

Second-Order Hydrodynamics in Next-to-Leading-Order QCD

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We compute the hydrodynamic relaxation times τ_π and τ_j for hot QCD at next-to-leading order in the coupling with kinetic theory. We show that certain dimensionless ratios of second-order to first-order transport coefficients obey bounds which apply whenever a kinetic theory description is possible; the computed values lie somewhat above these bounds. Strongly coupled theories with holographic duals strongly violate these bounds, highlighting their distance from a quasiparticle description.

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Introduction.—The quark-gluon plasma (QGP) produced at RHIC [1,2] and the LHC [3–6] appears to be an excellent fluid. Despite the small system size, viscous hydrodynamics does a good job describing many collective properties, spectra, and correlations [7,8]. To be causal and stable [9,10], such treatments must work to second order in the gradient expansion, requiring many more coefficients than the celebrated shear viscosity to entropy ratio η/s . In particular, a treatment of collective flow requires not only the shear viscosity η but also the shear relaxation rate τ_π , and baryon-number diffusion needs not just a diffusion coefficient D_q but also a diffusive relaxation time τ_j .

We would like to use experiments to constrain the properties of the QGP such as η/s , but the necessity to include higher-order coefficients could lead to a proliferation of fitting parameters. So one often assumes that the coefficients follow some simple relations, such as $\tau_\pi = K\eta/(\epsilon + \mathcal{P})$, with $(\epsilon + \mathcal{P})$ the enthalpy density and K a constant which we draw from some microscopic theory of relativistic plasmas. For instance, York and Moore showed that weakly coupled massless QCD treated to leading order (LO) in the gauge coupling yields $5 < K < 6$, nearly independent of coupling strength [11], while Baier *et al.* find that strongly coupled $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory has $K \simeq 2.62$ [12].

Recently, we extended previous perturbative results for the shear viscosity and baryon-number diffusion of hot QCD from leading [13] to next-to-leading order (NLO) [14]; see Fig. 1. How does a NLO treatment change K ? In this Letter, we will explore this issue. In addition to finding

concrete results for K and τ_j/D_q , we will also show very general bounds on these dimensionless ratios which follow as soon as we state that a theory is well described by relativistic kinetic theory. These bounds are badly violated by strongly coupled theories with holographic duals, with the interesting implication that these theories are *very far* from having quasiparticle descriptions.

Definitions.—Let us start by defining the objects of our investigation. In the Landau-Lifshitz fluid rest frame, the stress tensor has the form

$$\langle T^{ij}(x) \rangle = \delta^{ij} \langle \mathcal{P} \rangle + \pi^{ij}, \quad (1)$$

where the nonideal dissipative part can be gradient expanded. At first order,

$$\pi_1^{ij} = -\eta \left(\nabla_i u^j + \nabla_j u^i - \frac{2}{3} \delta^{ij} \nabla_l u^l \right) - \zeta \delta^{ij} \nabla_l u^l. \quad (2)$$

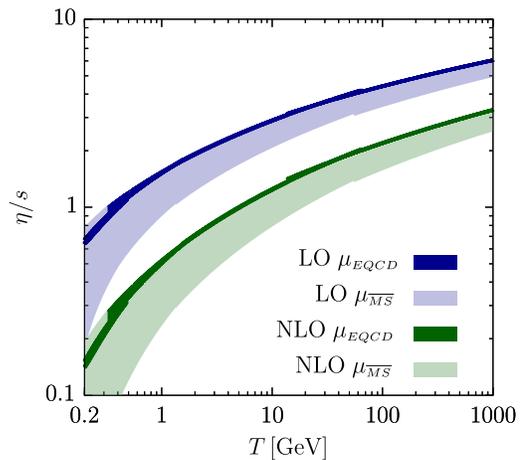


FIG. 1. η/s of QCD as a function of the temperature at LO and NLO for the different choices of the running coupling detailed in Fig. 3 and in the text later on. Figure taken from Ref. [14].

