CERN-TH.6301/91

WHY ARE THERE NO EXACT S-MATRICES FOR AFFINE TODA THEORIES BASED ON NONSIMPLY-LACED LIE ALGEBRAS?

G.W. Delius and M.T. Grisaru[*)]

Theoretical Physics Division, CERN, CH-1211 Geneva 23

and

D. Zanon

Dipartimento di Fisica dell'Università di Milano and
INFN, Sezione di Milano, I-20133 Milano, Italy

ABSTRACT

We discuss features of Toda theories based on nonsimply-laced algebras by treating as an explicit example the case of the $a_3^{(2)}$ theory. We review the failure of such theories to have exact, factorizable S-matrices and describe how by extending them to include fermions one obtains theories with satisfactory S-matrices. We explain why, although the bosonic theories have higher spin conserved currents at the quantum level, certain on-shell singularities of Feynman amplitudes lead to a breakdown of the charge conservation arguments used in the bootstrap construction of exact S-matrices.

Invited talk at the Conference
"Strings and Symmetries 91"
Stony Brook, May 1991

---

*) On leave from Brandeis University, Waltham, WA 02254, USA.

CERN-TH.6301/91
October 1991

1. Introduction

Two-dimensional Toda field theories are represented by lagrangians of the form

$$\mathcal{L} = -\frac{1}{2}\vec{\phi}\Box\vec{\phi} - \sum q_i e^{\vec{\alpha}_i \cdot \vec{\phi}} \tag{1.1}$$

Here the $\vec{\alpha}_i$ are the simple roots of a rank r Lie algebra augmented by (the negative of) a maximal root and $\vec{\phi} = (\phi_1, \phi_2, ...\phi_r)$ are bosonic fields describing r massive particles, with masses rigidly controled by properties of the Lie algebra [1]. The Kač labels q_i are chosen so as to set the vacuum expectation value of $\vec{\phi}$ to zero. These theories are classically integrable: the lagrangian above admits an infinite number of symmetries described by conserved currents $J_{\pm}^{(s)}$ of increasing spin s [2]. Assuming the symmetries survive quantization they can be used to prove that the n-body S-matrices of these theories factorize into products of two-body S-matrices satisfying Yang-Baxter equations as well as conservation of individual particle momenta, so that particles of mass m, m' only scatter into particles of the same mass. With additional, standard S-matrix assumptions, it then follows that these elastic S-matrices can be determined exactly [3].

In practice, the procedure that has been used in the recent literature [4, 5] starts with a knowledge of the mass spectrum and some information about the three-point couplings, postulates the symmetries, constructs a suitable S-matrix (this construction is essentially unique), and checks that it agrees in low orders of perturbation theory with that computed from the particular Toda lagrangian. (More deductive methods exist for the case of the sine-Gordon theory, as well as cases where the quantum group structure is sufficiently well understood [6].) This procedure has been successful for the case of simply-laced Lie algebras (equal length simple roots) but has failed for the nonsimply-laced Lie algebras (roots of different length). It is generally believed that Toda theories based on nonsimply-laced algebras, though classically integrable, do not possess factorizable S-matrices, both because no suitable matching has been found in perturbation theory, and because radiative corrections distort the mass spectrum of the Toda lagrangians in a manner that seems incompatible with the existence of any exact S-matrix [5].

For certain cases the situation can be improved by introducing fermions. The resulting theories are Toda theories naturally based on suitable Lie superalgebras [7]. Contributions from fermionic loops remove the above-mentioned distortions of the mass spectrum [8] and also lead to a singularity structure of Feynman diagrams (double poles, etc.) which is consistent with that determined by exact S-matrix considerations [9, 10]. However, this approach has not been extended to all cases. Furthermore, until now one had achieved little understanding of the reasons for the failure of the nonsimply-laced bosonic theories to maintain their classical integrability.

We present here, as a pedagogical example, the case of the Toda theory based on the nonsimply-laced affine Lie algebra $a_3^{(2)}$ and its fermionic extension based on the superalgebra $A^{(2)}(0,3)$. We review briefly the bootstrap ideas behind the S-matrix construction and describe it for the case of the fermionic theory. We then outline possible reasons for the failure of this construction for the bosonic theory.

