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Hidden geometrical structures in integrable models[1]

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The bootstrap equations for the ADE series of purely elastic scattering theories have turned out to be intimately connected with the geometry of root systems and the Coxeter element. An informal review of some of this material is given, mentioning also a couple of other contexts – the Pasquier models, and the simply-laced affine Toda field theories – where similar structures are encountered. The relevance of twisted Coxeter elements is indicated, and a construction of these elements inspired by the twisted foldings of the affine Toda models is described.

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1. An example

To provide some motivation, this first section is devoted to the scaling region of the 2D Ising model in a magnetic field. However, the physical relevance of the discussion will not be immediately apparent.

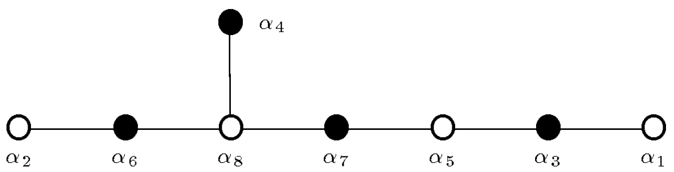
So, the starting point is a certain set Φ of 240 vectors ('roots') in eight-dimensional space, each having length-squared 2, known as the E_8 root system. For each $\alpha \in \Phi$, there is a reflection r_α in the 7-dimensional hyperplane orthogonal to α, and a key feature of Φ is that these reflections generate a *finite* group, W say. In symbols,

$$\begin{aligned} r_\alpha(x) &= x - 2\frac{\alpha.x}{\alpha^2}\alpha\,; \\ |\langle\{r_\alpha\}_{\alpha\in\Phi}\rangle| &= |W| < \infty\,. \end{aligned} \tag{1.1}$$

This group is none other than the Weyl group of the Lie algebra E_8, more abstractly defined inside the group of inner automorphisms of the algebra as the quotient of the normaliser of a Cartan subalgebra (those inner automorphisms that map the subalgebra into itself) by the centraliser (those that leave it pointwise fixed). But for now this fact is best forgotten, as the geometrical characterisation just given is going to be the relevant one. Two further properties of W will be needed: first, W leaves the set Φ invariant, and second, W can be generated by the reflections for a subset $\Delta \subset \Phi$ of just eight so-called simple roots. Actually, there are 240 such subsets, but they are all equivalent in the sense of being conjugate under W, so we can focus on one, and label its elements $\alpha_1, \alpha_2, \ldots, \alpha_8$, and the corresponding reflections $r_1, r_2, \ldots, r_8$. All of W is encoded in the mutual inner products of these eight vectors, given equivalently by the Cartan matrix:

$$C_{ab} = 2\frac{\alpha_a.\alpha_b}{\alpha_b^2} \tag{1.2}$$

or the Dynkin diagram:



(1.3)

If two simple roots are joined by a single line, then they have inner product -1; otherwise, it is 0. The only unusual feature of (1.3) is the two-colouring: to maintain an element of suspense, this will not be explained immediately, but rather used to define a particular element w of the Weyl group, as follows:

$$w = r_3 r_4 r_6 r_7 r_1 r_2 r_5 r_8\,. \tag{1.4}$$

Those who have encountered these things before might recognize w as a Coxeter element, in a Steinberg ordering – that is, a product of all the generating reflections, in an ordering such that the 'white' reflections act first, followed by the 'black' ones. As an – at the moment completely unmotivated – exercise, consider computing the repeated action of w^{-1} on, say, the simple root α_1. Each r_a squares to the identity, so $w^{-1} = r_8r_5r_2r_1r_7r_6r_4r_3$. By (1.1) and (1.2), the action of a simple reflection r_a on any simple root α_b is just $r_a\alpha_b = \alpha_b - C_{ba}\alpha_a$. In terms of the Dynkin diagram, r_a negates α_a, adds α_a to each α_b joined to α_a by a link, and leaves all of the other simple roots unchanged. So, working in from the right of w^{-1} , r_3 sends α_1 to $\alpha_1{+}\alpha_3$, and then r_4, r_6 and r_7 all leave these two roots alone. Then comes r_1, which negates α_1 while at the same time transforming α_3 into $\alpha_3{+}\alpha_1$, so that the total is now just α_3. Next is r_2, which does nothing, and then r_5 on α_3 produces $\alpha_3{+}\alpha_5$. These last two roots are both orthogonal to α_8, and hence are unchanged by the final action of r_8. Thus

