + \frac{eC_0B_0}{2\pi p_0}\right)\alpha\right\}\frac{C_0}{2\pi}\Delta p . \tag{20}$$

In Appendix D an expression is derived that links the Δp to $\Delta\dot{\theta}$ via the optical properties of the lattice.

$$\Delta p = \frac{m_0\gamma C_0}{2\pi h\eta}\frac{\mathrm{d}\Delta\theta}{\mathrm{d}t}, \tag{21}$$

where η is the fractional change in revolution frequency per unit of fractional change in momentum spread and h is the harmonic number The substitution of (21) into (20) yields,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta\theta = \frac{h\eta}{m_0\gamma}\left(\frac{2\pi}{C_0}\right)^2\left\{1 + \left(1 + \frac{eC_0B_0}{2\pi p_0}\right)\alpha\right\}^{-1}\Delta S . \tag{22}$$

Since $\Delta\theta$ differs from $\Delta\Theta$ by a constant, $\Delta\theta = h\Delta\Theta$ from (16), the variables $(\Delta\theta, \Delta S)$ will also preserve phase-space area and can be considered as canonically conjugate. Equations (15) and (22) are therefore the first-order canonically conjugate equations for the particle motion relative to the synchronous particle. They can be combined to give a single second-order equation describing the phase oscillations.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0\gamma}{h\eta}\left[\frac{C_0}{2\pi}\right]^2\left\{1 + \left(1 + \frac{eC_0B_0}{2\pi p_0}\right)\alpha\right\}\frac{\mathrm{d}}{\mathrm{d}t}\Delta\theta\right) = \frac{e\hat{u}}{2\pi}\left(\cos\theta - \cos\theta_0\right). \tag{23}$$

When the guide field is purely magnetic the cyclotron relation, $p = -eB_0\rho_0$ can be applied to give

$$-1 = \frac{eC_0B_0}{2\pi p_0} . \tag{24}$$

The substitution of (24) into (23) simplifies the equation to

Magnetic guide field
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0\gamma C_0}{2\pi h\eta}\frac{\mathrm{d}}{\mathrm{d}t}\Delta\theta\right) = \frac{e\hat{u}}{C_0}\left(\cos\theta - \cos\theta_0\right) , \tag{25}$$

which is the usual form of the phase equation. The expression for ΔS is also simplified for a magnetic guide field by the substitution of (24) into (20) to give,

Magnetic guide field
$$\Delta S = \frac{C_0}{2\pi}\Delta \mathrm{p} = R_0\Delta p . \tag{26}$$

Thus ΔS and Δp differ only by a constant in this case and $(\Delta\theta, \Delta p)$ and $(\Delta\Theta, \Delta p)$ like $(\Delta\theta, \Delta S)$ will conserve phase-space area and be conjugate. The choice of variables depends on the problem. For example, $(\Delta\Theta, \Delta p)$ is a convenient choice for the analysis of the onset of coherent instabilities in a coasting beam.

The reservation made above, that the guide field should be magnetic, may seem a little unnecessary, but suppose for a moment that the guiding force was provided by a radial electric field. This would not appear in the above azimuthal motion equations, but B_0 would be zero and the phase equation would become,

Electric guide field
$$\frac{\mathrm{d}}{\mathrm{d}t}\left((1+\alpha)\frac{m_0\gamma C_0}{2\pi h\eta}\frac{\mathrm{d}}{\mathrm{d}t}\Delta\theta\right)=\frac{e\hat{u}}{C_0}(\cos\theta-\cos\theta_0)\ . \tag{27}$$

This equation differs from (25) because the betatron acceleration force has been removed. The factor $(1+\alpha)$, between the two equations, is independent of time, so that the damping law will be the same whichever equation is used. However, the amplitudes and frequencies of the synchrotron oscillations are different in the two cases, although these differences are likely to be small, especially in large strong focusing machines (remember, $\alpha \cong Q_x^{-2}$).

Those readers interested in further details could try the papers in Refs. [11] and [12] and for a combined Hamiltonian treatment of the longitudinal and transverse motions Refs. [13] and [14].