2. The $a_3^{(2)}$ and the $A^{(2)}(0,3)$ theories

We consider the lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi_1 \Box \phi_1 - \frac{1}{2}\phi_2 \Box \phi_2 - e^{-\phi_1-\phi_2} - e^{-\phi_1+\phi_2} - e^{2\phi_1} \tag{2.1}$$

The simple roots are the two-dimensional vectors $\vec{\alpha}_1 = (-1,1)$ and $\vec{\alpha}_2 = (2,0)$ while the maximal root is $\vec{\alpha}_m = \vec{\alpha}_1 + \vec{\alpha}_2 = (1,1)$. It describes two particles of masses $m_1 = \sqrt{6}$ and $m_2 = \sqrt{2}$. We note for future reference that the three-point couplings are $-\phi_1^3$ and $\frac{1}{3}\phi_1\phi_2^2$. In our conventions the lagrangian enters the functional integral as $\exp\left(\frac{i}{2\pi\hbar}\int \mathcal{L}\right)$. We use light-cone coordinates with $\Box = 2\partial_+\partial_-$.

Using the classical field equations it is straightforward to check the conservation $\partial_- J_+^{(s)} + \partial_+ J_-^{(s)} = 0$ of spin two and spin four currents, where the spin two current is the stress tensor, while

$$J_+^{(4)} \equiv J_{++++} = (\partial_+\phi_1)^2(\partial_+\phi_2)^2 + (\partial_+^2\phi_2)^2 + 2\partial_+\phi_1\partial_+\phi_2\partial_+^2\phi_2 \tag{2.2}$$

with a suitable expression for $J_-^{(4)}$. They define corresponding spin one and spin three charges

$$Q^{(1)} = \int dx^+ T_{++} \quad , \quad Q^{(3)} = \int dx^+ J_{++++} \tag{2.3}$$

whose presence would guarantee the existence of factorizable elastic S-matrices[11]. Unitarity, analyticity, crossing and bootstrap assumptions would then allow one to construct explicitly these S-matrices. However, this does not seem to be the case. First, identifying the location of simple poles of the S-matrix with the masses of the theory would require that these masses stay in the ratio $m_1/m_2 = \sqrt{3}$ to all orders of perturbation theory. However, one-loop calculations [8] change the ratio by *

$$\delta\left(\frac{m_1}{m_2}\right) = -\frac{\pi\hbar}{18} \tag{2.4}$$

Second, such S-matrices have higher order poles which are normally interpreted as anomalous threshold singularities [12, 5, 9]. Their position and coefficients are again rigidly controlled. No successful matching with singularities of Feynman diagrams computed from the lagrangian in Eq. (2.1) has been possible. There is little doubt that the S-matrix of the $a_3^{(2)}$ Toda theory does not have all of the simple properties predicted by the existence of the charges in Eq. (2.3).

The situation changes dramatically when fermions are added. The Toda lagrangian obtained from the Lie superalgebra $A^{(2)}(0,3)$

$$\begin{aligned}\mathcal{L} \;=\;& -\frac{1}{2}\phi_1\Box\phi_1 - \frac{1}{2}\phi_2\Box\phi_2 - \frac{i}{2}\psi_-\partial_+\psi_- + \frac{i}{2}\psi_+\partial_-\psi_+ \\ & - e^{-\phi_1-\phi_2} - e^{-\phi_1+\phi_2} - e^{2\phi_1} - \psi_+\psi_- e^{\phi_1}\end{aligned} \tag{2.5}$$

*We have multiplied the result in [8] by a factor of 2π to account for a different normalization in the functional integral.