$$w^{-1}\alpha_1 = \alpha_3 + \alpha_5 \,. \tag{1.5}$$

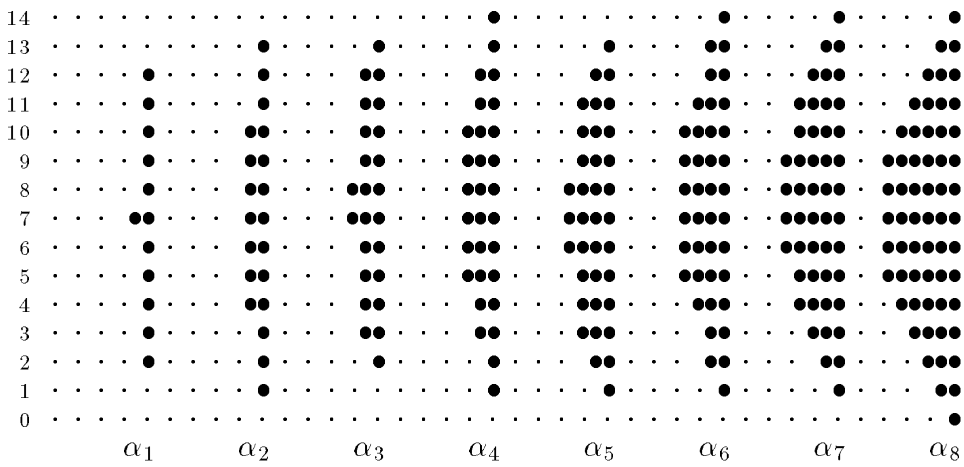
Nothing apart from growing tedium prevents us from carrying on further, and calculating $w^{-2}\alpha_1$, $w^{-3}\alpha_1$ and so on. The sequence must ultimately repeat, since $|\Phi|$ is finite; in fact, α_1 appears on its own again after 30 steps. The first 14 of these steps are shown in the following table, where the multiplicity of α_a in $w^{-p}\alpha_1$ is given by the number of blobs (•) that appear in the $a^{\rm th}$ position of the $p^{\rm th}$ row.

Images of α_1 under w^{-1} : (1.6)

14	•		•					
13					•		•	
12				•		•	•	•
11		•				•	•	•
10			•	•	•		•	•
9	•		•		•	•	•	•
8		•		•	•	•	•	•
7				•		•	•	••
6		•	•		•	•	•	•
5	•		•	•	•		•	•
4					•	•	•	•
3		•		•		•	•	•
2							•	•
1			•		•			
0	•							
	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8

This was a deceptively easy case – the orbits become much more complicated, although they always contain exactly 30 elements. Worst of all is that for α_8:

Images of α_8 :



(1.7)

To give an example, $w^{-7}\alpha_8 = 2\alpha_1+2\alpha_2+3\alpha_3+3\alpha_4+4\alpha_5+4\alpha_6+5\alpha_7+6\alpha_8$. This root is made from a total of $2+2+3+3+4+4+5+6 = 29$ simple roots, which is equivalent to saying that its 'height' is 29. In fact this is the largest possible height in E_8, and furthermore it occurs just once for a given choice of simple roots. Hence $w^{-7}\alpha_8$ should be equal to ψ, the highest root of E_8, and entering its coefficients on the Dynkin diagram (1.3) to see that this is indeed the case provides a quick check on the calculation. One last point before moving on: in compiling these two tables, there was no need to invent a notation for the negative of a simple root – an 'antiblob', perhaps – as all the roots from 0 to 14 were *positive*-linear combinations of the simple roots. This will be relevant later.

But first, Zamoldchikov's ideas about the scaling region of the Ising model in a magnetic field. At criticality, the continuum limit of the model is well-known to be described by a $c=\frac{1}{2}$ conformal field theory [1]; Zamolodchikov's proposal [2] was to probe the neighbourhood of this point via a study of actions

$$S_{pert} = S_{CFT} + \lambda\int d^2x\,\phi(x)\,, \tag{1.8}$$

where S_{CFT} is a notional action for the conformal theory, inside of which ϕ is some (relevant, spinless) field, the coupling constant λ ensuring that all the dimensions match up. There are just two possibilities for this simplest form of S_{pert} in the case of the Ising model, one for each of the two spinless relevant fields in the $c=\frac{1}{2}$ operator algebra. These are usually labelled σ and ϵ, and have dimensions $(\frac{1}{16},\frac{1}{16})$ and $(\frac{1}{2},\frac{1}{2})$ respectively. From their identifications with the scaling limits of the local magnetisations (spins) and energy densities on the lattice, perturbing by σ corresponds to switching on a magnetic field, and perturbing by ϵ to changing the temperature T away from its critical value T_c.

Zamolodchikov was able to extract much non-perturbative information about the theories defined through (1.8), essentially because the perturbative expansions for certain quantities of interest truncate after finitely many terms. In particular, his 'counting argument' led to the conclusion that both the ϵ and σ perturbations of the Ising model preserve certain higher-spin conserved charges, and hence are integrable. For the σ perturbation, these spins are the numbers $1, 7, 11, 13, 17, 19, 23, 29$, repeated modulo 30.

The next stage is to study the long-distance behaviour of the theory described by S_{pert}. In the ultraviolet, the theory is well-approximated by the original conformal theory; in the infrared, the model might be massive, or alternatively it might undergo a crossover to another conformal field theory. However, in the latter case the central charge of the infrared model is constrained by Zamolodchikov's c-theorem [3] to be less than that of the original conformal theory. Here, c started at $\frac{1}{2}$, below which there is no unitary central charge, so this possibility is ruled out. Hence the infrared limit is massive, something that should in any case have been expected from knowledge of the phase diagram of the lattice model. Now after a Wick rotation, this massive model can also be examined in Minkowski space, where it should have an S-matrix. Better than that, since the perturbation was integrable, the S-matrix should be factorizable [4], and perhaps even findable...