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We will concentrate on shear viscosity η and not discuss bulk viscosity ζ further. At second order, the coefficients relevant for a conformal theory have been introduced in Ref. [12]. Here, we will only deal with second-order relaxation, whose coefficient τ_π is defined as [15,16]

$$\tau_\pi \partial_i \pi^{ij} = \pi_1^{ij} - \pi^{ij}. \quad (3)$$

When there are additional conserved global charges Q_α such as baryon or lepton number, the associated charge density $n_\alpha \equiv j^0$ and current density \mathbf{j} satisfy, at first order in the gradients, a diffusion equation,

$$\langle \mathbf{j}_1 \rangle = -D_\alpha \nabla \langle n_\alpha \rangle = -k_{\mu_\alpha} \nabla \frac{\mu_\alpha}{T}, \quad (4)$$

where D_α is the diffusion coefficient. Here, we have rewritten the current with a gradient of the associated chemical potential μ_α . The associated transport coefficient k_{μ_α} is related to D_α through the susceptibility χ_α :

$$D_\alpha = \frac{k_{\mu_\alpha}}{T\chi_\alpha}, \quad \chi_\alpha \equiv \frac{\partial n_\alpha}{\partial \mu_\alpha}. \quad (5)$$

If we were to write Eq. (2) as a gradient of the charges $T^{0j} = (\epsilon + \mathcal{P})u^j$, we would naturally see that the associated relaxation coefficient is $\eta/(\epsilon + \mathcal{P})$. Analogous to Eq. (3), the second-order relaxation of \mathbf{j} reads [18–20]

$$\tau_j \partial_j \mathbf{j} = \mathbf{j}_1 - \mathbf{j}. \quad (6)$$

In Refs. [13,21], it was shown how the first-order transport coefficients can be determined from a linearized kinetic theory. In Ref. [21], the collision operator defining the kinetic theory of QCD was determined at leading logarithmic accuracy, in Ref. [13] at LO, and in Refs. [14,22] at (almost) NLO. The kinetic theory expression for τ_π was derived in Ref. [11], leading to its LO determination.

First, we summarize the main findings of Refs. [11,13,21]. We start from a generic kinetic theory of the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_p \cdot \frac{\partial}{\partial \mathbf{x}} \right) f^a(\mathbf{p}, \mathbf{x}, t) = -C_a[f], \quad (7)$$

where $f^a(\mathbf{p}, \mathbf{x}, t) = dN^a/d^3\mathbf{x}d^3\mathbf{p}$ is the phase space distribution function for the excitation (gluon, quark, antiquark) of index a . If u^i , μ vary with space, then the local-equilibrium form of f^a does as well [23], $f_0^a = [\exp(-\beta u^\mu P_\mu - q_\alpha^a \beta \mu) \mp 1]^{-1}$. The gradients on the left-hand side of Eq. (7), which we treat as perturbatively small, give rise to a source of departure from equilibrium $X^i = \nabla_i \mu_\alpha$ for flavor diffusion ($\ell = 1$) and $X^{ij} \propto \pi_1^{ij}/\eta$ for shear ($\ell = 2$). This determines the linearized departure from equilibrium via a linearized version of Eq. (7),

$$S^a(\mathbf{p}) = (\mathcal{C}f_1)^a(\mathbf{p}), \quad (8)$$

where $S^a = \beta q^a X_{i\dots j} I_{i\dots j}(\mathbf{p}) f_0^a [1 \pm f_0^a]$, with $q^a = q_\alpha^a$ for number diffusion and p for shear. f_1 is the linearized departure from equilibrium, $f^a(\mathbf{p}) = f_0^a(p) + f_1(\mathbf{p})f_0[1 \pm f_0]$ and $I_{i\dots j} \propto p_i \dots p_j$ (see Refs. [13,21]). At linear order $f_1^a \propto X_{i\dots j}$, allowing us to define the scalar function $\chi(p)$ [24]

$$f_1^a(\mathbf{p}) \equiv \beta^2 X_{i\dots j} I_{i\dots j}(\hat{\mathbf{p}}) q^a \chi^a(p). \quad (9)$$

The linearized collision operator \mathcal{C} is worked out in detail for the case of weakly coupled QCD in Ref. [13] at LO and in Ref. [14] at NLO.

General bounds.—To determine η , D_α , τ_π , and τ_j , we will need to solve Eq. (8) to linear order in f_1 but to subleading order in gradients, which will depend in detail on the form of the collision operator. However, we can already make some generic statements about the solution, which will allow us to place bounds on certain dimensionless ratios which hold automatically for all systems described by relativistic kinetic theory, regardless of the details of \mathcal{C} . To see this, let us first define an inner product on the Hilbert space of functions of momentum,

$$(g, h) \equiv \beta^2 \sum_a \nu_a \int_p (q^a)^2 f_0^a(p) [1 \pm f_0^a(p)] g^a(\mathbf{p}) h^a(\mathbf{p}), \quad (10)$$

with ν_a the degeneracy of species a and $\int_p \equiv \int (d^3\mathbf{p}/(2\pi)^3)$. Basic considerations such as stability ensure that the linearized collision operator \mathcal{C} is a linear, real, symmetric, positive semidefinite operator under this inner product and strictly positive in the channels we consider. In terms of this inner product, the first-order transport coefficients become [13,21]

$$\eta = \frac{1}{15}(\chi, 1), \quad k_{\mu_\alpha} = \frac{T}{3}(\chi, 1). \quad (11)$$