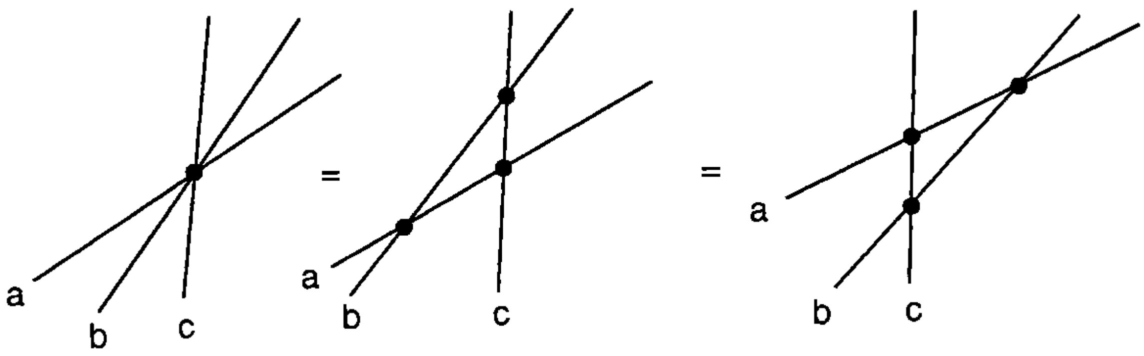

Figure 1: Factorization of the S-matrix

has the same bosonic mass spectrum, couplings, and tree level amplitudes as the one in Eq. (2.1), and there exist corresponding conserved spin two and spin four currents, but the addition of the fermions leads to very different behavior at the loop level. First, radiative corrections due to the fermions, when added to those due to the bosons, restore the classical mass ratios [8]. Second, one can construct exact S-matrices for this system [9] which agree in perturbation theory with those calculated from the lagrangian above (at least up to the one-loop level). In the next section we outline this construction.

3. The S-matrix of the $A^{(2)}(0,3)$ theory

In a theory with higher spin conserved currents the factorization and elasticity of the scattering amplitudes follow from the following arguments [3]: by Lorentz invariance, single particle states of momentum $p = (p_+, p_-)$ are eigenstates of the spin s charge with eigenvalues proportional to $(p_+)^s$:

$$Q^{(s)}|p\rangle = \omega\, p_+^s |p\rangle \tag{3.1}$$

(throughout we work with charges of one definite chirality, hence the dependence on p_+. Similar statements can be made about the other chirality). Acting on a product of wave-packets the operator $\exp iQ^{(s)}$, $s > 1$, displaces them relative to each other so that if $Q^{(s)}$ commutes with the S-operator a multiparticle scattering amplitude is equal to one where well-separated wave-packets scatter pairwise [13], as illustrated in Fig. 1. The same argument implies the validity of the Yang-Baxter equations, as indicated by the second equality in the figure.

The elasticity is based on the charge conservation relation for the process $p_a + p_b + ... \rightarrow p_f + p_g + ...$

$$\omega_a p_{a+}^s + \omega_b p_{b+}^s + ... = \omega_f p_{f+}^s + \omega_g p_{g+}^s + ... \tag{3.2}$$

which can be satisfied, generically, only if the outgoing momenta are at most a permutation of the incoming momenta.

Since we will have to come back to Eq. (3.2) let us describe one standard derivation of this statement [14]: from the current conservation equation it follows that

$$-\int d^2x e^{ik_-x^-}\partial_- J_+^{(s+1)}(x) = \int d^2x e^{ik_-x^-}\partial_+ J_-^{(s+1)}(x) = \int d^2x \partial_+\left(e^{ik_-x^-}J_-^{(s+1)}(x)\right) = 0 \quad (3.3)$$

The relation

$$\langle out|\int d^2x e^{ik_-x^-}\partial_- J_+^{(s+1)}(x)|in\rangle = 0 \quad (3.4)$$

has the graphical interpretation shown in Fig. 2: One inserts the current operator carrying momentum $k = (0, k_-)$ in all possible ways in the Feynman diagrams for the process $\langle out|in\rangle$ and the result must be zero.

When inserted in external lines with on-shell momentum p for which $\langle p|Q^{(s)}|p\rangle \neq 0$ one gets from Eq. (3.1) a factor ωp_+^{s+1}, an additional factor k_-, and a factor

$$\frac{1}{(p+k)^2 - m^2} = \frac{1}{p_+k_-} \quad (3.5)$$

When the current is inserted inside the Feynman diagram *one gets contributions which do not have a pole singularity in the limit* $k_- \to 0$ but still have the numerator factor k_-. Therefore, in this limit one finds $(\sum \omega_i p_{i+}^s)\langle out|in\rangle = 0$, implying that scattering amplitudes vanish for processes where the charge is not conserved.

As we shall discuss later, the *italicized statement above is not true for the nonsimply laced bosonic theory.* But for the moment we shall assume it holds for the fermionic theory and proceed with the derivation of the S-matrix [9].