Easiest of all would be for the S-matrix to be diagonal – the theory would then consist of a collection of scalar particles, perhaps with different masses, which never mix under scattering. In this way, the joys/sorrows of the Yang-Baxter equation would be avoided, and the $2\to 2$ S-matrix would boil down to a collection of meromorphic functions, one for each pair of particle types in the model. The critical Ising model being the simplest unitary conformal field theory, one might expect its integrable deformations to have S-matrices of this simplest possible form. Indeed, at $T\neq T_c$ (the ϵ perturbation) the model is a theory of free massive fermions, which certainly have a diagonal S-matrix. Perturbing by σ, things are not quite so simple, but in fact the S-matrix is again diagonal. A sequence of ingenious arguments, combining general principles with certain physical inputs specific to the Ising model, led Zamolodchikov to propose the following expression for the S-matrix element for the scattering of two of the lightest particles, as a function of their relative rapidity θ:

$$S_{11} = -(2)(10)(12)(18)(20)(28)\,, \tag{1.9}$$

where an abbreviated notation has been used:

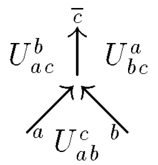
$$(x) \equiv \frac{\sinh\left(\frac{\theta}{2} + \frac{i\pi x}{60}\right)}{\sinh\left(\frac{\theta}{2} - \frac{i\pi x}{60}\right)}\ . \tag{1.10}$$

Note, (x) has a pole at $i\pi x/30$, so (1.9) exhibits the poles in $S_{11}(\theta)$ between 0 and $i\pi$.

How to proceed from here? The key notion is that of a *bootstrap equation* [5,2]. Assume that an S-matrix element $S_{ab}(\theta)$ has a simple pole at $\theta{=}iU^c_{ab}$, say, with residue a positive-real multiple of i. Transforming back to the Mandlestam variable $s=(p_a{+}p_b)^2$ reveals a positive-residue pole below threshold, which should still correspond to the formation of a (forward channel) bound state. If this is assigned the charge-conjugated label $\overline{c}$, then its presence as a bound state in a b scattering corresponds to the non-vanishing of the three-point coupling C^{abc}. Kinematic considerations (the conservation of momentum and the fact that a, b and c should all be on-shell at the pole) then give a relation between the particle masses and the 'fusing angle' U^a_{bc}:

$$m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos U^c_{ab}\,. \tag{1.11}$$

That the process happens below threshold requires $m_c < m_a{+}m_b$, consistent with U^c_{ab} being real. Hence all relative momenta are euclidean, and the process can be drawn in the plane:



(1.12)

Rotating the diagram by $\pm 2\pi/3$ gives pictures for a c and b c scattering, and the corresponding fusing angles have also been marked in; the triplet of angles satisfies

$$U^a_{bc} + U^b_{ac} + U^c_{ab} = 2\pi\,. \tag{1.13}$$

Now imagine that (1.12) is just part of a larger diagram, involving at least one further particle d. Depending on its impact parameter, the world-line of particle d will either cross those of a and b, or else that of c, corresponding to the interaction with d happening either before or after the fusing of a with b. (The idea is that the momenta of all particles have been slightly 'smudged' about the values given, so that a description in terms of localised wavepackets propagating freely in between interactions is valid.) Now our system is integrable, which among other things means that the values of impact parameters should be irrelevant – and so, irrespective of what happens elsewhere in the larger diagram, we should certainly equate the two possible contributions to the part of the amplitude from particle d. This translates into a bootstrap relation between S-matrix elements, expected to hold whenever a positive-residue simple pole in $S_{ab}(\theta)$ has led us to deduce the non-vanishing of the three-point coupling C^{abc}:

$$S_{d\overline{c}}(\theta) = S_{da}(\theta{-}\overline{U}^b_{ac})S_{db}(\theta{+}\overline{U}^a_{bc})\,, \tag{1.14}$$

where $\overline{U}=\pi-U$. (To see why, shift the diagram (1.12) to the frame for which the rapidity of particle c is zero, whereupon the rapidities of a and b become $-\overline{U}^{b}_{ac}$ and $\overline{U}^{a}_{bc}$ respectively. Then imagine d to traverse the picture either above or below the fusing, and equate the results.) This 'derivation' is in the spirit of a discussion of the Yang-Baxter equation given by Shankar and Witten [6], compared to which there are (at least) two further dodgy points: first, some rapidities are necessarily unphysical, and second, the way in which particles a and b fuse to form c has been left imprecise. The second objection is potentially the more serious, especially since there do appear to be situations – for example, breather-breather scattering below the two-breather threshold in the sine-Gordon model [7], or certain amplitudes in the non simply-laced affine Toda theories [8] – where the competing presence of a number of anomalous threshold poles at the same point may prevent the direct identification of a simple pole in the S-matrix with a single bound state. The bootstrap equation (1.14) is therefore best taken as a 'working axiom', to be re-examined in the event that it contradicts other information. For the case in hand, there don't seem to be any problems.