The enthalpy density and charge susceptibility can be easily obtained as

$$\epsilon + \mathcal{P} = \frac{T}{3}(1, 1), \quad \chi_\alpha = T(1, 1). \quad (12)$$

τ_π , τ_j require inserting f_1 into the left-hand side of Eq. (7) and using the time derivative to find f_2 at one space-derivative, one time-derivative order. As shown in Ref. [11], the properties of the inner product and of \mathcal{C} then turn the evaluation into the inner product of the first-order departure from equilibrium χ with itself:

$$\eta \tau_\pi = \frac{\beta}{15}(\chi, \chi). \quad (13)$$

The same analysis can be applied to τ_j and we find

$$k_{\mu\alpha}\tau_j = \frac{1}{3}(\chi, \chi). \quad (14)$$

It is then insightful to consider these dimensionless ratios,

$$\frac{\tau_\pi}{\eta/(\epsilon + \mathcal{P})} = 5 \frac{(\chi, \chi)(1, 1)}{(\chi, 1)^2}, \quad \frac{\tau_j}{D_\alpha} = 3 \frac{(\chi, \chi)(1, 1)}{(\chi, 1)^2}, \quad (15)$$

which also have the same number of powers of the collision operator ($\chi \propto \mathcal{C}^{-1}$) in the numerator as in the denominator. The triangle inequality implies

$$\frac{\tau_\pi}{\eta/(\epsilon + \mathcal{P})} \geq 5, \quad \frac{\tau_j}{D_\alpha} \geq 3. \quad (16)$$

These results apply to any kinetic theory description of these transport coefficients, as long as the enthalpy density or the charge susceptibility are also consistently computed within the kinetic theory. We remark that the $\ell = 1, 2$ departures from equilibrium contributing to these transport coefficients do not by construction contribute to the ($\ell = 0$) thermodynamical functions $\epsilon + \mathcal{P}$ or χ_α .

In contrast, strong-coupling results from the AdS/CFT correspondence in $\mathcal{N} = 4$ SYM theory give for τ_π [12] and for the relaxation of a $U(1)$ current in SYM theory [25]

$$\left. \frac{\tau_\pi}{\eta/(\epsilon + \mathcal{P})} \right|_{\text{AdS}} = 4 - 2 \ln(2), \quad \left. \frac{\tau_j}{D_{U(1)}} \right|_{\text{AdS}} = \frac{\pi}{2}. \quad (17)$$

In both cases, these strong-coupling results are approximately half the minimum value attainable in kinetic theory. Finite-coupling corrections [26–30] to the first ratio show a

modest increase. We also note that our kinetic theory bounds in Eq. (16) can be shown to become in d spatial dimensions, $d + 2$ and d , respectively. It would be interesting to derive larger-dimension holographic results in comparison.

Second-order relaxation at (almost) NLO.—We now provide results for the second-order relaxation of the shear stress tensor and of the light quark current j_q in QCD. In Ref. [14], we have introduced in great detail a linearized collision operator to “(almost) NLO,” (Corrections which lie beyond the kinetic theory picture arise at still higher order.) $2 \leftrightarrow 2$ elastic scatterings and effective $1 \leftrightarrow 2$ inelastic scatterings contribute to the LO collision operator, the former taking the lion’s share. At NLO, we found all new scattering processes, and corrections to the LO processes, which are suppressed by a single power of the QCD coupling g . As we showed in detail, there are only a few such $\mathcal{O}(g)$ subleading effects. First, the rate of soft $2 \leftrightarrow 2$ scattering is modified; this can be described as an additional momentum-diffusion coefficient $\delta\hat{q}$. This modification and an $\mathcal{O}(g)$ correction to the in-medium dispersion also provide an $\mathcal{O}(g)$ shift in the $1 \leftrightarrow 2$ rate. Next, this $1 \leftrightarrow 2$ splitting rate must be corrected wherever one participant becomes “soft” ($p \sim gT$) or when the opening angle becomes less collinear. And finally, subtractions are needed because of the way the numerical implementation of the LO scattering kernel [13] already resums a small amount of the NLO effects. We were able to give a relatively simple determination of these effects by the use of light-cone techniques fostered by Ref. [31]. Unfortunately, these methods typically keep track of the incoming and outgoing momentum of a particle but lose track of the momentum which it transfers to the other participants. This momentum transfer also affects the