We are dealing with a system of three particles with masses

$$m_1^2 = 6 \quad , \quad m_2^2 = m_f^2 = 2 \quad (3.6)$$

The elasticity of the two-body amplitudes does allow transitions from states with particle 2 to states with fermions, but not with particle 1. We have therefore the following coupled amplitudes:

$$\begin{array}{ll} B = \langle 22|22\rangle \quad , \quad F = \langle ff|ff\rangle \quad , \quad D = \langle ff|22\rangle \\ T = \langle 2f|2f\rangle \quad , \quad R = \langle 2f|f2\rangle \end{array} \quad (3.7)$$

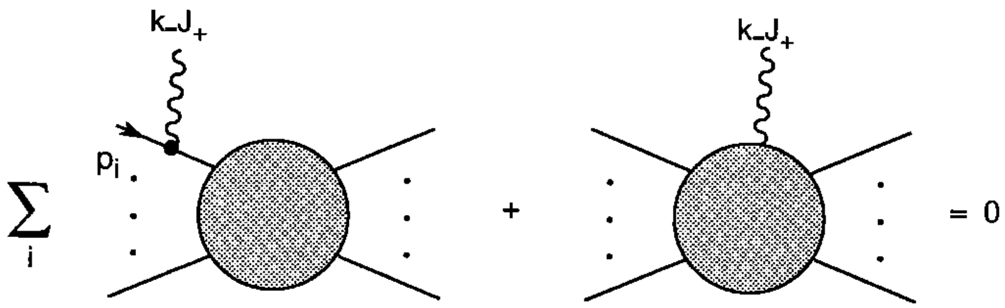

Figure 2: Illustrating the derivation of charge conservation

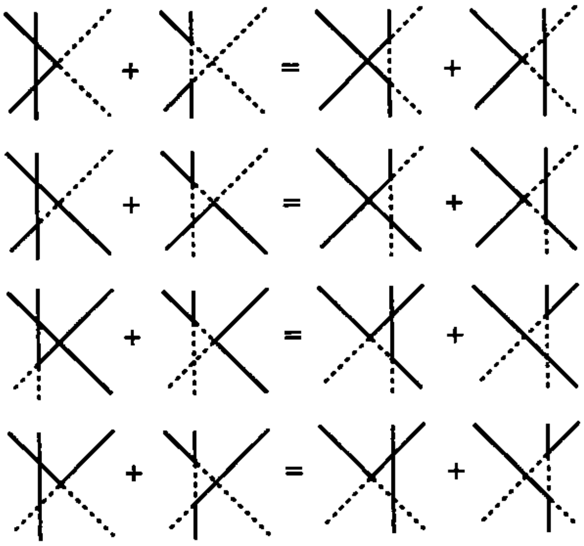

Figure 3: Yang-Baxter diagrams for the amplitudes in Eq. (3.7). Solid lines represent bosons, dashed lines represent fermions.

as well as amplitudes $\langle 12|12\rangle$, $\langle 1f|1f\rangle$, and $\langle 11|11\rangle$. The amplitudes satisfy Yang-Baxter equations, as depicted in Fig. 3, corresponding to relations such as

$$B(\theta)R(\theta+\theta')B(\theta') - D(\theta)F(\theta+\theta')D(\theta') = R(\theta)B(\theta+\theta')R(\theta') + T(\theta)R(\theta+\theta')T(\theta) \quad (3.8)$$

etc. We have introduced rapidity variables defined by $p_{\pm} = me^{\pm\theta}$.

Setting one of the variables to zero in such equations, as well as in equations obtained by taking one derivative, one ends up with differential equations which can be solved for ratios of the amplitudes. One finds, after using some tree-level information, etc.

$$\begin{aligned} R(\theta) &= \gamma \frac{i}{sinh\frac{3}{2}\theta} T(\theta) \\ D(\theta) &= \gamma \frac{-i}{cosh\frac{3}{2}\theta} T(\theta) \\ B(\theta) &= \left(1 + \gamma \frac{2}{sinh 3\theta}\right) T(\theta) \\ F(\theta) &= \left(-1 + \gamma \frac{2}{sinh 3\theta}\right) T(\theta) \end{aligned} \quad (3.9)$$

where γ is an integration constant. We write $\gamma^2 = sin^2 \pi\Omega$, and also $\eta = -\frac{3i\theta}{2\pi}$. Unitarity and crossing symmetry determine then

$$\begin{aligned} T(\theta) &= \Pi(\theta)\Pi(i\pi - \theta) \\ \Pi(\theta) &= \prod_{l=1}^{\infty} \frac{\Gamma(\eta - \Omega + 1 + 3(l-1))\Gamma(\eta + \Omega + 1 + 3(l-1))}{\Gamma(\eta - \Omega + 1 + 3(l-\frac{1}{2}))\Gamma(\eta + \Omega + 1 + 3(l-\frac{1}{2}))} \end{aligned}$$