The S-matrix element (1.9) has forward-channel poles from the blocks (2), (12) and (20), with fusing angles of $\pi/15$, $2\pi/5$ and $2\pi/3$ respectively. Via (1.11), these correspond to bound state masses $m_3{=}m_1\sin 2\pi/30$, $m_2{=}2m_1\sin 3\pi/10$ and m_1, where m_1 is the common mass of the two incident particles. The simplest possibility for the third of these is just another copy of the (then self-conjugate) particle 1, implying $C^{111}{\neq}0$. Using this 'ϕ^3' property in the bootstrap equation then implies a rather strong constraint on $S_{11}(\theta)$:

$$S_{11}(\theta)=S_{11}(\theta{-}i\pi/3)S_{11}(\theta{+}i\pi/3)\,. \tag{1.15}$$

It is instructive to check the intricate cancellations which ensure that the expression (1.9) passes this test. But to get something new, the other bound state poles should be exploited, which have been provisionally associated with the second and third lightest particles in the model. For the $1\;1\to 2$ fusing, the bootstrap equation predicts

$$\begin{aligned}S_{12}(\theta)&=S_{11}(\theta{-}i\pi/5)S_{11}(\theta{+}i\pi/5)\\&=(6)(8)(12)(14)(16)(18)(22)(24)\,,\end{aligned} \tag{1.16}$$

while $1\;1\to 3$ yields

$$\begin{aligned}S_{13}(\theta)&=S_{11}(\theta{-}i\pi/30)S_{11}(\theta{+}i\pi/30)\\&=(1)(3)(9)(11)^2(13)(17)(19)^2(21)(27)(29)\,.\end{aligned} \tag{1.17}$$

The forward channel poles are proliferating: between them, S_{12} and S_{13} have eight, five of which are explicable in terms of bound states of types 1, 2 and 3, while the remaining three require the introduction of two new species, call them 4 and 5. The poles also imply the non-vanishing of further three-point couplings: C^{122}, C^{123} and C^{124} from S_{12}, and C^{134} and C^{135} from S_{13}. This information can be fed back into (1.14) to give more bootstrap equations, which on the one hand provide further tests for the existing set of S-matrix elements, and on the other add to this set by predicting new ones.

Now iterate! The procedure just outlined is sufficiently well-defined that there is nothing (apart, again, from growing tedium) to prevent its continuation, resulting in an increasing collection of particles, S-matrix elements and non-vanishing three-point couplings, all bound together by the bootstrap equations (1.14). But it is worth persevering: as Zamolodchikov [2] discovered, quite remarkably the process closes in on itself after a total of eight particle types have been encountered. Tables of the 36 meromorphic functions which form their mutual two-particle S-matrix elements (recall, $S_{ab}{=}S_{ba}$), and the corresponding non-vanishing three-point couplings, were given in refs. [9,10], and it is a finite, though lengthy, task to verify that the system is closed: all forward-channel poles are satisfactorily explained, and all the corresponding bootstrap equations are obeyed.

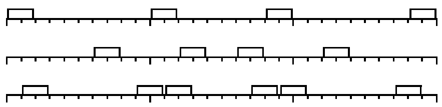
However, all this turns out to be unnecessary [11]. The iteration of the bootstrap equations for the spin-perturbed Ising model is simply a more complicated way of doing the Weyl group computation presented in the first half of this section, while the fact that these equations are implied by the pole structure of the very functions that they constrain, when viewed from this perspective, follows from simple properties of the E_8 root system. Some sort of connection was perhaps to have been expected, given that (a) the $c{=}\frac{1}{2}$ conformal field theory can be realised as the coset model $E_8^{(1)}{\times}E_8^{(1)}/E_8^{(2)}$, within which the field σ corresponds to the branching (id, id, adj); (b) the spins of the conserved charges given earlier are precisely the exponents of E_8, repeated modulo the Coxeter number; and (c) the masses $m_1, m_2, \ldots, m_8$ can be formed into the Perron-Frobenius eigenvector for the incidence matrix of the E_8 Dynkin diagram [12,9]. Nevertheless, the way it works is quite striking, and this introduction finishes with a visual demonstration of the result.

The key is to rewrite the S-matrix elements in a slightly different way. Notice first that in all three S-matrix elements given above, each block (x) with x not equal to 2 or 28 can be paired off with another block, either $(x{-}2)$ or $(x{+}2)$. Even the blocks (2) and (28) in $S_{11}(\theta)$ can be made to obey this rule, if they are paired formally with the 'dummy' blocks $(0){\equiv}1$ and $(30){\equiv}-1$. Since this feature persists for all of the other S-matrix elements, it is possible to economise on the previous formulae by introducing a new, larger, building block [10]:

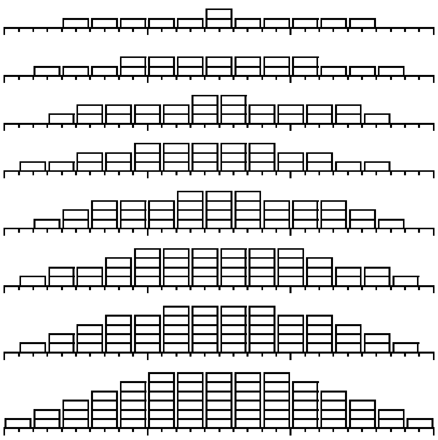
$$\{x\} = (x{-}1)(x{+}1)\,. \tag{1.18}$$

The earlier expressions become:

$$\begin{array}{lcll} S_{11} & = & \{1\}\{11\}\{19\}\{29\} & = \\ S_{12} & = & \{7\}\{13\}\{17\}\{23\} & = \\ S_{13} & = & \{2\}\{10\}\{12\}\{18\}\{20\}\{28\} & = \end{array} \tag{1.19}$$



The 'brick wall' notation for S-matrix elements, used here and in ref. [13], represents each factor $\{x\}$ in a product by a brick, centred at x. The connection with the Weyl group data compiled earlier can be seen by taking these three pieces of wall, rotating them by 90°, and comparing them with the first three columns of (1.6). To be sure that this isn't a coincidence, the scattering amplitudes of the heaviest particle, 8, can also be checked:



(1.20)

The functions represented here have become quite complicated – for example,

$$S_{88} = \{1\}\{3\}^2\{5\}^3\{7\}^4\{9\}^5\{11\}^6\{13\}^6\{15\}^6\{17\}^6\{19\}^6\{21\}^5\{23\}^4\{25\}^3\{27\}^2\{29\} \tag{1.21}$$

Nevertheless, every feature is perfectly reproduced by the Coxeter orbit of α_8, as listed in table (1.7). This is part of the 'hidden geometrical structure' advertised in the title, and the next section will outline some of the uses to which it can be put.