4. CHOICE OF VARIABLES (Θ, W)

In much of the literature, the canonically conjugate variables (Θ, W) are used. The angle variable Θ has already been defined in (8) and the action variable W is defined [15] as,

$$W=\int_{E_0}^{E}\frac{\mathrm{d}E}{\Omega(E)}\ . \tag{28}$$

The well-known relativistic expression (29) for the total energy of the particle, provides the link to the variables used earlier.

$$E^2=c^2p^2+E_0^2\ . \tag{29}$$

When differentiated (29) becomes,

$$2E\frac{\mathrm{d}E}{\mathrm{d}p}=2c^2p$$

$$\frac{\mathrm{d}E}{\mathrm{d}p}=v\ . \tag{30}$$

The substitution of (30) into (28) gives the relationship between W and p, i.e.,

$$W=\int_{p_0}^{p}R(p)\mathrm{d}p\ , \tag{31}$$

which shows more clearly why W was called an action variable earlier (dimensions of angular momentum). The equivalent form of (31) for a small change in W about the central orbit is,

$$\Delta W=\Delta E/\Omega_0=R_0\,\Delta p\ . \tag{32}$$

Equation (32) yields the same result as (26) for ΔS in a machine with pure magnetic bending. Thus ΔW is equivalent to ΔS in this particular case.

5. CHOICE OF VARIABLES (τ, E)

Another set of canonically conjugate variables that can be used is time and energy and since this pair of variables ($\Delta\tau$, ΔE) appears frequently in the literature, it is worth giving the relations between this choice and ($\Delta\theta$, Δp):

$$\Delta\tau = \frac{C_0}{2\pi c h\sqrt{(1-\gamma^{-2})}}\Delta\theta \qquad = \frac{C_0}{2\pi c h \beta}\,\Delta\theta$$

$$\Delta E = \sqrt{(1-\gamma^{-2})}\;c\Delta p \qquad = \beta c\;\Delta p\;. \tag{33}$$

The variables ($\Delta\tau$, ΔE) are better adapted to the description of beam transfers between machines than ($\Delta\theta$, Δp).

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APPENDIX A

GENERALISED POTENTIAL

The generalised potential ($\phi - \boldsymbol{A}.\boldsymbol{v}$) has the somewhat unusual feature of a velocity-dependent term. In a scalar electric field $\boldsymbol{E}$, the force on a particle is found by applying the operator $-e\nabla$ to the scalar potential. In an analogous way, the Lorentz force on a charged particle in an electromagnetic field is found by applying the operator $-e[\nabla - d/dt(\partial/\partial \boldsymbol{v})]$ to the generalised potential. This is demonstrated below.

$$\boldsymbol{F} = e\left\{-\nabla U + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \boldsymbol{v}}\right)\right\} \qquad \text{where } U = \phi - \boldsymbol{A}.\boldsymbol{v}$$

$$\boldsymbol{F} = e\left\{-\nabla\phi + \nabla(\boldsymbol{A}.\boldsymbol{v}) + \frac{\mathrm{d}}{\mathrm{d}t}(-\boldsymbol{A})\right\} .$$

The operator $\frac{\mathrm{d}}{\mathrm{d}t}$ can be expanded to $\frac{\partial}{\partial t} + (\boldsymbol{v}.\nabla)$ so that,

$$\boldsymbol{F} = e\left\{-\nabla\phi - \frac{\partial}{\partial t}\boldsymbol{A} + \nabla(\boldsymbol{A}.\boldsymbol{v}) - (\boldsymbol{v}.\nabla)\boldsymbol{A}\right\} .$$

Since

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} ,$$

$$\boldsymbol{F} = e\{\boldsymbol{E} + \nabla(\boldsymbol{A}.\boldsymbol{v}) - (\boldsymbol{v}.\nabla)\boldsymbol{A}\} .$$

The triple vector product can be written as, $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{b}(\boldsymbol{c}.\boldsymbol{a}) - (\boldsymbol{a}.\boldsymbol{b})\boldsymbol{c}$, which when matched to the above equation gives

$$\boldsymbol{v} \times \nabla \times \boldsymbol{A} = \nabla(\boldsymbol{A}.\boldsymbol{v}) - (\boldsymbol{v}.\nabla)\boldsymbol{A}$$

so that

$$\boldsymbol{F} = e\{\boldsymbol{E} + \boldsymbol{v} \times \nabla \times \boldsymbol{A}\} = e\{\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\}$$

i.e., the Lorentz force on a moving charge.