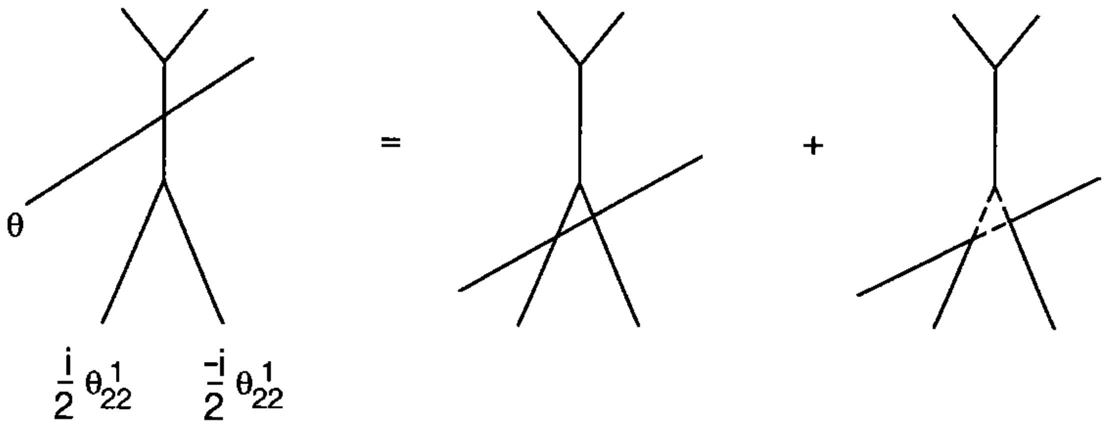

Figure 4: Illustrating Eq. (3.12)

$$\times\frac{\Gamma(\eta+1+3(l-\frac{1}{2}))\Gamma(\eta+3(l-\frac{1}{2}))}{\Gamma(\eta+1+3(l-1))\Gamma(\eta+3(l-1))} \tag{3.10}$$

One can check that on the physical sheet $0 \leq Im\theta \leq \pi$, $T(\theta)$ has no poles whereas D, B, and F have a pole at $\theta_{22}^{1} = \frac{1}{3}\pi i$ which corresponds to the existence of an intermediate state particle of mass $m^2 = 6$. In the bootstrap approach one identifies this with the particle described by ϕ_1. The amplitudes involving this particle are then determined by looking again at Yang-Baxter equations in the vicinity of this pole. As illustrated in Fig. 4 one obtains, denoting $S_{12}^{21} = \langle 12|12\rangle$

$$\begin{aligned} Res[B(\theta_{22}^{1})]S_{12}^{21}(\theta) &= Res[B(\theta_{22}^{1})]B(\theta+\frac{1}{2}\theta_{22}^{1})B(\theta-\frac{1}{2}\theta_{22}^{1}) \\ &\quad -Res[D(\theta_{22}^{1})]R(\theta+\frac{1}{2}\theta_{22}^{1})R(\theta-\frac{1}{2}\theta_{22}^{1}) \end{aligned} \tag{3.11}$$

which leads to

$$S_{12}^{21} = \left\{\frac{3}{2}\right\} \tag{3.12}$$

We are using a standard notation [5]

$$\{x\} = \frac{(x-1)(x+1)}{(x-1+2\Omega)(x+1-2\Omega)} \quad , \quad (x) = \frac{sinh(\frac{\theta}{2}+\frac{i\pi}{2h}x)}{sinh(\frac{\theta}{2}-\frac{i\pi}{2h}x)} \tag{3.13}$$

where h, the Coxeter number of the Lie algebra, is here equal to 3. In a similar fashion one finds the remaining two-body amplitudes

$$\begin{aligned} S_{11}^{11} &= \{1\}\{2\} \\ S_{1f}^{f1} &= \left\{\frac{3}{2}\right\} \end{aligned} \tag{3.14}$$

The factorization assumption determines then all n-particle amplitudes.

That the above amplitudes do indeed describe scattering for the theory given by the Toda lagrangian Eq. (2.5) is not immediately obvious. One used as input some information gleaned from the tree level amplitudes but even at the tree level complete agreement with all the two-body amplitudes was not guaranteed. A much more stringent test is agreement at the one-loop level. The amplitudes we have obtained above have double-pole singularities, and an important check is to ascertain that these singularities match the anomalous threshold singularities of one-loop Feynman diagrams constructed from the Toda lagrangian. This we have done [9].