2. More details

The picture outlined above is not special to E_8: in fact, it is found for any simply-laced Lie algebra $g = A$, D or E. To explain the general formalism, some more notation is needed: this is the task of the first part of this section. For more on the mathematical background, see refs. [14–17].

Most of the definitions given above can be carried straight over, on replacing 8 by r, the rank of g, and 30 by h, the Coxeter number. However, although the two Coxeter orbits exhibited above were both based on simple roots, this is not always possible, nor is it the most convenient choice. It seems best to follow the treatment of Kostant [15]: if the chosen Coxeter element is

$$w = r_1 r_2 \ldots r_r \,, \tag{2.1}$$

then, for $a = 1 \ldots r$, define

$$\phi_a = r_r r_{r-1} \ldots r_{a+1} \alpha_a \,, \tag{2.2}$$

the simple reflections acting in reverse order on the simple root α_a. A useful property of these roots is their relation to the fundamental weights:

$$\phi_a = (1 - w^{-1})\lambda_a \,. \tag{2.3}$$

(This can be checked using the duality between the weights λ_a and the simple co-roots $\alpha_a^\vee \equiv (2/\alpha_a^2)\alpha_a$, which implies, via (1.1), that $r_a\lambda_b = \lambda_b - \delta_{ab}\alpha_b$.) The eigenvalues of w are $\exp(2\pi i s/h)$, where s runs over the exponents of g; therefore, 1 is not an eigenvalue, and so $R \equiv (1-w^{-1})^{-1}$ is well-defined. Now the highest weights of different representations lie in different Weyl-orbits, and in particular the fundamental weights lie in distinct orbits of the Coxeter element w. Translated via the action of R into a statement about roots, this implies that the ϕ_a all lie in different orbits of W. Hence, defining $\Gamma_a = \{w^p\phi_a\}_{p=0}^{h-1}$ to be the orbit of ϕ_a under w, all of these orbits are distinct. Inside Φ, all orbits of a Coxeter element have exactly h elements (note, the same statement is not always true for weights, though via (2.3) it does hold for the λ_a). Thus the union of the Γ_a contains $r.h$ roots, which is in fact the full set (recall for E_8 we had $|\Phi|$=240=8.30), and all of the orbits of w have been captured in this way. Furthermore, the ϕ_a have a distinguished position in the orbits: they are the (unique) positive roots which become negative under the action of w [15]. (Recall that the positive roots Φ^+ are the positive-linear combinations of the simple roots, the remainder of Φ being Φ^-, the negative-linear combinations: $\Phi = \Phi^+ \cup \Phi^-$.)

It will also be convenient to assume that w is written in the 'Steinberg ordering' [14] mentioned earlier. If the Dynkin diagram of g has been two-coloured as in (1.3), then this requires that

$$w = \prod_{\bullet'} r_{\bullet'} \prod_{\circ'} r_{\circ'} = w_{\{\bullet\}} w_{\{\circ\}} \,, \tag{2.4}$$

where the symbol $\bullet'$ stands for an arbitrary black, and $\circ'$ an arbitrary white, simple root. (We could swap black and white; this would just send w to its inverse.) Note that since roots of like colour are orthogonal, their reflections commute and as a consequence $w_{\{\bullet\}}^2 = w_{\{\circ\}}^2 = 1$, and w has been written as a product of two involutions. This looks to be a rather special choice, but in fact it isn't: Carter [18] has shown that *any* element of the Weyl group can be written as the product of two involutions as in (2.4), and that the Coxeter elements are exactly those for which the inner products of the roots defining the two involutions have mutual inner products given by the Dynkin diagram of the algebra. Thus rather than thinking that a particular choice of Coxeter element has been made, it could equally be said that the set of simple roots has been changed at this point, to one better-suited to current needs. In any event, with (2.4) in place, an integer $u(\alpha,\beta)$ can be defined modulo $2h$ for each pair of roots $\alpha,\beta \in \Phi$:

$$\begin{aligned} u(\alpha,\beta) &= -u(\beta,\alpha) & \quad u(w\alpha,\beta) &= u(\alpha,\beta)+2 \\ u(\phi_\bullet,\phi_{\bullet'}) &= u(\phi_\circ,\phi_{\circ'}) = 0 & \quad u(\phi_\circ,\phi_\bullet) &= 1. \end{aligned} \tag{2.5}$$

As explained in [13], this definition has the geometrical meaning that $\pi u(\alpha,\beta)/h$ is the signed angle between the projections of α and β into the $\exp(2\pi i/h)$ eigenspace of w – in terms of which, it has a sense independent of the Steinberg ordering (2.4).