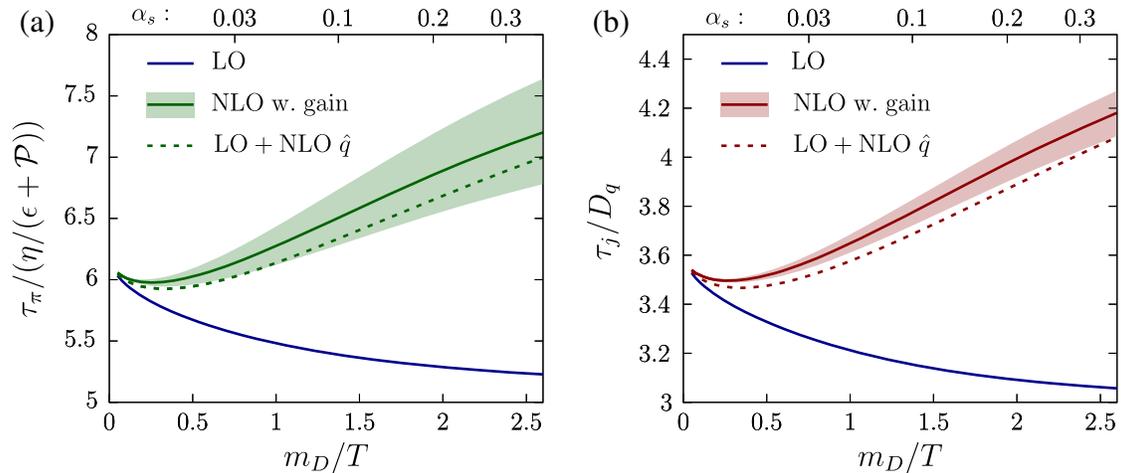


FIG. 2. The second- to first-order ratio of the relaxation coefficients for (a) shear stress $\tau_\pi/[\eta/(\epsilon + \mathcal{P})]$ and (b) for quark number diffusion τ_j/D_q as a function of m_D/T for QCD with three light flavors. (The corresponding value of (α_s) is shown on the upper horizontal axis.) The LO result for τ_π is from Ref. [11] and that for τ_j is also new. The uncertainty from the unknown gain terms is shown by the bands; it is estimated as specified in Ref. [14] by the LO value for the gain terms, times m_D/T , times a constant in the interval $[-2, 2]$. The dashed lines represent an estimate in which we include only the NLO \hat{q} to the LO collision operator.

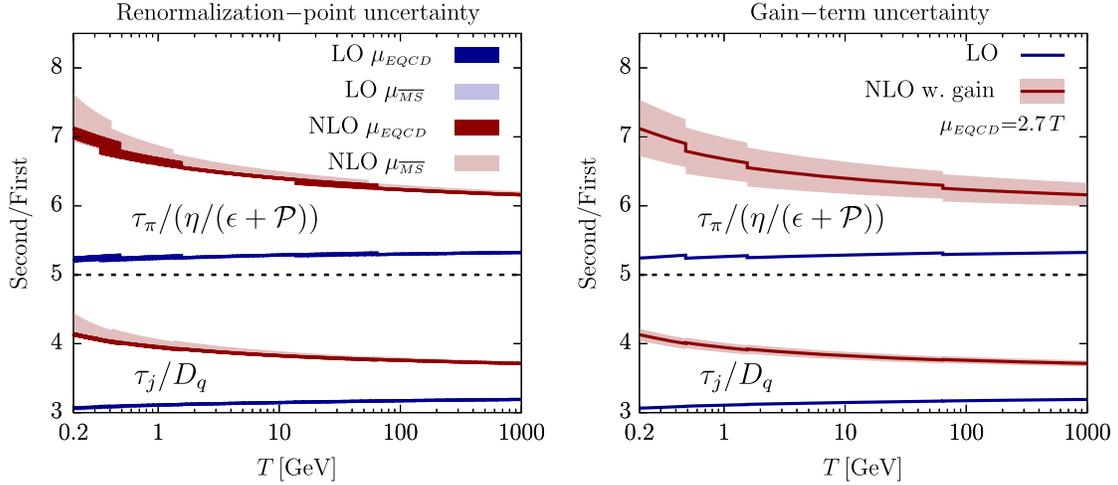


FIG. 3. The second- to first-order ratio of the relaxation coefficients for shear stress $\tau_\pi/[\eta/(\epsilon + \mathcal{P})]$ (values above 5) and for quark number diffusion τ_j/D_q (values below 5) as a function of T . On the left, we plot different choices of the running coupling: the solid bands fix the coupling using the two-loop EQCD value with $\mu_{EQCD} = (2.7 \leftrightarrow 4\pi)T$, while the shaded bands use the standard \overline{MS} two-loop coupling with $\mu_{\overline{MS}} = (\pi \leftrightarrow 4\pi)T$. On the right, we plot instead in the shaded red bands the estimated uncertainty due to the gain terms. All curves in this plot are obtained using the effective EQCD coupling with $\mu_{EQCD} = 2.7T$.