4. Why does the $a_3^{(2)}$ Toda theory fail to have a simple S-matrix?

Affine Toda theories for simply-laced algebras such as for instance that for $a_1^{(1)}$, given by the lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi_1\Box\phi_1 - \frac{1}{2}\phi_2\Box\phi_2 - e^{-\phi_1-\sqrt{3}\phi_2} - e^{-\phi_1+\sqrt{3}\phi_2} - e^{2\phi_1} \tag{4.1}$$

have exact, factorizable S-matrices. Compared to the theory we are discussing the coefficients in the exponentials (and therefore masses and coupling constants) are slightly different and this difference is crucial.

Since the arguments that lead to the factorization and elasticity of the S-matrix rely heavily on the existence of conserved charges, it might be suspected that the conservation of the spin four current we have exhibited earlier (or some other higher spin current) breaks down at the quantum level due to anomalies. But this is not the case.

The quantum conservation laws can be studied by conventional methods. One can either rely on the BPHZ procedures used for the sine-Gordon system [15], or on OPE (massless perturbation theory) methods which treat the complete exponentials in the interaction lagrangian as a perturbation. The calculations are straightforward. Using the Gell-Mann-Low formula one computes

$$\langle J_+^{op}(x)...\rangle = \left\langle T\left(J_+^{(4)}(x)... \exp\left(\frac{i}{2\pi\hbar}\int \mathcal{L}_{int}\right)\right)\right\rangle \tag{4.2}$$

and checks whether

$$\partial_-\langle J_+^{op}(x)...\rangle = \partial_+(\text{local expression}) \tag{4.3}$$

For the present case this is not so, but we found that by adding local terms which renormalize the current in Eq. (2.2), the quantum operator is indeed conserved. At the one-loop level it is given by:

$$\begin{aligned} J_+^{(4)} &= (1+\frac{\hbar}{2})(\partial_+\phi_1)^2(\partial_+\phi_2)^2 + (1+\frac{23}{12}\hbar)(\partial_+^2\phi_2)^2 - \frac{\hbar}{12}(\partial_+^2\phi_1)^2 \\ &\quad -\frac{\hbar}{12}(\partial_+\phi_1)^4 - \frac{\hbar}{12}(\partial_+\phi_2)^4 + (2+3\hbar)\partial_+\phi_1\partial_+\phi_2\partial_+^2\phi_2 \end{aligned} \tag{4.4}$$

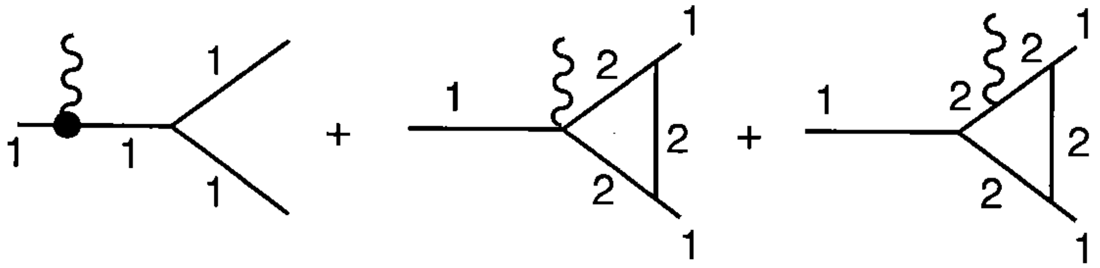

Figure 5: Diagrams showing the breakdown of the naive charge conservation law.

Since there are arguments [11] to claim that the conservation of two currents (e.g. the stress tensor and the spin four currents) are enough to insure the factorization properties of the S-matrix the problem lies elsewhere: although the current is indeed conserved, *it is not the case that the sum of the corresponding charges of incoming particles equals the sum of the charges of outgoing particles.* The argument presented earlier breaks down because in two dimensions, and for the particular mass ratios of the theory at hand, *Feynman diagrams where the current is inserted inside a loop may have pole singularities as* $k_- \to 0$.

We will illustrate this on the simple case of the three-point function which is interesting in its own right. Let us consider the vertex function with three external bosons ϕ_1, with momenta evaluated on shell (this means they must be complex, but it does not affect the argument). We emphasize that this vertex function is not zero, since the theory does have a ϕ_1^3 coupling.