At last, enough notation is in place to give the formula which encodes the observations of the last section:

$$S_{ab} = \prod_{\beta\in\Gamma_b^+} \{1+u(\phi_a,\beta)\}^{(\lambda_a,\beta)}, \tag{2.6}$$

where $\Gamma_b^+ = \Gamma_b \cap \Phi^+$ is the positive part of the w-orbit Γ_b. To see that that this really does come to the same thing as before, note first that for w given by (2.4), the definition (2.2) reduces to

$$\phi_\bullet = w_{\{\circ\}}\alpha_\bullet\,, \qquad \phi_\circ = \alpha_\circ\,, \tag{2.7}$$

and that both α_1 and α_8 were coloured white on (1.3). Also, the weights λ_a are dual to the simple roots α_b in the simply-laced cases, so that the exponent in (2.6) is simply counting the number of times that α_a appears in the expansion of β in simple roots. That all the roots shown in (1.6) and (1.7) were positive-linear combinations of simple roots corresponds to the root β running over only positive roots in (2.6). Finally, the colour-dependent difference of ±1 in the value of $u(\phi_a,\phi_b)$ implied by the last line of (2.5) exactly accounts for the small shifts in some of the brick walls in (1.19) and (1.20) as compared to the orbit tables (1.6) and (1.7) – for example, the fact that S_{13} in (1.19) is symmetrically placed relative to S_{12}, even though the second and third columns of (1.6) did not line up quite so nicely. Physically, this is necessary to ensure that (2.6) has the correct crossing symmetry, and that the poles turn up in the correct places.

Assuming that (2.6) is the correct general formula, the first task should be to examine its analytic structure. The following basically follows [13]; an alternative discussion can be found in [19]. As β runs through Γ_b, $1+u(\phi_a,\beta)$ remains between 0 and h, a fact which can be seen geometrically [20], or alternatively traced back to ϕ_b being the only positive root in Γ_b to become negative on the action of w, together with certain symmetries of the orbit between positive and negative roots [13,19]. From the definitions (1.10), (1.18) of the blocks $\{x\}$, S_{ab} might therefore have poles at any of the points $-i\pi/h \leq i\pi u(\phi_a,\beta)/h \leq i\pi + i\pi/h$, but it is possible to show (see, for example, sect. 3 of [20]) that any blocks which could contribute poles at the extremal locations, outside the physical strip $0 \leq \mathrm{Im}\theta \leq \pi$, are necessarily raised to the power zero in (2.6). Hence in looking for poles in S_{ab}, attention can be restricted to points $i\pi u(\phi_a,\beta)/h$ inside the physical strip. Remembering from (2.5) that $u(\phi_a, w\beta) = u(\phi_a,\beta) - 2$, it follows from (1.18) that a such a pole can only receive direct contributions from the blocks associated with β and $w\beta$, and that all other blocks only multiply this by a positive real amount, which can be ignored for bootstrap purposes. That is, near $i\pi u(\phi_a,\beta)/h \equiv \theta_0$ the relevant part of (2.6) is

$$\begin{aligned} S_{ab} &= \ldots \{u(\phi_a,\beta)-1\}^{(\lambda_a,w\beta)}\{u(\phi_a,\beta)+1\}^{(\lambda_a,\beta)}\ldots \\ &\approx \ldots \left(\frac{i}{\theta-\theta_0}\right)^{(\lambda_a,w\beta)}\left(\frac{-i}{\theta-\theta_0}\right)^{(\lambda_a,\beta)}\ldots \end{aligned} \tag{2.8}$$

where in the second line the dominant contributions at θ_0 have been extracted using (1.18). Therefore the residue is a positive-real multiple of

$$i^{(\lambda_a,w\beta)-(\lambda_a,\beta)} = i^{-((1-w^{-1})\lambda_a,\beta)} = i^{-(\phi_a,\beta)}, \tag{2.9}$$

where (2.3) was used in the second equality. In terms of the pictures (1.19) and (1.20), a $+i$ residue for the leading singularity occurs whenever there is a downhill step in the wall of bricks, reading from left to right. Empirically, this rule is also correct for assigning forward/crossed channels to the odd-order poles of higher orders, and the treatment in terms of root systems does not notice the distinction. The way in which this works in perturbation theory is however quite complicated [21]. But ignoring these subtleties, and continuing to focus on the forward-channel poles (the others can be treated in a similar way), there is only one way that a residue of $+i$ can emerge from (2.9): we must have

$$(\phi_a,\beta) = -1\,. \tag{2.10}$$

In turn, this holds if and only if $\phi_a+\beta = -\gamma$, say, is another root. For the 'if', note that $\phi_a+\beta = -\gamma \in \Phi$ implies $2 = \gamma^2 = \phi_a^2+2(\phi_a,\beta)+\beta^2$, and hence $(\phi_a,\beta) = -1$, all simply-laced roots having length-squared 2. Conversely, if (2.10) holds then, from (1.1), $\phi_a+\beta = r_\beta\phi_a$, and is therefore a root by the closure of Φ under W. The situation in $\mathbf{R}^r$ can be drawn as follows:

$$\begin{tikzcd}[column sep=small] & -\gamma=\phi_a+\beta & \\ & {} \arrow[u] & \\ \phi_a \arrow[ur] & & \beta \arrow[ul] \end{tikzcd} \tag{2.11}$$