departure from equilibrium of the other particle or particles which receive the momentum, generating, in the effective Fokker-Planck approach applicable for these soft scatterings, a *gain term*. This is an effect which we failed to account for at NLO, hence, the “almost” NLO. However, we estimated that this missing part is most likely small. Finally, we found out that η/s and D_q at NLO become smaller than their LO counterparts by a factor of 4 at the couplings of relevance for heavy ion collisions; see Fig. 1. The large $\delta\hat{q}$ contribution is by far the main contribution responsible for this behavior.

We now use this (almost) NLO collision operator to determine τ_π and τ_j using Eqs. (8) and (15). We solve Eq. (8) with the same variational method as in Ref. [14], which also details the NLO operator $\delta\mathcal{C}$. In Fig. 2, we plot our results for the second-order coefficients τ_π and τ_j normalized as in Eq. (15) as functions of the Debye mass $m_D \sim gT$ over the temperature. The LO results for τ_π were originally obtained in Ref. [11]. Those for τ_j are new and consistent with the leading-log estimate in Ref. [19]. The plot shows that both LO results in solid blue decrease with increasing coupling, approaching the minimum values [Eq. (16)], while the NLO results in solid green and red, respectively, start to differ significantly from the LO at $m_D \gtrsim 0.5T$, where they start growing, getting in the ballpark of $3/2$ of the minima when $\alpha_s \sim 0.3$. The dashed green or red curves are the results obtained by adding only $\delta\hat{q}$ to the LO collision operator, showing that also in this case it dominates NLO corrections. The bands are obtained by varying the estimate for the unknown gain terms within a range reasonably encompassing their probable size (and sign), as described in Ref. [14]. Intuitively, the LO results approach the bound at increasing coupling because the

log-enhanced $2 \leftrightarrow 2$ processes, which force $\chi(p) \propto p$, become less effective at larger couplings, while the other processes drive $\chi(p)$ to a constant, saturating Eq. (15). At NLO, the large $\delta\hat{q}$ drives $\chi(p)$ towards p^2 , which is further from the bound.

Figure 3 presents the more phenomenologically relevant dependence of these second-order coefficients on the temperature. Since only a NNLO treatment would directly include running-coupling effects, this requires that we pick a prescription for relating the running coupling to the temperature. We do so by either using the \overline{MS} coupling in the range $\pi T < \mu_{\overline{MS}} < 4\pi T$ (leading to the larger, light-shaded bands in the left plot) or via the effective electrostatic QCD (EQCD) coupling with $2.7T < \mu_{EQCD} < 4\pi T$ as in Ref. [32] (narrow, dark-shaded bands in the left plot). The discontinuities in the plot occur where we change prescriptions for the number of light fermion species. The right plot in the figure indicates the errors due to the uncertainties from our ignorance of the gain terms which we discussed above.

Conclusions.—Viscous hydrodynamical studies of heavy ion collisions require second-order hydrodynamical coefficients τ_π, τ_j which can be understood as relaxation times towards the first-order behavior. While the hydrodynamic coefficients such as η/s and $\tau_\pi T$ vary by orders of magnitude as a function of the temperature and differ substantially between LO and NLO calculations (see Fig. 1), we have shown that simple dimensionless ratios, Eq. (15), are remarkably robust, varying at most by 40% as a function of coupling or temperature and between LO and NLO determinations. Furthermore, and more remarkably, we have shown that in *any* theory which can be described by kinetic theory of ultrarelativistic particles, these dimensionless ratios

obey inequalities shown in Eq. (16). These inequalities hold regardless of the details of the collision operators, and they give the hydrodynamics practitioner a simple prescription for how to estimate the relation between first-order and second-order transport coefficients.

It is also remarkable that the bounds we have found fail by a full factor of 2 when we compared them to the results within strongly coupled theories with holographic duals. We conclude that such strongly coupled theories are very far from having a kinetic description. This provides a useful counterpoint to the frequent unspoken assumption that the QGP should have a kinetic description.

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