In the center of mass frame where two of the particles have rapidity $\pm i\theta$ and the third one has zero rapidity, the charge conservation laws in Eq.(3.2) (for the spin one momentum and spin three charge) would give the two equations

$$\begin{aligned} 2m_1 cos\theta &= m_1 \\ 2\omega_1 m_1^3 cos3\theta &= \omega_1 m_1^3 \end{aligned} \qquad (4.5)$$

The first equation implies $\theta = \pi/3$ and when this value is substituted in the second equation it implies that $\omega_1 = 0$, the charge of particle 1 vanishes. This is indeed the case at the classical level, since the spin four current does not have a term quadratic in ϕ_1. Quite generally by the above argument, if a particle has a cubic self-coupling, only certain higher spin charges can be nonzero. However, as we shall discuss presently, at the quantum level the charge does not vanish and we must conclude that there is some inconsistency in the arguments that lead to the second equation in Eq. (4.5).

It is not difficult to find the source of the problem. Consider, up to the one-loop level, the argument of section 3 as applied to the present situation. Since the classical current does not contain a ϕ_1^2 term one only gets the diagrams in Fig. 5, where the blob on the ϕ_1 line indicates one-loop corrections to the current. It turns out that the triangle diagrams with particle 2 inside have pole singularities as $k_- \to 0$. (In fact, on the mass shell the vertex function itself is infinite, as can be seen by standard dual diagram arguments [9]. When the current is inserted

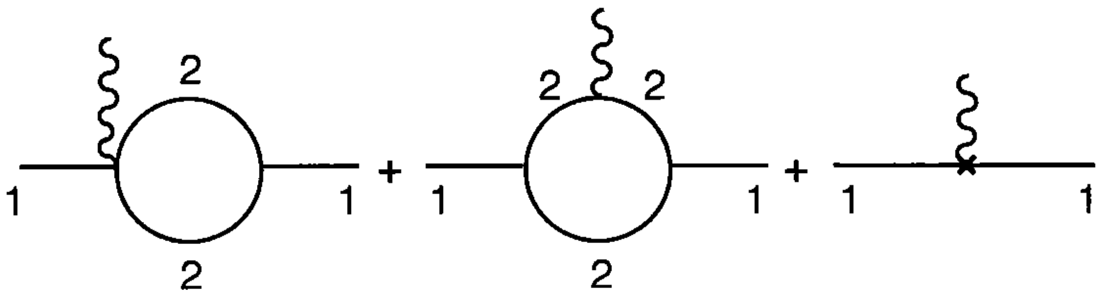

Figure 6: Diagrams for the calculation of the charge of particle 1.

the corresponding expression is finite, but the infinity shows itself in the limit of zero momentum k_-.) Consequently the argument that the charge of one of the external particles equals the sum of the charges of the other two, as would be suggested by the first diagram, breaks down and a detailed analysis shows that the amount of breaking is indeed related to the residue of the pole in the triangle diagram.

Although we have not yet studied the general situation, it seems clear that the above pathology will affect the higher point functions and lead to a breakdown of some of the arguments in favor of the existence of exact, factorizable S-matrices. (We note that triangle pole singularities are a common occurence in theories with perfectly good S-matrices, but pathologies will not be present unless the relevant particles have nonzero charges.)

As we have mentioned above, quantum corrections to the charge of particle 1, and therefore ω_1, are not zero. We have computed them at the one-loop level using the renormalized current of Eq. (4.4) and evaluating the matrix element

$$\langle p| \int dx^+ J_+^{(4)}(x)|p\rangle \equiv \omega_1\, p_+^3\, \langle p|p\rangle \tag{4.6}$$

as indicated in Fig. 6. We find

$$\delta\omega_1 = -\hbar \frac{\pi}{18\sqrt{3}} \tag{4.7}$$

where $\delta\omega_1 = \omega_1 - \omega_1(classical)$ indicates the deviation of the charge from its (zero) classical value.

Let us consider now the vertex function $\langle 2,2|1\rangle$ and go through the charge conservation arguments for it. We observe that, at least at the one-loop level, this vertex function does not have on-shell singularities so that the standard arguments should hold. In analogy with Eq. (4.5) we have

$$\begin{aligned} 2m_2 cos\theta &= m_1 \\ 2\omega_2 m_2^3 cos3\theta &= \omega_1 m_1^3 \end{aligned} \tag{4.8}$$

This time, at the classical level, from the first equation with $m_1 = \sqrt{6}$, $m_2 = \sqrt{2}$ we obtain $\theta = \pi/6$ which when substituted in the second equation again would require $\omega_1 = 0$. However, at the quantum level $\omega_1 \neq 0$, but also, *at the quantum level the masses receive radiative corrections and the mass ratio is no longer* $\sqrt{3}$. From the above value of ω_1 we can determine the mass corrections, and compare with the values computed in Ref. [8]

Indeed, taking variations of the second equation above and examining $O(\hbar)$ terms (on the left hand side we can use the classical value $\omega_2 = \frac{1}{2}$) we obtain

$$\delta\theta = -\sqrt{3}\delta\omega_1 \tag{4.9}$$

Taking variations of the first equation, we obtain then

$$\delta\left(\frac{m_1}{m_2}\right) = -\frac{\pi\hbar}{18} \tag{4.10}$$

a result in agreement with Eq. (2.4).