This picture is reminiscent of (1.12), and with good reason [11]: projecting down from $\mathbf{R}^r$ onto the two-dimensional eigenspace of w for the eigenvalue $\exp(2\pi i/h)$, the relative angles become exactly the fusing angles, and the line-lengths exactly the masses, for a fusing of two particles of types a and b to form a bound state of type $\bar{c}$, if c is the label for the orbit containing γ (so, for example, $U_{ab}^c = \pi u(\phi_a,\beta)/h$). Thus, $C^{abc} \neq 0$ should be deduced in any situation where $\beta \in \Gamma_b^+$, $\gamma \in \Gamma_c$ can be found such that $\phi_a+\beta+\gamma = 0$. The picture emerging is that each particle type should be associated with an orbit of the

Coxeter element, and it turns out that the antiparticle is associated with the negative orbit: $\Gamma_{\overline{a}} = -\Gamma_a = w_{\{\bullet\}}\Gamma_a = w_{\{\circ\}}\Gamma_a$. These three different ways of conjugating the charge allows an element $\widetilde{w} = -w_{\{\bullet\}}$ to be defined, which leaves whole orbits unchanged while mixing around their elements in such a way that when it acts on the three roots in (2.11) simultaneously, the orientation of the projected 'momentum picture', (1.12), is reversed – it implements parity. But also, it is possible to show (see [13]) that *all* triplets ($\alpha \in \Gamma_a, \beta \in \Gamma_b, \gamma \in \Gamma_c$) of roots satisfying $\alpha+\beta+\gamma = 0$ are conjugate to each other under the combined action of w and $\widetilde{w}$ – there are in fact $2h$ of them, of which ($\phi_a, \beta \in \Gamma_b^+, \gamma \in \Gamma_c$) was but one example. Hence, and by a somewhat tortuous route, to the fusing rule for the non-vanishing of three-point couplings in the ADE-related purely elastic scattering theories:

$$\boxed{\begin{array}{rl} C^{abc} \neq 0 & \text{iff } \exists \text{ roots } \alpha \in \Gamma_a, \beta \in \Gamma_b, \gamma \in \Gamma_c \text{ with } \quad \alpha+\beta+\gamma = 0, \\ ie & \text{iff } \ 0 \in \Gamma_a + \Gamma_b + \Gamma_c. \end{array}} \tag{2.12}$$

Given this rule, it isn't too hard to prove that (2.6) satisfies (1.14), and by considering projections of the root triangles onto the other eigenspaces of w, consistent solutions to the bootstrap equations for the conserved charges [2] can be constructed for spins equal, modulo the Coxeter number, to an exponent; for more details of all this, and also of how to establish the various other properties expected of (2.6), see refs. [11,13,19].

3. Other examples, and twisted Coxeter elements

This section gives very brief mention to two other situations where very similar machinery is encountered. The first of these is a calculation by Saleur and Bauer [22] of the partition functions of the Pasquier models [23] on a cylinder. They found that if the heights were constrained to be equal to a and b (two nodes on the relevant Dynkin diagram) at the two ends of the cylinder, then in the continuum limit the partition function could be expanded in Virasoro characters as

$$Z_{ab}(l,l') \sim \sum_{\lambda=1}^{h-1} V_{ab}^{\lambda}\chi_{1,\lambda}(q), \tag{3.1}$$

where

$$V_{ab}^{\lambda} = \sum_{s\in\{\substack{\text{exponents}\\ \text{of } g}\}} \frac{\sin(\pi s\lambda/h)}{\sin(\pi s/h)} q_a^{(s)} q_b^{(s)}. \tag{3.2}$$

Here, $\chi_{1,\lambda}$ is a Virasoro character from the first row of the Kac table for the central charge $c = 1 - 6/h(h-1)$ of the g model, $q = \exp(-\pi l/l')$ is the modular parameter for a cylinder of circumference l and width l', and $q^{(s)}$ is an eigenvector of the Cartan matrix of g, with eigenvalue $2-2\cos\pi s/h$. Subsequently, the V_{ab}^{λ} were also studied in the context of general models based on graphs, and fusion algebras [24,25]. The connection with the material of section 2 comes from the observation that the sum (3.2), if non-zero, simply gives the expansion of the inner product between a root and a weight of g, in a basis of eigenplanes of the Coxeter element [13]. More precisely,

$$V_{ab}^{1+u(\phi_a,\beta)} = (\lambda_a, \beta)\,, \tag{3.3}$$

and (3.1) can be rewritten in a way very reminiscent of (2.6):

$$Z_{ab}^{(G)} \sim \sum_{\beta\in\Gamma_b^+} (\lambda_a, \beta)\chi_{1,1+u(\phi_a,\beta)}\,. \tag{3.4}$$

One consequence is that the tables (1.6) and (1.7) can also be thought of as lists of partition functions. More important is that the positivity of the root β in (3.4) establishes the positivity of the V_{ab}^{λ}'s, expected from their appearance in (3.1) as multiplicities, in a general way. Previously this had only been checked case-by-case.

The S-matrices for the (simply-laced) affine Toda field theories [26,27,9,10] are rather more obviously related to the earlier discussion. Essentially, the only problem with (2.6) in this context is its lack of a coupling-constant dependence, and this is easily remedied by replacing the block $\{x\}$ defined in (1.18) by a slightly more complicated object, namely

$$\{x\}_B = \frac{(x-1)(x+1)}{(x-1+B)(x+1-B)}\,, \tag{3.5}$$

where B contains the coupling constant: $B(\beta) = 2\beta^2/(\beta^2+4\pi)$. After this modification, all the earlier discussion of physical-strip pole structure, fusing rules and so on goes through unchanged. However an affine Toda theory also has a Lagrangian, which can be expanded perturbatively to find the classical three-point couplings. Case-by-case, their non-vanishing was known to be the same as that deduced from the quantum S-matrices; but more recently, a general group-theoretic proof that the classical couplings obey the rule (2.12) has been given [28]. The connection between (2.12) and a previously-observed Clebsch-Gordon selection rule has also been established [29].