APPENDIX B

REPRESENTATION OF ELECTROMAGNETIC FIELDS BY POTENTIAL FUNCTIONS

Electomagnetic fields are frequently represented by scalar and/or vector potential functions [16]. The most usual forms are:

$$\boldsymbol{B} = \nabla \mathrm{x} \boldsymbol{A}_0 \quad \text{and} \quad \boldsymbol{E} = -\nabla\phi_0 - \frac{\partial \boldsymbol{A}_0}{\partial t} .$$

Firstly, it should be noted that neither $\boldsymbol{A}_0$ nor ϕ_0 are unique. Any function of the form $\nabla\psi$ can be added to $\boldsymbol{A}_0$ without altering $\boldsymbol{B}$,

$$\nabla \mathrm{x}(\boldsymbol{A}_0 + \nabla\psi) = \nabla \mathrm{x} \boldsymbol{A}_0 + \nabla \mathrm{x} \nabla\psi = \nabla \mathrm{x} \boldsymbol{A}_0 = \boldsymbol{B} .$$

When the electric field is included, a new ϕ is needed to match the new $\boldsymbol{A}$ in order that $\boldsymbol{E}$ is unchanged.

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial}{\partial t}(\boldsymbol{A}_0 + \nabla\psi) = -\nabla(\phi + \frac{\partial\psi}{\partial t}) - \frac{\partial \boldsymbol{A}_0}{\partial t} .$$

Thus the relationships between the new variables [ϕ, $\boldsymbol{A}$] and the original [ϕ_0, $\boldsymbol{A}_0$] must be

$$\phi = \phi_0 - \frac{\partial\psi}{\partial t} \qquad \text{and} \qquad \boldsymbol{A} = \boldsymbol{A}_0 + \nabla\psi .$$

All transformations of this form will leave $\boldsymbol{E}$ and $\boldsymbol{B}$ unchanged. Secondly, the above formulation is not completely general. In current- and charge-free regions Maxwell's equations are symmetric between $\boldsymbol{B}$ and $\boldsymbol{E}$. Thus general solutions will be of the form,

$$\boldsymbol{B} = \nabla \mathrm{x} \boldsymbol{A} - \mu\frac{\partial \boldsymbol{A}^*}{\partial t} - \mu\nabla\phi^* \quad \text{and} \quad \boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} - \frac{1}{\varepsilon}\nabla \mathrm{x} \boldsymbol{A}^* .$$

$\boldsymbol{A}^*$ and ϕ^* are the potentials set up by sources outside the region of interest while $\boldsymbol{A}$ and ϕ are due to the sources inside the region of interest. The formulation used is a matter of convenience and the nature of the problem.

A simple example is the magnetic field in the current-free gap of an accelerator magnet. This can equally well be expressed using a scalar potential ϕ^*, or a vector potential $\boldsymbol{A}$. The former implies that the source of the field is totally external to the region of interest. A second example is the electric field on the axis of an rf cavity. If one considers the full volume of the cavity then Faraday's law says that the azimuthal magnetic field concentrated on the outer cavity wall will induce an electric field on the axis. The electric field can therefore be derived from $\boldsymbol{A}$ where $\boldsymbol{A}$ describes the rf magnetic field in the cavity. However, the field distribution in a cavity is such that the axis is virtually free of rf magnetic field. Hence if only the paraxial region is considered the electric field can be represented by a time-dependent scalar potential ϕ. This is the choice made for the simple accelerating gap. The time-variation of ϕ does imply currents and magnetic fields, but these are considered as totally external to the region of interest. Although it would be unnecessarily complicated, it would also be possible to consider say half the volume of the cavity. The electric field arising from the rf magnetic field in the region of interest would then be derived from $\boldsymbol{A}$ and that arising from the fields outside the volume of interest by ϕ.