We have thus obtained a verification of our calculations, but more importantly, we can draw the conclusion that in situations such as this, if the higher spin charge of some particles gets corrections at the quantum level, this implies corresponding corrections for mass ratios and is the reason why no exact, factorizable S-matrices have been found. Presumably this is the case for many Toda theories based on nonsimply-laced Lie algebras.

5. Acknowledgments

We thank Silvia Penati for the collaboration on which some of the work presented here is based. We also thank Bernard de Wit for discussions and suggestions which were instrumental in our present understanding of the situation for nonsimply-laced algebras.

References

[1] M.D. Freeman, Phys. Lett. **261B** (1991) 57;
A. Fring, H.C. Liao and D.I. Olive, Phys. Lett. **B266** (1991) 82.

[2] A.V. Mikhailov, M.A. Olshanetsky and A.M. Perelomov, Comm. Math. Phys. **79** (1981) 473;
D. Olive and N. Turok, Nucl. Phys. **B257**[FS14] (1985) 277;
A. Bilal and J.L. Gervais, Nucl. Phys. **314** (1989) 646.

[3] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. (NY) **120** (1979) 253.

[4] A.E. Arinshtein, V.A. Fateev and A.B. Zamolodchikov, Phys. Lett. **87B** (1979) 389;
J.L. Cardy and G. Mussardo, Phys. Lett. **225B** (1989) 275;

P.G.O. Freund, T.R. Klassen and E. Melzer, Phys. Lett. **229B** (1989) 243;
P. Christe and G. Mussardo, Nucl. Phys. **B330** (1990) 465; Int. J. Mod. Phys. **A5** (1990) 4581:
C. Destri and H.J. de Vega, Phys. Lett. **233B** (1989) 336;
G. Mussardo and G. Sotkov, Santa Barbara preprint UCSBTH–89–64 (December 1989);
T.R. Klassen and E. Meltzer, Nucl. Phys. **B338** (1990) 485.

[5] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, Phys. Lett. **227B** (1989) 441; Nucl. Phys. **B338** (1990) 689.

[6] D. Bernard and A. LeClair, 'Quantum Group Symmetries and Non-Local Currents in 2D QFT', in Comm. Math. Phys. (to appear)

[7] M.A. Olshanetsky, Comm. Math. Phys. **88** (1983) 63.

[8] M.T. Grisaru, S. Penati and D. Zanon, Phys. Lett. **253B** (1991) 357.

[9] G.W. Delius, M.T. Grisaru, S. Penati and D. Zanon, Phys. Lett. **256B** (1991) 164; Nucl. Phys. **B359** (1991) 125; see also C. Destri, H.J. De Vega and V.A. Fateev, Phys. Lett **256B** (1991) 173.

[10] M.T. Grisaru, S. Penati and D. Zanon, "Threshold and anomalous threshold singularities in the $C^{(2)}(n+1)$ Toda theory", Nucl. Phys. **B** (to appear).

[11] S. Parke, Nucl. Phys. **B174** (1980) 166.

[12] S. Coleman and H.J. Thun, Comm. Math. Phys. **61** (1978) 31;
C.J. Goebel, Prog. Theor. Phys. Suppl. **86** (1986) 261.

[13] R. Shankar and E. Witten, Phys. Rev. **D17** (1978) 2134.

[14] S. Weinberg, Phys. Rev. **135** (1964) 1049;
S.N. Vergeles and V.M. Gryanik, Sov. J. Nucl. Phys **23** (1976) 704.

[15] P.P. Kulish and E.R. Nisimov, Theor. Math. Phys. **29** (1976) 922;
R. Flume, Phys. Lett. **62B** (1976) 93; Phys. Lett. **68B** (1977) 487;
J.H. Lowenstein and E.R. Speer, Comm. Math. Phys. **63** (1978) 97.