The quantum theory of the non simply-laced affine Toda models is considerably more complicated than that of the simply-laced versions [8], and to find a geometrical understanding remains a challenge. However, in the classical domain, the conserved charge bootstrap, and the treatment [28] of the three-point couplings, go through unharmed, providing a uniform description of all the untwisted affine theories, based on the Coxeter elements of the underlying (non-affine) Weyl groups. One gap has been the twisted non simply-laced cases, and it seems worth pointing out that in fact the necessary concept has already been introduced by Steinberg, and is described in an article by Springer [30]. It is the 'twisted Coxeter element', defined as follows. Recall first that whenever a Dynkin diagram has an automorphism, σ say, then the automorphism group of the root lattice Φ is larger than W, the Weyl group – there are also 'outer automorphisms', induced by σ. The twisted Coxeter element for σ lies in $W\sigma$, and is defined by first choosing one simple reflection from each σ-orbit in the simple roots (note, σ, being a diagram automorphism, maps the simple roots to themselves) and then forming the product w' of these reflections. Then $w_\sigma \equiv w'\sigma$ is a twisted Coxeter element, a particular outer automorphism of the non-affine root system Φ. Combining the information in [30] with a remark in an article by Kac [31] is enough to see that the arguments of [28] will go through in this case as well, leading to a characterisation of the twisted affine couplings in terms of the orbits of the twisted Coxeter element. The properties of w_σ are sufficiently simple (for example, all orbits in Φ have the same length) that it is tempting to try to generalise (2.6) to this case. One problem is that the orbits are longer, and hence there are fewer of them – certainly less than r, the dimension of the space in which Φ sits. Thus, if a particle type is associated to each orbit, the S-matrix formula (2.6) cannot be used as it stands since the term (λ_a, β) in the exponent links particle a to a fundamental weight, of which there are now too many. The relation (2.3) provides a hint as to one way out, suggesting the expression

$$S_{ab} = \prod_{\beta \in \widetilde{\Gamma}_b^+} \{1 + u(\widetilde{\phi}_a, \beta)\}^{((1-w_\sigma^{-1})^{-1}\widetilde{\phi}_a, \beta)} \tag{3.6}$$

where $\widetilde{\Gamma}_b^+$ is now the positive part of a w_σ-orbit, and $\widetilde{\phi}_a$ is the (unique for suitably-chosen simple roots [30]) positive root in $\widetilde{\Gamma}_a$ which becomes negative on the action of w_σ. Unfortunately, although (3.6) produces functions which satisfy all the bootstrap requirements in terms of bound-state structure and bootstrap consistency, it cannot be the right answer for the quantum theory. In fact, it yields sub-matrices of the simply-laced S-matrices. To see why this should be so, an alternative characterisation of the twisted Coxeter element can be used. Recall from [32] that each twisted affine Dynkin diagram can be obtained as a folding of some simply-laced affine diagram, a fact useful in establishing the classical properties of the twisted affine Toda theories. For example, the $d_4^{(3)}$ diagram can be obtained by folding that for $e_6^{(1)}$, so that the solutions of the $d_4^{(3)}$ affine Toda theory can be found simply by imposing a certain symmetry on the solutions of the $e_6^{(1)}$ model – a process known as reduction. Now let w be a Coxeter element for the non-affine root system E_6, and let P be the w-eigenplane for the exponent 4 of E_6, that is for the eigenvalue $\exp 2\pi i/3$; it is also the eigenplane for the exponent 8. The orthogonal complement of this plane, $P^{\perp}$, is four-dimensional, and its intersection with Φ, the set of E_6 roots, is a set of roots for D_4, made up from the w-orbits of the two self-conjugate particles in the $e_6^{(1)}$ affine Toda model. Furthermore, the action of w in this embedded D_4 is precisely that of a D_4 order-three twisted Coxeter element. An inner automorphism of E_6 has induced an outer automorphism of D_4, in a way which explains why the three point couplings in the twisted folding were observed to be simply a subset of those of the parent theory [10], and also explains why the S-matrix predicted by (3.6) for this case is just that for the scattering of the two self-conjugate particles in the E_6 theory. Although I do not know a general proof, a case-by-case check shows that this phenomenon generalises to all the twisted foldings.

While (3.6) does not reproduce the formulae of [8] for the twisted affine Toda theories, it only fails at the last hurdle, not accounting correctly for some quantum effects. In this respect the situation here is in better shape than for the untwisted non simply-laced models, where the simplest guess would be just to apply formula (2.6) with w a Coxeter element of the relevant non simply-laced algebra. Since the roots and weights are then no longer dual, the exponents cease to be integers and the analyticity properties of $S_{ab}(\theta)$ are drastically changed. In fact, for the non simply-laced theories, the quantum duality between strong and weak couplings is expected to relate the untwisted and twisted theories [8,33,34]. It seems that the rigid structure organising the simply-laced cases is being deformed into something rather richer, and a deeper understanding of this would be very interesting.

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