APPENDIX C

ACTIVE COMPONENT OF THE FIELD

Consider an accelerating gap of length L_g with an applied accelerating voltage,

$$u(t) = \hat{u}\cos[\int \omega(t)\mathrm{d}t] \ .$$

The frequency ω is assumed to be quasi-constant, but it is written in integral form in order to account for the slow variations needed during the acceleration process. For convenience, the longitudinal field is expressed as a function of the azimuthal coordinate Θ $(= 2\pi s/C_0)$, so that

$$E(t) = u(t)/L_g \qquad \text{for} \qquad |\Theta| \leq \pi L_g / C_0$$

$$E(t) = 0 \qquad \text{for} \qquad (\pi L_g / C_0) \leq |\Theta| \leq \pi \ .$$

This field has a spatial periodicity of 2π in Θ and can be Fourier analysed with the result,

$$E(t) = \frac{\hat{u}}{C}\left[\cos(\int \omega \mathrm{d}t) + \sum_{n=1}^{\infty}[\cos(\int \omega \mathrm{d}t + n\Theta) + \cos(\int \omega \mathrm{d}t - n\Theta)]\right]$$

where C is the machine circumference. This equation comprises two sets of counter-rotating waves. All the wave components act as a.c. fields on the particles (with zero average effect) except the one that satisfies the condition,

$$\int \omega \mathrm{d}t - h\Theta = \text{constant}, \qquad \text{or} \qquad h\frac{\mathrm{d}}{\mathrm{d}t}\Theta = \omega \qquad \text{written as} \qquad h\Omega_0 = \omega$$

where Ω_0 is the angular frequency of a particle with velocity v_0 running on a closed orbit of circumference C_0 according to the relationship,

$$\Omega_0 = 2\pi v_0 / C_0 \ .$$

Such a particle is called an equilibrium particle or synchronous particle. It is assumed that the guide field is increased to match the energy gain of the equilibrium particle from the rf gap. In this way, the synchronous particle's closed orbit will remain constant. Thus the only component of interest for analysing the longitudinal motion is

$$E = \frac{\hat{u}}{C}\cos\left(h\Theta - \int \omega \mathrm{d}t\right) \quad \textbf{Active wave component}$$

where h represents the number of rf cycles per particle revolution and is called the harmonic number. It should be noted that although it was convenient to assume a short single gap for the acceleration, it is immaterial for the analysis how the active wave component is set up. There could be many gaps or even travelling wave structures.

APPENDIX D

MOMENTUM DISPERSION OF REVOLUTION FREQUENCY

It is important to establish the relationship between the deviations of the particle's revolution frequency and momentum with respect to the synchronous particle in terms of the lattice properties. The revolution frequency is given by $\Omega = 2\pi v/C$, which yields by logarithmic differentiation,

$$\Delta\Omega / \Omega_0 = \Delta v / v_0 - \Delta C / C_0 \ .$$

Simple relativity theory gives,

$$\Delta v / v_0 = \gamma^{-2} \Delta p / p_0 \ ,$$

where $\gamma = m/m_0$. From the definition of the momentum compaction

$$\Delta C / C_0 = \alpha \ \Delta p / p_0 \ .$$

The combination of the above yields

$$\Delta\Omega / \Omega_0 = \left(\gamma^{-2} - \alpha\right) \ \Delta p / p_0 \ .$$

This is frequently rewritten as,

$$\eta = \frac{\Delta\Omega}{\Omega} / \frac{\Delta p}{p_0} = \left(\gamma^{-2} - \alpha\right)$$

where η is the fractional change in revolution frequency per unit of fractional change in momentum spread. The above expression is now rearranged to give

$$\Delta p = p_0 \eta^{-1} \frac{\Delta\Omega}{\Omega} \ .$$

It follows from (16) that $\Delta\Omega$ can be replaced by $\Delta\dot{\theta}/h$ and if ω is constant, or slowly varying compared to the particle oscillations about the synchronous particle, Ω_0 can be replaced by ω/h, so that

$$\Delta p = p_0 \eta^{-1} \Delta\dot{\theta} / \omega \ .$$

The particle momentum is given by,

$$p_0 = m v_0 = m C_0 \Omega_0 / (2\pi) = m_0 \gamma C_0 \omega / (2\pi h) \ .$$

The replacement of p_0 in the above yields,

$$\Delta p = \frac{m_0 \gamma C_0}{2\pi h \eta} \Delta\dot{\theta} \ .$$